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FACTORIZATION OF THE POLAR CURVE AND THE NEWTON POLYGON

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Abstract

Using the Newton polygon we prove a factorization theorem for the local polar curves. Then we give some applications to the polar invariants and pencils of plane curve singularities.

Introduction

Let $\mathbf{C}\{X, Y\}$ be the ring of complex power series in two variables X, Y. We denote by $\operatorname{ord} f$ and $\operatorname{in} f$ respectively the order and the initial form of a nonzero power series $f \in \mathbf{C}\{X, Y\}$. By definition $\operatorname{ord} 0 = +\infty$ and $\operatorname{in} 0 = 0$. Let f be a nonzero power series without constant term. If $f = f_1^{m_1} \cdots f_r^{m_r}$ is a decomposition of f into irreducible pairwise different factors $f_i \in \mathbf{C}\{X, Y\}$ then we put $f_{\text{red}} = f_1 \cdots f_r$. Let $t(f) = \operatorname{ord}(\operatorname{in} f)_{\text{red}}$. Then $t(f)$ is the number of tangents to the local curve $f = 0$. In the sequel we use the convention that a sum (resp. a product) over the empty set equals zero (resp. one).

Write

$$\operatorname{in} f = (\text{a monomial}) \prod_{i=1}^{s} (X - c_i Y)^{m_i}$$

with pairwise different c_i. We put

$$d(f) = \sum_{i=1}^{s} (m_i - 1)$$

and call $d(f)$ *degeneracy* of f. If $s = 0$ then $\operatorname{in} f$ reduces to a monomial and $d(f) = 0$. Note that $d(f) = 0$ if and only if all tangents to $f = 0$ different from the axes are of multiplicity 1.

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Assume that f has an isolated singularity at $(0,0) \in \mathbf{C}^2$ (this is equivalent to the conditions $\operatorname{ord} f > 1$ and $f = f_{\mathrm{red}}$) and suppose that the line $bX - aY = 0$ is not tangent to $f = 0$. The generic polar of f is by definition the series $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$.

Let us consider the factorization $\partial f = \prod_{i=1}^{u} h_i$ with irreducible $h_i \in \mathbf{C}\{X,Y\}$ and put $(f,h)_0 = \dim_{\mathbf{C}} \mathbf{C}\{X,Y\}/(f,h)$. According to Teissier [Te1] the quotients $(f,h_i)_0/\operatorname{ord} h_i$ are called polar invariants of the singularity f. The multiplicity m_q of the polar quotient q is defined to be $m_q = \sum_{i \in I_q} \operatorname{ord} h_i$ where $I_q = \{i : (f,h_i)_0/\operatorname{ord} h_i = q\}$.

Teissier's collection $\{(q, m_q)\}$ of polar invariants and their multiplicities is a topological invariant of the singularity (see [Te1] and [Te2]). There are several theorems on the factorization of the polar curve that enable calculation of Teissier's collection (see [M], [D], [G], [LP]). The aim of this note is to study the factorization of the polar curve ∂f associated with the Newton polygon $\mathcal{N}_f$ of f. The main result (Theorem 1.1) is a refinement of the factorization theorem given in [LP]. Using our theorem we calculate the minimal polar invariant (Theorem 2.1) and prove a bound on the number of special values of the pencil $(f - tl^N : t \in \mathbf{C})$ (Theorem 3.2). This bound is analogous to the estimation due to Le Van Thanh and Mutsuo Oka (see [LO], Main result) given in the global affine context.

1. Main result

Let $f \in \mathbf{C}\{X,Y\}$ be a nonzero power series without constant term. Write $f = \sum c_{\alpha\beta} X^{\alpha} Y^{\beta} \in \mathbf{C}\{X,Y\}$ and $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbf{N}^2 : c_{\alpha\beta} \neq 0\}$. The Newton polygon $\mathcal{N}_f = \mathcal{N}(f)$ is the set of the compact faces of the boundary of the convex hull $\Delta(f)$ of the set $\operatorname{supp} f + \mathbf{N}^2$. We call $\Delta(f)$ the Newton diagram of f. For every $S \in \mathcal{N}_f$ we denote by $|S|_1$ and $|S|_2$ the lenghts of the projection of S on the horizontal and vertical axes. We call $|S|_1/|S|_2$ *inclination* of the segment S. The power series f is elementary if $\mathcal{N}_f$ contains only one segment with vertices on the axes. Let $\|S\| = \min\{|S|_1, |S|_2\}$ and denote a_S, b_S the distances from S to the axes. Thus the vertices of S are $(a_S, |S|_2 + b_S)$ and $(|S|_1 + a_S, b_S)$. Let $\alpha/\alpha(S) + \beta/\beta(S) = 1$ be the equation of the line containing S. Clearly $\alpha(S), \beta(S)$ are rational numbers and $\alpha(S)/\beta(S) = |S|_1/|S|_2$. A segment $S \in \mathcal{N}_f$ is *exceptional* if $1 = |S|_1 < |S|_2$ and $a_S = 0$ or $1 = |S|_2 < |S|_1$ and $b_S = 0$. A segment $S \in \mathcal{N}_f$ (necessarily unique) is principal if $|S|_1 = |S|_2$. We set $\mathcal{N}_f^* = \mathcal{N}_f \backslash \{\text{exceptional segments}\}$ and $\mathcal{N}_f^{**} = \mathcal{N}_f^* \backslash \{\text{principal segment}\}$. For every segment $S \in \mathcal{N}_f^*$ we define $\varepsilon(S) \in \{-1, 0, 1\}$ by putting $\varepsilon(S) = -1$ if $|S|_1 < |S|_2$ and $a_S = 0$ or $|S|_2 < |S|_1$ and $b_S = 0$. If $|S|_1 = |S|_2$ then $\varepsilon(S) = 1 - (\text{number of vertices of } S \text{ lying on the axes})$. We put $\varepsilon(S) = 0$ for all remaining cases. A segment $S \in \mathcal{N}_f^{**}$ is of the first kind if $\varepsilon(S) = 0$, it is of the second kind if $\varepsilon(S) = -1$.

