A. BARROS, A. BRASIL AND L. A. M. SOUSA JR.
KODAI MATH. J.
27 (2004), 45–56

A NEW CHARACTERIZATION OF SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR IN S^{n+p}

Abdênago Alves de Barros, Aldir Chaves Brasil Jr. and Luis Amancio Machado de Sousa Jr.

Abstract

In this work we will consider compact submanifold M^n immersed in the Euclidean sphere S^{n+p} with parallel mean curvature vector and we introduce a Schrödinger operator $L = -\Delta + V$, where Δ stands for the Laplacian whereas V is some potential on M^n which depends on n, p and h that are respectively, the dimension, codimension and mean curvature vector of M^n. We will present a gap estimate for the first eigenvalue μ_1 of L, by showing that either $\mu_1 = 0$ or $\mu_1 \le -n(1 + H^2)$. As a consequence we obtain new characterizations of spheres, Clifford tori and Veronese surfaces that extend a work due to Wu [W] for minimal submanifolds.

1. Introduction

Let M^n be a closed Riemannian manifold, i.e. M^n is compact without boundary, and denote by S^{n+p} the Euclidean sphere of sectional curvature one. For an immersion $\psi : M^n \to S^{n+p}$ we will denote by A its second fundamental form whereas h stands for its mean curvature vector and the mean curvature is defined by $H = |h|$. We introduce on M^n the traceless tensor $\Phi = A - hg$, where g stands for the induced metric on M and we consider $\Phi_h(X, Y) = \langle \Phi(X, Y), h \rangle$ for any tangent vector fields X, Y on M^n. It is easy to check that $|\Phi|^2 = |A|^2 - nH^2$. Moreover, $|\Phi|^2 = 0$ if, and only if, $\psi(M^n)$ is totally umbilic. Now we define constants $B_{p,h}$ and $\rho = \rho(n, p, h)$ as follows

$$B_{p,h} = \begin{cases} \frac{1}{(2-1/p)}, & \text{if } p = 1 \text{ or } h = 0 \\ \frac{1}{(2-1/(p-1))}, & \text{if } p \neq 1 \end{cases}$$

and

$$\rho = B_{p,h}\left\{ n(1 + H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} |\Phi_h| \right\}.$$

1991 *Mathematics Subject Classification*: [2000] Primary 53C42, 53A10.
Key words and phrases: Mean curvature vector, first eigenvalue, Clifford torus.
We would like to thank FINEP and FAPERJ for financial support.
Received November 11, 2002; revised August 19, 2003.

When $\psi : M^n \to S^{n+1}$ is a hypersurface H. Alencar and M. do Carmo [AC] have classified tori with constant mean curvature such that $|\Phi|^2 \le \rho$. They work was inspired by the ideas of the earlier papers due to J. Simons [Si], S. S. Chern, M. do Carmo and S. Kobayashi [CdCK] and B. Lawson [L]. For codimension bigger than one, supposing in addition that h is a parallel vector, W. Santos ([S], p. 405) and H. Xu ([X], p. 494) have generalized, independently, the work due to H. Alencar and M. do Carmo by showing that $|\Phi|^2 \le \rho$ implies either $|\Phi|^2 = 0$ or $|\Phi|^2 = \rho$. Moreover, they have described all such M^n by showing that M^n is a sphere in the first case and either one of the Clifford tori or one of the Veronese surface in the second case. On the other hand, introducing the Schrödinger operator

$$L = -\Delta - \left(2 - \frac{1}{p}\right)|A|^2,$$

where Δ stands for the Laplacian on M^n, C. Wu [W] has proved the following result concerning a minimal submanifold of S^{n+p}.

THEOREM 1 [C. Wu]. *Let M^n be an n-dimensional closed minimally immersed submanifold in a unit sphere S^{n+p} and let μ_1 be the first eigenvalue of L. If $\mu_1 \ge -n$ then either $\mu_1 = 0$ and M^n is totally geodesic, or $\mu_1 = -n$ and M^n is the Veronese surface in S^4 or the Clifford torus in S^{n+1}.*

The purpose of this paper is to extend the above result for closed submanifold in a unit sphere with parallel mean curvature vector. Before announcing our main result one introduces the following operator:

$$L_2 = -\Delta - B_{p,h}^{-1}|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|\Phi_h|.$$

We point out that $H = 0$ yields $L_2 = L$, where L is the operator considered by C. Wu on [W]. Taking into account this fact we generalize Wu's result according to the following theorem:

