

On unconditional well-posedness for the periodic modified Korteweg–de Vries equation

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Abstract. We prove that the modified Korteweg–de Vries equation is unconditionally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/3$. For this we gather the smoothing effect first discovered by Takaoka and Tsutsumi with an approach developed by the authors that combines the energy method, with Bourgain’s type estimates, improved Strichartz estimates and the construction of modified energies.

1. Introduction.

We consider the initial value problem (IVP) associated to the modified Korteweg–de Vries (mKdV) equation

$$\partial_t u + \partial_x^3 u \mp \partial_x(u^3) = 0 \tag{1.1}$$

$$u(\cdot, 0) = u_0, \tag{1.2}$$

where $u = u(x, t)$ is a real valued function, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $t \in \mathbb{R}$.

In [1] Bourgain introduced the Fourier restriction norm method and proved that (1.1) is locally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. Note that, by a change of variable, Bourgain substituted the mKdV equation (1.1) by the renormalized mKdV equation

$$v_t + \partial_x^3 v \mp 3(v^2 - P_0(v_0^2))v_x = 0, \quad v(\cdot, 0) = v_0,$$

where $P_0 w$ denotes the mean value of w . This result was then proved to be sharp if one requires moreover the smoothness or the uniform continuity on bounded sets of the solution-map associated with the renormalized equation (see [2], [11], [5]). This obstruction is related to the resonant term $\sum_{k \in \mathbb{Z}} |\widehat{v}(k)|^2 \widehat{v}(k) e^{ikx}$ that appears in the nonlinear part of this equation. However, in [20], Takaoka–Tsutsumi proved that (1.1) is locally well posed in $H^s(\mathbb{T})$ for $s > 3/8$. For this, they first establish a smoothing effect on the difference $|\mathcal{F}_x(v(t))(k)|^2 - |\widehat{v}_0(k)|^2$ and then work in a Bourgain’s space depending on the initial data in order to treat the resonant term. This was improved in [19] where the local well-posedness was pushed down to $H^s(\mathbb{T})$ with $s > 1/3$.

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The local well-posedness results proved in these papers mean the following: for any initial data $u_0 \in H^s(\mathbb{T})$ there exists a time $T = T(\|u_0\|_{H^s}) > 0$ only depending on $\|u_0\|_{H^s}$ and a solution u that satisfies the equation at least in some weak sense and is unique in some function space (called resolution space) $X_T \hookrightarrow C([0, T]; H^s(\mathbb{T}))$ that can depend on the initial data. Moreover, for any $R > 0$, the flow-map $u_0 \mapsto u$ is continuous from the ball centered at the origin with radius R of $H^s(\mathbb{T})$ into $C([0, T(R)]; H^s(\mathbb{T}))$.

On the other hand, in [7], Kappeler and Topalov introduced the following notion of solutions which a priori does not always corresponds to the solution in the sense of distributions: *A continuous curve $\gamma : (a, b) \rightarrow H^\beta(\mathbb{T})$ with $0 \in (a, b)$ and $\gamma(0) = u_0$ is called a solution of the mKdV equation in $H^\beta(\mathbb{T})$ with initial data u_0 if and only if for any C^∞ -sequence of initial data $\{u_{0,n}\}$ converging to u_0 in $H^\beta(\mathbb{T})$ and for any $t \in]a, b[$, the sequence of emanating solutions $\{u_n\}$ of the mKdV equation satisfies: $u_n(t) \rightarrow \gamma(t)$ in $H^\beta(\mathbb{T})$.*

Note that a solution in the sense of this definition is necessarily unique. With this notion of solution they proved the global well-posedness of the defocusing mKdV equation (with a + sign in front of the nonlinear term) in $H^s(\mathbb{T})$ $s \geq 0$, with a solution-map which is continuous from $L^2(\mathbb{T})$ into $C(\mathbb{R}; L^2(\mathbb{T}))$. Their proof is based on the inverse scattering method and thus depends in a crucial way of the complete integrability of this equation. It is worth noticing that, by Sobolev embedding theorem, their solutions of the defocusing mKdV equation satisfy the equation in the distributional sense as soon as $s \geq 1/6$. In [14] Molinet proved that, actually, the solutions constructed by Kappeler–Topalov always satisfy the equation at least in a weak sense. He also proved that the flow-map cannot be continuously extended in $H^s(\mathbb{T})$ as soon as $s < 0$. Therefore the result of Kappeler–Topalov is in some sense optimal. However it is not known to hold for the focusing equation. Moreover, it uses the integrability of the equation and is thus not suitable to solve perturbations of the defocusing mKdV equation. Also, the question of the existence of a resolution space where the uniqueness holds remains open at this low regularity level.

Another interesting question about uniqueness, even in higher regularity, is to know whether uniqueness also holds in some larger spaces that contain weak solutions. This kind of question was first raised by Kato [8] in the Schrödinger equation context. We refer to such uniqueness in $L^\infty(]0, T[; H^s)$, without intersecting with any auxiliary function space as *unconditional uniqueness*. This ensures the uniqueness of the weak solutions to the equation at the H^s -regularity. This is useful, for instance, to pass to the limit on perturbations of the equation as the perturbative coefficient tends to zero (see for instance [15] for such an application).

Unconditional uniqueness was proved for the mKdV equation to hold in $H^{1/2}(\mathbb{T})$ by Kwon and Oh ([12]) following an approach developed in [3]. In this paper we push down the local well-posedness and the unconditional uniqueness for the mKdV equation to $H^{1/3}(\mathbb{T})$.

To obtain our unconditional uniqueness result we gather the approach developed in [17] based on the construction of modified energies with some ideas of [20] and [19] to derive the smoothing effect. On the one hand, the absence of very small frequencies enables to simplify some estimates on the nonlinear term with respect to [17]. On the other hand, because of true resonances, we need to derive a smoothing effect as in [20].

Actually this is the obtention of the smoothing effect that limits us to the Sobolev index $s \geq 1/3$ (see Remark 4.3). It is also worth noticing that we do not succeed to get an estimate on the $L_T^\infty H^s$ -norm of the difference of two solutions with different initial data—this seems to be related to the fact that the flow-map is not Lipschitz below $s = 1/2$. Instead we will establish an a priori estimate in $L_T^\infty H^{s'}$, for some $s' < s$, on the difference of two solutions emanating from the same initial datum. This estimate will lead to the unconditional uniqueness result. It will be also sufficient to prove the well-posedness result thanks to the smoothing effect which ensures that, given a sequence of solutions $\{u_n\} \subset L^\infty(0, T; H^s(\mathbb{T}))$ to (2.1) associated with a sequence of initial data $\{u_{0,n}\}$ relatively compact in $H^s(\mathbb{T})$, the set $\{u_n(t), t \in [0, T]\}$ is relatively compact in $H^s(\mathbb{T})$.

2. Notations, functions spaces and statement of the result.

We will not work directly with the mKdV equation but with the renormalized mKdV equation defined by

$$u_t + \partial_x^3 u \mp \partial_x(u^3 - 3P_0(u^2)u) = 0. \tag{2.1}$$

We explain how to come back to the mKdV equation (1.1) in Subsection 6.3. In the sequel of this paper, we choose to take the sign “+” since this sign will not play any role in our analysis. Let us start by giving our notion of solution.

DEFINITION 2.1. Let $T > 0$ and $s \geq 1/6$. We will say that $u \in L^\infty(0, T; H^s(\mathbb{T}))$ is a solution to (1.1) (resp. (2.1)) associated with the initial datum $u_0 \in H^s(\mathbb{T})$ if u satisfies (1.1) and (1.2) (resp. (2.1) and (1.2)) in the distributional sense, i.e. for any test function $\phi \in C_c^\infty(] - T, T[\times \mathbb{T})$, there holds

$$\int_0^\infty \int_{\mathbb{T}} [(\phi_t + \partial_x^3 \phi)u + \phi_x F(u)] dx dt + \int_{\mathbb{T}} \phi(0, \cdot)u_0 dx = 0 \tag{2.2}$$

where $F(u) = u^3$ (resp. $F(u) = u^3 - 3P_0(u^2)u$).

REMARK 2.1. Note that for $u \in L^\infty(0, T; H^s(\mathbb{T}))$, with $s \geq 1/6$, u^3 is well-defined and belongs to $L^\infty(0, T; L^1(\mathbb{T}))$. Therefore (2.2) forces $u_t \in L^\infty(0, T; H^{-3}(\mathbb{T}))$ and ensures that (1.1) (resp. (2.1)) is satisfied in $L^\infty(0, T; H^{-3}(\mathbb{T}))$. In particular, $u \in C([0, T]; H^{-3}(\mathbb{T}))$ and (2.2) forces the initial condition $u(0) = u_0$. Note that, since $u \in L^\infty(0, T; H^s(\mathbb{T}))$, this actually ensures that $u \in C_w([0, T]; H^s(\mathbb{T}))$ and $u \in C([0, T]; H^{s'}(\mathbb{T}))$ for any $s' < s$. Finally, we notice that this also ensures that u satisfies the Duhamel formula associated with (1.1) and (1.2) (resp. (2.1) and (1.2)).

DEFINITION 2.2. Let $s \geq 1/6$. We will say that the Cauchy problem associated with (1.1) (resp. (2.1)) is unconditionally locally well-posed in $H^s(\mathbb{T})$ if for any initial data $u_0 \in H^s(\mathbb{T})$ there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u \in C([0, T]; H^s(\mathbb{T}))$ to (1.1) (resp. (2.1)) emanating from u_0 . Moreover, u is the unique solution to (1.1) (resp. (2.1)) associated with u_0 that belongs to $L^\infty(]0, T[; H^s(\mathbb{T}))$. Finally, for any $R > 0$, the solution-map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{T})$ with radius R centered at the

origin into $C([0, T(R)]; H^s(\mathbb{T}))$.

THEOREM 2.3. *The mKdV equation (1.1) and the renormalized mKdV equation (2.1) are unconditionally locally well-posed in $H^s(\mathbb{T})$ for $s \geq 1/3$.*

2.1. Notation.

Throughout this paper, \mathbb{N} denotes the set of non negative integer numbers. For any positive numbers a and b , the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, α_+ , respectively α_- , will denote a number slightly greater, respectively lesser, than α .

For real numbers $a_1, a_2, a_3 > 0$, we define the quantities $a_{\max} \geq a_{\text{med}} \geq a_{\min}$ to be the maximum, sub-maximum and minimum of a_1, a_2 and a_3 . Usually, we use k_i, j_i to denote integers and $N_i = 2^{k_i}, L_i = 2^{j_i}$ to denote dyadic numbers. For $f = f(x) \in L^2(\mathbb{T})$, we denote its Fourier transform by $\hat{f} : \mathbb{Z} \rightarrow \mathbb{C}$ by $\hat{f}(k) = \int_{\mathbb{T}} e^{-2i\pi kx} f(x) dx$ and for any integer $k \in \mathbb{N}$ we set

$$P_k f = \hat{f}(k) e^{2i\pi kx}, \quad P_{\sim k} = \sum_{|q| \sim k} \hat{f}(q) e^{2i\pi kqx} \quad \text{and} \quad P_{\lesssim k} u = \sum_{|q| \lesssim k} \hat{f}(q) e^{2i\pi qx}.$$

In particular,

$$P_0 f = \hat{f}(0) = \int_{\mathbb{T}} f(x) dx.$$

For $u = u(x, t) \in \mathcal{S}'(\mathbb{R}^2)$, $\mathcal{F}_{tx} u = (u)^{\wedge tx}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = \hat{u}$, respectively $\mathcal{F}_t u = (u)^{\wedge t}$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s , by

$$J_x^s u = \mathcal{F}_x^{-1}((1 + |k|^2)^{s/2} \mathcal{F}_x u) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}(|k|^s \mathcal{F}_x u).$$

We also denote by $U(t) = e^{-t\partial_x^3}$ the unitary group associated to the linear part of (1.1), i.e.,

$$U(t)u_0 = e^{-t\partial_x^3} u_0 = \mathcal{F}_x^{-1}(e^{itk^3} \mathcal{F}_x(u_0)(k)).$$

Throughout the paper, we fix a smooth cutoff function χ such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi|_{[-1, 1]} = 1 \quad \text{and} \quad \text{supp}(\chi) \subset [-2, 2].$$

We set $\phi(k) := \chi(k) - \chi(2k)$. For any $l \in \mathbb{N}$, we define

$$\phi_{2^l}(k) := \phi(2^{-l}k),$$

and

$$\psi_{2^l}(k, \tau) = \phi_{2^l}(\tau - k^3).$$

By convention, we also denote

$$\phi_0(k) = \chi(2k) \quad \text{and} \quad \psi_0(k, \tau) := \chi(2(\tau - k^3)).$$

Any summations over capitalized variables such as N , L , K or M are presumed to be dyadic. We work with non-homogeneous dyadic decompositions i.e., these variables range over numbers of the form $\{2^k : k \in \mathbb{N}\} \cup \{0\}$. We call those numbers *nonhomogeneous dyadic numbers*. Then, we have that $\sum_N \phi_N(k) = 1$

$$\text{supp}(\phi_N) \subset I_N := \left\{ \frac{N}{2} \leq |k| \leq 2N \right\}, \quad N \geq 1, \quad \text{and} \quad \text{supp}(\phi_0) \subset I_0 := \{|k| \leq 1\}.$$

Finally, let us define the Littlewood–Paley multipliers P_N , R_K and Q_L by

$$P_N u = \mathcal{F}_x^{-1}(\phi_N \mathcal{F}_x u), \quad R_K u = \mathcal{F}_t^{-1}(\phi_K \mathcal{F}_t u) \quad \text{and} \quad Q_L u = \mathcal{F}^{-1}(\psi_L \mathcal{F} u),$$

$$P_{\geq N} := \sum_{K \geq N} P_K, \quad P_{\leq N} := \sum_{K \leq N} P_K, \quad Q_{\geq L} := \sum_{K \geq L} Q_K \quad \text{and} \quad Q_{\leq L} := \sum_{K \leq L} Q_K.$$

Sometimes, for the sake of simplicity and when there is no risk of confusion, we also denote $u_N = P_N u$.

2.2. Function spaces.

For $1 \leq p \leq \infty$, $L^p(\mathbb{T})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ denotes the space of all distributions of $(C^\infty(\mathbb{T}))'$ whose usual norm $\|u\|_{H^s} = \|J_x^s u\|_{L^2}$ is finite.

If B is one of the spaces defined above, $1 \leq p \leq \infty$ and $T > 0$, we define the space-time spaces $L_t^p B_x$, and $L_T^p B_x$ equipped with the norms

$$\|u\|_{L_t^p B_x} = \left(\int_{\mathbb{R}} \|f(\cdot, t)\|_B^p dt \right)^{1/p}, \quad \|u\|_{L_T^p B_x} = \left(\int_0^T \|f(\cdot, t)\|_B^p dt \right)^{1/p}$$

with obvious modifications for $p = \infty$. For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to linear KdV group as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|u\|_{X^{s,b}} := \left(\sum_k \int_{\mathbb{R}} \langle \tau - k^3 \rangle^{2b} \langle k \rangle^{2s} |\mathcal{F}_{xt} u(k, \tau)|^2 d\tau \right)^{1/2}, \quad (2.3)$$

where $\langle x \rangle := (1 + |x|^2)^{1/2}$. It is easy to check that

$$\|u\|_{X^{s,b}} \sim \|U(-t)u\|_{H_{x,t}^{s,b}} \quad \text{where} \quad \|u\|_{H_{x,t}^{s,b}} = \|J_x^s J_t^b u\|_{L_{x,t}^2}. \quad (2.4)$$

We defined the function space Z^s , with $s \geq 0$, by

$$Z^s = X^{s-11/10,1} \cap L_t^\infty H_x^s. \quad (2.5)$$

Finally, we will also use restriction in time versions of these spaces. Let $T > 0$ be a positive time and B be a normed space of space-time functions. The restriction space B_T will be the space of functions $u : \mathbb{R} \times]0, T[\rightarrow \mathbb{R}$ or \mathbb{C} satisfying

$$\|u\|_{B_T} := \inf\{\|\tilde{u}\|_B \mid \tilde{u} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ or } \mathbb{C}, \tilde{u}|_{\mathbb{R} \times]0, T[} = u\} < \infty.$$

2.3. Extension operator.

The aim of this subsection is to construct a bounded linear operator from $X_T^{s-11/10,1} \cap L_T^\infty H_x^s$ into Z^s with a bound that does not depend on s and T . For this we follow [13] and introduce the extension operator ρ_T defined by

$$\rho_T(u)(t) := U(t)\chi(t)U(-\mu_T(t))u(\mu_T(t)), \quad (2.6)$$

where χ is the smooth cut-off function defined in Section 2.1 and μ_T is the continuous piecewise affine function defined by

$$\mu_T(t) = \begin{cases} 0 & \text{for } t \notin]0, 2T[, \\ t & \text{for } t \in [0, T], \\ 2T - t & \text{for } t \in [T, 2T]. \end{cases} \quad (2.7)$$

LEMMA 2.4. *Let $0 < T \leq 1$ and $s \in \mathbb{R}$. Then,*

$$\begin{aligned} \rho_T : X_T^{s-11/10,1} \cap L_T^\infty H_x^s &\longrightarrow Z^s \\ u &\mapsto \rho_T(u) \end{aligned}$$

is a bounded linear operator, i.e.,

$$\|\rho_T(u)\|_{L_t^\infty H_x^s} + \|\rho_T(u)\|_{X^{s-11/10,1}} \lesssim \|u\|_{L_T^\infty H_x^s} + \|u\|_{X_T^{s-11/10,1}}, \quad (2.8)$$

for all $u \in X_T^{s-11/10,1} \cap L_T^\infty H_x^s$.

Moreover, the implicit constant in (2.8) can be chosen independent of $0 < T \leq 1$ and $s \in \mathbb{R}$.