Let $\operatorname{in}(f,S) = \sum_{(\alpha,\beta) \in S} X^{\alpha} Y^{\beta}$. Clearly $X^{a_S} Y^{b_S}$ is the monomial of the highest degree dividing $\operatorname{in}(f,S)$. Thus we can write $\operatorname{in}(f,S) = X^{a_S} Y^{b_S} \operatorname{in}(f,S)^{\circ}$ in $\mathbf{C}\{X,Y\}$. Note that $\mathcal{N}(\operatorname{in}(f,S)^{\circ}) = \{S'\}$ where S' is the segment with vertices $(|S|_1, 0)$ and $(0, |S|_2)$. We define the degeneracy $d(f,S)$ of f on S by putting $d(f,S) = \operatorname{ord}\operatorname{in}(f,S)^\circ - \operatorname{ord}\operatorname{in}(f,S)^\circ_{\mathrm{red}}$. Note that $d(f,S) = 0$ if and only if f is nondegenerate on S that is if the polynomial $\operatorname{in}(f,S)$ has no critical points in the set $(\mathbf{C}\backslash\{0\}) \times (\mathbf{C}\backslash\{0\})$. Recall that a series is *nondegenerate* if it is nondegenerate on every segment of its Newton polygon. If $S \in \mathcal{N}_f^{**}$ is of the second kind then we let $v_S = X$ if $|S|_1 < |S|_2$ and $v_S = Y$ if $|S|_2 < |S|_1$. Let S be a segment of a Newton polygon. We call a power series S-elementary if it is elementary and its unique segment is parallel to S. A line $l \subset \mathbf{R}^2$ is a barrier of $\Delta(f)$ if it has an equation $v_1\alpha + v_2\beta = w$ where $v_1, v_2, w > 0$ are integers such that $v_1\alpha + v_2\beta \geq w$ for $(\alpha,\beta) \in \Delta(f)$ with equality for at least one point $(\alpha,\beta) \in \Delta(f)$. Let us state the main result

THEOREM 1.1. *Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$. Then for every line $bX - aY = 0$ not tangent to the curve $f = 0$ there is a factorization of the polar $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$:*

$$\partial f = AB \prod_{S \in \mathcal{N}_f^{**}} A_S B_S \quad \text{in } \mathbf{C}\{X,Y\}$$

such that

(i) $\operatorname{ord} A = t(f) - 1$, $\operatorname{ord} B = d(f)$. *If h is an irreducible factor of AB then $(f,h)_0/\operatorname{ord} h \geq \operatorname{ord} f$ with equality if and only if h divides A.*

(ii) $\operatorname{ord} A_S = \|S\| + \varepsilon(S) - d(f,S)$, $\operatorname{ord} B_S = d(f,S)$. *If h is an irreducible factor of $A_S B_S$ then $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S), \beta(S))$ with equality if and only if h divides A_S.*

(iii) *If* $\operatorname{ord} B_S > 0$ *then B_S is S-elementary. If* $\operatorname{ord} A_S > 0$ *and S is of the first kind then A_S is S-elementary.*

(iv) *If* $\operatorname{ord} A_S > 0$ *and S is of the second kind then there is a factorization $A_S = A'_S A''_S$ such that*

- · *if* $\operatorname{ord} A'_S > 0$ *then A'_S is S-elementary.*
- ·· *If* $\operatorname{ord} A''_S > 0$ *then every barier of the Newton diagram of f parallel to a segment of $\mathcal{N}(A''_S)$ passes through the vertex of S lying on the vertical (resp. horizontal) axis if $V_S = X$ (resp. $V_S = Y$). If $|S|_1 < |S|_2$ (resp. $|S|_2 < |S|_1$) then for every $T \in \mathcal{N}(A''_S)$: $|T|_1/|T|_2 < |S|_1/|S|_2$ (resp. $|S|_1/|S|_2 < |T|_1/|T|_2$).*

The proof of Theorem 1.1 is given in Section 6 of this note.

COROLLARY 1.2. *Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$ such that $\mathcal{N}_f^{**} \neq \emptyset$. Then*

(i) *Let $S \in \mathcal{N}_f^{**}$ be of the first kind. Then* $\max(\alpha(S), \beta(S))$ *is a polar invariant of the curve $f = 0$. Its multiplicity is at least $\|S\| - d(f,S)$.*

(ii) *If $S \in \mathcal{N}_f^{**}$ is of the second kind then* $\max(\alpha(S), \beta(S))$ *is a polar invariant of f if and only if* $\operatorname{ord}\operatorname{in}(f,S)^\circ_{\mathrm{red}} > 1$. *Its multiplicity is at least $\|S\| - d(f,S) - 1$. If* $\operatorname{ord}\operatorname{in}(f,S)^\circ_{\mathrm{red}} = 1$ *then there is a polar invariant strictly greater than* $\max(\alpha(S), \beta(S))$.

Proof. Fix a segment $S \in \mathcal{N}_f^{**}$. It is easy to check that $\operatorname{ord} A_S = \|S\| + \varepsilon(S) - d(f,S) = \operatorname{ord}\operatorname{in}(f,S)^{\circ}_{\mathrm{red}} + \varepsilon(S)$. If S is of the first kind then $\varepsilon(S) = 0$ and (i) follows. If S is of the second kind then $\max(\alpha(S), \beta(S))$ is a polar invariant if and only if $\operatorname{ord} A_S = \operatorname{ord}\operatorname{in}(f,S)^{\circ}_{\mathrm{red}} - 1 > 0$.

Example 1.3. Let $f(X,Y) = (Y - X^2)^2 + X^5$. Then $\mathcal{N}_f = \{S\}$ where S is the segment with vertices $(0,2)$ and $(4,0)$. Clearly $\operatorname{in}(f,S)^{\circ}_{\mathrm{red}} = Y - X^2$ is of order 1. According to Corollary 1.2 we can only say that the curve $f = 0$ has a polar invariant greater than $\max(\alpha(S), \beta(S)) = \max(2,4) = 4$. Taking the new system of coordinates $X_1 = X$, $Y_1 = Y - X^2$ we get $f_1(X_1, Y_1) = Y_1^2 + X_1^5$ and using 1.2 to f_1 in coordinates (X_1, Y_1) we get that there is a unique polar invariant equal to 5.

Example 1.4. Let $f(X,Y) = Y^{11} + XY^8 - 2X^2Y^6 + X^3Y^4 - 2X^4Y^3 + X^5Y^2 - 2X^7Y + X^9$. Then $\mathcal{N}_f = \{E, S, U, T\}$ where $|E|_1/|E|_2 < |S|_1/|S|_2 < |U|_1/|U|_2 < |T|_1/|T|_2$. Here E is exceptional, U is principal and $\mathcal{N}_f^{**} = \{S, T\}$ where S is of the first kind ($\varepsilon(S) = 0$) and T is of the second kind ($\varepsilon(T) = -1$). According to Theorem 1.1 there is a factorization $\partial f = ABA_SB_SA_TB_T$ in $\mathbf{C}\{X,Y\}$ where $\operatorname{ord} A = t(f) - 1 = 2$, $\operatorname{ord} B = d(f) = 1$, $\operatorname{ord} A_S = \|S\| - d(f,S) = 1$, $\operatorname{ord} B_S = d(f,S) = 1$, $\operatorname{ord} A_T = \|T\| - 1 - d(f,T) = 0$, $\operatorname{ord} B_T = d(f,T) = 1$. We may assume that $A_T = 1$ in $\mathbf{C}\{X,Y\}$ for A_T is a unit. The polar $\partial f = 0$ consists of the curve $A = 0$ of order 2 transverse to the curve $f = 0$ and of four nonsingular branches $A_S = 0$, $B = 0$, $B_S = 0$ and $B_T = 0$. The polar invariants are $\operatorname{ord} f = 7$ (of multiplicity 2), $(f, A_S)_0 = \max(\alpha(S), \beta(S)) = 10$ and the numbers $(f,B)_0 > 7$, $(f,B_S)_0 > 10$ and $(f,B_T)_0 > \max(\alpha(T), \beta(T)) = 9$. The theorem does not give information as to whether the invariants $(f,B)_0$, $(f,B_S)_0$ and $(f,B_T)_0$ are equal or not.