THEOREM 2. *Let M^n be a closed submanifold of S^{n+p} with mean curvature vector h parallel and let μ_1 be the first eigenvalue of L_2. Then either $\mu_1 = 0$ and M^n is totally umbilic, or $\mu_1 \le -n(1 + H^2)$. Moreover, $\mu_1 = -n(1 + H^2)$ if, and only if, $|\Phi|^2 = \rho$; in this case M^n is either the Veronese surface or the Clifford torus.*

2. Preliminaries

Throughout this section we will introduce some basic facts and notations that will appear on this paper. A Riemannian manifold of dimension k will be denoted by M^k. Now let M^n be a closed submanifold immersed in a unit Euclidean sphere S^{n+p}. We use the following standard convention of index:

$$1 \leq A, B, C, \ldots, \leq n+p, 1 \leq i, j, k, \ldots, \leq n, n+1 \leq \alpha, \beta, \gamma, \ldots, \leq n+p.$$

We consider an adapted orthonormal local frame $\{e_A\}$ and its associated connection forms $\{\omega_A\}$ on S^{n+p}. Restricting those forms to M we get

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, h_{ij}^\alpha = h_{ji}^\alpha, \tag{2.1}$$

$$A = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes \omega_\alpha \tag{2.2}$$

and

$$h = \frac{1}{n} \sum_{i,\alpha} h_{ii}^\alpha e_\alpha. \tag{2.3}$$

If R_{ijkl} and $R_{\alpha\beta kl}$ stand for the tensor of curvature and normal curvature, respectively, then Gauss, Ricci and Codazzi equations can be read, respectively, as follows:

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha); \tag{2.4}$$

$$R_{\alpha\beta kl} = \sum_{i,j} (h_{ik}^\alpha h_{jl}^\beta - h_{il}^\alpha h_{jk}^\beta) \tag{2.5}$$

and

$$h_{ijk}^\alpha = h_{ikj}^\alpha \tag{2.6}$$

On the other hand, the traceless tensor Φ previously considered can be given by

$$\Phi = \sum_{i,j,\alpha} \Phi_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\Phi_{ij}^\alpha = h_{ij}^\alpha - (1/n) \operatorname{tr} H_\alpha \delta_{ij}$ and $H_\alpha = (h_{ij}^\alpha)$. Denoting by $N(T)$ the squared of the norm of a symmetric operator T, we have $N(A) = |A|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$, whereas $N(\Phi) = |\Phi|^2 = \sum_\alpha \operatorname{tr}(\Phi_\alpha^2) = |A|^2 - nH^2$. We note also that the gradient of Φ, denoted by $\nabla\Phi$, verifies

$$|\nabla\Phi|^2 = \sum_{i,j,\alpha} |\nabla\Phi_{ij}^\alpha|^2 = \sum_{i,j,k,\alpha} (\Phi_{ijk}^\alpha)^2, \tag{2.7}$$

while the gradient of $|\Phi|^2$ satisfies the following identity:

$$|\nabla|\Phi|^2|^2 = 4 \sum_k \left(\sum_{i,j,\alpha} \Phi_{ij}^\alpha \Phi_{ijk}^\alpha \right)^2 \tag{2.8}$$

We will use also the following notation $\langle \Delta\Phi, \Phi \rangle = \sum_{i,j,\alpha} \Phi_{ij}^{\alpha} \Delta \Phi_{ij}^{\alpha}$, which gives

$$\frac{1}{2}\Delta|\Phi|^2 = \langle \Delta\Phi, \Phi \rangle + |\nabla\Phi|^2. \tag{2.9}$$

3. Proof of Theorem 2

In order to show our theorem we will need some auxiliary results. At first we will show two general lemmas. The first one can be read as follows:

Lemma 1. *Let M^n be a Riemannian manifold isometrically immersed into a Riemannian manifold N^{n+p}. Consider $\Psi = \sum_{i,j,\alpha} \Psi_{ij}^{\alpha} \omega_i \otimes \omega_j \otimes e_\alpha$ a traceless symmetric tensor satisfying Codazzi equation. Then the following inequality holds*

$$|\nabla|\Psi|^2|^2 \leq \frac{4n}{(n+2)}|\Psi|^2|\nabla\Psi|^2,$$

where $|\Psi|^2 = \sum_{i,j,\alpha}(\Psi_{ij}^{\alpha})^2$ and $|\nabla\Psi|^2 = \sum_{i,j,k,\alpha}(\Psi_{ijk}^{\alpha})^2$. In particular the conclusion holds for the tensor Φ defined in the introduction.