PROOF. On one hand, the unitarity of the free group $U(\cdot)$ in $H^s(\mathbb{T})$ easily leads to

$$\|\rho_T(u)\|_{L_t^\infty H^s} \lesssim \|u(\mu_T(\cdot))\|_{L_t^\infty H^s} \lesssim \|u\|_{L_T^\infty H^s} \vee \|u_0\|_{H^s}.$$

On the other hand, the definition of the $X^{s,b}$ -norm and the continuity of μ_T lead to

$$\begin{aligned} &\|\rho_T(u)\|_{X^{s-11/10,1}} \\ &= \|\chi U(-\mu_T(\cdot))u(\mu_T(\cdot))\|_{H_{x,t}^{s-11/10,1}} \\ &\lesssim \|\chi U(-\mu_T(\cdot))u(\mu_T(\cdot))\|_{L_t^2 H^{s-11/10}} + \left\| \partial_t \left(\chi U(-\mu_T(\cdot))u(\mu_T(\cdot)) \right) \right\|_{L_t^2 H^{s-11/10}} \\ &\lesssim \|u(0)\|_{H^{s-11/10}} + \|U(\cdot)u\|_{L^2([0,T]; H^{s-11/10})} + \|U(\cdot)(u_t + \partial_x^3 u)\|_{L^2([0,T]; H^{s-11/10})} \\ &\quad + \left\| U(T-\cdot) \left(-u_t(T-\cdot) - \partial_x^3 u(T-\cdot) \right) \right\|_{L^2([T,2T]; H^{s-11/10})} \\ &\lesssim \|u(0)\|_{H^{s-11/10}} + \|u\|_{X_T^{s-11/10,1}}. \end{aligned}$$

Now, since it is well-known (see for instance [6]), that $X_T^{s-11/10,1} \hookrightarrow C([0, T]; H^{s-11/10}(\mathbb{R}))$, we infer that $u \in C([0, T]; H^{s-11/10}(\mathbb{R})) \cap L_T^\infty H_x^s$ and we claim that

$$\|u(0)\|_{H^s} \leq \|u\|_{L_T^\infty H_x^s}. \quad (2.9)$$

Indeed, if $\|u(0)\|_{H^s} > \|u\|_{L_T^\infty H_x^s}$ there would exist $\epsilon > 0$ and a decreasing sequence $\{t_n\} \subset (0, T)$ tending to 0 such that for any $n \in \mathbb{N}$, $\|u(t_n)\|_{H^s} \leq \|u(0)\|_{H^s} - \epsilon$. The continuity of u with values in $H^{s-1}(\mathbb{R})$ then ensures that $u(t_n) \rightharpoonup u(0)$ in $H^s(\mathbb{R})$, which forces $\|u(0)\|_{H^s} \leq \liminf \|u(t_n)\|_{H^s}$ and yields a contradiction.

Gathering the two above estimates, we thus infer that for any $(T, s) \in]0, +\infty[\times \mathbb{R}$, ρ_T is a bounded linear operator from $L_T^\infty H^s \cap X_T^{s-11/10,1}$ into $L_t^\infty H^s \cap X^{s-11/10,1}$ with a bound that does not depend on (T, s) . \square

3. A priori estimates on solutions.

3.1. Preliminaries.

DEFINITION 3.1. Let η be a (possibly complex-valued) bounded function on \mathbb{Z}^3 . We define the *pseudo-product* Π_η (that will also be denoted by Π when there is no risk of confusion) in Fourier variable by

$$\mathcal{F}_x(\Pi_\eta(f, g, h))(k) = \sum_{k_1+k_2+k_3=k} \eta(k_1, k_2, k_3) \widehat{f}(k_1) \widehat{g}(k_2) \widehat{h}(k_3). \quad (3.1)$$

Moreover for any dyadic integer $M \geq 1$ and any $j \in \{1, 2, 3\}$, we also denote $\Pi_{\eta, M}^j$ (or Π_M^j when there is no risk of confusion) the operator defined in Fourier variable by

$$\mathcal{F}_x(\Pi_{\eta, M}^j(f, g, h))(k) = \sum_{k_1+k_2+k_3=k} \eta(k_1, k_2, k_3) \phi_M \left(\sum_{\substack{1 \leq q \leq 3 \\ q \neq j}} k_q \right) \widehat{f}(k_1) \widehat{g}(k_2) \widehat{h}(k_3). \quad (3.2)$$

The following technical lemma corresponds to integration by parts for some pseudo-products (cf. [17]).

LEMMA 3.2. Assume that the (possibly complex-valued) bounded measurable function η satisfies on \mathbb{Z}^3

$$\eta(k_1, k_2, k_3) = \eta(k_2, k_1, k_3) = \Lambda(k_1, k_2) \Theta(|k_2 + k_3|, |k_1 + k_3|, |k_1 + k_2|) \quad (3.3)$$

for some functions Θ on \mathbb{Z}^3 and Λ on \mathbb{Z}^2 with $\Lambda(k_1, k_2) = \Lambda(k_2, k_1)$, $\forall (k_1, k_2) \in \mathbb{Z}^2$. Let N and M be two nonhomogeneous dyadic numbers satisfying $N \gg 1$. For any real-valued functions $f_1, f_2, g \in L^2(\mathbb{R})$, we define

$$T_{\eta, M, N}(f_1, f_2, g, g) = \int_{\mathbb{T}} \partial_x P_N \Pi_{\eta, M}^3(f_1, f_2, g) P_N g \, dx. \quad (3.4)$$

Then, for $M \ll N$, it holds

$$T_{\eta, M, N}(f_1, f_2, g, g) = M \int_{\mathbb{T}} \Pi_{\eta_2, M}^3(f_1, f_2, P_{\sim N} g) P_N g \, dx, \quad (3.5)$$

where η_2 is a function of (k_1, k_2, k_3) whose l^∞ -norm is uniformly bounded in N and M .

PROOF. From Plancherel's identity we may rewrite $T_{\eta,M,N}$ as

$$\begin{aligned} T_{\eta,M,N}(f_1, f_2, g, g) &= \sum_{\substack{(k_1, k_2, k_3, k) \in \mathbb{Z}^4 \\ k_1 + k_2 + k_3 - k = 0}} (ik)\eta(k_1, k_2, k_3)\phi_M(k_1 + k_2)\phi_N(k)^2 \widehat{f}_1(k_1)\widehat{f}_2(k_2)\widehat{g}(k_3)\overline{\widehat{g}(k)}. \end{aligned}$$

We decompose $T_{M,N}(f_1, f_2, g, g)$ as follows:

$$\begin{aligned} T_{\eta,M,N}(f_1, f_2, g, g) &= M \sum_{\substack{(k_1, k_2, k_3, k) \in \mathbb{Z}^4 \\ k_1 + k_2 + k_3 - k = 0}} (ik)\eta(k_1, k_2, k_3)\phi_M(k_1 + k_2) \left(\frac{\phi_N(k) - \phi_N(k_3)}{M} \right) \widehat{f}_1(k_1)\widehat{f}_2(k_2)\widehat{g}(k_3)\overline{\widehat{g}_N(k)} \\ &\quad + \sum_{\substack{(k_1, k_2, k_3, k) \in \mathbb{Z}^4 \\ k_1 + k_2 + k_3 - k = 0}} (ik)\eta(k_1, k_2, k_3)\phi_M(k_1 + k_2)\widehat{f}_1(k_1)\widehat{f}_2(k_2)\widehat{g}_N(k_3)\overline{\widehat{g}_N(k)} \\ &=: I_1 + I_2. \end{aligned} \tag{3.6}$$

We notice that I_1 can be rewritten as

$$I_1 = M \int_{\mathbb{T}} \Pi_{\eta_1, M}^3(f_1, f_2, P_{\sim N}g) P_N g \, dx$$

with

$$\eta_1(k_1, k_2, k_3) = i \frac{\phi_N(k_1 + k_2 + k_3) - \phi_N(k_3)}{M} (k_1 + k_2 + k_3) \mathbb{1}_{\text{supp } \phi_M}(k_1 + k_2)$$

and that it clearly follows from the mean value theorem and the frequency localization that η_1 is uniformly bounded in M and N .

Next, we deal with I_2 . Since g is real-valued, we have $\overline{\widehat{g}_N(k)} = \widehat{g}_N(-k)$, so that

$$I_2 = \sum_{\substack{(k_1, k_2, k_3, k) \in \mathbb{Z}^4 \\ k_1 + k_2 + k_3 - k = 0}} (ik)\eta(k_1, k_2, k_3)\phi_M(k_1 + k_2)\widehat{f}_1(k_1)\widehat{f}_2(k_2)\overline{\widehat{g}_N(-k_3)}\widehat{g}_N(-k).$$

Performing the change of variables $(\hat{k}_3, \hat{k}) = (-k, -k_3)$ so that $\hat{k} = k_1 + k_2 + \hat{k}_3$, we get

$$I_2 = \sum_{\substack{(k, k_1, \hat{k}_2, \hat{k}_3) \in \mathbb{Z}^4 \\ k_1 + k_2 + \hat{k}_3 - \hat{k} = 0}} (-i\hat{k}_3)\eta(k_1, k_2, -\hat{k})\phi_M(k_1 + k_2)\widehat{f}_1(k_1)\widehat{f}_2(k_2)\overline{\widehat{g}_N(\hat{k})}\widehat{g}_N(\hat{k}_3).$$

Now, observe that $|k_1 - \hat{k}| = |k_2 + \hat{k}_3|$ and $|k_2 - \hat{k}| = |k_1 + \hat{k}_3|$. Thus, according to (3.3)

$$\eta(k_1, k_2, -\hat{k}) = \eta(k_2, k_1, -\hat{k}) = \Lambda(k_1, k_2)\Theta(|k_2 + \hat{k}_3|, |k_1 + \hat{k}_3|, |k_1 + k_2|) = \eta(k_1, k_2, \hat{k}_3)$$

so that

$$I_2 = \sum_{\substack{(\hat{k}, k_1, k_2, \hat{k}_3) \in \mathbb{Z}^4 \\ k_1 + k_2 + \hat{k}_3 - \hat{k} = 0}} i(k_1 + k_2)\eta(k_1, k_2, \hat{k}_3)\phi_M(k_1 + k_2)\widehat{f}_1(k_1)\widehat{f}_2(k_2)\overline{\widehat{g}_N(\hat{k})}\widehat{g}_N(\hat{k}_3) - I_2.$$

Setting

$$\tilde{\eta}_1(k_1, k_2, k_3) = \frac{i(k_1 + k_2)}{M} \mathbb{1}_{\text{supp } \phi_M}(k_1 + k_2) \phi_N(k_3),$$

this leads to

$$T_{\eta, M, N}(f_1, f_2, g, g) = M \int_{\mathbb{T}} \Pi_{\eta \eta_2, M}^3(f_1, f_2, P_{\sim N} g) P_N g \, dx \quad (3.7)$$

where $\eta_2 = \eta_1 + (1/2)\tilde{\eta}_1$ is also uniformly bounded function in M and N and completes the proof of the Lemma. \square

The following proposition gives suitable estimates for the pseudo-products Π_M .

PROPOSITION 3.3. *Let M be a dyadic number with $M \geq 1$ and η is a bounded measurable function. Then for all $j = 1, 2, 3$ and all $f_i \in L^2(\mathbb{T})$ it holds that*

$$\left| \int_{\mathbb{T}} \Pi_{\eta, M}^j(f_1, f_2, f_3) f_4 \, dx \right| \lesssim M \prod_{i=1}^4 \|f_i\|_{L^2}, \quad (3.8)$$

where the implicit constant only depends on the L^∞ -norm of the function η .

PROOF. By symmetry we can assume that $j = 3$. Since the norms in the right-hand side only see the size of the modulus of the Fourier transform, we can assume that all the functions have non negative Fourier transform. By using Plancherel's formula, Hölder and Bernstein inequalities, we get that

$$\begin{aligned} & \left| \int_{\mathbb{T}} \Pi_{\eta, M}^3(f_1, f_2, f_3) f_4 \, dx \right| \\ & \lesssim \sum_{(k_1, k_2, k) \in \mathbb{Z}^3} \phi_M(k_1 + k_2) \widehat{f}_1(k_1) \widehat{f}_2(k_2) \widehat{f}_3(k - k_1 - k_2) \widehat{f}_4(k) \, dk_1 \, dk_2 \, dk \\ & = \int_{\mathbb{T}} P_M(f_1 f_2) f_3 f_4 \, dx \\ & \lesssim \|P_M(f_1 f_2)\|_{L_x^\infty} \|f_3\|_{L_x^2} \|f_4\|_{L_x^2} \\ & \lesssim M \prod_{i=1}^4 \|f_i\|_{L_x^2}. \end{aligned} \quad \square$$

Before stating the main result of this subsection, let us define the resonance function of order 3 by

$$\begin{aligned} \Omega_3(k_1, k_2, k_3) &= k_1^3 + k_2^3 + k_3^3 - (k_1 + k_2 + k_3)^3 \\ &= -3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3). \end{aligned} \quad (3.9)$$

PROPOSITION 3.4. *Assume that $0 < T \leq 1$, η is a bounded measurable function and u_i are functions in Z_T^0 where Z^0 is defined in (2.5). Assume also that $N \geq 2^8$, $N_1, N_2, N_3 \geq 1$, $M \geq 1$ and $j \in \{1, 2, 3\}$. We define*

$$G_{\eta,M}^T(u_1, u_2, u_3, u_4) := \int_{]0,T[\times \mathbb{T}} \Pi_{\eta,M}^j(u_1, u_2, u_3) u_4 dx dt. \quad (3.10)$$

Then for any $K \gg 1$ it holds

$$|G_{\eta^1|_{|\Omega_3| \geq K}, M}(P_{\lesssim N_1} u_1, P_{\lesssim N_2} u_2, P_{\lesssim N_3} u_3, P_N u_4)| \lesssim T^{1/8} M \frac{N_{\max}^{11/10}}{K} \prod_{i=1}^4 \|u_i\|_{Z_T^0}, \quad (3.11)$$

where $N_{\max} = \max(N_1, N_2, N_3)$. Moreover, the implicit constant in estimates (3.11) only depend on the L^∞ -norm of the function η .

REMARK 3.1. Sometimes, when there is no risk of confusion, we also denote

$$G_M^T(u_1, u_2, u_3, u_4) = G_{\eta,M}^T(u_1, u_2, u_3, u_4).$$

To prove Proposition 3.4, we need the following technical lemmas derived in [16]. For any $0 < T \leq 1$, let us denote by 1_T the characteristic function of the interval $]0, T[$. One of the main difficulty in the proof of Proposition 3.4 is that the operator of multiplication by 1_T does not commute with Q_L . To handle this situation, we follow the arguments introduced in [16] and use the decomposition

$$1_T = 1_{T,R}^{\text{low}} + 1_{T,R}^{\text{high}}, \quad \text{with } \mathcal{F}_t(1_{T,R}^{\text{low}})(\tau) = \chi(\tau/R) \mathcal{F}_t(1_T)(\tau), \quad (3.12)$$

for some $R > 0$ to be fixed later.

LEMMA 3.5. *Let L be a nonhomogeneous dyadic number. Then the operator $Q_{\leq L}$ is bounded in $L_t^\infty L_x^2$ uniformly in L . In other words,*

$$\|Q_{\leq L} u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2}, \quad (3.13)$$

for all $u \in L_t^\infty L_x^2$ and the implicit constant appearing in (3.13) does not depend on L .

LEMMA 3.6. *For any $R > 0$ and $T > 0$ it holds*

$$\|1_{T,R}^{\text{high}}\|_{L^1} \lesssim T \wedge R^{-1} \quad (3.14)$$

and

$$\|1_{T,R}^{\text{high}}\|_{L^\infty} + \|1_{T,R}^{\text{low}}\|_{L^\infty} \lesssim 1. \quad (3.15)$$

LEMMA 3.7. *Assume that $T > 0$, $R > 0$ and $L \gg R$. Then, it holds*

$$\|Q_L(1_{T,R}^{\text{low}} u)\|_{L^2} \lesssim \|Q_{\sim L} u\|_{L^2}, \quad (3.16)$$

for all $u \in L^2(\mathbb{R}^2)$.

PROOF OF PROPOSITION 3.4. We take the extensions $\tilde{u}_i = \rho_T(u_i)$ of u_i defined in (2.7) and to lighten the notations we denote $P_{\lesssim N_i} \tilde{u}_i$ by \tilde{u}_i , $i = 1, 2, 3$ and $P_N \tilde{u}_4$ by \tilde{u}_4 . We first notice that (3.9) ensures that for $K > 2^8 N_{\max}^3$, the set $|\Omega_3| \geq K$ is empty

and thus (3.11) holds trivially. We can thus assume that $K \leq 2^8 N_{\max}^3$. We set

$$R = \left(\frac{K}{N_{\max}^{11/10}} \right)^{8/7} = K \left(\frac{K}{N_{\max}^{44/5}} \right)^{1/7} \leq K \left(2^8 N_{\max}^{-29/5} \right)^{1/7} \ll K, \quad (3.17)$$

since $N \geq 2^8$ ensures that $N_{\max} \geq 2^5$, and decompose 1_T as $1_T = 1_{T,R}^{\text{high}} + 1_{T,R}^{\text{low}}$. The contribution of the first term is easily estimated thanks to Hölder's inequality in time, (3.8), (3.14) and (3.15) by

$$\begin{aligned} & \left| G_{\eta^{\sharp}|\Omega_3| \geq K, M} (1_{T,R}^{\text{high}} \check{u}_1, \check{u}_2, \check{u}_3, \check{u}_4) \right| \\ & \leq T^{1/8} \|1_{T,R}^{\text{high}}\|_{L^{8/7}} \left\| \int_{\mathbb{T}} \Pi_M^j(\check{u}_1, \check{u}_2, \check{u}_3) \check{u}_4 dx \right\|_{L_t^\infty} \\ & \lesssim T^{1/8} R^{-7/8} M \prod_{i=1}^4 \|\check{u}_i\|_{L_t^\infty L_x^2} \\ & \lesssim T^{1/8} \frac{MN_{\max}^{11/10}}{K} \prod_{i=1}^4 \|u_i\|_{L_T^\infty L_x^2}. \end{aligned}$$

To deal with the contribution of $1_{T,R}^{\text{low}}$, we first note (3.17) ensures that we are in the hypotheses of Lemma 3.7. Now, by the definition of Ω_3 , we may decompose the contribution of $1_{T,R}^{\text{low}}$ as

$$\begin{aligned} & G_{\eta^{\sharp}|\Omega_3| \geq K, M} (1_{T,R}^{\text{low}} \check{u}_1, \check{u}_2, \check{u}_3, \check{u}_4) \\ & = G_{\eta^{\sharp}|\Omega_3| \geq K, M} (Q_{\gtrsim K} (1_{T,R}^{\text{low}} \check{u}_1), \check{u}_2, \check{u}_3, \check{u}_4) \\ & \quad + G_{\eta^{\sharp}|\Omega_3| \geq K, M} (Q_{\ll K} (1_{T,R}^{\text{low}} \check{u}_1), Q_{\gtrsim K} u_2, u_3, \check{u}_4) \\ & \quad + G_{\eta^{\sharp}|\Omega_3| \geq K, M} (Q_{\ll K} (1_{T,R}^{\text{low}} \check{u}_1), Q_{\ll K} \check{u}_2, Q_{\gtrsim K} \check{u}_3, \check{u}_4) \\ & \quad + G_{\eta^{\sharp}|\Omega_3| \geq K, M} (Q_{\ll K} (1_{T,R}^{\text{low}} \check{u}_1), Q_{\ll K} \check{u}_2, Q_{\ll K} \check{u}_3, Q_{\gtrsim K} \check{u}_4). \end{aligned} \quad (3.18)$$

The first term of the right-hand side of the above equality can be estimated thanks to (3.8) and (3.16) by

$$\begin{aligned} \left| G_{\eta^{\sharp}|\Omega_3| \geq K, M} (Q_{\gtrsim K} (1_{T,R}^{\text{low}} \check{u}_1), \check{u}_2, \check{u}_3, \check{u}_4) \right| & \lesssim T^{1/2} M \|Q_{\gtrsim K} (1_{T,R}^{\text{low}} \check{u}_1)\|_{L_{x,t}^2} \prod_{j=2}^4 \|\check{u}_j\|_{L_T^\infty L_x^2} \\ & \lesssim T^{1/2} \frac{MN_{\max}^{11/10}}{K} \|\check{u}_1\|_{X^{-11/10,1}} \prod_{j=2}^4 \|\check{u}_j\|_{L_t^\infty L_x^2} \\ & \lesssim T^{1/2} \frac{MN_{\max}^{11/10}}{K} \prod_{i=1}^4 \|u_i\|_{Z_T^0}. \end{aligned}$$

The other terms can be controlled in exactly the same way. Note that we use (3.13) and not (3.16) for these terms. \square

3.2. Uniform estimates on solutions.

The preceding subsection enables us to easily get an uniform H^s -bound for solutions to (2.1). This is the aim of this subsection where we do not attempt to get the lowest propagated regularity since we will be forced to take $s \geq 1/3$ in the estimate on the difference.