Here is an improved version of the main result of [LP].

COROLLARY 1.5 (see [LP], Theorem 1.1). *Let $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ be a power series with an isolated singularity at $(0,0) \in \mathbf{C}^2$. Then for every line $bX - aY = 0$ not tangent to the curve $f = 0$ there is a factorization of the polar $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$:*

$$\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S \quad \textit{in } \mathbf{C}\{X,Y\}$$

such that

(i) $\operatorname{ord} g_S = \|S\| + \varepsilon(S)$. *If h is an irreducible factor of g_S then* $(f,h)_0/\operatorname{ord} h \geq \max(\alpha(S), \beta(S))$.

(ii) *The following conditions are equivalent*:

(α) $(f,h)_0/\operatorname{ord} h = \max(\alpha(S), \beta(S))$ *for every irreducible factor h of g_S*,

(β) *the power series f is nondegenerate on S.*

(iii) *One has* $\operatorname{ord} g = t(f) - 1 + d(f)$. *Moreover* $(f,h)_0/\operatorname{ord} h = \operatorname{ord} f$ *for every irreducible factor* h *of* g *if and only if* $d(f) = 0$.

Proof. We put $g = AB$ and $g_S = A_S B_S$. Then we use Theorem 1.1 (i) and (ii).

Note that $d(f) = 0$ if and only if the Newton polygon $\mathcal{N}_f$ has no principal segment or the Newton polygon $\mathcal{N}_f$ has a principal segment and f is nondegenerate on it. Therefore Corrolary 1.5 enables the calculation of Teissier's collection of a nondegenerate singularity by means of its Newton polygon.

Example 1.6. Let $f(X,Y) = Y^8 + X^3Y^3 + Y^4Y^2 + X^6Y$. Then $\mathcal{N}_f = \{S, U, T\}$ where $|S|_1/|S|_2 < |U|_1/|U|_2 < |T|_1/|T|_2$ and f is nondegenerate. We have $\varepsilon(S) = -1$, $\varepsilon(T) = 0$ and $\max(\alpha(S), \beta(S)) = \max(\alpha(T), \beta(T)) = 8$. The segment U is principal. Therefore $\partial f = g g_S g_T$ where $\operatorname{ord} g = t(f) - 1 = 2$, $\operatorname{ord} g_S = \|S\| - 1 = 2$, $\operatorname{ord} g_T = \|T\| = 1$. Moreover if h is a prime divisor of g_S or g_T then $(f,h)_0/\operatorname{ord} h = 8$. The polar invariants are 6 (of multiplicity $t(f) - 1 = 2$) and 8 (of multiplicity $\operatorname{ord} g_S g_T = 2 + 1 = 3$).

2. Contact exponent and minimal polar invariant

Let $f = f_1 \cdots f_r$ be an isolated singularity with branches $f_i = 0$ and let $l = 0$ be a smooth curve (that is l is a series of order 1). Then we consider the contact exponent of $l = 0$ with $f = 0$

$$\delta(f,l) = \min_{i=1}^{r} \left\{ \frac{(f_i, l)_0}{\operatorname{ord} f_i} \right\}$$

and the contact exponent of $f = 0$:

$$\delta(f) = \sup\left\{ \delta(f,l) \colon \begin{array}{l} l = 0 \text{ runs over the set of nonsingular} \\ \text{curves different from the branches } f_i = 0 \end{array} \right\}$$

(see [BK] pp. 640–661 for Hironaka's theory of maximal contact).

Note that $\delta(f) \geq 1$ and $\delta(f) = 1$ if and only if $t(f) > 1$.

Theorem 2.1. *Let* $f = f(X,Y) \in \mathbf{C}\{X,Y\}$ *be a power series with an isolated singularity at* $(0,0) \in \mathbf{C}^2$. *Then*

(i) *if* $t(f) > 1$ *then the minimal polar invariant of* $f = 0$ *is equal to* $\operatorname{ord} f$ *and its multiplicity is* $t(f) - 1$.

(ii) *Suppose that* $t(f) = 1$ *and* $\delta(f, Y) = \delta(f)$. *Let* F *be the first segment of the Newton polygon* $\mathcal{N}_f$. *Then the minimal polar invariant of* $f = 0$ *is equal to* $\alpha(F)$ *and its multiplicity is* $\|F\| + \varepsilon(F) - d(F,f)$.

(iii) *The minimal polar invariant of the singularity* $f = 0$ *is equal to* $(\operatorname{ord} f)\delta(f)$.

Proof. Part (i) of the theorem follows from Theorem 1.1 (i). To check (ii) observe that from the assumptions it follows that the axis $X = 0$ is transverse to the curve $f = 0$. The Newton diagram of f has the vertex $(0, \operatorname{ord} f)$ and lies strictly above the line $\alpha + \beta = \operatorname{ord} f$. Hence all segments of $\mathcal{N}_f$ have the inclination strictly greater than 1. In particular $|F|_1 > |F|_2$. Recall that $|F|_1/|F|_2 = \delta(f, Y) = \delta(f)$ and consider two cases.

CASE 1. The power series f is not elementary. Then the segment F is of the first kind and $\alpha(F)$ is a polar invariant of $f = 0$. Using Theorem 1.1 we check that all polar invariants of f different from $\alpha(F)$ are strictly greater than $\alpha(F)$. Thus $\alpha(F)$ is the minimal polar invariant of f and its multiplicity equals $\|F\| + \varepsilon(F) - d(f, F)$.

CASE 2. The power series f is elementary. Then F is the unique segment of $\mathcal{N}_f$. Using the criterion of maximal contact (see [BK], Lemma 5, p. 649) we get that $\operatorname{in}(f, F)$ is not of the form $(bY - aX^k)^m$, $ab \neq 0$. Therefore by our main result $\alpha(F) = \max(\alpha(F), \beta(F))$ is the minimal polar invariant of f and its multiplicity is $\|F\| + \varepsilon(F) - d(f, F)$.

To check (iii) we note that $\alpha(F)/\operatorname{ord} f = \alpha(F)/\beta(F) = |F|_1/|F|_2 = \delta(f)$ and use (ii).

Example 2.2. Suppose that f is an irreducible power series with characteristic $\beta_0, \beta_1, \ldots, \beta_g$ (see for example [M]). If the axis $Y = 0$ has the maximal contact with $f = 0$ then $\mathcal{N}_f = \{F\}$ where F is the segment with vertices $(0, \beta_0)$ and $(\beta_1, 0)$. Let $e_1 = \operatorname{GCD}(\beta_0, \beta_1)$. Then $\operatorname{in}(f, F) = (bX^{\beta_1/e_1} - cY^{\beta_0/e_1})^{e_1}$ with $bc \neq 0$ and an easy calculation shows that the minimal polar invariant equals $\max(\beta_0, \beta_1) = \beta_1$ and is of multiplicity $\beta_0/e_1 - 1$ (here $\varepsilon(F) = -1$). Thus we have got the first of Merle's formulas [M].