Proof. First we fix e_α and define Ψ^α as the α-component of Ψ. Now we take an orthonormal frame $\{e_i^\alpha\}$ of eigenfunctions of Ψ^α with correspondent eigenvalues μ_i^α. Hence we have

$$|\nabla|\Psi^\alpha|^2|^2 = 4\sum_k \left(\sum_{i,j} \Psi_{ij}^{\alpha}\Psi_{ijk}^{\alpha}\right)^2 = 4\sum_k \left(\sum_i \mu_i^\alpha \Psi_{iik}^{\alpha}\right)^2. \tag{3.1}$$

By using Cauchy-Schwarz inequality we have

$$|\nabla|\Psi^\alpha|^2|^2 \leq 4\sum_i (\mu_i^\alpha)^2 \sum_{i,k}(\Psi_{iik}^{\alpha})^2.$$

This can be rewritten as

$$4|\Psi^\alpha|^2\left(\sum_i (\Psi_{iii}^{\alpha})^2 + \sum_{i,k,i\neq k}(\Psi_{iik}^{\alpha})^2\right) \geq |\nabla|\Psi^\alpha|^2|^2. \tag{3.2}$$

Now we fix an index i. Taking into account that $\mathrm{tr}(\Psi^\alpha) = 0$, we conclude that $\Psi_{iii}^{\alpha} = -\sum_{k,k\neq i}\Psi_{kki}^{\alpha}$. By using Cauchy-Schwarz inequality again we have

$$\sum_i (\Psi_{iii}^{\alpha})^2 = \sum_i \left(\sum_{k,k\neq i}\Psi_{kki}^{\alpha}\right)^2 \leq (n-1)\sum_{k,i,k\neq i}(\Psi_{iik}^{\alpha})^2. \tag{3.3}$$

Hence we obtain from inequalities (3.2) and (3.3) that

$$|\nabla|\Psi^\alpha|^2|^2 \leq 4n|\Psi^\alpha|^2 \sum_{i,k,i\neq k}(\Psi_{iik}^{\alpha})^2. \tag{3.4}$$

On the other hand $\Psi_{ik}^{\alpha} = \Psi_{ki}^{\alpha}$ implies $\Psi_{iki}^{\alpha} = \Psi_{kii}^{\alpha}$. In view of Codazzi equation we obtain

$$\Psi_{iik}^{\alpha} = \Psi_{iki}^{\alpha} = \Psi_{kii}^{\alpha}. \tag{3.5}$$

Since $|\nabla\Psi^{\alpha}|^2 = \sum_{i,j,k}(\Psi_{ijk}^{\alpha})^2$ and

$$\sum_{i,j,k}(\Psi_{ijk}^{\alpha})^2 = \sum_{i}(\Psi_{iii}^{\alpha})^2 + \sum_{i,k;i\neq k}((\Psi_{iik}^{\alpha})^2 + (\Psi_{iki}^{\alpha})^2 + (\Psi_{kii}^{\alpha})^2) + 6\sum_{i<j<k}(\Psi_{ijk}^{\alpha})^2$$

we may use (3.5) to conclude

$$|\Psi^{\alpha}|^2|\nabla\Psi^{\alpha}|^2 = |\Psi^{\alpha}|^2\left(\sum_{i}(\Psi_{iii}^{\alpha})^2 + 3\sum_{i,k,i\neq k}(\Psi_{iik}^{\alpha})^2 + 6\sum_{i<j<k}(\Psi_{ijk}^{\alpha})^2\right).$$

It follows from this last equation the next inequality

$$|\Psi^{\alpha}|^2|\nabla\Psi^{\alpha}|^2 \geq 2|\Psi^{\alpha}|^2\sum_{i,k,i\neq k}(\Psi_{iik}^{\alpha})^2 + |\Psi^{\alpha}|^2\left(\sum_{i,k,i\neq k}(\Psi_{iik}^{\alpha})^2 + \sum_{i}(\Psi_{iii}^{\alpha})^2\right).$$

Combining the first term of the right hand side of this last inequality with (3.4) and the second term with (3.2) we derive

$$|\Psi^{\alpha}|^2|\nabla\Psi^{\alpha}|^2 \geq \frac{1}{2n}|\nabla|\Psi^{\alpha}|^2|^2 + \frac{1}{4}|\nabla|\Psi^{\alpha}|^2|^2.$$

Since $|\nabla|\Psi|^2|^2 = \sum_{\alpha}|\nabla|\Psi^{\alpha}|^2|^2$ and $|\Psi^{\alpha}|^2 \leq |\Psi|^2$ it follows from the last inequality that

$$|\nabla|\Psi|^2|^2 \leq \frac{4n}{n+2}|\Psi|^2|\nabla\Psi|^2,$$

which finishes the proof of the Lemma 1.