We first prove refined Strichartz estimates. The following linear estimate in Bourgain's space is established in [1],

$$\|u\|_{L^4([0,1] \times \mathbb{T})} \lesssim \|u\|_{X^{0,1/3}} \quad \forall u \in X^{0,1/3}.$$

We will make use of a Strichartz estimate which follows directly from the above estimate (see for instance [6]),

$$\|U(t)\varphi\|_{L^4([0,T] \times \mathbb{T})} \lesssim T^{1/6} \|\varphi\|_{L^2}, \quad \forall T \in]0,1[, \quad \forall \varphi \in L^2(\mathbb{T}), \quad (3.19)$$

where the implicit constant does not depend on T .

LEMMA 3.8. *Let $0 < T < 1$ and let $u \in L^\infty(0, T; H^s(\mathbb{T}))$ be a solution to (2.1) emanating from $u_0 \in H^s(\mathbb{T})$. For $s > 11/35$ it holds*

$$\|u\|_{L_T^4 L_x^{20}} \lesssim \|u\|_{L_T^\infty H_x^s} (1 + \|u\|_{L_T^\infty H_x^s}^2) \quad (3.20)$$

and for $s > 9/28$,

$$\|D_x^{5/24} u\|_{L_T^4 L_x^4} \lesssim \|u\|_{L_T^\infty H^s} (1 + \|u\|_{L_T^\infty H^s}^2). \quad (3.21)$$

PROOF. Let u be a solution to (2.1) defined on a time interval $[0, T]$. We use a nonhomogeneous Littlewood–Paley decomposition, $u = \sum_N u_N$ where $u_N = P_N u$ and N is a dyadic integer. Since, (3.20) and (3.21) are obvious for $N \lesssim 1$, it suffices to control $\|u_N\|_{L_T^4 L_x^{20}}$ for any $N \gg 1$. For such N , Sobolev and Bernstein inequalities lead to

$$\|u_N\|_{L_T^4 L_x^{20}} \lesssim N^{1/5} \|u_N\|_{L_{T,x}^4}.$$

Let δ be a nonnegative number to be fixed later. We chop out the interval in small intervals of length $N^{-\delta}$. In other words, we have that $[0, T] = \bigcup_{j \in J} I_j$ where $I_j = [a_j, b_j]$, $|I_j| \sim N^{-\delta}$ and $\#J \sim N^\delta$. Since u_N is a solution to the integral equation

$$u_N(t) = e^{-(t-a_j)\partial_x^3} u_N(a_j) + \int_{a_j}^t e^{-(t-t')\partial_x^3} P_N \partial_x (u^3 - 3P_0(u^2)u)(t') dt'$$

for $t \in I_j$, we deduce from (3.19) that

$$\begin{aligned} & \|u_N\|_{L_T^4 L_x^{20}} \\ & \lesssim \left(\sum_j \|D_x^{-\delta/6+1/5} u_N(a_j)\|_{L_x^2}^4 \right)^{1/4} \end{aligned}$$

$$\begin{aligned}
& + \left(\sum_j \left(\int_{I_j} \|D_x^{-\delta/6+6/5} P_N(u^3 - 3P_0(u^2)u)(t')\|_{L_x^2} dt' \right)^4 \right)^{1/4} \\
& \lesssim N^{\delta/4} \|D_x^{-\delta/6+1/5} u_N\|_{L_T^\infty L_x^2} \\
& + \left(\sum_j |I_j|^3 \int_{I_j} \|D_x^{-\delta/6+6/5} P_N(u^3 - 3P_0(u^2)u)(t')\|_{L_x^2}^4 dt' \right)^{1/4} \\
& \lesssim N^{0-} \left[\|D_x^{\delta/12+1/5+} u_N\|_{L_T^\infty L_x^2} + \|D_x^{(-11/12)\delta+6/5+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2} \right]. \tag{3.22}
\end{aligned}$$

Now, to prove (3.20) we notice that Sobolev's inequalities and the fractional Leibniz rule lead to

$$\begin{aligned}
& \|D_x^{(-11/12)\delta+6/5+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2} \\
& \lesssim \|D_x^{-11\delta/12+1/p+7/10+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^p} \\
& \lesssim \|u\|_{L_T^\infty L_x^q}^2 \|D_x^{\kappa+} u\|_{L_T^\infty L_x^2} \tag{3.23}
\end{aligned}$$

for all $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ satisfying $2/q + 1/2 = 1/p$ and $0 < \kappa = (-11/12)\delta + 7/10 + 1/p < 1$. Thus, the Sobolev embedding yields

$$\|D_x^{(-11/12)\delta+6/5+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2} \lesssim \|u\|_{L_T^\infty H_x^{\kappa+}}^3, \tag{3.24}$$

if we choose κ satisfying $\kappa = 1/2 - 1/q = 3/4 - 1/2p$. This implies that

$$\kappa = -\frac{11}{12}\delta + \frac{7}{10} + \frac{1}{p} = -\frac{11}{12}\delta + \frac{11}{5} - 2\kappa \Rightarrow \kappa = -\frac{11}{36}\delta + \frac{11}{15}. \tag{3.25}$$

Then, we choose δ such that $\delta/12 + 1/5 = \kappa$ which leads to

$$\delta = \frac{48}{35}, \quad \kappa = \frac{11}{35}, \quad p = \frac{70}{61} \quad \text{and} \quad q = \frac{70}{13}.$$

Therefore, we conclude gathering (3.22)–(3.25) that if u is a solution to (2.1) and (1.2) defined on the time interval $[0, T]$, then for $N \gg 1$,

$$\|P_N u\|_{L_T^4 L_x^{20}} \lesssim N^{0-} \left[\|D_x^{11/35+} P_N u_0\|_{L_x^2} + \|D_x^{11/35+} u\|_{L_T^\infty L_x^2}^3 \right], \tag{3.26}$$

which proves (3.20) with $s > 11/35$, by summing over N .

To prove (3.21) we proceed in the same way. We eventually obtain for $N \gg 1$,

$$\|D_x^{5/24} u_N\|_{L_T^4 L_x^4} \lesssim \|D_x^{\delta/12+5/24+} u_N\|_{L_T^\infty L_x^2} + \|D_x^{(-11/12)\delta+29/24+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2}$$

with

$$\|D_x^{(-11/12)\delta+29/24+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2} \lesssim \|u\|_{L_x^q}^2 \|D_x^{\kappa+} u\|_{L_T^\infty L_x^2},$$

for all $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ satisfying $2/q + 1/2 = 1/p$ and $0 < \kappa = (-11/12)\delta +$

$17/24 + 1/p < 1$. Thus, the Sobolev embedding yields

$$\|D_x^{(-11/12)\delta+29/24+} P_N(u^3 - 3P_0(u^2)u)\|_{L_T^4 L_x^2} \lesssim \|u\|_{L_T^\infty H_x^{\kappa+}}^3,$$

if we choose κ satisfying $\kappa = 1/2 - 1/q = 3/4 - 1/2p$. This implies that

$$\kappa = -\frac{11}{12}\delta + \frac{17}{24} + \frac{1}{p} = -\frac{11}{12}\delta + \frac{53}{24} - 2\kappa \quad \Rightarrow \quad \kappa = -\frac{11}{36}\delta + \frac{53}{72}.$$

Then, choosing δ such that $\delta/12 + 5/24 = \kappa$ which leads to

$$\delta = \frac{19}{14}, \quad \kappa = \frac{9}{28}, \quad p = \frac{7}{6} \quad \text{and} \quad q = \frac{28}{5},$$

we obtain (3.21) with $s = 9/28+$. \square

LEMMA 3.9. *Assume that $0 < T \leq 1$, $s > 11/35$ and $u \in L^\infty(0, T; H^s(\mathbb{T}))$ is a solution to (2.1) emanating from $u_0 \in H^s(\mathbb{T})$. Then,*

$$\|u\|_{Z_T^s} \lesssim \|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^\infty H_x^s}^3 (1 + \|u\|_{L_T^\infty H_x^s})^4. \quad (3.27)$$

PROOF. By using Lemma 2.4, it is clear that we only have to estimate the $X_T^{s-11/10,1}$ -norm of u to prove (3.27). Now, using the Duhamel formula associated to (2.1), the standard linear estimates in Bourgain's spaces and the fractional Leibniz rule (c.f. Theorem A.12 in [10]), we have that

$$\begin{aligned} \|u\|_{X_T^{s-11/10,1}} &\lesssim \|u_0\|_{H^{s-11/10}} + \|\partial_x(u^3)\|_{X_T^{s-11/10,0}} + \|P_0(u^2)u_x\|_{X_T^{s-11/10,0}} \\ &\lesssim \|u_0\|_{H^{s-11/10}} + \|J_x^{s-1/10}(u^3)\|_{L_T^2 L_x^2} + \|u\|_{L_T^\infty L_x^2}^2 \|u\|_{L_T^2 H_x^{s-1/10}} \\ &\lesssim \|u\|_{L_T^\infty H^s} + \|J_x^s(u^3)\|_{L_T^2 L_x^{5/3}} + \|u\|_{L_T^\infty L_x^2}^2 \|u\|_{L_T^2 H_x^s} \\ &\lesssim \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^4 L_x^{20}}^2 \|J_x^s u\|_{L_T^\infty L_x^2} + \|u\|_{L_T^\infty L_x^2}^2 \|u\|_{L_T^2 H_x^s}, \end{aligned} \quad (3.28)$$

which leads to (3.27) by using (3.20). \square

To prove the main result of this section we need to define some subsets of \mathbb{Z}^3 . In the sequel we set

$$D = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : (k_1 + k_2)(k_1 + k_3)(k_2 + k_3) \neq 0\}, \quad (3.29)$$

and

$$\begin{aligned} D^1 &= \left\{ (k_1, k_2, k_3) \in D : \min_{1 \leq i \neq j \leq 3} (|k_i + k_j|) \lesssim 2^{-9} |k_1 + k_2 + k_3| \right\}, \\ D^2 &= D \setminus D^1. \end{aligned} \quad (3.30)$$

To bound from below the resonance function $|\Omega_3|$ (see (3.9)) on D^1 and D^2 we will make a frequent use of the following lemma.

LEMMA 3.10. *On D^1 , it holds*

$$|k_1| \sim |k_2| \sim |k_3| \sim |k| \quad \text{and} \quad \max_{1 \leq i \neq j \leq 3} (|k_i + k_j|) \gtrsim |k|,$$

where $|k| = |k_1 + k_2 + k_3|$, whereas on D^2 it holds

$$\min_{1 \leq i \neq j \leq 3} (|k_i + k_j|) \gtrsim \max_{1 \leq i \leq 3} |k_i|.$$

PROOF. To prove the first assertion, we assume without loss of generality that $|k_2 + k_3| \geq |k_1 + k_3| \geq |k_1 + k_2|$. On D^1 , this forces $|k_2| \sim |k_3| \sim |k_1 + k_2 + k_3|$. On one hand $|k_1| \ll |k_1 + k_2 + k_3|$ would imply $|k_1 + k_3| \sim |k_1 + k_2 + k_3|$ which can not hold. On the other hand, $|k_1| \gg |k_1 + k_2 + k_3|$ would imply $\max(|k_2|, |k_3|) \sim |k_1| \gg |k_1 + k_2 + k_3|$ which is in contradiction with the preceding deduction. Therefore $|k_1| \sim |k_1 + k_2 + k_3|$. Finally, either $k_2 k_3 \geq 0$ and then $|k_2 + k_3| \gtrsim |k_1 + k_2 + k_3|$ or $k_2 k_3 < 0$ and then $\max(|k_1 + k_2|, |k_1 + k_3|) \gtrsim |k_1 + k_2 + k_3|$.

To prove the second assertion, we first notice that this assertion is trivial when $\max_{1 \leq i \leq 3} |k_i| \sim |k|$ where $k = k_1 + k_2 + k_3$. We thus can assume that $\max_{1 \leq i \leq 3} |k_i| \gg |k|$. By symmetry, we can assume that $|k_1| \geq |k_2| \geq |k_3|$. This forces $|k_2| \sim |k_1| \gg |k|$. Therefore $|k_1 + k_3| = |k - k_2| \sim |k_1|$ and $|k_2 + k_3| = |k - k_1| \sim |k_1|$. \square

Let us also set

$$m_{\min} = \min_{1 \leq i \neq j \leq 3} |k_i + k_j| \quad (3.31)$$

and

$$\begin{aligned} A_1 &= \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : |k_2 + k_3| = m_{\min}\}, \\ A_2 &= \{(k_1, k_2, k_3) \in \mathbb{Z}^3 / A_1 : |k_1 + k_3| = m_{\min}\}, \\ A_3 &= \{(k_1, k_2, k_3) \in \mathbb{Z}^3 / (A_1 \cup A_2) : |k_1 + k_2| = m_{\min}\} = \mathbb{Z}^3 / (A_1 \cup A_2). \end{aligned} \quad (3.32)$$

Then, it is clear from the definition of those sets that

$$\sum_{j=1}^3 \mathbb{1}_{A_j}(k_1, k_2, k_3) = 1, \quad \forall (k_1, k_2, k_3) \in \mathbb{Z}^3. \quad (3.33)$$

However, we will not work directly with this partition because of a lack of symmetry. Note that $\chi_{A_1}(k_1, k_2, k_3) = \chi_{A_1}(k_1, k_3, k_2)$ and $\chi_{A_3}(k_1, k_2, k_3) = \chi_{A_3}(k_2, k_1, k_3)$ but $\chi_{A_2}(k_1, k_2, k_3) \neq \chi_{A_2}(k_3, k_2, k_1)$ on some set of \mathbb{Z}^3 . Therefore, in order to apply Lemma 3.2, we have to symmetrize this partition¹. For this we set

$$\mathbb{1}_{\bar{A}_j}(k_1, k_2, k_3) = \mathbb{1}_{A_j}(k_3, k_2, k_1).$$

Then, for all $(k_1, k_2, k_3) \in \mathbb{Z}^3$,

¹Note that one does not need such symmetrization on the real line since the sets $|k_i + k_j| = |k_{i'} + k_{j'}|$ with $(i, j) \neq (i', j')$ are of measure zero.

$$1 = \frac{1}{2} \sum_{j=1}^3 \left(\mathbb{1}_{A_j} + \mathbb{1}_{\tilde{A}_j} \right) (k_1, k_2, k_3) = \sum_{j=1}^3 \Theta_j(k_1, k_2, k_3) \quad (3.34)$$

where

$$\Theta_1 = \frac{1}{2} (\mathbb{1}_{A_1} + \mathbb{1}_{\tilde{A}_3}), \quad \Theta_2 = \frac{1}{2} (\mathbb{1}_{A_2} + \mathbb{1}_{\tilde{A}_2}) \quad \text{and} \quad \Theta_3 = \frac{1}{2} (\mathbb{1}_{A_3} + \mathbb{1}_{\tilde{A}_1}). \quad (3.35)$$

Note that this new partition satisfies the following symmetry property: for all $j \in \{1, 2, 3\}$ and $(k_1, k_2, k_3) \in \mathbb{Z}^3$,

$$\Theta_j(k_1, k_2, k_3) = \Theta_j(k_{\sigma_j(1)}, k_{\sigma_j(2)}, k_{\sigma_j(3)}) \quad (3.36)$$

where $\sigma_j \in \mathcal{S}_3$ is defined by $\sigma_j(j) = j$ and $\sigma_j(i) \neq i$ for $i \neq j$.

We are now in position to prove the main result of this section.