The reasoning like that in the proof of Theorem 2.1 shows

THEOREM 2.3. *Suppose that $f = 0$ has exactly one polar invariant. If $\delta(f, Y) = \delta(f)$ and $(f, X)_0 = \operatorname{ord} f$ then $\mathcal{N}_f = \{F\}$ and f is nondegenerate on F. The segment F has vertices $(0, n)$ and $(m, 0)$ or $(0, n)$ and $(m, 1)$ with $m \geq n$.*

If two isolated singularities $f = 0$ and $g = 0$ have the same Newton diagram and are nondegenerate then they are topologically equivalent. On the other hand for every isolated singularity there is a system of coordinates such that in Theorem 2.3. Therefore we get the following classification result due to Eggers.

COROLLARY 2.4 ([E], p. 16). *If $f = 0$ has exactly one polar invariant then $f = 0$ is topologically equivalent to a plane curve singularity of type $Y^n - X^m = 0$ or of type $Y^n - YX^m = 0$.*

3. Special values of plane curve pencils

When studying the singularities at infinity of a polynomial in two complex variables of degree $N > 0$ one considers the pencils of plane curves of the form $f_t = f - tl^N$, $t \in \mathbf{C}$ where f and $l = bX - aY$ are coprime (such pencils are called in [C] Iomdin Lê deformations). Let $\mu_0(f) = (\partial f/\partial X, \partial f/\partial Y)_0$ be the Milnor number of the local curve $f = 0$. Recall that $\mu_0(f) = +\infty$ if and only if f has a multiple factor. The number $t_0 \in \mathbf{C}$ is a special value of the pencil $(f_t, t \in \mathbf{C})$ if $\mu_0(f_{t_0}) > \inf\{\mu_0(f_t) : t \in \mathbf{C}\}$. The set of special values is finite. Using our main result we will give a bound on the number of special values in terms of the Newton diagram of the series. Let $r(f,S)$ be the number of irreducible factors of $\mathrm{in}(f,S)^\circ$ and put $r(S) = \mathrm{GCD}(|S|_1, |S|_2)$.

Lemma 3.1. *One has* $r(S) - r(f,S) = (r(S)/\|S\|)d(f,S)$. *In particular* $r(f,S) \le r(S)$ *with equality if and only if* f *is nondegenerate on* S.

Proof. Write

$$\mathrm{in}(f,S)^\circ = \prod_{i=1}^{r}(b_i X^{|S|_1/r(S)} - a_i Y^{|S|_2/r(S)})^{m_i}$$

with pairwise linearly independent $(a_i, b_i) \in \mathbf{C}^2$. Then $r(f,S) = r$ and $r(S) = \sum_{i=1}^{r} m_i$. Now

$$\begin{aligned} d(f,S) &= \mathrm{ord}\,\mathrm{in}(f,S)^\circ - \mathrm{ord}\,\mathrm{in}(f,S)^\circ_{\mathrm{red}} \\ &= \sum_{i=1}^{r} m_i \frac{\|S\|}{r(S)} - \sum_{i=1}^{r} \frac{\|S\|}{r(S)} = \frac{\|S\|}{r(S)}(r(S) - r(f,S)) \end{aligned}$$

and the lemma follows.

The following result is a local counterpart of the Le Van Thanh and Oka theorem giving an estimation for the number of critical values at infinity (see [LO], Main Theorem).

Let $q(S) = \max(\alpha(S), \beta(S))$ for any $S \in \mathcal{N}_f^*$. We put $l = bX - aY$ and suppose that the line $l = 0$ is not tangent to $f = 0$.

Theorem 3.2. *Let* $N \ne \mathrm{ord}\, f$ *be a strictly positive integer. The number of nonzero special values of the pencil* $(f - tl^N : t \in \mathbf{C})$ *is less than or equal to*

$$\sum_{S: q(S) < N} (r(S) - r(f,S)) + \sum_{S: q(S) = N} r(f,S).$$

Recall that a sum over the empty set equals zero. If f is a nondegenerate power series then the sum above reduces to

$$\sum_{S:q(S)=N} r(S).$$

Note also the bound for all series with the given Newton polygon.

Corollary 3.3. *The number of the nonzero special values of* $(f - tl^N$: $t \in \mathbf{C})$ $(N \neq \operatorname{ord} f)$ *is less than or equal to*

$$\sum_{S:q(S)\leq N} r(S).$$

In connection with the above corollary recall the following well-known fact: the number of branches of the curve $f = 0$ different from the axes is less or equal to $\sum_S r(S)$ (with equality for nondegenerate curves).

To get Theorem 3.2 from the main result we need a few lemmas. The lemma below is a local version of the description of critical values at infinity given in [LO] (pp. 410–411). Let f/h be a meromorphic fraction with coprime $f, h \in \mathbf{C}\{X, Y\}$ and let $p = p(X, Y) \in \mathbf{C}\{X, Y\}$ be irreducible power series such that p does not divide h. Let $(x(u), y(u)) \in \mathbf{C}\{u\}^2$, $(x(0), y(0)) = (0, 0)$ be a parametrization of the branch $p = 0$. Then we put

$$\left(\frac{f}{h}\right)(p) = \frac{f(x(u), y(u))}{l(x(u), y(u))}\bigg|_{u=0} \in \mathbf{C} \cup \{\infty\}.$$

Lemma 3.4. *The set of nonzero special values of the pencil* $(f - tl^N : t \in \mathbf{C})$ *is equal to the set*

$$\{(f/l^N)(p)\colon p \text{ is irreducible factor of } j(f,l) \text{ such that } (f,p)_0/(l,p)_0 = N\}.$$

Proof. See [MM] Théorème 1 or [GB-P] Proposition 2.2.

Let $r_0(\phi)$ be the number of irreducible factors of the series ϕ.

Lemma 3.5. *Suppose that* ϕ *is* S*-elementary. Then*

$$r_0(\phi) \leq (\operatorname{ord} \phi)\frac{r(S)}{\|S\|}.$$

Proof. Let $r = r_0(\phi)$. Then $\phi = \prod_{i=1}^r \phi_i$ with irreducible ϕ_i. The power series ϕ_i are S-elementary. Therefore the unique segment of $\mathcal{N}(\phi_i)$ joins the points $(k_i|S|_1/r(S), 0)$ and $(0, k_i|S|_2/r(S))$ for an integer $k_i \geq 1$. Consequently

$$\operatorname{ord} \phi_i = \min\left(\frac{|S|_1}{r(S)}k_i, \frac{|S|_2}{r(S)}k_i\right) \geq \frac{\|S\|}{r(S)}$$

for all $i = 1, \ldots, r$. We get

$$\text{ord } \phi = \sum_{i=1}^{r} \text{ord } \phi_i \geq \frac{\|S\|}{r(S)} r_0(\phi)$$

and the lemma follows.