Now we consider the differentiable function $f_{\varepsilon} = (|\Phi|^2 + \varepsilon)^{1/2}$ defined on M^n, where Φ is the traceless tensor previously defined in the introduction, ε is a positive number and we prove the following lemma concerning this function.

Lemma 2. *Let M^n be a Riemannian manifold immersed in S^{n+p} and let f_{ε} be the function above defined. Then the Laplacian of f_{ε} satisfies the inequality*

$$f_{\varepsilon}\Delta f_{\varepsilon} \geq \frac{2(|\Phi|^2 + \varepsilon)^{-1}}{(n+2)}|\Phi|^2|\nabla\Phi|^2 + \langle\Delta\Phi, \Phi\rangle.$$

Proof. Since $\Delta f_{\varepsilon} = \operatorname{div}(\nabla f_{\varepsilon})$ and $\nabla f_{\varepsilon} = (((|\Phi|^2 + \varepsilon)^{-1/2}/2)(\nabla|\Phi|^2)$ we have

$$f_{\varepsilon}\Delta f_{\varepsilon} = \frac{1}{2}\Delta|\Phi|^2 - \frac{(|\Phi|^2 + \varepsilon)^{-1}}{4}\langle\nabla|\Phi|^2, \nabla|\Phi|^2\rangle.$$

Using (2.9), we may conclude

$$f_\varepsilon \Delta f_\varepsilon = (|\Phi|^2 + \varepsilon)^{-1}\left(|\nabla\Phi|^2(|\Phi|^2 + \varepsilon) - \frac{1}{4}|\nabla|\Phi|^2|^2\right) + \langle \Delta\Phi, \Phi\rangle. \tag{3.6}$$

On the other hand, Lemma 1 yields $(1/4)|\nabla|\Phi|^2|^2 \leq (n/(n+2))|\Phi|^2|\nabla\Phi|^2$. Since $|\Phi|^2 + \varepsilon \geq |\Phi|^2$ we have

$$\left(|\nabla\Phi|^2(|\Phi|^2 + \varepsilon) - \frac{1}{4}|\nabla|\Phi|^2|^2\right) \geq |\Phi|^2|\nabla\Phi|^2\left(1 - \frac{n}{(n+2)}\right),$$

that is

$$\left(|\nabla\Phi|^2(|\Phi|^2 + \varepsilon) - \frac{1}{4}|\nabla|\Phi|^2|^2\right) \geq \frac{2|\Phi|^2|\nabla\Phi|^2}{(n+2)}. \tag{3.7}$$

Putting together equations (3.6) and (3.7) we have

$$f_\varepsilon \Delta f_\varepsilon \geq \frac{2(|\Phi|^2 + \varepsilon)^{-1}}{(n+2)}(|\Phi|^2|\nabla\Phi|^2) + \langle \Delta\Phi, \Phi\rangle,$$

which finishes the proof of the Lemma 2.

Now let L_2 be the Schrödinger operator considered in the introduction. We will prove the next proposition concerning to the first eigenvalue of L_2, which extends a result derived by C. Wu [W] in the minimal case.

Proposition 1. *Let M^n be a closed submanifold immersed in S^{n+p} with parallel mean curvature vector h in such way that M^n is not totally umbilic. If μ_1 is the first eigenvalue of L_2 then*

$$\mu_1 \leq -n(1 + H^2) - \frac{2}{(n+2)}\frac{\int_M |\nabla\Phi|^2 * 1}{\int_M |\Phi|^2 * 1},$$

where $*1$ stands for the form of volume of M^n.

Proof. If we define the set $\Gamma = \{f \in C^\infty(M) : f \neq 0\}$ then Rayleigh quotient yields $\mu_1 = \inf_{f \in \Gamma}(\int_M f L_2 f * 1/\int_M f^2 * 1)$. We consider now $f_\varepsilon = (|\Phi|^2 + \varepsilon)^{1/2}$ the differentiable function given in the previous lemma. Since M^n is not totally umbilic we get $\lim_{\varepsilon\to 0}\int_M f_\varepsilon^2 * 1 = \int_M |\Phi|^2 * 1 > 0$. Thus we may use f_ε as a test function to compute μ_1. On the other hand, since h is parallel W. Santos ([S], p. 405) has showed the following inequality

$$\langle \Delta\Phi, \Phi\rangle \geq \Lambda|\Phi|^2, \tag{3.8}$$

where $\Lambda = \{n(1 + H^2) - (n(n-2)/\sqrt{n(n-1)})|\Phi_h| - B_{p,h}^{-1}|\Phi|^2\}$.