PROPOSITION 3.11. *Assume that $0 < T \leq 1$, $s > 3/10$ and that $u \in Z_T^s$ is a solution to (2.1) with initial data $u_0 \in H^s(\mathbb{T})$. Then,*

$$\|u\|_{L_T^\infty H_x^s}^2 \lesssim \|u_0\|_{H^s}^2 + T^{1/8} \|u\|_{Z_T^s}^4. \quad (3.37)$$

PROOF. By using (2.1), we have

$$\frac{1}{2} \frac{d}{dt} \|P_N u(\cdot, t)\|_{L_x^2}^2 = -\Re \left[\int_{\mathbb{T}} \partial_x P_N \left(u^3 - 3P_0(u^2)u \right) P_N u \, dx \right]$$

which yields after integration in time between 0 and t and summation over N

$$\sum_N \|P_N u(t)\|_{H_x^s}^2 \lesssim \|u_0\|_{H^s}^2 + \sum_N N^{2s} \left| \Re \left[\int_{\mathbb{T} \times [0, t]} \partial_x P_N \left(u^3 - 3P_0(u^2)u \right) P_N u \, dx ds \right] \right|. \quad (3.38)$$

In the case where $N \lesssim 1$, we easily get

$$\begin{aligned} \left| \sum_N N^{2s} \int_{\mathbb{T} \times [0, t]} \partial_x P_N \left(u^3 - 3P_0(u^2)u \right) P_N u \, dx ds \right| &\lesssim \|u\|_{L_T^3 L_x^3}^3 \|P_N u\|_{L_T^\infty L_x^\infty} + \|u\|_{L_T^\infty L_x^2}^4 \\ &\lesssim \|u\|_{L_T^\infty H_x^{1/6}}^4 \lesssim \|u\|_{Z_T^{1/6}}^4. \end{aligned} \quad (3.39)$$

In the following, we can then assume that $N \gg 1$ and we use the classical decomposition of $N(u) := \partial_x(u^3 - 3P_0(u^2)u)$ in a resonant and a non resonant part by writing:

$$\begin{aligned} \mathcal{F}_x[N(u)](k) &= ik \left[\sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_1+k_3)(k_2+k_3) \neq 0}} \hat{u}(k_1)\hat{u}(k_2)\hat{u}(k_3) - 3\hat{u}(k)\hat{u}(k)\hat{u}(-k) \right] \\ &:= \left(\mathcal{F}_x \left[A(u, u, u) \right] (k) - \mathcal{F}_x \left[B(u, u, u) \right] (k) \right) \end{aligned}$$

i.e.

$$\partial_x(u^3 - 3P_0(u^2)u) = \partial_x(A(u, u, u) - B(u, u, u)). \quad (3.40)$$

Now, we notice that, since u is real-valued, we have

$$\int_{\mathbb{T}} \partial_x P_N B(u, u, u) P_N u = ik \sum_{k \in \mathbb{Z}} |\hat{u}(k)|^2 |\varphi_N(k) \hat{u}(k)|^2 \in i\mathbb{R}. \quad (3.41)$$

Therefore (3.38) and (3.40) lead to

$$\begin{aligned} \sum_{N \gg 1} \|P_N u(t)\|_{H_x^s}^2 &\lesssim \|u_0\|_{H^s}^2 + \left| \Re \left[\sum_{N \gg 1} N^{2s} \int_{\mathbb{T} \times [0, t]} \partial_x P_N (A(u, u, u)) P_N u \, dx ds \right] \right| \\ &\lesssim \|u_0\|_{H^s}^2 + \sum_{N \gg 1} J_{t, N}(u). \end{aligned}$$

By using the decomposition in (3.34), we get that $J_{t, N}(u) = |\Re(\sum_{l=1}^3 J_{t, N}^l(u))|$ with

$$J_{t, N}^l(u) = N^{2s} \sum_{M \geq 1} \int_{\mathbb{T} \times [0, t]} \partial_x P_N \Pi_{\Theta_l \mathbb{1}_D, M}^l(u, u, u) P_N u \, dx ds, \quad (3.42)$$

where D is defined in (3.29) and where $\Pi_{\eta, M}^l$ is defined in (3.2). Thus, by symmetry, it is enough to estimate $J_{t, N}^3(u)$ that will be still denoted $J_N^3(u)$ for sake of simplicity. We rewrite $J_N^3(u)$ as

$$\begin{aligned} N^{2s} &\left(\sum_{1 \leq M \leq N^{1/2}} \int_{[0, t]} T_{\Theta_3 \mathbb{1}_D, M, N}(u, u, u) ds + \sum_{M > N^{1/2}} \int_{[0, t]} T_{\Theta_3 \mathbb{1}_D, M, N}(u, u, u) ds \right) \\ &=: I_N^{\text{low}}(u) + I_N^{\text{high}}(u), \end{aligned} \quad (3.43)$$

where $T_{\eta, M, N}(u, u, u)$ is defined in (3.4). At this stage it worth noticing that $\Theta_3 \mathbb{1}_D$ satisfies the symmetry hypothesis (3.3) of Lemma 3.2.

• **Estimate for $I_N^{\text{low}}(u)$.**

According to (3.5) we have

$$\begin{aligned} I_N^{\text{low}, 1}(u) &= \sum_{1 \leq M \leq N^{1/2}} MN^{2s} \int_{\mathbb{T} \times [0, t]} \Pi_{\eta_3 \mathbb{1}_D, M}^3(u, u, u_{\sim N}) P_N u \, dx ds \\ &= \sum_{N_1, N_2 \geq 1} \sum_{1 \leq M \leq N^{1/2}} MN^{2s} \int_{\mathbb{T} \times [0, t]} \Pi_{\eta_3 \mathbb{1}_D, M}^3(P_{N_1} u, P_{N_2} u, u_{\sim N}) P_N u \, dx ds, \end{aligned}$$

where η_3 is a function of (k_1, k_2, k_3) whose L^∞ -norm is uniformly bounded in N and M . We now separate the contributions of $I_N^{\text{low}, 1}$ of D^1 and $I_N^{\text{low}, 2}$ of D^2 to I_N^{low} . On D^1 , Lemma 3.10 and (3.9) ensure that $|k_1| \sim |k_2| \sim |k_3| \sim N$ and $|\Omega_3| \gtrsim M^2 N$. Therefore, (3.11) leads to

$$\begin{aligned} |I_N^{\text{low},1}(u)| &\lesssim \sum_{1 \leq M \leq N^{1/2}} T^{1/8} N^{2s} \frac{M^2 N^{11/10}}{M^2 N} N^{-4s} \|P_{\sim N} u\|_{Z_T^s}^4 \\ &\lesssim T^{1/8} N^{-2s+1/10+} \|P_{\sim N} u\|_{Z_T^s}^4, \end{aligned}$$

which is acceptable for $s > 1/20$. On D^2 , Lemma 3.10 and (3.9) ensure that $|\Omega_3| \gtrsim MN_{\max}^2$ and (3.11) leads to

$$\begin{aligned} |I_N^{\text{low},2}(u)| &\lesssim \sum_{N_1, N_2 \geq 1} \sum_{1 \leq M \leq N^{1/2}} T^{1/8} N^{2s} \frac{M^2 N_{\max}^{11/10}}{MN_{\max}^2} N^{-2s} \|P_{\sim N} u\|_{Z_T^s}^2 \prod_{i=1}^2 N_i^{-s} \|P_{N_i} u\|_{Z_T^s} \\ &\lesssim T^{1/8} N^{-2/5} \|u\|_{Z_T^s}^4, \end{aligned}$$

which is acceptable.

• **Estimate for $I_N^{\text{high}}(u)$.**

We separate the contributions of $I_N^{\text{high},1}$ of D^1 and $I_N^{\text{high},2}$ of D^2 to I_N^{high} . On D^1 , (3.11) yields

$$\begin{aligned} |I_N^{\text{high},1}(u)| &\lesssim \sum_{M > N^{1/2}} T^{1/8} N^{2s} \frac{MN N^{11/10}}{M^2 N} N^{-4s} \|P_{\sim N} u\|_{Z_T^s}^4 \\ &\lesssim T^{1/8} N^{-2s+6/10} \|P_{\sim N} u\|_{Z_T^s}^4, \end{aligned}$$

which is acceptable for $s > 3/10$. On D^2 , noticing that $M > N^{1/2}$ forces $N_1 \vee N_2 \gtrsim N^{1/2}$, (3.11) leads to

$$\begin{aligned} |I_N^{\text{high},2}| &\lesssim \sum_{\substack{N_1, N_2 \geq 1 \\ N_1 \vee N_2 \gtrsim N^{1/2}}} \sum_{M > N^{1/2}} T^{1/8} N^{2s} \frac{MN N_{\max}^{11/10}}{MN_{\max}^2} N^{-2s} \|P_{\sim N} u\|_{Z_T^s}^2 \prod_{i=1}^2 N_i^{-s} \|P_{N_i} u\|_{Z_T^s} \\ &\lesssim T^{1/8} N^{1/10-s/2} \|u\|_{Z_T^s}^4, \end{aligned}$$

which is acceptable for $s > 1/5$. This concludes the proof of the proposition. \square

Combining Lemma 3.9 and Proposition 3.11 we can easily get an a priori estimate on the $H^{3/10}(\mathbb{T})$ -norm of smooth solutions to (2.1). This will be done in Section 6.

4. The smoothing estimate.

The aim of this section is to prove the proposition below that show a kind of smoothing effect first observed in [20]. This smoothing effect is the only way we know to treat some resonant terms involving B (see (3.40)) when estimating the difference of two solutions. Note that, by symmetry, the terms involving B do cancel in the proof of the energy estimate (3.37).

THEOREM 4.1. *Let $s \geq 1/3$ be fixed. For any solution $u \in Z_T^s$ of (2.1) and (1.2) and any $k \geq 2^9$ it holds*

$$\sup_{t \in]0, T[} k \left| |\widehat{u}(t, k)|^2 - |\widehat{u}_0(k)|^2 \right| \lesssim \sup_{N \gtrsim k} \left[\left(\frac{k}{N} \right)^{s^-} \|P_{\leq Nu}\|_{Z_T^s}^4 (1 + \|P_{\leq Nu}\|_{Z_T^s}^4) \right] \quad (4.1)$$

where the implicit constant does not depend on k .

4.1. Notations.

In this section we will widely use the following notations: $\vec{k}_{(3)} = (k_1, k_2, k_3)$. Let D , D^1 and D^2 be defined as in (3.29)–(3.30). We set

$$\Gamma_3(k) = \{ \vec{k}_{(3)} \in \mathbb{Z}^3 : k_1 + k_2 + k_3 = k \},$$

$$D(k) := \Gamma_3(k) \cap D, \quad D^1(k) = D(k) \cap D^1 \quad \text{and} \quad D^2(k) = D(k) \cap D^2,$$

$m_1 = |k_2 + k_3|$, $m_2 = |k_1 + k_3|$, $m_3 = |k_1 + k_2|$, $m_{\min} = \min(m_1, m_2, m_3)$ and $m_{\text{med}} = \text{med}(m_1, m_2, m_3)$.

$$D_M(k) = \{ \vec{k}_{(3)} \in D(k) : m_{\min} \sim M \} \quad \text{and} \quad D_M^i(k) = D_M(k) \cap D^i, \quad i = 1, 2.$$

M_1, M_2, M_3, M_{\min} and M_{med} are the dyadic integers associated with respectively m_1, m_2, m_3, m_{\min} and m_{med} .

For $i \in \{1, 2, 3, 4\}$, we set $\vec{k}_{i(3)} = (k_{i1}, k_{i2}, k_{i3})$,

$$m_{i,\min} = \min(|k_{i1} + k_{i2}|, |k_{i1} + k_{i3}|, |k_{i2} + k_{i3}|),$$

$$m_{i,\text{med}} = \text{med}\{|k_{i1} + k_{i2}|, |k_{i1} + k_{i3}|, |k_{i2} + k_{i3}|\}.$$

$M_{i,\min}$ and $M_{i,\text{med}}$ are the dyadic numbers associated with respectively $m_{i,\min}$ and $m_{i,\text{med}}$.

4.2. L^2 -multilinear space-time estimates.

4.2.1. L^2 -trilinear estimates.

LEMMA 4.2. *Let $f_j \in l^2(\mathbb{Z})$, $j = 1, \dots, 4$. Then it holds*

$$\mathcal{J}_k^3 := \sum_{\vec{k}_{(3)} \in \Gamma^3(k), k_4 = -k} \phi_M(k_1 + k_2) \prod_{j=1}^4 |f_j(k_j)|$$

$$\lesssim [(M^{1/2} \|f_3\|_{l^2}) \wedge (M \|f_3\|_{l^\infty})] \|f_4\|_{l^\infty} \prod_{j=1}^2 \|f_j\|_{l^2}. \quad (4.2)$$

PROOF. We can assume without loss of generality that $f_i \geq 0$ for $i = 1, \dots, 4$. Then, we get by using Hölder and Young’s inequalities that

$$\mathcal{J}_k^3 \leq |f_4(k)| \|\phi_M(f_1 * f_2)\|_{l^2} \|f_3\|_{l^2}$$

$$\leq M^{1/2} \|f_1 * f_2\|_{l^\infty} \|f_3\|_{l^2(\mathbb{Z})} \|f_4\|_{l^\infty}$$

$$\leq M^{1/2} \prod_{j=1}^3 \|f_j\|_{l^2} \|f_4\|_{l^\infty},$$

and

$$\begin{aligned} \mathcal{J}_k^3 &\leq |f_4(k)| \|\phi_M(f_1 * f_2)\|_{l^1(\mathbb{Z})} \|f_3\|_{l^\infty} \\ &\leq M \|f_1 * f_2\|_{l^\infty} \|f_3\|_{l^\infty} \|f_4\|_{l^\infty} \\ &\leq M \prod_{j=1}^2 \|f_j\|_{l^2} \prod_{i=3}^4 \|f_i\|_{l^\infty}, \end{aligned}$$

which proves (4.2). □

PROPOSITION 4.3. *Assume that $0 < T \leq 1$, $\eta : \mathbb{Z}^3 \rightarrow \mathbb{C}$ is a bounded function and u_i are functions in Z_T^0 . Assume also that $k \geq 2^9$, $M \geq 1$ and $j \in \{1, 2, 3\}$. We define*

$$J_{\eta, M}^{3, k, T}(u_1, u_2, u_3, u_4) := \int_{[0, T] \times \mathbb{T}} \Pi_{\eta, M}^j(u_1, u_2, u_3) P_k u_4 dx dt, \tag{4.3}$$

where $\Pi_{\eta, M}^j$ is defined in (3.2). Then for any $K \gg 1$ it holds

$$|J_{\eta^1|_{\Omega_3}^1|_{\geq K}, M}(P_{\lesssim N_1} u_1, P_{\lesssim N_2} u_2, P_{\lesssim N_3} u_3, u_4)| \lesssim \frac{M^{1/2} N_{\max}^{11/10}}{K} \prod_{i=1}^4 \|u_i\|_{Z_T^0}, \tag{4.4}$$

where $N_{\max} = \max(N_1, N_2, N_3)$. Moreover, the implicit constant in estimate (4.4) only depends on the L^∞ -norm of the function η .

PROOF. Keeping in mind that $l^2(\mathbb{Z}) \hookrightarrow l^\infty(\mathbb{Z})$, we proceed exactly as in the proof of Proposition 3.4 but with the help of (4.2) instead of (3.8). □

4.2.2. L^2 -quintic linear estimates.

We use the notations $\vec{k}_{(5)} = (k_1, k_2, \dots, k_6) \in \mathbb{Z}^6$ and for any $k \in \mathbb{Z}$,

$$\Gamma_5(k) = \left\{ \vec{k}_{(5)} \in \mathbb{Z}^6 : \sum_{i=1}^6 k_i = k \right\}.$$

Before stating our quintic space-time estimates, let us define the resonance function of order 5 for $\vec{k}_{(5)} = (k_1, \dots, k_6) \in \Gamma^5(0)$ by

$$\Omega^5(\vec{k}_{(5)}) = k_1^3 + k_2^3 + k_3^3 + k_4^3 + k_5^3 + k_6^3. \tag{4.5}$$

It is worth noticing that a direct calculus leads to

$$\Omega^5(\vec{k}_{(5)}) = \Omega^3(k_1, k_2, k_3) + \Omega^3(k_4, k_5, k_6). \tag{4.6}$$

In the sequel we set

$$\vec{k}_{1(3)} = (k_{11}, k_{12}, k_{13}).$$

LEMMA 4.4. *Let $f_j \in l^2(\mathbb{Z})$, $j = 1, \dots, 6$. Then it holds that*

$$\mathcal{J}_k^{5,1} := \sum_{\substack{\vec{k}_{(5)} \in \Gamma^5(0) \\ k_6 = -k}} \phi_M(k_1 + k_2) \phi_{M'}(k_4 + k_5) \prod_{j=1}^6 |f_j(k_j)| \lesssim [(M^{1/2}M') \wedge (MM'^{1/2})] \prod_{j=1}^6 \|f_j\|_{l^2} \quad (4.7)$$

and

$$\mathcal{J}_k^{5,1} \lesssim \|f_6\|_{l^2} \min \left(M' \prod_{i=1}^2 \|\langle \cdot \rangle^{1/4} f_i\|_{l^2} \prod_{j=3}^5 \|f_j\|_{l^2}, M \prod_{j=1}^3 \|f_j\|_{l^2} \prod_{i=4}^5 \|\langle \cdot \rangle^{1/4} f_i\|_{l^2} \right). \quad (4.8)$$

$$\mathcal{J}_k^{5,2} := \sum_{\substack{\vec{k}_{(5)} \in \Gamma^5(0) \\ k_6 = -k}} \phi_M((k_1 + k_2 + k_3) + k_4) \phi_{M'}(k_1 + k_2) \prod_{j=1}^6 |f_j(k_j)| \lesssim M^{1/2}M' \prod_{j=1}^6 \|f_j\|_{l^2}. \quad (4.9)$$

$$\tilde{\mathcal{J}}_k^5 := \sum_{\substack{\vec{k}_{(3)} \in \Gamma^3(k) \\ (k_4, k_5, k_6) \in \Gamma^3(-k)}} \phi_M(k_1 + k_2) \phi_{M'}(k_4 + k_5) \prod_{j=1}^6 |f_j(k_j)| \lesssim M^{1/2}M'^{1/2} \prod_{j=1}^6 \|f_j\|_{l^2}. \quad (4.10)$$

PROOF. We can assume without loss of generality that $f_i \geq 0$ for $i = 1, \dots, 6$. Proceeding in the same way as in the proof of Lemma 4.2, we get

$$\begin{aligned} \mathcal{J}_k^{5,1} &\leq |f_6(k)| \|f_3\|_{l^2} \\ &\quad \times \min \left(\|\phi_M(f_1 * f_2)\|_{l^2} \|\phi_{M'}(f_4 * f_5)\|_{l^1}, \|\phi_M(f_1 * f_2)\|_{l^1} \|\phi_{M'}(f_4 * f_5)\|_{l^2} \right) \\ &\leq [(M^{1/2}M') \wedge (MM'^{1/2})] \prod_{j=1}^6 \|f_j\|_{l^2}. \end{aligned}$$

In the same way,

$$\begin{aligned} \mathcal{J}_k^{5,1} &\leq |f_6(k)| \\ &\quad \times \min \left(\|f_1 * f_2 * f_3\|_{l^\infty} \|\phi_{M'}(f_4 * f_5)\|_{l^1}, \|\phi_M(f_1 * f_2)\|_{l^1} \|f_3 * f_4 * f_5\|_{l^\infty} \right) \\ &\lesssim |f_6(k)| \min \left(\|f_1 * f_2\|_{l^2} M' \prod_{i=3}^5 \|f_i\|_{l^2}, \|f_4 * f_5\|_{l^2} M \prod_{i=1}^3 \|f_i\|_{l^2} \right) \end{aligned}$$

which leads to the desired result by using that

$$\|f_1 * f_2\|_{l^2} \lesssim \prod_{i=1}^2 \|\langle \cdot \rangle^{1/4} f_i\|_{l^2}. \quad (4.11)$$