Lemma 3.6. *Let us keep the notation from Theorem* 1.1. *Then*
(i) *If* ord $B > 0$ *then* $\mathcal{N}_f$ *has a principal segment* U *and* ord $B = r(U) - r(f, U)$.
(ii) *If* ord $B_S > 0$ *then* $r_0(B_S) \leq r(S) - r(f, S)$.
(iii) *If* ord $A_S > 0$ *and* S *is of the first kind then* $r_0(A_S) \leq r(f, S)$.

Proof.
(i) It is easy to see that if $d(f) > 0$ then $\mathcal{N}_f$ has a principal segment U and $d(f) = r(U) - r(f, U)$. Use Theorem 1.1 (i).
(ii) Suppose that ord $B_S > 0$. Then by Theorem 1.1 (ii) we get ord $B_S = d(f, S)$. Now Lemmas 3.5 and 3.1 give

$$r_0(B_S) \leq (\text{ord } B_S) \frac{r(S)}{\|S\|} = d(f, S) \frac{r(S)}{\|S\|} = r(S) - r(f, S).$$

(iii) Suppose that ord $A_S > 0$ and S is of the first kind. Then ord $A_S = \|S\| - d(f, S)$ by Theorem 1.1 (ii) and using Lemmas 3.5 and 3.1 we get

$$r_0(A_S) \leq (\text{ord } A_S) \frac{r(S)}{\|S\|} = (\|S\| - d(f, S)) \frac{r(S)}{\|S\|} = r(f, S).$$

Lemma 3.7. *Suppose that* ord $A_S > 0$ *for a segment* $S \in \mathcal{N}_f^{**}$ *of the second kind. Let* $N = \max(\alpha(S), \beta(S))$. *Let* $A_S = A'_S A''_S$ *be the factorization of* A_S *such that in Theorem* 1.1 (iv). *Then*
(i) $r_0(A'_S) \leq r(f, S) - 1$,
(ii) *for every prime factor* p *of* A''_S: $(f/l^N)(p) = (f/l^N)(v_S)$.

Proof. By Theorem 1.1 (ii) we get ord $A_S = \|S\| - 1 - d(f, S)$ ($\varepsilon(S) = -1$ for the segments of second kind) and consequently, like in the proof of Lemma 3.7 we obtain

$$r_0(A'_S) \leq (\text{ord } A'_S) \frac{r(S)}{\|S\|} \leq (\text{ord } A_S) \frac{r(S)}{\|S\|} = r(f, S) - \frac{r(S)}{\|S\|} < r(f, S).$$

Since $r_0(A'_S)$ and $r(f, S)$ are integers we get $r_0(A'_S) \leq r(f, S) - 1$. To prove the second part of Lemma 3.7 assume that $S = F$ is the first segment of $\mathcal{N}_f$ (if $S = L$ is the last segment then the proof is similar). Then $v_S = v_F = X$. Let p be a prime factor of A''_F. We may assume that the branch $p = 0$ is different from the axis $X = 0$. Note that $|F_1|/|F_2| < 1$ and $N = \beta(F)$. Let $(x(u), y(u))$ be the injective parametrization of the branch $p = 0$. Put $m =$ ord $x(u)$ and $n =$ ord $y(u)$.

The series p is elementary, the unique segment of $\mathcal{N}_p$ joins the points $(n, 0)$ and $(0, m)$ and is of inclination $n/m \leq |F|_1/|F|_2 < 1$ by Theorem 1.1 (iv). The line supporting the Newton diagram of f of slope $-m/n$ passes through the point $(0, \beta(F)) = (0, N)$ and, consequently, has the equation $m\alpha + n\beta = nN$. It intersects the Newton diagram of f exactly at point $(0, N)$. Therefore

$$f(X, Y) = c_{0N} Y^N + \sum_{m\alpha + n\beta > nN} c_{\alpha\beta} X^\alpha Y^\beta \quad \text{with } c_{0N} \neq 0.$$

The line $l(X, Y) = bX - aY$ is not tangent to $f = 0$. Then $a \neq 0$ and

$$f(x(u), y(u)) = c_{0N} y(u)^N + \text{terms of order} > nN$$
$$l(x(u), y(u)) = (-a)^N y(u)^N + \text{terms of order} > nN$$

Consequently

$$\left(\frac{f}{l^N}\right)(p) = \frac{c_{0N}}{(-a)^N} = \left(\frac{f}{l^N}\right)(X).$$

Now we give the proof of Theorem 3.2.

Let

$$\partial f = AB \prod_{S \in \mathcal{N}_f^{**}} A_S B_S$$

be a factorization of ∂f such that in Theorem 1.1.

According to Lemmas 3.4 and 3.7 the number of nonzero special values of $(f - tl^N : t \in \mathbf{C})$ is equal to

$$\#\{(f/l^N)(p) \colon p \text{ is a prime factor of } \partial f \text{ and } (f, p)_0/\operatorname{ord} p = N\}$$
$$\leq \operatorname{ord} B + \sum_{S : q(S) < N} r_0(B_S) + \sum_{S : q(S) = N}^{I} r_0(A_S) + \sum_{S : q(S) = N}^{II} (r_0(A'_S) + 1)$$

where the symbols $\sum^I$ resp $(\sum^{II})$ mean that the summation is carried over the segments of the first kind (of the second kind). The theorem follows from Lemmas 3.6 and 3.7.

Remark 3.8. An obvious modification of the above proof shows that the pencil $(f - tl^{\operatorname{ord} f} : t \in \mathbf{C})$ has at most $t(f) - 1$ nonzero special values.

Example 3.9. Let $1 < n < m$ be integers such that $d = \mathrm{GCD}(m, n) < n$. Put weight $X = m$, weight $Y = n$ and let $f(X, Y) = (bX^{n/d} - aY^{m/d})^d +$ terms of weight $> mn$, $(ab \neq 0)$ be a power series with an isolated singularity at $0 \in \mathbf{C}^2$. Using Theorem 3.2 we check that the pencil $f_t - tY^m$, $t \in \mathbf{C}$ has at most one nonzero special value. One can prove that this value always exists.

4. Preliminary lemmas

Let $\varphi = \varphi(X, Y) \in \mathbf{C}\{X, Y\}$ be a nonzero power series without constant term and $\lambda = bX - aY$ be a linear form. Put $\partial\varphi = a(\partial\varphi/\partial X) + b(\partial\varphi/\partial Y)$ (we do not assume that $\mathrm{in}\,\varphi(a,b) \neq 0$!) and note that $\mathrm{ord}\,\partial\varphi = \mathrm{ord}\,\varphi - 1$ if and only if $\mathrm{in}\,\varphi \neq \mathrm{const.}\,\lambda^{\mathrm{ord}\,\varphi}$.