Therefore we may combine Lemma (2) and inequality (3.8) to obtain

$$f_\varepsilon \Delta f_\varepsilon \geq \frac{2}{(n+2)}(|\Phi|^2 + \varepsilon)^{-1}|\Phi|^2|\nabla\Phi|^2 + \Lambda|\Phi|^2.$$

Since $f_\varepsilon L_2 f_\varepsilon = -f_\varepsilon \Delta f_\varepsilon - B_{p,h}^{-1}|\Phi|^2 f_\varepsilon^2 - (n(n-2)/\sqrt{n(n-1)})|\Phi_h| f_\varepsilon^2$, we get

$$f_\varepsilon L_2 f_\varepsilon \leq -\frac{2(|\Phi|^2+\varepsilon)^{-1}(|\Phi|^2|\nabla\Phi|^2)}{(n+2)} - n(1+H^2)|\Phi|^2 + \frac{n(n-2)|\Phi|^2|\Phi_h|}{\sqrt{n(n-1)}} + B_{p,h}^{-1}(|\Phi|^2)^2 - \frac{n(n-2)|\Phi_h|}{\sqrt{n(n-1)}}(|\Phi|^2+\varepsilon) - B_{p,h}^{-1}(|\Phi|^2+\varepsilon)|\Phi|^2. \tag{3.9}$$

From where we obtain

$$f_\varepsilon L_2 f_\varepsilon \leq -n(1+H^2)|\Phi|^2 - \frac{2(|\Phi|^2+\varepsilon)^{-1}}{(n+2)}|\Phi|^2|\nabla\Phi|^2. \tag{3.10}$$

Since $\mu_1 \leq (\int_M f_\varepsilon L_2 f_\varepsilon * 1/\int_M f_\varepsilon^2 * 1)$ and H is constant inequality (3.10) yields

$$\mu_1 \leq -n(1+H^2)\frac{\int_M |\Phi|^2 * 1}{\int_M (|\Phi|^2+\varepsilon) * 1} - \frac{2}{(n+2)}\frac{\int_M (|\Phi|^2+\varepsilon)^{-1}|\Phi|^2|\nabla\Phi|^2 * 1}{\int_M (|\Phi|^2+\varepsilon) * 1}.$$

Making $\varepsilon \to 0$ on the last inequality, we obtain

$$\mu_1 \leq -n(1+H^2) - \frac{2}{(n+2)}\frac{\int_M |\nabla\Phi|^2 * 1}{\int_M |\Phi|^2 * 1},$$

which completes the proof of the desired result.

On the next proposition we consider the case when M^n is not pseudo-umbilical. By a pseudo-umbilical submanifold M^n into S^{n+p} we mean that h is an umbilic direction of the second fundamental form A of M^n. Now let $L_3 = -\Delta - (n/2\sqrt{n-1})|A|^2$ be a new Schrödinger operator and let us prove an estimate concerning to its first eigenvalue according to the following proposition.

Proposition 2. *Let M^n be a closed submanifold immersed in S^{n+p} with parallel non null mean curvature vector h in such way that M^n is not pseudo-umbilical. If μ_1 is the first eigenvalue of L_3 then*

$$\mu_1 \leq -n - \frac{2}{(n+2)}\frac{\int_M |\nabla\Phi^{n+1}|^2 * 1}{\int_M |\Phi^{n+1}|^2 * 1},$$

where $\Phi^{n+1} = \Phi_{ij}^{n+1} e_{n+1}$, $\Phi_{ij}^{n+1} = (h_{ij}^{n+1} - H\delta_{ij})$ and $e_{n+1} = h/H$.