To derive (4.9), we notice that

$$\begin{aligned} \mathcal{J}_k^{5,2} &\leq |f_6(k)| \|f_5\|_{l^2} \left\| \phi_M \left([\phi_{M'}(f_1 * f_2)] * f_3 * f_4 \right) \right\|_{l^2} \\ &\lesssim M^{1/2} \|f_6\|_{l^\infty} \|f_5\|_{l^2} \|f_4\|_{l^2(\mathbb{Z})} \|[\phi_{M'}(f_1 * f_2)] * f_3\|_{l^2} \end{aligned}$$

$$\lesssim M^{1/2} \|f_6\|_{l^\infty} \|f_5\|_{l^2} \|f_4\|_{l^2} \|f_3\|_{l^2} \|[\phi_{M'}(f_1 * f_2)]\|_{l^1},$$

which yields the desired estimate. Finally,

$$\begin{aligned} \tilde{\mathcal{J}}_k^5 &\leq \|\phi_M(f_1 * f_2)\|_{l^2} \|f_3\|_{l^2} \|[\phi_{M'}(f_4 * f_5)] * f_6\|_{l^\infty} \\ &\leq M^{1/2} \|f_1 * f_2\|_{l^\infty} \|f_3\|_{l^2} \|\phi_{M'}(f_4 * f_5)\|_{l^2} \|f_6\|_{l^2} \\ &\leq M^{1/2} M'^{1/2} \prod_{j=1}^6 \|f_j\|_{l^2}. \end{aligned} \quad \square$$

PROPOSITION 4.5. *Assume that $0 < T \leq 1$, $\eta : \mathbb{Z}^5 \rightarrow \mathbb{C}$ is a bounded function and u_i are functions in Z_T^0 . Assume also that $k \geq 2^9$, $M, M' \geq 1$ and $K \gg 1$. We define*

$$\begin{aligned} &\mathcal{J}_{\eta, M, M', K}^{5, k, T}(\vec{u}_{1(3)}, u_2, u_3, u_4) \\ &:= \sum_{\substack{\vec{k}_{(3)} \in D_M^{(k)} \\ |k_1| \gtrsim |k_2| \vee |k_3|}} \sum_{\substack{\vec{k}_{1(3)} \in D_{M'}^{(k_1)} \\ |\Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4)| \gtrsim K}} \int_{[0, T]} \eta(\vec{k}_{1(3)}, k_2, k_3) \prod_{j=1}^3 \widehat{u}_{1j}(k_{1j}) \prod_{i=2}^4 \widehat{u}_i(k_i) \end{aligned}$$

and

$$\begin{aligned} &\tilde{\mathcal{J}}_{\eta, M, K}^{5, k, T}(\vec{u}_{1(3)}, u_2, u_3, u_4) \\ &:= \sum_{\substack{\vec{k}_{(3)} \in D_M^{(k)} \\ |k_1| \gg |k_2| \vee |k_3|}} \sum_{\substack{\vec{k}_{1(3)} \in D^{(k_1)} \\ |\Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4)| \gtrsim K}} \int_{[0, T]} \eta(\vec{k}_{1(3)}, k_2, k_3) \prod_{j=1}^3 \widehat{u}_{1j}(k_{1j}) \prod_{i=2}^4 \widehat{u}_i(k_i) \end{aligned}$$

with $k_4 = -k$ and where $\vec{u}_{1(3)} := (u_{11}, u_{12}, u_{13})$.

Then

$$\begin{aligned} &|\mathcal{J}_{\eta, M, M', K}^{5, k, T}(P_{N_{11}} u_{11}, P_{N_{12}} u_{12}, P_{N_{13}} u_{13}, u_2, u_3, u_4)| \\ &\lesssim \frac{N_{1, \max}^{11/10}}{K} M^{1/2} M' \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \prod_{i=2}^4 \|u_i\|_{Z_T^0} \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} &|\tilde{\mathcal{J}}_{\eta, M, K}^{5, k, T}(P_{N_{11}} u_{11}, P_{N_{12}} u_{12}, P_{N_{13}} u_{13}, u_2, u_3, u_4)| \\ &\lesssim \frac{M}{K} N_{1, \max}^{11/10} N_{1, \min}^{1/2} \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \prod_{i=2}^4 \|u_i\|_{Z_T^0}. \end{aligned} \quad (4.13)$$

Moreover, the implicit constant in estimate (3.11) only depends on the L^∞ -norm of the function η .

PROOF. By symmetry, we may assume that $N_{1, \max} = N_{11}$ to prove (4.12). We proceed as in the proof of Proposition 3.4. Setting $R = K/N_{11}^{11/10} \ll K$ and using (4.7) and (4.9) we can easily estimate the contribution of $1_{T, R}^{\text{low}} P_{N_{11}} u_{11}$ by

$$\begin{aligned}
& \left| J_{\eta, M, M', K}^{5, k, T} (1_{T, R}^{\text{low}} P_{N_{11}} u_{11}, P_{N_{12}} u_{12}, P_{N_{13}} u_{13}, u_2, u_3, u_4) \right| \\
& \lesssim \sum_{\vec{k}_{(3)} \in D_M(k)} \sum_{\substack{\vec{k}_{1(3)} \in D_{M'}(k_1) \\ |\Omega_5(k_{1(3)}, k_2, k_3, k_4)| \gtrsim K}} \|1_{R, T}\|_{L_T^1} \\
& \quad \times \left\| \eta(k_{1(3)}, k_2, k_3) \widehat{P_{N_{11}} u_{11}}(k_{11}) \prod_{j=2}^3 \widehat{P_{N_{1j}} u_{1j}}(k_{1j}) \prod_{i=2}^4 \widehat{u}_i(k_i) \right\|_{L_T^\infty} \\
& \lesssim \frac{N_{11}^{11/10}}{K} M^{1/2} M' \prod_{j=1}^3 \|u_{1j}\|_{L_T^\infty L_x^2} \prod_{i=2}^4 \|u_i\|_{L_T^\infty L_x^2}.
\end{aligned}$$

Then we decompose the contribution of $1_{T, R}^{\text{high}} P_{N_{11}} u_{11}$ in the same way as in (3.18). The contribution of $Q_{\gtrsim M' N_{11}^2} 1_{T, R}^{\text{high}} P_{N_{11}} u_{11}$ can be estimated by using (4.7) and (4.9)

$$\begin{aligned}
& \left| J_{\eta, M, M', K}^{5, k, T} (Q_{\gtrsim M' N_{11}^2} 1_{T, R}^{\text{high}} P_{N_{11}} u_{11}, P_{N_{12}} u_{12}, P_{N_{13}} u_{13}, u_2, u_3, u_4) \right| \\
& \lesssim \frac{N_{11}^{11/10}}{K} \left((M^{1/2} M') \wedge (M M'^{1/2}) \right) \|u_{11}\|_{X_T^{-11/10, 1}} \prod_{j=1}^2 \|u_{1j}\|_{L_T^\infty L_x^2} \prod_{i=2}^4 \|u_i\|_{L_T^\infty L_x^2} \\
& \lesssim \frac{N_{11}^{11/10}}{K} M^{1/2} M' \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \prod_{i=2}^4 \|u_i\|_{Z_T^0},
\end{aligned}$$

and the other contributions can be estimated in the same way.

Now to prove (4.13) we use (4.2) instead of (4.7). Actually, since $|k_1| \gg |k_2| \vee |k_3|$ on the support of $\tilde{J}_{\eta, M, M', K}^{5, k, T}$, we know that $m_{\min} = |k_2 + k_3|$ and $|k_1| \sim k$. Therefore (4.2) and Bernstein inequalities lead to

$$\begin{aligned}
& \left| \sum_{\substack{\vec{k}_{(3)} \in D_M^2(k) \\ |k_1| \gg |k_2| \vee |k_3|}} \sum_{\substack{\vec{k}_{1(3)} \in D(k_1) \\ |\Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4)| \gtrsim K}} \eta(\vec{k}_{1(3)}, k_2, k_3) \prod_{j=1}^3 \widehat{P_{N_{1j}} u_{1j}}(k_{1j}) \prod_{i=2}^4 \widehat{u}_i(k_i) \right| \\
& \lesssim M \left\| \prod_{j=1}^3 P_{N_{1j}} u_{1j} \right\|_{L_x^1} \prod_{i=2}^4 \|u_i\|_{L_x^2} \\
& \lesssim M N_{1, \min}^{1/2} \prod_{j=1}^3 \|P_{N_{1j}} u_{1j}\|_{L_x^2} \prod_{i=2}^4 \|u_i\|_{L_x^2}.
\end{aligned}$$

With this estimate in hand, (4.13) follows from the same considerations as (4.12) by taking $R = K/N_{1, \max}^{11/10} \ll K$. \square

REMARK 4.1. Proceeding as in Proposition 4.5 but with the help of (4.10) we get in the same way

$$\begin{aligned}
& \left| \sum_{\vec{k}_{(3)} \in D_M(k)} \sum_{\substack{\vec{k}_{4(3)} \in D_{M'}(-k) \\ |\Omega_5(k_1, k_2, k_3, k_{4(3)})| \gtrsim K}} \int_{[0, T]} \eta(k_2, k_3, k_{4(3)}) \prod_{j=1}^3 \widehat{P_{N_{4j}} u_{4j}}(k_{4j}) \prod_{i=1}^3 \widehat{u}_i(k_i) \right| \\
& \lesssim \frac{N_{4, \max}^{11/10}}{K} M^{1/2} M'^{1/2} \prod_{j=1}^3 \|P_{4j} u_{4j}\|_{Z_T^0} \prod_{i=1}^3 \|u_i\|_{Z_T^0}
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
& \left| \sum_{\substack{\vec{k}_{(3)} \in D_M^2(k) \\ |k_1| \gg |k_2| \vee |k_3|}} \sum_{\substack{\vec{k}_{4(3)} \in D(-k) \\ |\Omega_5(k_1, k_2, k_3, k_{4(3)})| \gtrsim K}} \int_{[0, T]} \eta(k_2, k_3, k_{4(3)}) \prod_{j=1}^3 \widehat{u}_{4j}(k_{4j}) \prod_{i=1}^3 \widehat{u}_i(k_i) \right| \\
& \lesssim \frac{N_{4, \max}^{11/10}}{K} M^{1/2} N_{4, \min}^{1/2} \prod_{j=1}^3 \|P_{4j} u_{4j}\|_{Z_T^0} \prod_{i=1}^3 \|u_i\|_{Z_T^0}.
\end{aligned} \tag{4.15}$$

The above estimates will be actually also needed in the proof of Proposition 4.9.

4.2.3. L^2 -7-linear estimates.

We use the notations : $\vec{k}_{(7)} = (k_1, k_2, \dots, k_8) \in \mathbb{Z}^8$ and for any $k \in \mathbb{Z}$,

$$\Gamma_7(k) = \left\{ \vec{k}_{(7)} \in \mathbb{Z}^8 : \sum_{i=1}^8 k_i = k \right\}.$$

The proof of the following lemma follows from exactly the same considerations as the ones used in the proof of Lemma 4.4.

LEMMA 4.6. *Let $f_j \in l^2(\mathbb{Z})$, $j = 1, \dots, 8$. Setting*

$$\mathcal{J}_k^{7,1} := \sum_{\substack{\vec{k}_{(7)} \in \Gamma^7(0) \\ k_8 = -k}} \phi_M \left(\sum_{q=1}^6 k_q \right) \phi_{M_1}(k_1 + k_2) \phi_{M_2}(k_4 + k_5) \prod_{j=1}^8 |f_j(k_j)|$$

and

$$\mathcal{J}_k^{7,2} := \sum_{\substack{\vec{k}_{(7)} \in \Gamma^7(0) \\ k_8 = -k}} \phi_M \left(\sum_{q=4}^7 k_q \right) \phi_{M_1}(k_1 + k_2) \phi_{M_2}(k_4 + k_5) \prod_{j=1}^8 |f_j(k_j)|$$

it holds

$$\mathcal{J}_k^{7,1} + \mathcal{J}_k^{7,2} \lesssim \min \left(M^{1/2} M_1 M_2 \prod_{j=1}^8 \|f_j\|_{l^2}, M^{1/2} M_1 M_2^{1/2} \prod_{j=4}^5 \|\langle \xi \rangle^{1/4} f_j\|_{l^2} \prod_{\substack{j=1 \\ j \notin \{4,5\}}}^8 \|f_j\|_{l^2} \right). \tag{4.16}$$

Similarly, by setting

$$\tilde{\mathcal{J}}_{k,i}^{\tau} := \sum_{\substack{\vec{k}(\tau) \in \Gamma^{\tau}(0) \\ k_8 = -k}} \psi_i(\vec{k}(\tau)) \prod_{j=1}^8 |f_j(k_j)|$$

with

$$\psi_1(\vec{k}(\tau)) = \phi_M \left(\sum_{q=1}^6 k_q \right) \phi_{M_1} \left(\sum_{q=1}^4 k_q \right) \phi_{M_2}(k_1 + k_2),$$

$$\psi_2(\vec{k}(\tau)) = \phi_M \left(\sum_{q=1}^6 k_q \right) \phi_{M_1}(k_4 + k_5) \phi_{M_2}(k_1 + k_2),$$

$$\psi_3(\vec{k}(\tau)) = \phi_M(k_6 + k_7) \phi_{M_1} \left(\sum_{q=1}^4 k_q \right) \phi_{M_2}(k_1 + k_2)$$

and

$$\psi_4(\vec{k}(\tau)) = \phi_M(k_6 + k_7) \phi_{M_1}(k_4 + k_5) \phi_{M_2}(k_1 + k_2),$$

it holds

$$\sum_{i=1}^4 \tilde{\mathcal{J}}_k^{\tau,i} \lesssim \min \left(M^{1/2} M_1 M_2 \prod_{j=1}^8 \|f_j\|_{l^2}, M^{1/2} M_1 M_2^{1/2} \prod_{j=1}^2 \|\langle \xi \rangle^{1/4} f_j\|_{l^2} \prod_{j=3}^8 \|f_j\|_{l^2} \right). \quad (4.17)$$

PROPOSITION 4.7. *Assume that $0 < T \leq 1$, $\eta : \mathbb{Z}^7 \rightarrow \mathbb{C}$ is a bounded measurable function. Assume also that u_{1i}, u_{2i} with $i = 1, 2, 3$, and u_3, u_4 are functions in Z_T^0 and that $k \geq 2^9$, $M \geq 1$ and $M_1 \geq 1$. For $i = 1, 2$, we set $\vec{u}_{i(3)} := (u_{i1}, u_{i2}, u_{i3})$ and define*

$$\begin{aligned} & J_{\eta, M, M_1}^{\tau, k, T, i}(\vec{u}_{1(3)}, \vec{u}_{2(3)}, u_3, u_4) \\ & := \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\vec{k}_{1(3)} \in D_{M_1}^1(k_1)} \sum_{\substack{\vec{k}_{2(3)} \in D^i(k_2) \\ \Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4) \neq \Omega_3(\vec{k}_{2(3)})}} \\ & \int_{[0, T]} \eta(\vec{k}_{1(3)}, \vec{k}_{2(3)}, k_3) \prod_{j=1}^3 \hat{u}_{1j}(k_{1j}) \prod_{m=1}^3 \hat{u}_{2m}(k_{2m}) \prod_{q=3}^4 \hat{u}_q(k_q) \end{aligned}$$

with $k_4 = -k$.

Then

$$|J_{\eta, M, M_1}^{\tau, k, T, 1}(\vec{u}_{1(3)}, \vec{u}_{2(3)}, u_3, u_4)| \lesssim M^{1/2} M_1 k^{1/10} \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \|u_{2j}\|_{Z_T^0} \prod_{q=3}^4 \|u_q\|_{Z_T^0} \quad (4.18)$$

and

$$|J_{\eta, M, M_1}^{7, k, T, 2}(\vec{u}_{1(3)}, \vec{u}_{2(3)}, u_3, u_4)| \lesssim M^{1/2} M_1 k^{-9/10+} \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \|u_{2j}\|_{Z_T^0} \prod_{q=3}^4 \|u_q\|_{Z_T^0}. \tag{4.19}$$

Moreover, the implicit constant in estimate (3.11) only depends on the L^∞ -norm of the function η .

PROOF. We define the resonance function of order 7 for $\vec{k}_{(7)} = (k_1, \dots, k_8) \in \Gamma^7(0)$ by

$$\Omega^7(\vec{k}_{(7)}) = \sum_{i=1}^8 k_i^3. \tag{4.20}$$

Again a direct calculation shows that

$$\Omega_7(\vec{k}_{1(3)}, \vec{k}_{2(3)}, k_3, k_4) = \Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4) + \Omega_3(\vec{k}_{2(3)}) \tag{4.21}$$

and thus $|\Omega_7(\vec{k}_{1(3)}, \vec{k}_{2(3)}, k_3, k_4)| \gtrsim |\Omega_3(\vec{k}_{2(3)})|$ on the support of $J_{\eta, M, M_1}^{7, k, T, i}$ with $i \in \{1, 2\}$. Moreover, we deduce from Lemma 3.10 that $|\Omega_3(\vec{k}_{2(3)})| \gtrsim M_2^2 k$ on the support of $J_{\eta, M, M_1}^{7, k, T, 1}$ and that $|\Omega_3(\vec{k}_{2(3)})| \gtrsim M_2 k^2$ on the support of $J_{\eta, M, M_1}^{7, k, T, 2}$.