It is easy to check the following two properties:

(i) if $\partial\varphi \equiv 0 \pmod{\varphi}$ then $\varphi \equiv 0 \pmod{\lambda}$,

(ii) if $\varphi = \lambda^k\psi$ in $\mathbf{C}\{X, Y\}$, ψ without constant term and $\psi \not\equiv 0 \pmod{\lambda}$ then $\partial\varphi = \lambda^k\partial\psi$ and $\partial\psi \not\equiv 0 \pmod{\lambda}$.

Lemma 4.1. *Let $k \geq 0$ be the greatest integer such that λ^k divides φ and let $\varphi = \lambda^k\varphi_1^{m_1}\cdots\varphi_s^{m_s}$ with $s > 0$ irreducible and pairwise coprime $\varphi_i \in \mathbf{C}\{X, Y\}$. Then $\partial\varphi = \lambda^k\varphi_0\varphi_1^{m_1-1}\cdots\varphi_s^{m_s-1}$ in $\mathbf{C}\{X, Y\}$ and φ, φ_0 are coprime.*

Proof. Differentiating the product $\varphi = \lambda^k\varphi_1^{m_1}\cdots\varphi_s^{m_s}$ we get

$$\partial\varphi = \lambda^k\varphi_0\varphi_1^{m_1-1}\cdots\varphi_s^{m_s-1}$$

where $\varphi_0 = m_1(\partial\varphi_1)\varphi_2\cdots\varphi_s + \cdots + \varphi_1\cdots\varphi_{s-1}m_s(\partial\varphi_s)$. If φ_i $(i \neq 0)$ were a factor of φ_0 then φ_i would be a factor of $\partial\varphi_i$. This implies $\varphi_i \equiv 0 \pmod{\lambda}$ by property (i), which is impossible because λ does not divide φ_i. To check that λ does not divide φ_0 we use property (ii).

Remark. It is easy to check that $\mathrm{ord}\,\varphi_0 = \sum_{i=1}^{s}\mathrm{ord}\,\varphi_i - (\mathrm{ord}\,\varphi - \mathrm{ord}\,\partial\varphi)$.

The following is well-known (see Section 5, Lemma 5.2).

Lemma 4.2. *If $\mathrm{in}\,f = \varphi_1^{m_1}\cdots\varphi_t^{m_t}$ with φ_i linear pairwise linearly independent then $f = f_1\cdots f_t$ in $\mathbf{C}\{X, Y\}$ and $\mathrm{in}\,f_i = \varphi_i^{m_i}$ for $i = 1, \ldots, t$.*

Using the above lemmas we will prove

Lemma 4.3. *Let $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$ with $(a,b) \in \mathbf{C}^2$ such that $\mathrm{in}\,f(a,b) \neq 0$. Then $\partial f = A\tilde{A}$ in $\mathbf{C}\{X, Y\}$ where $\mathrm{ord}\,A = t(f) - 1$, $\mathrm{ord}\,\tilde{A} = \mathrm{ord}\,f - t(f)$ and for every irreducible factor h of ∂f: $(f,h)_0/\mathrm{ord}\,h = \mathrm{ord}\,f$ if and only if h divides A.*

Proof. Let $\mathrm{in}\,f = \varphi_1^{m_1}\cdots\varphi_t^{m_t}$, φ_i linear and $t = t(f)$. Then $\mathrm{in}(\partial f) = \partial(\mathrm{in}\,f) = \varphi_0\varphi_1^{m_1-1}\cdots\varphi_t^{m_t-1}$ in $\mathbf{C}\{X, Y\}$ with coprime φ_0, $\mathrm{in}\,f$. By Lemma 4.2 we get a factorization $\partial f = g_0g_1\cdots g_t$ where $\mathrm{in}\,g_0 = \varphi_0$ and $\mathrm{in}\,g_i = \varphi_i^{m_i-1}$ for $i = 1, \ldots, t$. By Remark to Lemma 4.1 we get

$$\mathrm{ord}\,g_0 = \sum_{i=1}^{t}\mathrm{ord}\,\varphi_i - 1 = t - 1 = t(f) - 1.$$

Put $A = g_0$, $\tilde{A} = g_1 \cdots g_t$. Let h be an irreducible factor of ∂f, if h divides A then the curves $f = 0$, $h = 0$ are transverse and $(f, h)_0 = (\operatorname{ord} f)(\operatorname{ord} h)$. If h divides $\tilde{A}$ they are not, thus $(f, h)_0 > (\operatorname{ord} f)(\operatorname{ord} h)$.

5. Newton polygon and factorization of power series

Let us keep the notation introduced in Section 1. The following two lemmas are well-known.

Lemma 5.1. *Let $f = f(X, Y)$ be a nonzero power series without constant term. Then there is a factorization*

$$f = uX^{d_1} Y^{d_2} \prod_{S \in \mathcal{N}_f} f_S \quad \text{in } \mathbf{C}\{X, Y\}$$

where u is a unit, such that

(i) *$\mathcal{N}(f_S) = \{S'\}$ where S' is the segment with vertices $(|S|_1, 0)$ and $(0, |S|_2)$,*

(ii) $\operatorname{in}(f_S, S') = \text{const.} \operatorname{in}(f, S)^\circ$.

Lemma 5.2. *Suppose that $\mathcal{N}(f) = \{S\}$ where S is a segment with vertices on the axes. Suppose that $\operatorname{in}(f, S) = \psi_1 \cdots \psi_m$ with coprime ψ_i. Then there is a factorization*

$$f = f_1 \cdots f_m$$

such that

(i) *$\mathcal{N}(f_i) = \{S^{(i)}\}$ where $S^{(i)}$ is a segment parallel to S,*

(ii) *$\operatorname{in}(f_i, S^{(i)}) = \psi_i$ for $i = 1, \ldots, m$.*

6. Proof of the main result

We will prove our theorem for polars $\partial f = a(\partial f/\partial X) + b(\partial f/\partial Y)$ such that $ab \operatorname{in} f(a, b) \neq 0$. If $a = 0$ or $b = 0$ but $\operatorname{in} f(a, b) \neq 0$ then the proof needs some modifications (see [LP], p. 318). By Lemma 5.1 we may write

(1) $\partial f = uX^{\delta_1} Y^{\delta_2} \prod_{T \in \mathcal{N}(\partial f)} (\partial f)_T$ in $\mathbf{C}\{X, Y\}$ where u is a unit and

(2) $(\partial f)_T$ is an elementary power series; $\mathcal{N}((\partial f)_T) = \{T'\}$ where T' is the segment with vertices $(|T|_1, 0)$ and $(0, |T|_2)$,

(3) $\operatorname{in}((\partial f)_T, T') = \text{const.} \operatorname{in}(\partial f, T)^\circ$.

The proposition below is already proved in [LP] but it is not stated there explicitly.

Proposition 6.1. *Suppose that $f \in \mathbf{C}\{X, Y\}$ has an isolated singularity at $(0, 0) \in \mathbf{C}^2$. Then there is a factorization*

$$\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S \quad \text{in } \mathbf{C}\{X, Y\}$$

such that

(i) $\operatorname{ord} g = d(f) + t(f) - 1$, $\operatorname{ord} g_S = \|S\| + \varepsilon(S)$ *for* $S \in \mathcal{N}_f^{**}$,

(ii) *If* $S \in \mathcal{N}_f^{**}$ *is a segment of the first kind then there is a segment* $T \in \mathcal{N}_{\partial f}$ *(necessarily unique) parallel to* S. *We have* $g_S = (\partial f)_T$.