Proof. The proof is similar to the previous proposition. Indeed, let us consider $|\nabla(\Phi^{n+1})|^2 = \sum_{i,j,k}(\Phi_{ijk}^{n+1})^2$ and $\langle\Delta(\Phi^{n+1}), \Phi^{n+1}\rangle = \sum_{i,j}\Phi_{ij}^{n+1}\Delta\Phi_{ij}^{n+1}$. Now it is enough to define $g_\varepsilon = (|\Phi^{n+1}|^2+\varepsilon)^{1/2}$ and to proceed as before. Following the same computation as that one of the Lemma 2 we have

$$g_\varepsilon \Delta g_\varepsilon = |\nabla(\Phi^{n+1})|^2 - \frac{1}{4}(|\Phi^{n+1}|^2 + \varepsilon)^{-1} |\nabla|\Phi^{n+1}|^2|^2 + \langle \Delta(\Phi^{n+1}), \Phi^{n+1} \rangle. \tag{3.11}$$

On the other hand a similar result like that one of Lemma 1 is also true for Φ^{n+1}, that is,

$$|\nabla|\Phi^{n+1}|^2|^2 \le \frac{4n}{(n+2)} |\Phi^{n+1}|^2 |\nabla(\Phi^{n+1})|^2. \tag{3.12}$$

From (3.11) and (3.12) it follows that

$$g_\varepsilon \Delta g_\varepsilon \ge \frac{2(|\Phi^{n+1}|^2 + \varepsilon)^{-1}}{(n+2)} |\Phi^{n+1}|^2 |\nabla(\Phi^{n+1})|^2 + \langle \Delta(\Phi^{n+1}), \Phi^{n+1} \rangle. \tag{3.13}$$

Since $\operatorname{tr} H_{n+1} = \sum_i h_{ii}^{n+1} = nH$ and H is constant we have $\sum_i \Delta h_{ii}^{n+1} = 0$. This yields

$$\sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} = \sum_{i,j} \Phi_{ij}^{n+1} \Delta \Phi_{ij}^{n+1} + H \sum_i \Delta h_{ii}^{n+1} = \langle \Delta(\Phi^{n+1}), \Phi^{n+1} \rangle. \tag{3.14}$$

We use also the following inequality obtained by Z. Hou ([H], p. 39)

$$\sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} \ge n|\Phi^{n+1}|^2 \left(1 - \frac{|A|^2}{2\sqrt{n-1}}\right). \tag{3.15}$$

From (3.13), (3.14) and (3.15) we have

$$g_\varepsilon \Delta g_\varepsilon \ge |\Phi^{n+1}|^2 \left(\frac{2(|\Phi^{n+1}|^2 + \varepsilon)^{-1}}{(n+2)} |\nabla(\Phi^{n+1})|^2 + n - \frac{n|A|^2}{2\sqrt{n-1}} \right).$$

Since $g_\varepsilon L_3 g_\varepsilon = -g_\varepsilon \Delta g_\varepsilon - (n/2\sqrt{(n-1)})|A|^2 g_\varepsilon^2$ we obtain

$$\begin{aligned} g_\varepsilon L_3 g_\varepsilon \le{} & -n|\Phi^{n+1}|^2 + \frac{n|A|^2 |\Phi^{n+1}|^2}{2\sqrt{n-1}} - \frac{n|A|^2}{2\sqrt{n-1}} (|\Phi^{n+1}|^2 + \varepsilon) \\ & - \frac{2(|\Phi^{n+1}|^2 + \varepsilon)^{-1}}{(n+2)} (|\Phi^{n+1}|^2 |\nabla(\Phi^{n+1})|^2), \end{aligned}$$

that is,

$$g_\varepsilon L_3 g_\varepsilon \le -n|\Phi^{n+1}|^2 - \frac{2(|\Phi^{n+1}|^2 + \varepsilon)^{-1}}{(n+2)} (|\Phi^{n+1}|^2 |\nabla(\Phi^{n+1})|^2).$$

Since M^n is not pseudo-umbilical $\lim_{\varepsilon \to 0} \int_M g_\varepsilon^2 * 1 = \int_M |\Phi^{n+1}|^2 > 0$. Therefore using again the characterization of μ_1 given by Rayleigh quotient we obtain

$$\mu_1 \le -\frac{2}{(n+2)} \frac{\int_M |\Phi^{n+1}|^2 (n(n+2)/2 + (|\Phi^{n+1}|^2 + \varepsilon)^{-1} |\nabla(\Phi^{n+1})|^2) * 1}{\int_M (|\Phi^{n+1}|^2 + \varepsilon) * 1}.$$

Making $\varepsilon \to 0$ in the last inequality we have

$$\mu_1 \le -n - \frac{2}{(n+2)} \frac{\int_M |\nabla(\Phi^{n+1})|^2 * 1}{\int_M |\Phi^{n+1}|^2 * 1},$$

which concludes the proof of the proposition.

We consider now the case when M^n is pseudo-umbilical and has codimension $p \ge 2$. Introducing the Schrödinger operator $L_4 = -\Delta - (3/2)|\Phi|^2$ we derive the following proposition.