Estimates (4.18)–(4.19) follow then from the same considerations as in the proof of Propositions 4.3 and 4.5 by making use of (4.16). Note that we loose a factor k^{0+} in (4.19) by resuming over $M_{2, \min}$. \square

REMARK 4.2. Setting

$$\begin{aligned} & \tilde{J}_{\eta, M, M_1}^{7, k, T, i}(\vec{u}_{11(3)}, u_{12}, u_{13}, u_2, u_3, u_4) \\ & := \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\vec{k}_{1(3)} \in D_{M_1}^1(k_1)} \sum_{\substack{\vec{k}_{11(3)} \in D^i(k_2) \\ \Omega_5(\vec{k}_{1(3)}, k_2, k_3, k_4) \neq \Omega_3(\vec{k}_{11(3)})}} \\ & \int_{[0, T]} \eta(\vec{k}_{11(3)}, k_{12}, k_{13}, k_2, k_3) \prod_{j=1}^3 \widehat{u}_{11j}(k_{11j}) \prod_{m=2}^3 \widehat{u}_{1m}(k_{1m}) \prod_{q=2}^4 \widehat{u}_q(k_q) \end{aligned}$$

and proceeding as in the proof of the preceding proposition but with the help of (4.17) we obtain

$$|\tilde{J}_{\eta, M, M_1}^{7, k, T, 1}(\vec{u}_{11(3)}, u_{12}, u_{13}, u_2, u_3, u_4)| \lesssim M^{1/2} M_1 k^{1/10} \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \|u_{2j}\|_{Z_T^0} \prod_{q=3}^4 \|u_q\|_{Z_T^0} \tag{4.22}$$

and

$$|\tilde{J}_{\eta, M, M_1}^{7, k, T, 2}(\vec{u}_{11(3)}, u_{12}, u_{13}, u_2, u_3, u_4)| \lesssim M^{1/2} M_1 k^{-9/10+} \prod_{j=1}^3 \|u_{1j}\|_{Z_T^0} \|u_{2j}\|_{Z_T^0} \prod_{q=3}^4 \|u_q\|_{Z_T^0}. \tag{4.23}$$

These estimates are actually also needed in the proof of Proposition 4.9.

4.3. Proof of Theorem 4.1.

To prove Theorem 4.1 we construct a modified energy in the same way as in [17]. Note that this way of construction of modified energies has much in common with the I-method [4].

Theorem 4.1 will be a direct consequence of Lemma 4.8 and Proposition 4.9 below.

DEFINITION OF THE MODIFIED ENERGY. For $t \geq 0$, we define the modified energy at the mode $k \geq 2^9$ by

$$\mathcal{E}_k(t) = \mathcal{E}_k(u(t)) = \frac{k}{2} |\widehat{u}(t, k)|^2 + \alpha \mathcal{E}_k^{3,1}(t) + \beta \mathcal{E}_k^{3,2}(t) + \gamma \mathcal{E}_k^5(t) \quad (4.24)$$

where α, γ and β are real constants to be determined later.

In the sequel of this subsection, to simplify the formula, we set $k_4 = -k$.

$\mathcal{E}_k^{3,1}, \mathcal{E}_k^{3,2}, \mathcal{E}_k^5$ are then defined as follows:

$$\begin{aligned} \mathcal{E}_k^{3,1}(u) &= k^2 \mathfrak{R} \left[\sum_{M < k^{7/12}} \sum_{\vec{k}_{(3)} \in D_M^1(k)} \frac{1}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^4 \widehat{u}(k_j) \right], \\ \mathcal{E}_k^{3,2}(u) &= k^2 \mathfrak{R} \left[\sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ |k_{\text{med}}| \ll k^{2/3}}} \frac{1}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^4 \widehat{u}(k_j) \right] \end{aligned}$$

where $\vec{k}_{(3)} = (k_1, k_2, k_3)$ and the dyadic decompositions in N_j are nonhomogeneous,

$$\begin{aligned} \mathcal{E}_k^5(u) &= k^2 \mathfrak{R} \left[\sum_{i=1}^4 \sum_{1 \leq M < k^{7/12}} \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{i(3)} \in D^1(k_i) \\ \Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)}) \Omega^5(\vec{k}_{i(3)})} \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \prod_{q=1}^3 \widehat{u}(k_{i,q}) \right] \end{aligned} \quad (4.25)$$

with the notation

$$\vec{k}_{j(5)} = (\vec{k}_{j(3)}, \vec{k}_{j(3)}) \in \Gamma^5(k), \quad (4.26)$$

where $\vec{k}_{j(3)}$ is defined by

$$\vec{k}_{1(3)} = (k_2, k_3, k_4), \quad \vec{k}_{2(3)} = (k_1, k_3, k_4), \quad \vec{k}_{3(3)} = (k_1, k_2, k_4), \quad \vec{k}_{4(3)} = (k_1, k_2, k_3).$$

Next, we show that if $s > 1/4$, then the non quadratic part of $\mathcal{E}_k(u)$ is controlled by the H^s -norm of u .

LEMMA 4.8. *Let $s > 1/4$. For any $u \in H^s(\mathbb{T})$ it holds*

$$|\mathcal{E}_k^{3,1}(u)| + |\mathcal{E}_k^{3,2}(u)| \lesssim \|P_{\lesssim k} u\|_{H^s}^4 \quad (4.27)$$

and

$$|\mathcal{E}_k^5(u)| \lesssim \|P_{\lesssim k} u\|_{H^s}^6. \quad (4.28)$$

PROOF. Since $|\Omega_3(\vec{k}_{(3)})| \geq M_{\min} M_{\text{med}} k$ on $D^1(k)$, (4.2) leads to

$$|\mathcal{E}_k^{3,1}(u)| \lesssim \sum_{1 \leq M_{\min} \leq M_{\text{med}}} \frac{k^2 M_{\min}^{1/2} k^{-4s}}{M_{\min} M_{\text{med}} k} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^4 \lesssim k^{1-4s} \|P_{\lesssim k} u\|_{H_x^s}^4$$

which is acceptable. In the same way, on $D^2(k)$, it holds $|\Omega_3(\vec{k}_{(3)})| \geq M_{\min} k^2$ and thus (4.2) leads this time to

$$|\mathcal{E}_k^{3,2}(u)| \lesssim \sum_{M \geq 1} \frac{k^2 M^{1/2} k^{-2s}}{M k^2} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^2 \|P_{\lesssim k} u\|_{L_t^\infty L_x^2}^2 \lesssim k^{-2s} \|P_{\lesssim k} u\|_{H_x^s}^4.$$

Now, fix $i \in \{1, 2, 3, 4\}$. For $\vec{k}_{(3)} \in D^1(k)$ and $\vec{k}_{i(3)} \in D^1(k_i)$ we have

$$|k_1| \sim |k_2| \sim |k_3| \sim |k_4| \sim |k_{i,1}| \sim |k_{i,2}| \sim |k_{i,3}| \sim k.$$

Moreover, for $\vec{k}_{i(5)}$ (see (4.26)) such that $\Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})$ we have

$$|\Omega^5(\vec{k}_{i(5)})| = |\Omega^3(\vec{k}_{(3)}) + \Omega^3(\vec{k}_{i(3)})| \sim |\Omega^3(\vec{k}_{i(3)})| \gg |\Omega^3(\vec{k}_{(3)})| > 0.$$

Therefore $\mathcal{E}_k^5(u)$ is well defined and according to (4.7)–(4.10),

$$\begin{aligned} |\mathcal{E}_k^5(u)| &\lesssim \sum_{i=1}^4 \sum_{1 \leq M_{\min} \leq M_{\text{med}}} \sum_{1 \leq M_{i,\min} \leq M_{i,\text{med}}} \frac{k^3 M_{\min}^{1/2} M_{i,\min} k^{-6s}}{M_{\min} M_{\text{med}} M_{i,\min} M_{i,\text{med}} k^2} \|P_{\sim k} u\|_{H_x^s}^6 \\ &\lesssim k^{1-6s} \|P_{\lesssim k} u\|_{H_x^s}^6 \end{aligned}$$

which is acceptable. \square

PROPOSITION 4.9. *Let $s \geq 1/3$ and $u \in Z_T^s$ be a solution to (2.1). Then for $k \geq 2^9$,*

$$|\mathcal{E}_k(t) - \mathcal{E}_k(0)| \lesssim \sup_{N \gtrsim k} \left[\left(\frac{k}{N} \right)^{s-} \|P_{\lesssim N} u\|_{Z_T^s}^4 (1 + \|P_{\lesssim N} u\|_{Z_T^s}^4) \right]. \quad (4.29)$$

PROOF. Since u is real valued we can restrict ourself to positive k . As above, we differentiate \mathcal{E}_k with respect to time and then integrate between 0 and t to get

$$\begin{aligned} \mathcal{E}_k(t) &= \mathcal{E}_k(0) - k \mathfrak{R} \left[\int_0^t P_k \partial_x (A(u, u, u) - B(u, u, u)) P_k u \right] + \alpha \int_0^t \frac{d}{dt} \mathcal{E}_k^{3,1}(t') dt' \\ &\quad + \beta \int_0^t \frac{d}{dt} \mathcal{E}_k^{3,2}(t') dt' + \gamma \int_0^t \frac{d}{dt} \mathcal{E}_k^5(t') dt' \\ &=: \mathcal{E}_k(0) + I_k + \alpha J_k^1 + \beta J_k^2 + \gamma K_k. \end{aligned} \quad (4.30)$$

As in the preceding section, since u is real-valued, the contribution of $\partial_x B(u, u, u)$ is purely imaginary and thus vanishes. Recalling that we set $k_4 = -k$, we can thus rewrite I_k in Fourier variables as

$$I_k = k^2 \Im \left[\int_0^t \sum_{\vec{k}_{(3)} \in D(k)} \widehat{u}(k_1) \widehat{u}(k_2) \widehat{u}(k_3) \widehat{u}(k_4) \right]$$

with $D(k)$ defined as in the beginning of this section. We denote by I_k^1 and I_k^2 the contributions to I_k of respectively $D^1(k)$ and $D^2(k)$. Finally, we decompose I_k^1 and I_k^2 in the following way:

$$\begin{aligned} I_k^1 &= \left(\sum_{M \geq k^{7/12}} + \sum_{M < k^{7/12}} \right) k^2 \Im \left[\int_0^t \sum_{\vec{k}_{(3)} \in D_M^1(k)} \widehat{u}(k_1) \widehat{u}(k_2) \widehat{u}(k_3) \widehat{u}(k_4) \right] \\ &= I_k^{1,\text{high}} + I_k^{1,\text{low}} \end{aligned}$$

and

$$\begin{aligned} I_k^2 &= \left(\sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ k_{\text{med}} \geq k^{2/3}}} + \sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ k_{\text{med}} < k^{2/3}}} \right) k^2 \Im \left[\int_0^t \widehat{u}(k_1) \widehat{u}(k_2) \widehat{u}(k_3) \widehat{u}(k_4) \right] \\ &= I_k^{2,\text{high}} + I_k^{2,\text{low}}. \end{aligned}$$

• **Estimate for $I_k^{1,\text{high}}$.**

In view of Lemma 3.10, on D^1 it holds $|k_1| \sim |k_2| \sim |k_3| \sim |k|$ and (3.9) ensures that $|\Omega_3| \gtrsim M^2 k$. Therefore (4.4) leads to

$$\begin{aligned} |I_k^{1,\text{high}}| &\lesssim k^2 \sum_{M \geq k^{7/12}} \frac{M^{1/2} k^{11/10}}{M^2 k} k^{-4s} \prod_{i=1}^4 \|P_{\sim k} u\|_{Z^s} \\ &\lesssim k^{-4s+49/40} \|P_{\lesssim k} u\|_{Z^s}^4, \end{aligned}$$

which is acceptable for $s > 49/160$.

• **Estimate for $I_k^{1,\text{low}} + \alpha J_k$.**

By (2.1), we can rewrite $(d/dt)\mathcal{E}_k^{3,1}$ as the sum of the “linear” contribution²

$$\sum_{M < k^{7/12}} k^2 \Re \left[\sum_{\vec{k}_{(3)} \in D_M^1(k)} \frac{i(k_1^3 + k_2^3 + k_3^3 + k_4^3)}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^4 \widehat{u}(k_j) \right]$$

and the “nonlinear” contribution

²By “linear” contribution, we mean the contribution of the linear part when substituting u_t by using the equation.

$$\sum_{i=1}^4 \sum_{M < k^{7/12}} k^2 \Re \left[\sum_{\vec{k}_{(3)} \in D_M^1(k)} \frac{-ik_i}{\Omega^3(\vec{k}_{(3)})} \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \left(3|\widehat{u}(k_i)|^2 \widehat{u}(k_i) + \sum_{\vec{k}_{i(3)} \in D(k_i)} \prod_{q=1}^3 \widehat{u}(k_{i,q}) \right) \right].$$

Using the resonance relation (3.9), we see by choosing $\alpha = 1$ that $I_k^{1,\text{low}}$ is canceled out by the linear contribution of $\int_0^t (d/dt) \mathcal{E}_N^3$. Hence,

$$I_k^{1,\text{low}} + J_k = c \sum_{j=0}^1 \tilde{A}_k^j,$$

where, by symmetry,

$$\begin{aligned} \tilde{A}_k^0 &= \sum_{1 \leq M < k^{7/12}} k^3 \Im \left[\int_0^t \sum_{\vec{k}_{(3)} \in D_M^1(k)} \frac{1}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^3 \widehat{u}(k_j) \right. \\ &\quad \left. \times \left(3|\widehat{u}(-k)|^2 \widehat{u}(-k) + \sum_{\vec{k}_{4(3)} \in D(-k)} \prod_{q=1}^3 \widehat{u}(k_{4,q}) \right) \right] \\ &= A_k^{0,0} + A_k^0 \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_k^1 &= 3 \sum_{1 \leq M < k^{7/12}} k^2 \Im \left[\int_0^t \sum_{\vec{k}_{(3)} \in D_M^1(k)} \frac{k_1}{\Omega^3(\vec{k}_{(3)})} \prod_{j=2}^4 \widehat{u}(k_j) \right. \\ &\quad \left. \times \left(3|\widehat{u}(k_1)|^2 \widehat{u}(k_1) + \sum_{\vec{k}_{1(3)} \in D(k_1)} \prod_{q=1}^3 \widehat{u}(k_{1,q}) \right) \right] \\ &= A_k^{1,0} + A_k^1. \end{aligned}$$

It thus remains to treat the terms \tilde{A}_k^j corresponding to the nonlinear contribution of $(d/dt) \mathcal{E}_N^3$. Since $|k_i| \sim k$, A_k^0 and A_k^1 can be treated almost in the same way. Actually, some estimates on A_k^0 are easier thanks to (4.10) and (4.14). We thus only consider \tilde{A}_k^1 . First, by using (4.2), $A_k^{1,0}$ can be easily estimated by

$$\begin{aligned} |A_k^{1,0}| &\lesssim \sum_{1 \leq M_{\min} \leq M_{\text{med}}} \frac{k^3 M_{\min}^{1/2}}{M_{\min} M_{\text{med}} k} k^{-6s} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^6 \\ &\lesssim k^{2-6s} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 1/3$.

• **Estimate for A_k^1 .**

By symmetry we can assume that $M_{11} \leq M_{12} \leq M_{13}$. We set $N_{1,\text{max}} = \max(N_{11}, N_{12}, N_{13})$.

Case 1: $|\Omega_3(\vec{k}_{(3)})| \gtrsim |\Omega_3(\vec{k}_{1(3)})|$. Then we must have

$$M_{11} \lesssim \frac{M_{\min} M_{\text{med}} k}{M_{12} M_{13}}.$$

Case 1-1: $\vec{k}_{1(3)} \in D^2(k_1)$. In this region it holds $M_{12} \gtrsim k$ and thus $M_{11} \lesssim M_{\min} M_{\text{med}}/k$. On account of (4.7)–(4.9) we get

$$\begin{aligned} |A_k^1| &\lesssim \sum_{1 \leq M_{\min} \leq \min(M_{\text{med}}, k^{7/12})} \sum_{1 \leq M_{11} \lesssim M_{\min} M_{\text{med}}/k} \sum_{N_{1, \max} \gtrsim k} \frac{k^3 M_{\min}^{1/2} M_{11}}{M_{\min} M_{\text{med}} k} k^{-3s} N_{1, \max}^{-s} \\ &\quad \times \|P_{\sim k} u\|_{L_t^\infty H_x^s}^3 \|P_{N_{1, \max}} u\|_{L_t^\infty H_x^s} \|P_{\lesssim N_{1, \max}} u\|_{L_t^\infty L_x^2}^2 \\ &\lesssim \sum_{N_{1, \max} \gtrsim k} \sum_{1 \leq M_{\min} < k^{7/12}} M_{\min}^{1/2} k^{1-4s} \left(\frac{k}{N_{1, \max}}\right)^s \|P_{\lesssim N_{1, \max}} u\|_{L_t^\infty H_x^s}^6 \\ &\lesssim \sum_{N_{1, \max} \gtrsim k} \sum_{1 \leq M_{\min} < k^{7/12}} \left(\frac{M_{\min}}{k^{7/12}}\right)^{1/2} k^{31/24-4s} \left(\frac{k}{N_{1, \max}}\right)^s \|P_{\lesssim N_{1, \max}} u\|_{L_t^\infty H_x^s}^6 \\ &\lesssim \sum_{N_{1, \max} \gtrsim k} k^{-4s+31/24} \left(\frac{k}{N_{1, \max}}\right)^s \|P_{\lesssim N_{1, \max}} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 31/96$.

Case 1-2: $\vec{k}_{1(3)} \in D^1(k_1)$. Then it holds

$$|k_{11}| \sim |k_{12}| \sim |k_{13}| \sim k.$$

Since $|\Omega_3(\vec{k}_{(3)})| \gtrsim |\Omega_3(\vec{k}_{1(3)})|$, we must have $M_{\min} M_{\text{med}} \gtrsim M_{11}^2$. Therefore (4.7)–(4.9) lead to

$$\begin{aligned} |A_k^1| &\lesssim \sum_{1 \leq M_{\min} \leq M_{\text{med}}} \frac{k^3 M_{\min}^{1/2} (M_{\min} M_{\text{med}})^{1/2} k^{-6s}}{M_{\min} M_{\text{med}} k} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^6 \\ &\lesssim k^{2-6s} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 1/3$.

Case 2: $|\Omega_3(\vec{k}_{(3)})| \ll |\Omega_3(\vec{k}_{1(3)})|$. Then, by (4.6), $|\Omega_5| \sim |\Omega_3(\vec{k}_{1(3)})| \gg |\Omega_3(\vec{k}_{(3)})|$.