(iii) *Suppose that* $S \in \mathcal{N}_f^{**}$ *is a segment of the second kind. Then*

(α) *for every* $T \in \mathcal{N}(g_S)$ *the power series* $(\partial f)_T$ *divides* g_S.

(β) *If* $|S|_1 < |S|_2$ *(resp.* $|S|_2 < |S|_1$*) then for every* $T \in \mathcal{N}(g_S) : |T|_1/|T|_2 \leq |S|_1/|S|_2$ *(resp.* $|S|_1/|S|_2 \leq |T|_1/|T|_2$*).*

(γ) *Every barier of the Newton diagram of* f *parallel to a segment of* $\mathcal{N}(g_S)$ *passes through the vertex of* S *lying on the axis* $v_S = 0$.

(δ) *If there is no segment of* $\mathcal{N}(g_S)$ *parallel to a segment of* $\mathcal{N}_f^{**}$ *then* $d(f, S) = 0$.

Proof. If $\mathcal{N}_f^{**} = \emptyset$ then $d(f) + t(f) = \operatorname{ord} f$, thus we may assume that $\mathcal{N}_f^{**} \neq \emptyset$. According to [LP], Theorem 1.1 p. 310 there is a factorization

(4) $\partial f = v \prod_{S \in \mathcal{N}_f^{**}} g_S$ in $\mathbf{C}\{X, Y\}$ such that

(5) $\operatorname{ord} g_S = \|S\| + \varepsilon(S)$ for every $S \in \mathcal{N}_f^{*}$,

(6) if $\operatorname{ord} v > 0$ then $\operatorname{ord} v = 1$ and $(f, v)_0 = \operatorname{ord} f$.

Moreover, by the definition of v given in [LP], p. 317 we have

(7) $\operatorname{ord} v > 0$ if and only if $\operatorname{in} f = \text{const.}\, X^{\alpha_0} Y^{\beta_0}$ for some $\alpha_0, \beta_0 > 0$.

To define the power series g we consider two cases.

Case 1. The initial form $\operatorname{in} f$ is not a monomial. Then there exists the principal segment $U \in \mathcal{N}_f^{*}$ and $\operatorname{in} f = \operatorname{in}(f, U)$. It is easy to see that $\operatorname{ord} \operatorname{in}(f, U)^{\circ}_{\mathrm{red}} = t(f) - 1 - \varepsilon(U)$. Consequently

$$d(f) = d(f, U) = \|U\| - \operatorname{ord} \operatorname{in}(f, U)^{\circ}_{\mathrm{red}} = \|U\| + \varepsilon(U) - (t(f) - 1)$$

and $\operatorname{ord} g_U = \|U\| + \varepsilon(U) = d(f) + t(f) - 1$ by (5). Put $g = v g_U$. Note that v is a unit by (7) hence $\operatorname{ord} g = \operatorname{ord} g_U = d(f) + t(f) - 1$.

Case 2. The initial form $\operatorname{in} f$ is a monomial. If $\operatorname{in} f = \text{const.}\, X^{\operatorname{ord} f}$ or $\operatorname{in} f = \text{const.}\, Y^{\operatorname{ord} f}$ then $d(f) + t(f) - 1 = 0$. We put $g = v$. By (7) v is a unit and consequently $\operatorname{ord} g = 0$. If $\operatorname{in} f = \text{const.}\, X^{\alpha_0} Y^{\beta_0}$ with $\alpha_0 > 0$ and $\beta_0 > 0$ then $d(f) + t(f) - 1 = 1$. We put $g = v$. By (6) and (7) we get $\operatorname{ord} g = 1$.

By the definition of the series g we can rewrite (4) in the form

(8) $\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S$, $\operatorname{ord} g = d(f) + t(f) - 1$

The conditions (ii) and (iii) (α), (β), (γ) follow immediately from the definition of g_S given in [LP] p. 317. To check (iii) (δ) we observe that by [LP], Lemma 2.3 the initial form $\operatorname{in}(f, S)$ is the sum of two monomials. Thus $d(f, S) = 0$.

Proposition 6.2. *Let* $S \in \mathcal{N}_f^{**}$ *and* $T \in \mathcal{N}_{\partial f}$ *be parallel. Then there is a factorization*

$$(\partial f)_T = A_S B_S \quad \textit{in } \mathbf{C}\{X, Y\}$$

such that

(i) *If* $\operatorname{ord} A_S > 0$ *then* A_S *is* S-*elementary. Suppose that* $\operatorname{ord} A_S > 0$ *and let* $\mathcal{N}(A_S) = \{\tilde{S}\}$. *Then the power series* $\operatorname{in}(A_S, \tilde{S})$ *and* $\operatorname{in}(f, S)$ *are coprime.*

(ii) *We have* $\operatorname{ord} B_S = d(f, S)$. *If* $\operatorname{ord} B_S > 0$ *then* B_S *is* S-*elementary. Suppose that* $\operatorname{ord} B_S > 0$ *and let* $\mathcal{N}(B_S) = \{\tilde{\tilde{S}}\}$. *Then the power series* $\operatorname{in}(B_S, \tilde{\tilde{S}})$ *divides* $\operatorname{in}(f, S)$.

Proof. Let $\partial = a(\partial/\partial X) + b(\partial/\partial X)$ with $ab \operatorname{in} f(a,b) \neq 0$. We may assume that $|S|_1 < |S|_2$. By [LP], Theorem 2.1 (5), p. 313 we get

(9) $\operatorname{in}(\partial f, T) = a(\partial/\partial X) \operatorname{in}(f, S)$.

Let us consider the factorization

(10) $\operatorname{in}(f, S) = X^{a_S} Y^{b_S} \psi_1^{k_1} \cdots \psi_m^{k_m}$ with irreducible, pairwise coprime $\psi_i \in \mathbf{C}\{X, Y\}$

and let $\lambda = Y$. We apply Lemma 4.1 to $\operatorname{in}(f, S)$ and λ:

(11) $(\partial/\partial X) \operatorname{in}(f, S) = Y^{b_S} \psi_0 X^{\max(a_S - 1, 0)} \psi_1^{k_1 - 1} \cdots \psi_m^{k_m - 1}$

where ψ_0 and $\operatorname{in}(f, S)$ are coprime. By (2), (3) and (11) we get

(12) $\operatorname{in}((\partial f)_T, T') = \psi_0 \psi_1^{k_1 - 1} \cdots \psi_m^{k_m - 1}$.