Proposition 3. *Let M^n be a closed submanifold immersed in S^{n+p} such that M^n is pseudo-umbilical with parallel mean curvature vector h. If M^n is not totally umbilic, $p \ge 2$ and μ_1 is the first eigenvalue of L_4, then*

$$\mu_1 \le -n(1+H^2) - \frac{2}{(n+2)} \frac{\int_M |\nabla\Phi|^2 * 1}{\int_M |\Phi|^2 * 1}.$$

Proof. Taking into account that M^n is pseudo-umbilical we may use the following inequality due to Hou ([H], p. 42)

$$\langle \Delta\Phi, \Phi \rangle \ge |\Phi|^2 \left(n(1+H^2) - \frac{3}{2}|\Phi|^2 \right). \tag{3.16}$$

Therefore considering again $f_\varepsilon = (|\Phi|^2 + \varepsilon)^{1/2}$ the Lemma 2 yields

$$f_\varepsilon \Delta f_\varepsilon \ge |\Phi|^2 \left(\frac{2(|\Phi|^2+\varepsilon)^{-1}|\nabla\Phi|^2}{(n+2)} + n(1+H^2) - \frac{3}{2}|\Phi|^2 \right).$$

Since $f_\varepsilon L_4 f_\varepsilon = -f_\varepsilon \Delta f_\varepsilon - (3/2)|\Phi|^2 f_\varepsilon^2$ we get

$$f_\varepsilon L_4 f_\varepsilon \le -n(1+H^2)|\Phi|^2 - \frac{2(|\Phi|^2+\varepsilon)^{-1}}{(n+2)} |\Phi|^2 |\nabla\Phi|^2. \tag{3.17}$$

On the other hand since M^n is not totally umbilic we have

$$\lim_{\varepsilon \to 0} \int_M f_\varepsilon^2 * 1 = \int_M |\Phi|^2 * 1 > 0.$$

Hence we may use f_ε as a test function to estimate μ_1. Taking into account that $\mu_1 \le \int_M f_\varepsilon L_4 f_\varepsilon * 1 / \int_M f_\varepsilon^2 * 1$ and H is constant we derive from (3.17) that

$$\mu_1 \le -n(1+H^2) - \frac{2}{(n+2)} \frac{\int_M |\nabla\Phi|^2 * 1}{\int_M |\Phi|^2 * 1},$$

which completes the proof of the Proposition 3.

We point out now that to derive the Theorem 2 it is enough to apply the Proposition 1 with the result obtained independently by W. Santos and H. Xu. In fact, from that proposition we get $\mu_1 = 0$ if, and only if, M^n is totally umbilic, otherwise $\mu_1 \leq -n(1+H^2)$. Suppose now that $|\Phi|^2 = \rho \neq 0$. Then $L_2 = -\Delta - n(1+H^2)$ and $\mu_1 = -n(1+H^2)$. On the other hand, it follows again from Proposition 1 that $|\nabla\Phi| = 0$ provided $\mu_1 = -n(1+H^2)$. Therefore Φ and $|\Phi_h|$ are constants. Hence we conclude that

$$\mu_1 = -(B_{p,h})^{-1}|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|\Phi_h|.$$

From where we have $|\Phi|^2 = \rho$. In order to complete the rest of the proof of the theorm we may apply a theorem due to Santos ([S], p. 405) or Xu ([X], p. 494) presented in the introduction that describes all submanifolds M^n immersed in the Euclidean sphere S^{n+p} with parallel mean curvature vector h and $|\Phi|^2 = \rho$. More precisely, they have proved:

> If $\rho = 0$ then M^n is a sphere, otherwise M^n is either one of the Clifford tori or one of the Veronese surfaces in S^{n+p}.

4. Applications

In this section we will present two applications of our main theorem. We point out now that $|\Phi|^2 = \rho$ has two main consequences: either $L_2 = -\Delta$, or $L_2 = -\Delta - n(1+H^2)$. In fact, the former case comes from $\rho = 0$, whereas the last one comes from $\rho \neq 0$. Hence, we may derive from the theorem due to Santos ([S]) or Xu ([X]) and the Theorem 2 the following theorem:

THEOREM 3. *Let M^n be a closed submanifold of S^{n+p}, $p \geq 2$, with non null parallel mean curvature vector h and let $L_3 = -\Delta - (n|A|^2/2\sqrt{n-1})$ be the operator with first eigenvalue μ_1. If $n \geq 3$ and M^n is not pseudo-umbilical then $\mu_1 \leq -n$. Moreover, $\mu_1 = -n$ if, and only if, M^n is the Clifford torus $S^1(r) \times S^{n-1}(s) \hookrightarrow S^{n+1} \hookrightarrow S^{n+p}$, with $s^2 = \sqrt{n-1}(1+\sqrt{n-1})^{-1}$ and $r^2 = (1+\sqrt{n-1})^{-1}$. If $n = 2$ then M^2 is a totally umbilical sphere $S^2(1/(1+H^2))$.*

Proof. By using Proposition 2 we infer that if M^n is not pseudo-umbilical then $\mu_1 \leq -n$. We note that $|A|^2 = 2\sqrt{n-1}$ implies $L_3 = -\Delta - n$. From where we conclude $\mu_1 = -n$. Conversely, if $\mu_1 = -n$, Proposition 2 shows that $|\nabla(\Phi^{n+1})| = 0$. Hence $|\Phi^{n+1}|^2$ and $|A^{n+1}|^2 = |\Phi^{n+1}|^2 + nH^2$ are constants. Using (3.15) and the assumption $|A^{n+1}|^2$ is constant we derive

$$|A|^2 \geq 2\sqrt{n-1}.$$

In fact, since $(1/2)\Delta|A^{n+1}|^2 = \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1}$ we have

$$0 = \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \geq n|\Phi^{n+1}|^2\left(1 - \frac{|A|^2}{2\sqrt{n-1}}\right),$$

which gives the desired inequality.

On the other hand if $\Gamma = \{f \in C^\infty(M) : f \neq 0\}$ then Rayleigh quotient yields $\mu_1 = \inf_{f\in\Gamma}(\int_M fL_3 f * 1/\int_M f^2 * 1)$. Since $|A|^2 \geq 2\sqrt{n-1}$ and $\mu_1 = -n$ we conclude that $|A|^2 = 2\sqrt{n-1}$. Thus, if $n \geq 3$ then M^n is isometric to a Clifford torus according to a result due to Hou ([H], p. 40).

For $n = 2$ the same result yields that M^2 is a totally umbilical sphere in S^{2+p}. This completes the proof of the theorem.

Finally we treat the case when M^n is pseudo-umbilical with parallel mean curvature vector. More precisely we have the following theorem.

Theorem 4. *Let M^n be a closed submanifold immersed in S^{n+p} with parallel mean curvature vector h and $p \geq 2$. Suppose in addition that M^n is also pseudo-umbilical and let μ_1 be the first eigenvalue of $L_4 = -\Delta - (3/2)|\Phi|^2$. If M^n is totally umbilical, then $\mu_1 = 0$. Otherwise $\mu_1 \leq -n(1+H^2)$. Furthermore, if $\mu_1 = -n(1+H^2)$ we have*: a) *Either M^n is the Clifford torus*

$$S^k(r) \times S^{n-k}(s) \hookrightarrow S^{n+1}\left(\frac{1}{\sqrt{1+H^2}}\right) \hookrightarrow S^{n+2} \hookrightarrow S^{n+p}$$

b) *Or else, M^n is the Veronese surface $M^2 \hookrightarrow S^4(1/\sqrt{1+H^2}) \hookrightarrow S^5 \hookrightarrow S^{n+p}$.*

Proof. From Proposition 3 we get $\mu_1 = 0$ if, and only if, M^n is totally umbilic, otherwise $\mu_1 \leq -n(1+H^2)$. Now suppose $|\Phi|^2 = (2/3)n(1+H^2)$, then the operator L_3 becomes $L_3 = -\Delta - n(1+H^2)$ while $\mu_1 = -n(1+H^2)$. Conversely, if $\mu_1 = -n(1+H^2)$ it follows from Proposition 3 that $|\nabla\Phi|^2 = 0$. Hence Φ is constant, and so, in view of (3.16) we obtain $|\Phi|^2 = (2/3)n(1+H^2)$. Now the conclusion of the theorem is a consequence of the Proposition 2 of Z. Hou ([H], p. 42).

Acknowledgement. The authors would like to thank the referee for many valuable suggestions.

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Departamento de Matemática
UFC-Fortaleza-CE-Br-60455-760
e-mail: abbarros@mat.ufc.br
http://www.mat.ufc.br/posgrad.html/

Departamento de Matemática
UFC-Fortaleza-CE-Br-60455-760
e-mail: aldir@mat.ufc.br
http://www.mat.ufc.br/posgrad.html/

Departamento de Matemática
e Estatística-UNIRIO-Rio
de Janeiro-RJ-Br-22290-240
e-mail: amancio@impa.br