Case 2-1: $\vec{k}_{1(3)} \in D^2(k_1)$. By symmetry we may assume $k_{11} \geq k_{12} \geq k_{13}$. Then $|\Omega_3(\vec{k}_{1(3)})| \sim M_{1, \min} N_{11}^2$ and according to (4.12) we obtain

$$\begin{aligned} |A_k^1| &\lesssim \sum_{\substack{N_{11} \gtrsim k \\ 1 \leq N_{13} \leq N_{12} \leq N_{11}}} \sum_{\substack{1 \leq M_{\min} \leq M_{\text{med}} \\ 1 \leq M_{1, \min} \leq N_{11}}} \frac{k^3 M_{\min}^{1/2} M_{1, \min}}{M_{\min} M_{\text{med}} k (M_{1, \min} N_{11}^2)} N_{11}^{11/10} \\ &\quad \times k^{-3s} N_{11}^{-s} \|P_{\sim k} u\|_{Z^s}^3 \|P_{N_{11}} u\|_{Z^s} \|P_{\lesssim N_{12}} u\|_{Z^0} \|P_{\lesssim N_{13}} u\|_{Z^0} \\ &\lesssim \sum_{N_{11} \gtrsim k} k^{-4s+11/10} \left(\frac{k}{N_{11}}\right)^s \|P_{\lesssim N_{11}} u\|_{Z^s}^6 \end{aligned} \quad (4.31)$$

which is acceptable for $s > 11/40$.

Case 2-2: $\vec{k}_{1(3)} \in D^1(k_1)$. Then we must have

$$|k_{11}| \sim |k_{12}| \sim |k_{13}| \sim |k_1| \sim |k_2| \sim |k_3| \sim k \quad (4.32)$$

and

$$\Omega^3(\vec{k}_{1(3)}) \sim M_{1,\min} M_{1,\text{med}} k.$$

We call $A_k^{1,\text{low}}$ this contribution to A_k^1 and $A_k^{0,\text{low}}$ the same contribution to A_k^0 . Using the equation and the resonance relation (4.5), we can rewrite $K_k := \int_0^t (d/dt) \mathcal{E}_k^5$ as

$$\begin{aligned} \Re & \left[\int_0^t \sum_{i=1}^4 \sum_{1 \leq M < k^{7/12}} k^2 \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{i(3)} \in D^1(k_i) \\ \Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})}} \frac{ik_i}{\Omega^3(\vec{k}_{(3)})} \int_0^t \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \prod_{q=1}^3 \widehat{u}(k_{iq}) \right. \\ & + \int_0^t \sum_{i=1}^4 \sum_{1 \leq M < k^{7/12}} k^2 \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{i(3)} \in D^1(k_i) \\ \Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)}) \Omega^5(\vec{k}_{i(5)})} \prod_{q=1}^3 \widehat{u}(k_{iq}) \\ & \times \sum_{\substack{m=1 \\ m \neq i}}^4 \prod_{\substack{j=1 \\ j \neq i, m}}^4 \widehat{u}(k_j) (-ik_m) \left(3|\widehat{u}(k_m)|^2 \widehat{u}(k_m) + \sum_{\vec{k}_{m(3)} \in D(k_m)} \prod_{p=1}^3 \widehat{u}(k_{mp}) \right) \\ & + \int_0^t \sum_{i=1}^4 \sum_{1 \leq M < k^{7/12}} k^2 \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{i(3)} \in D^1(k_i) \\ \Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{i(3)})}} \frac{k_i}{\Omega^3(\vec{k}_{(3)}) \Omega^5(\vec{k}_{i(5)})} \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \\ & \times \sum_{m=1}^3 \prod_{\substack{q=1 \\ q \neq m}}^3 \widehat{u}(k_{i,q}) (-ik_{im}) \left(3|\widehat{u}(k_{im})|^2 \widehat{u}(k_{im}) + \sum_{\vec{k}_{i,m(3)} \in D(k_{i,m})} \prod_{p=1}^3 \widehat{u}(k_{i,m,p}) \right) \left. \right] \\ & := H_k^1 + \widetilde{H}_k^2 + \widetilde{H}_k^3. \end{aligned}$$

• **Estimate for $A_k^{1,\text{low}} + A_k^{0,\text{low}} + \gamma K_k$.**

By choosing $\gamma = c$, the above calculations lead to

$$A_k^{1,\text{low}} + A_k^{0,\text{low}} + \gamma K_k = c \left(\widetilde{H}_k^2 + \widetilde{H}_k^3 \right). \quad (4.33)$$

Because of (4.32), \widetilde{H}_k^2 and \widetilde{H}_k^3 can be estimated in the same way³. It thus suffices to consider \widetilde{H}_k^2 for any fixed couple $(i, m) \in \{1, 2, 3, 4\}^2$ with $i \neq m$. By symmetry, we can restrict ourselves to $(i, m) \in \{(1, 2), (1, 4)\}$. Since the case $m = 4$ is easier (see (4.10)), we only consider the case $(i, m) = (1, 2)$. We thus have to bound

$$k^2 \Im \left[\int_0^t \sum_{1 \leq M < k^{7/12}} \sum_{\vec{k}_{(3)} \in D_M^1(k)} \sum_{\substack{\vec{k}_{1(3)} \in D^1(k_1) \\ \Omega^3(\vec{k}_{(3)}) \ll \Omega^3(\vec{k}_{1(3)})}} \frac{k_1 k_2}{\Omega^3(\vec{k}_{(3)}) \Omega^5(\vec{k}_{1(3)}, k_2, k_3)} \prod_{j=3}^4 \widehat{u}(k_j) \right]$$

³Actually one needs to use (4.22) and (4.23) instead of (4.18) and (4.19) to treat \widetilde{H}_k^3 .

$$\begin{aligned} & \times \prod_{q=1}^3 \widehat{u}(k_{1,q}) \left(3|\widehat{u}(k_2)|^2 \widehat{u}(k_2) + \sum_{\vec{k}_{2(3)} \in D(k_2)} \prod_{p=1}^3 \widehat{u}(k_{2p}) \right) \Big] \\ =: & H_k^{3,0} + H_k^3. \end{aligned}$$

First by (4.7)–(4.9), we easily get

$$\begin{aligned} |H_k^{3,0}| & \lesssim \sum_{M \geq 1} \sum_{M_{1,\min} \geq 1} \frac{k^4}{M^2 k M_{1,\min}^2} M^{1/2} M_{1,\min} k^{-8s} \|P_{\sim k} u\|_{H^s}^8 \\ & \lesssim k^{2-8s} \|P_{\sim k} u\|_{H^s}^8, \end{aligned}$$

which is acceptable for $s \geq 1/4$.

• **Estimate for H_k^3 .**

Now, to bound H_k^3 we separate different contributions.

Case 1: $|\Omega_5(\vec{k}_{1(5)})| \gtrsim |\Omega_3(\vec{k}_{2(3)})|$. Then we must have $|\Omega_3(\vec{k}_{2(3)})| \lesssim |\Omega_3(\vec{k}_{1(3)})|$ since $|\Omega_5(\vec{k}_{1(5)})| \sim |\Omega_3(\vec{k}_{1(3)})|$. This forces $M_{2,\min} \lesssim M_{1,\text{med}}$.

Case 1-1: $N_{2,\text{med}} \sim N_{2,\text{max}}$. Then (4.16) leads to

$$\begin{aligned} |H_k^3| & \lesssim \sum_{\substack{1 \leq M \leq M_{\text{med}} \\ 1 \leq M_{1,\min} \leq M_{1,\text{med}} \lesssim k}} \sum_{N_{2,\text{max}} \gtrsim k} \frac{k^4 M^{1/2} M_{1,\min} M_{1,\text{med}}}{M M_{\text{med}} k M_{1,\min} M_{1,\text{med}} k} \\ & \quad \times k^{-5s} N_{2,\text{max}}^{-2s} \|P_{\sim k} u\|_{L_t^\infty H_x^s}^5 \|P_{\sim N_{2,\text{max}}} u\|_{L_t^\infty H_x^s}^2 \|P_{\lesssim N_{2,\text{max}}} u\|_{L_t^\infty L_x^2} \\ & \lesssim \sum_{N_{2,\text{max}} \gtrsim k} k^{-7s+2+} \left(\frac{k}{N_{2,\text{max}}} \right)^{2s} \|P_{\lesssim N_{2,\text{max}}} u\|_{L_t^\infty H_x^s}^8 \end{aligned}$$

which is acceptable for $s > 2/7$.

Case 1-2: $N_{2,\text{med}} \ll N_{2,\text{max}}$. On account of (4.32), it holds $N_{2,\text{max}} \sim k$ and $|\Omega_3(\vec{k}_{2(3)})| \sim M_{2,\min} k^2$. The inequality $|\Omega_3(\vec{k}_{2(3)})| \lesssim |\Omega_3(\vec{k}_{1(3)})|$ then ensures that

$$M_{2,\min} \lesssim \frac{M_{1,\min} M_{1,\text{med}}}{k}.$$

Estimate (4.16) thus leads for $s \geq 1/4$ to

$$\begin{aligned} |H_k^3| & \lesssim \sum_{\substack{1 \leq M \leq M_{\text{med}} \\ 1 \leq M_{1,\min} \leq M_{1,\text{med}} \lesssim k}} \frac{k^4 M^{1/2} M_{1,\min} (M_{1,\min} M_{1,\text{med}} k^{-1})^{1/2}}{M M_{\text{med}} k M_{1,\min} M_{1,\text{med}} k} k^{-6s} \\ & \quad \times \|P_{\sim k} u\|_{L_t^\infty H_x^s}^6 \|P_{\lesssim k} u\|_{L_t^\infty H_x^{1/4}}^2 \\ & \lesssim \sum_{\substack{1 \leq M \leq M_{\text{med}} \\ 1 \leq M_{1,\min} \leq M_{1,\text{med}} \lesssim k}} \frac{k^{3/2} k^{-6s}}{M^{1/2} M_{\text{med}}} \left(\frac{M_{1,\min}}{M_{1,\text{med}}} \right)^{1/2} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^8 \\ & \lesssim k^{-6s+3/2+} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^8, \end{aligned}$$

which is acceptable for $s > 1/4$.

Case 2: $|\Omega_5(\vec{k}_{1(5)})| \ll |\Omega_3(\vec{k}_{2(3)})|$. Then, $|\Omega_7| \sim |\Omega_3(\vec{k}_{2(3)})|$ on account of (4.21).

Case 2-1: $\vec{k}_{2(3)} \in D^2(k_2)$. Then (4.19) gives

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{M \geq 1 \\ M_{1,\min} \geq 1}} \sum_{N_{2,\max} \gtrsim k} \frac{k^4 M^{1/2} M_{1,\min}}{M^2 k M_{1,\min}^2 k} k^{-5s} N_{2,\max}^{-s} k^{-9/10+} \\ &\quad \times \|P_{\sim k} u\|_{Z^s}^5 \|P_{N_{2,\max}} u\|_{Z^s} \|P_{\lesssim N_{2,\max}} u\|_{Z^0}^2 \\ &\lesssim \sum_{N_{2,\max} \gtrsim k} k^{-6s+11/10+} \left(\frac{k}{N_{2,\max}}\right)^s \|P_{\lesssim N_{2,\max}} u\|_{Z^s}^6, \end{aligned}$$

which is acceptable for $s > 11/60$.

Case 2-2: $\vec{k}_{2(3)} \in D^1(k_2)$. Then we have

$$|k_{21}| \sim |k_{22}| \sim |k_{23}| \sim k.$$

Therefore (4.18) leads to

$$\begin{aligned} |H_k^3| &\lesssim \sum_{\substack{M \geq 1 \\ M_{1,\min} \geq 1}} \frac{k^4 M^{1/2} M_{1,\min}}{M^2 k M_{1,\min}^2 k} k^{-8s} k^{1/10} \|P_{\sim k} u\|_{Z^s}^8 \\ &\lesssim k^{21/10-8s} \|P_{\sim k} u\|_{Z^s}^8, \end{aligned}$$

which is acceptable for $s \geq 21/88$.

• Estimate for $I_k^{2,\text{high}}$.

By symmetry, in the sequel we assume $|k_1| \geq |k_2| \geq |k_3|$. We note from Lemma 3.10 that on $D^2(k)$ it holds $|\Omega_3(\vec{k}_{(3)})| \gtrsim M_{\min} k_1^2$.

Case 1: $|k_1| \sim |k_2| \geq |k_3|$.

Case 1-1: $|k_3| \gtrsim k$. Then by (4.4) it holds

$$\begin{aligned} |I_k^{2,\text{high}}| &\lesssim \sum_{N_1 \gtrsim k} \sum_{M \geq 1} \frac{k^2 M^{1/2} N_1^{11/10}}{M N_1^2} k^{-2s} N_1^{-2s} \|P_k u\|_{Z^s} \|P_{\gtrsim k} P_{\lesssim N_1} u\|_{Z^s} \|P_{\sim N_1} u\|_{Z^s}^2 \\ &\lesssim \sum_{N_1 \gtrsim k} k^{11/10-4s} \left(\frac{k}{N_1}\right)^{2s} \|P_k u\|_{Z^s} \|P_{\lesssim N_1} u\|_{Z^s}^3, \end{aligned}$$

which is acceptable for $s \geq 11/40$.

Case 1-2: $|k_3| \ll k$. It forces $m_{\min} = |k_1 + k_2| \sim k$ and thus (4.4) leads to

$$\begin{aligned} |I_k^{2,\text{high}}| &\lesssim \sum_{N_1 \gtrsim k} \sum_{M \sim k} \frac{k^2 M^{1/2} N_1^{11/10}}{M N_1^2} k^{-s} N_1^{-2s} \|P_k u\|_{Z^s} \|P_{\sim N_1} u\|_{Z^s}^2 \|P_{\lesssim N_1} u\|_{Z^0} \\ &\lesssim \sum_{N_1 \gtrsim k} k^{3/5-3s} \left(\frac{k}{N_1}\right)^{2s} \|P_{\lesssim N_1} u\|_{Z^s}^5, \end{aligned}$$

which is acceptable for $s \geq 1/5$.

Case 2: $|k_1| \gg |k_2| \geq |k_3|$. Then $|k_1| \sim k$ and $m_{\min} = |k_2 + k_3|$. In this region we will make use of the fact that $|k_2| \gtrsim k^{2/3}$.

Case 2-1: $|k_3| \ll |k_2|$. Then it holds $m_{\min} = |k_2 + k_3| \sim |k_2| \gtrsim k^{2/3}$ and thus by (4.4),

$$\begin{aligned} |I_k^{2,\text{high}}| &\lesssim \sum_{M \geq k^{2/3}} \frac{k^2 M^{1/2} k^{11/10}}{M k^2} k^{-2s} k^{-2s/3} \|P_{\sim k} u\|_{Z^s}^2 \|P_{\gtrsim k^{2/3}} P_{\lesssim k} u\|_{Z^s} \|P_{\lesssim k} u\|_{Z^0} \\ &\lesssim k^{(13/30 - 8s/3)} \|P_{\lesssim k} u\|_{Z^s}^4, \end{aligned}$$

which is acceptable for $s > 13/80$.

Case 2-2: $|k_2| \sim |k_3|$. Then (4.4) leads to

$$\begin{aligned} |I_k^{2,\text{high}}| &\lesssim \sum_{M \geq 1} \frac{k^2 M^{1/2} k^{11/10}}{M k^2} k^{-2s} k^{-4s/3} \|P_{\sim k} u\|_{Z^s}^2 \|P_{\gtrsim k^{2/3}} P_{\lesssim k} u\|_{Z^s}^2 \\ &\lesssim k^{11/10 - 10s/3} \|P_{\lesssim k} u\|_{Z^s}^4, \end{aligned}$$

which is acceptable for $s > 33/100$.

• **Estimate for $I_k^{2,\text{low}} + \beta J_k^2$.**

By (2.1), we can rewrite $J_k^2 = \int_0^t (d/dt) \mathcal{E}_k^{3,2}$ as the sum of the ‘‘linear’’ contribution

$$k^2 \Re \int_0^t \left[\sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ |k_{\text{med}}| \ll k^{2/3}}} \frac{i(k_1^3 + k_2^3 + k_3^3 + k_4^3)}{\Omega^3(\vec{k}_{(3)})} \prod_{j=1}^4 \widehat{u}(k_j) \right]$$

and the ‘‘nonlinear’’ contribution

$$\begin{aligned} &k^2 \Re \int_0^t \left[\sum_{i=1}^4 \sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ |k_{\text{med}}| \ll k^{2/3}}} \frac{-i k_i}{\Omega^3(\vec{k}_{(3)})} \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \left(3|\widehat{u}(k_i)|^2 \widehat{u}(k_i) + \sum_{\vec{k}_{i(3)} \in D(k_i)} \prod_{q=1}^3 \widehat{u}(k_{i,q}) \right) \right] \\ &= k^2 \Im \int_0^t \left[\sum_{i=1}^4 \sum_{\substack{\vec{k}_{(3)} \in D^2(k) \\ |k_{\text{med}}| \ll k^{2/3}}} \frac{k_i}{\Omega^3(\vec{k}_{(3)})} \prod_{\substack{j=1 \\ j \neq i}}^4 \widehat{u}(k_j) \left(3|\widehat{u}(k_i)|^2 \widehat{u}(k_i) + \sum_{\vec{k}_{i(3)} \in D(k_i)} \prod_{q=1}^3 \widehat{u}(k_{i,q}) \right) \right] \\ &= \sum_{i=1}^4 (B_k^{i,0} + B_k^i). \end{aligned}$$

Using the resonance relation (3.9), we see by choosing $\beta = 1$ that $I_k^{2,\text{low}}$ is canceled out by the linear contribution of $\int_0^t (d/dt) \mathcal{E}_k^{3,2}$. Hence,

$$I_k^{2,\text{low}} + J_k^2 = \sum_{i=1}^4 (B_k^{i,0} + B_k^i).$$

Note that since $|k_{\text{med}}| \ll k^{2/3}$, we must have $|k_{\text{max}}| \sim k$. In the sequel, by symmetry we

assume that $k_{\max} = k_1$. This forces $|k_1| \sim k$ and $M_{\min} = M_1 \lesssim k^{2/3}$.

• **Estimate for $B_k^{i,0}$, $i = 1, 2, 3, 4$.**

Let N_i be the dyadic variable associated to the dyadic decomposition with respect to k_i . Note that $N_i \lesssim k$. According to (4.2) it holds

$$\begin{aligned} |B_k^{i,0}| &\lesssim \sum_{M \geq 1} \sum_{1 \leq N_i \leq k} \frac{k^2 M^{1/2} N_i}{M k^2} k^{-s} N_i^{-3s} \|P_{\sim k} u\|_{L_t^\infty H_x^s} \|P_{N_i} u\|_{L_t^\infty H_x^s}^3 \|P_{\lesssim k} u\|_{L_T^\infty L_x^2}^2 \\ &\lesssim k^{-s} k^{\max(0, 1-3s)} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 1/4$.