Clearly the power series $\psi_1, \ldots, \psi_m$ and ψ_0 (if $\operatorname{ord} \psi_0 > 0$) are S-elementary. According to Lemma 5.2 we get the factorization

(13) $(\partial f)_T = g_0 g_1 \cdots g_m$ in $\mathbf{C}\{X, Y\}$ such that

· $\operatorname{ord} g_0 > 0$ if and only if $\operatorname{ord} \psi_0 > 0$. If $\operatorname{ord} g_0 > 0$ then g_0 is S-elementary, $\mathcal{N}(g_0) = \{S^{(0)}\}$ and $\operatorname{in}(g_0, S^{(0)}) = \psi_0$,

·· $\operatorname{ord} g_j > 0$ if and only if $k_j > 1$ (for $j = 1, \ldots, m$). If $\operatorname{ord} g_j > 0$ then g_j is S-elementary, $\mathcal{N}(g_j) = \{S^{(j)}\}$ and $\mathcal{N}(g_j, S^{(j)}) = \psi_j^{k_j - 1}$.

Let $A_S = g_0$ and $B_S = g_1 \cdots g_m$. By (13) we get $(\partial f)_T = A_S B_S$. Suppose that $\operatorname{ord} A_S > 0$. Then $\mathcal{N}(A_S) = \{\tilde{S}\}$ where $\tilde{S} = S^{(0)}$ and $\operatorname{in}(A_S, \tilde{S}) = \operatorname{in}(g_0, S^{(0)}) = \psi_0$, consequently $\operatorname{in}(A_S, \tilde{S})$ and $\operatorname{in}(f, S)$ are coprime. On the other hand

$$\begin{aligned}
\operatorname{ord} B_S &= \sum_{j=1}^{m} \operatorname{ord} g_j = \sum_{j=1}^{m} (k_j - 1) \operatorname{ord} \psi_j \\
&= \sum_{j=1}^{m} k_j \operatorname{ord} \psi_j - \sum_{j=1}^{m} \psi_j \\
&= \operatorname{ord} \operatorname{in}(f, S)^\circ - \operatorname{in}(f, S)^\circ_{\mathrm{red}} = d(f, S).
\end{aligned}$$

Suppose that $\operatorname{ord} B_S > 0$. The power series B_S is S-elementary as a product of S-elementary power series. If $\mathcal{N}(B_S) = \{\tilde{\tilde{S}}\}$ then $\operatorname{in}(B_S, \tilde{\tilde{S}}) = \prod_{j=1}^{m} \psi_j^{k_j - 1}$ divides $\operatorname{in}(f, S)$.

Proposition 6.3. *Let* $S \in \mathcal{N}_f^{**}$ *and let* h *be an irreducible factor of* g_S. *Then* $(f, h)_0 / \operatorname{ord} h \geq \max(\alpha(S), \beta(S))$. *The inequality* $(f, h)_0 / \operatorname{ord} h > \max(\alpha(S), \beta(S))$ *holds if and only if the following two conditions are fulfiled*

(a) *the power series h is S-elementary,*
(b) *if $\tilde{S}$ is the unique segment of $\mathcal{N}_h$ then the system of equations* $\mathrm{in}(h, \tilde{S}) = 0$, $\mathrm{in}(f, S) = 0$ *has a solution in* $(\mathbf{C}\backslash\{0\}) \times (\mathbf{C}\backslash\{0\})$.

Proof. Proposition 6.3 follows from [LP], Lemma 3.2, p. 316 which remains true when we replace the phrase "if the pair f, h is nondegenerate" by "if and only if the pair f, h is nondegenerate".

Now we can give the proof of Theorem 1.1. Let us consider the factorization

$$\partial f = g \prod_{S \in \mathcal{N}_f^{**}} g_S \quad \text{in } \mathbf{C}\{X, Y\}$$

such that in Proposition 6.1.

Let $S \in \mathcal{N}_f^{**}$ be a segment of the first kind. Then by Proposition 6.1 (ii) we have $g_S = (\partial f)_T$ where $T \in \mathcal{N}_{\partial f}$ is a segment parallel to S. Let $g_S = (\partial f)_T = A_S B_S$ be the factorization of $(\partial f)_T$ such that in Proposition 6.2. Then $\mathrm{ord}\, B_S = d(f, S)$ by Proposition 6.2 (ii) and consequently $\mathrm{ord}\, A_S = \mathrm{ord}\, g_S - \mathrm{ord}\, B_S = \|S\| + \varepsilon(S) - d(f, S)$ by Proposition 6.1 (i). Moreover if $\mathrm{ord}\, A_S > 0$ (resp. $\mathrm{ord}\, B_S > 0$) then A_S (resp. B_S) is S-elementary. Let h be an irreducible factor of $A_S B_S = g_S$. Then $(f, h)_0/\mathrm{ord}\, h \geq \max(\alpha(S), \beta(S))$ by Proposition 6.3. Using Proposition 6.2 we check that h divides B_S if and only if h fulfils conditions (a) and (b) from Proposition 6.3. Thus $(f, h)_0/\mathrm{ord}\, h > \max(\alpha(S), \beta(S))$ if and only if h is a divisor of B_S. Summing up, we have checked that the factorization $g_S = A_S B_S$ where $S \in \mathcal{N}_f^{*}$ is of the first kind, satisfies all conditions stated in Theorem 1.1. Now suppose that $S \in \mathcal{N}_f^{**}$ is of the second kind. We consider two cases.

Case 1. There is no segment $T \in \mathcal{N}_{\partial f}$ parallel to S. Then $d(f, S) = 0$ by 6.1 (iii) (δ) and we put $A_S = A'_S = 1$, $A''_S = g_S$ and $B_S = 1$. Using Proposition 6.1 we check that the factorization $g_S = A'_S A''_S B_S$ has all properties needed in Theorem 1.1.

Case 2. There is a segment $T \in \mathcal{N}_{\partial f}$ parallel to S. Then by Proposition 6.2 we get

$$(\partial f)_T = A'_S B_S \quad \text{in } \mathbf{C}\{X, Y\}.$$

On the other hand $(\partial f)_T$ divides g_S by Proposition 6.1 (iii) and we can write

$$g_S = A''_S (\partial f)_T \quad \text{in } \mathbf{C}\{X, Y\}.$$

Thus we get $g_S = A_S B_S$ with $A_S = A'_S A''_S$. As in the case of the segment of the first kind we check that the factorizations $g_S = A_S B_S$, $A_S = A'_S A''_S$ have all properties stated in Theorem 1.1. To finish the proof it suffices to check that there is a factorization $g = AB$ in $\mathbf{C}\{X, Y\}$ with $\mathrm{ord}\, A = t(f) - 1$ such that h is an irreducible factor of g with property $(f, h)_0/\mathrm{ord}\, h = \mathrm{ord}\, f$ if and only if h

divides A. To this purpose we apply Lemma 4.3 to the series ∂f and observe that if h is an irreducible factor of $\prod_{S \in \mathcal{N}_f^{**}} g_S$ then $(f, h)_0/\mathrm{ord}\, h \geq \max(\alpha(S), \beta(S))$ for a segment $S \in \mathcal{N}_f^{**}$ and $\max(\alpha(S), \beta(S)) > \mathrm{ord}\, f$.

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