• **Estimate for B_k^2 and B_k^3 .**

By symmetry, these two contributions can be treated in exactly the same way. So we only consider B_k^2 . By symmetry we can also assume that $|k_{21}| \geq |k_{22}| \geq |k_{23}|$. For $i = 1, 2, 3$, let N_{2i} be the dyadic variable associated to the dyadic decomposition with respect to k_{2i} . Recall also that $|k_2| \ll k^{2/3}$ on the contribution of B_k^2 .

Case 1: $|k_{21}| \geq k$. Then we must have $N_{22} \sim N_{21}$. On account of (4.2) and Hölder and Sobolev inequalities, we have for $s \geq 1/4$,

$$\begin{aligned} |B_k^2| &\lesssim \sum_{1 \leq M \lesssim k^{2/3}} \sum_{N_{21} \geq k} \frac{k^2 k^{2/3} M^{1/2}}{M k^2} k^{-2s} N_{21}^{-2s} \\ &\quad \times \|P_{\sim k} u\|_{L_T^\infty H_x^s}^2 \|P_{N_{21}} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim k^{2/3}} u\|_{L_T^\infty L_x^2} \|P_{\leq N_{21}} u\|_{L_T^\infty L_x^\infty} \\ &\lesssim \sum_{1 \leq M \lesssim k^{2/3}} \sum_{N_{21} \geq k} k^{2/3} k^{-2s} N_{21}^{-2s+1/4+} \\ &\quad \times \|P_{\sim k} u\|_{L_T^\infty H_x^s}^2 \|P_{N_{21}} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim k^{2/3}} u\|_{L_T^\infty H_x^{1/4}} \|P_{\leq N_{21}} u\|_{L_T^\infty H_x^{1/4}} \\ &\lesssim \sum_{N_{21} \geq k} k^{-4s+11/12+} \left(\frac{k}{N_{21}}\right)^{2s-(1/4+)} \|P_{\lesssim N_{21}} u\|_{L_T^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 1/4$.

Case 2: $|k_{21}| < k$. Using that $N_{2,\max} \gtrsim N_2$, similar considerations as in (4.2) together with (4.11), Hölder and Sobolev inequalities, we get for $s \geq 1/4$,

$$\begin{aligned} |B_k^2| &\lesssim \sum_{M \ll k^{2/3}} \sum_{N_2 \ll k^{2/3}} \frac{k^2 N_2 M^{1/2}}{M k^2} k^{-2s} N_2^{1/4-s} \|P_{\sim k} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim k} u\|_{L_T^\infty H_x^{1/4}}^4 \\ &\lesssim k^{-2s} k^{\max(0, 5/6-(2/3)s)} \|P_{\lesssim k} u\|_{L_T^\infty H_x^s}^6, \end{aligned}$$

which is acceptable for $s > 5/16$.

• **Estimate for B_k^1 .**

By symmetry we can assume that

$$|k_{11}| \geq |k_{12}| \geq |k_{13}|.$$

For $i = 1, 2, 3$, let $N_{1,i}$ be the dyadic variable associated to the dyadic decomposition with respect to $k_{1,i}$.

Case 1: $|\Omega_3(\vec{k}_{(3)})| \not\sim |\Omega_3(\vec{k}_{1(3)})|$.

Case 1-1: $M_{1,\text{med}} \geq 2^{-6}|k_{11}|$. Then $|\Omega_5| \gtrsim |\Omega_3(\vec{k}_{1(3)})| \gtrsim k_{11}^2$ and (4.13) leads, for $s > 1/4$ to

$$\begin{aligned} |B_k^1| &\lesssim \sum_{M \lesssim k} \sum_{N_{11} \gtrsim k} \sum_{N_{13} \leq N_{12} \leq N_{11}} \frac{k^2 k M N_{13}^{1/2-s}}{M k^2 N_{11}^2} k^{-s} N_{11}^{-s} N_{12}^{-s} N_{11}^{11/10} \\ &\quad \times \|P_k u\|_{Z^s} \|P_{N_{11}} u\|_{Z^s} \|P_{N_{12}} u\|_{Z^s} \|P_{N_{13}} u\|_{Z^s} \|P_{\lesssim k} u\|_{Z^0}^2 \\ &\lesssim \sum_{N_{11} \gtrsim k} k^{-2s+1/10+} \left(\frac{k}{N_{11}}\right)^{s+9/10} \|P_{\lesssim N_{11}} u\|_{Z^s}^6 \end{aligned}$$

which acceptable for $s > 1/4$.

Case 1-2: $M_{1,\text{med}} \leq 2^{-6}|k_{11}|$. Then $|k_{13}| \sim |k_{12}| \sim |k_{11}| \sim k$ and (4.12) leads to

$$\begin{aligned} |B_k^1| &\lesssim \sum_{M \geq 1} \sum_{M_{1,\text{min}} \geq 1} \frac{k^2 k M^{1/2} M_{1,\text{min}}}{M k^2 M_{1,\text{min}}^2 k} k^{-4s} k^{11/10} \|P_{\sim k} u\|_{Z^s}^4 \|u\|_{Z^0}^2 \\ &\lesssim k^{11/10-4s} \|P_{\sim k} u\|_{Z^s}^4 \|u\|_{Z^0}^2, \end{aligned}$$

which acceptable for $s > 11/40$.

Case 2: $|\Omega_3(\vec{k}_{(3)})| \sim |\Omega_3(\vec{k}_{1(3)})|$.

Case 2-1: $|k_{11}| \sim |k_{12}|$. Then we claim that $|k_{13}| \gtrsim k$. Indeed, recalling that $|k_{11} + k_{12} + k_{13}| \sim k$, $|k_{13}| \ll k$ would imply that $|k_{11} + k_{12}| \sim k$ and thus $|\Omega_3(\vec{k}_{1(3)})| \sim k k_{11}^2 \gtrsim k^3 \gg |\Omega_3(\vec{k}_{(3)})|$ since $|\Omega_3(\vec{k}_{(3)})| \leq k^{8/3}$. Therefore, $|\Omega_3(\vec{k}_{1(3)})| \gtrsim M_{1,\text{min}}^2 k_{11}$ and $|\Omega_3(\vec{k}_{(3)})| \sim M_1 k^2$ force $M_{1,\text{min}} \lesssim (M_1 k)^{1/2} \lesssim (N_2 k)^{1/2}$ and (4.7) leads, for $s \geq 1/4$, to

$$\begin{aligned} |B_k^1| &\lesssim \sum_{\substack{1 \leq M_1 \leq N_2 \leq k^{2/3} \\ 1 \leq M_{1,\text{min}} \leq (N_2 k)^{1/2}}} \sum_{N_{11} \geq N_{13} \gtrsim k} \frac{k^2 k M_{1,\text{min}}^{1/2} M_1}{M_1 k^2} k^{-s} N_{13}^{-s} N_{11}^{-2s} N_2^{-1/4} \\ &\quad \times \|P_k u\|_{L_t^\infty H_x^s} \|P_{\sim N_{11}} u\|_{L_t^\infty H_x^s}^2 \|P_{N_{13}} u\|_{L_t^\infty H_x^s} \|P_{N_2} u\|_{L_T^\infty H_x^{1/4}} \|P_{\lesssim k^{2/3}} u\|_{L_T^\infty L_x^2} \\ &\lesssim k^{-4s+5/4+} \left(\frac{k}{N_{11}}\right)^{2s} \|P_{\lesssim N_{11}} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s \geq 5/16$.

Case 2-2: $|k_{11}| \gg |k_{12}| \geq |k_{13}|$. Then $k/2 \leq |k_{11}| \leq 2k$, $M_{1,\text{min}} = M_{11}$ and $|\Omega_3(\vec{k}_{1(3)})| \sim M_{11} k^2$. Therefore, $|\Omega_3(\vec{k}_{(3)})| \sim |\Omega_3(\vec{k}_{1(3)})|$ forces $M_1 \sim M_{11}$.

Case 2-2-1: $k_{11} \neq k$. Let $k' \in \{k_{12}, k_{13}, k_2, k_3\}$ such that $|k'| = \max(|k_{12}|, |k_{13}|, |k_2|, |k_3|)$. Since $|k'| \ll |k|$, it holds

$$|\Omega_3(k_{11}, -k, k')| \geq \frac{k^2}{2}. \quad (4.34)$$

Case 2-2-1-1: $|\Omega_5| \gtrsim k^2$. Then, for $1/4 \leq s \leq 1/2$, (4.13) leads to

$$\begin{aligned} |B_k^1| &\lesssim \sum_{M_1 \lesssim k^{2/3}} \sum_{1 \leq N_{13} \leq N_{12} \lesssim k} \frac{k^3 M_1 N_{13}^{1/2-s}}{M_1 k^4} k^{-2s} N_{12}^{-s} k^{11/10} \\ &\quad \times \|P_{\sim k} u\|_{Z^s}^2 \|P_{N_{12}} u\|_{Z^s} \|P_{N_{13}} u\|_{Z^s} \|P_{\lesssim k} u\|_{Z^0}^2 \\ &\lesssim k^{-2s+1/10+} \|P_{\lesssim k} u\|_{Z^s}^6, \end{aligned}$$

which is acceptable for $s \geq 1/20$.

Case 2-2-1-2: $|\Omega_5| \ll k^2$. Recall that we have $M_{11} \sim M_1$ in this region. Let (z_1, z_2, z_3) be such that $\{z_1, z_2, z_3\} := \{k_{12}, k_{13}, k_2, k_3\} \setminus \{k'\}$. It then follows from (4.34) and (4.6) that $|\Omega_3(z_1, z_2, z_3)| \gtrsim k^2$ in this region. Since $|k_2| \vee |k_3| \ll k^{2/3}$ this forces $|k_{12}| \wedge |k_{13}| \gtrsim k^{2/3}$. For $s \geq 1/4$, (4.8) thus yields

$$\begin{aligned} |B_k^1| &\lesssim \sum_{M_{1,1} \sim M_1 \lesssim k^{2/3}} \frac{k^3 M_{1,1}}{M_1 k^2} k^{-2s} k^{-4s/3} \\ &\quad \times \|P_{\sim k} u\|_{L_t^\infty H_x^s}^2 \|P_{\gtrsim k^{2/3}} P_{\lesssim k} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim k^{2/3}} u\|_{L_T^\infty H_x^{1/4}}^2 \\ &\lesssim k^{(-10/3)s+1+} \|P_{\lesssim k} u\|_{L_T^\infty H_x^s}^6, \end{aligned}$$

which is acceptable for $s > 3/10$.

Case 2-2-2: $k_{11} = k$. This is the more complicated case. Following [20] we first notice that

$$\left| \frac{k^2}{(k_1 + k_2)(k_1 + k_3)} - 1 \right| = \left| \frac{(k_2 + k_3)k - k_2 k_3}{(k_1 + k_2)(k_1 + k_3)} \right| \lesssim \frac{|k_2| \vee |k_3|}{|k|}. \quad (4.35)$$

We decompose the contribution of this region to B_k^1 as

$$\begin{aligned} B_k^1 &= \mathfrak{I} \int_0^t \left[\sum_{|k_2| \vee |k_3| \leq k^{2/3}} \left[\left(\frac{k^2}{(k_1 + k_2)(k_1 + k_3)} - 1 \right) + 1 \right] \frac{k_1}{k_2 + k_3} \right. \\ &\quad \left. \times \widehat{u}(k) \widehat{u}(k_{12}) \widehat{u}(k_{13}) \widehat{u}(k_2) \widehat{u}(k_3) \widehat{u}(-k) \right] \\ &= C_k^1 + C_k^2. \end{aligned}$$

It is also worth noticing that, since $k_{12} + k_{13} + k_2 + k_3 = 0$ in this region, we must have

$$(k_{12} + k_{13}) = -(k_2 + k_3) \implies M_{1,1} = M_1. \quad (4.36)$$

• Estimate for C_k^1 .

Case 1: $|k_2| \vee |k_3| \gg |k_2| \wedge |k_3|$. By symmetry we can assume that $|k_2| \gg |k_3|$, which forces $M_1 \sim |k_2|$. According to (4.35) and (4.8), C_k^1 can be easily estimated for $s \geq 1/4$ by

$$|C_k^1| \lesssim \sum_{N_2 \sim M_1 \lesssim k^{2/3}} \frac{k}{M_1} \frac{N_2}{k} M_1 k^{-2s} N_2^{-s}$$

$$\begin{aligned} & \times \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{N_2} u\|_{L_T^\infty H_x^s} \|P_{\lesssim k} u\|_{L_T^\infty H_x^{1/4}}^2 \|P_{\lesssim k} u\|_{L_T^\infty L_x^2} \\ & \lesssim k^{(-8/3)s+2/3} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6, \end{aligned}$$

which is acceptable for $s \geq 1/4$.

Case 2: $|k_2| \sim |k_3|$. Then $|k_2| \wedge |k_3| \gtrsim M_1$ and since $M_1 = M_{1,1}$ we also have $|k_{12}| \gtrsim M_1$. Therefore, according to (4.35)–(4.36) and (4.8), C_k^1 can be easily estimated for $s < 1/2$ by

$$\begin{aligned} |C_k^1| & \lesssim \sum_{\substack{M_1 \lesssim N_2 \lesssim k^{2/3} \\ N_{1,2} \gtrsim M_1}} \frac{k}{M_1} \frac{N_2}{k} M_1^{1/2} M_1 k^{-2s} N_2^{-2s} N_{12}^{-s} \\ & \quad \times \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{N_2} u\|_{L_T^\infty H_x^s}^2 \|P_{N_{12}} u\|_{L_T^\infty H_x^s} \|P_{\lesssim k} u\|_{L_T^\infty L_x^2} \\ & \lesssim \sum_{M_1 \lesssim N_2 \lesssim k^{2/3}} M_1^{1/2-s} N_2^{1-2s} k^{-2s} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \\ & \lesssim k^{1-4s} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s > 1/4$.

• **Estimate for C_k^2 .**

Rewriting k_1 as $k_1 = k_{11} + k_{12} + k_{13} = k + k_{12} + k_{13}$ we decompose C_k^2 as the sum of three terms $C_k^{21} + C_k^{22} + C_k^{23}$.

• **Estimate on C_k^{22} and C_k^{23} .**

We only consider C_k^{22} which is the contribution of k_{12} since C_k^{23} can be treated in exactly the same way. We proceed as for C_k^1 .

Case 1: $|k_{12}| \gg |k_{13}|$. This forces $M_{11} \sim |k_{12}|$. According to (4.36) and (4.8), C_k^{22} can be easily estimated for $s \geq 1/4$ by

$$\begin{aligned} |C_k^{22}| & \lesssim \sum_{N_{12} \sim M_1 \lesssim k^{2/3}} \frac{N_{12}}{M_1} M_1^{1/2} k^{-2s} N_{12}^{-s} \\ & \quad \times \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{N_{12}} u\|_{L_T^\infty H_x^s} \|P_{\lesssim k} u\|_{L_T^\infty H_x^{1/4}}^2 \|P_{\lesssim k} u\|_{L_T^\infty L_x^2} \\ & \lesssim k^{(-8/3)s+2/3} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s > 1/4$.

Case 2: $|k_{12}| \sim |k_{13}|$. Then $|k_{12}| \wedge |k_{13}| \gtrsim M_1$ and since $M_1 = M_{1,1}$ we also have $|k_2| \vee |k_3| \gtrsim M_1$. Therefore, according to (4.7), C_k^{22} can be easily estimated for $s < 1/2$ by

$$\begin{aligned} |C_k^{22}| & \lesssim \sum_{\substack{M_1 \lesssim N_2 \lesssim k^{2/3} \\ M_1 \lesssim N_{12} \ll k}} \frac{N_{12}}{M_1} M_1^{1/2} M_1 k^{-2s} N_2^{-s} N_{12}^{-2s} \\ & \quad \times \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{N_2} u\|_{L_T^\infty H_x^s} \|P_{N_{12}} u\|_{L_T^\infty H_x^s} \|P_{\lesssim k} u\|_{L_T^\infty L_x^2} \\ & \lesssim \sum_{\substack{M_1 \lesssim k^{2/3} \\ M_1 \lesssim N_{12} \ll k}} M_1^{1/2-s} N_{12}^{1-2s} k^{-2s} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \\ & \lesssim k^{4/3-(14/3)s} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s > 2/7$.

• **Estimate for C_k^{21} .**

We first notice that since $|k_1| \gg |k_2| \vee |k_3|$ and $|k_{11}| \gg |k_{12}| \vee |k_{13}|$, $(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) \neq 0$ if and only if $k_2 + k_3 \neq 0$ and similarly, $(k_{11} + k_{12})(k_{11} + k_{13})(k_{12} + k_{13}) \neq 0$ if and only if $k_{12} + k_{13} = -(k_2 + k_3) \neq 0$. We can thus rewrite C_k^{21} as

$$C_k^{21} = k|u(k)|^2 \mathfrak{J} \int_0^t \left[\sum_{\substack{|k_2| \vee |k_3| \leq k^{2/3}, k_2 + k_3 \neq 0 \\ k_{12} + k_{13} = -k_2 - k_3, |k_{12}| \vee |k_{13}| \ll k}} \frac{1}{k_2 + k_3} \widehat{u}(k_{12}) \widehat{u}(k_{13}) \widehat{u}(k_2) \widehat{u}(k_3) \right].$$

We now separate the contributions $C_k^{21, \text{low}}$ and $C_k^{21, \text{high}}$ of the regions $|k_{12}| \vee |k_{13}| \leq k^{2/3}$ and $|k_{12}| \vee |k_{13}| > k^{2/3}$. Let us start by bounding $C_k^{21, \text{high}}$. In the region $|k_{13}| \sim |k_{12}|$, it can be bounded for $s \geq 1/4$, thanks to Sobolev inequalities, by

$$\begin{aligned} |C_k^{21, \text{high}}| &\lesssim \sum_{M_{1,1} = M_1 \leq k^{2/3}} \frac{k M_{1,1}^{1/2}}{M_1} k^{-2s} k^{-4s/3} \\ &\quad \times \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{\lesssim k} P_{\gtrsim k^{2/3}} u\|_{L_T^\infty H_x^s}^2 \|P_{\leq k^{2/3}} u\|_{L_T^\infty H^1}^2 \\ &\lesssim k^{1-10s/3} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6 \end{aligned}$$

which is acceptable for $s > 3/10$.

On the other hand, in the region $|k_{12}| \not\sim |k_{13}|$, we must have $|k_{12}| \gg |k_{13}|$ and thus $M_{1,1} \sim |k_{12}|$. Moreover, (4.36) forces $|k_2| \vee |k_3| \sim |k_{12}|$. Therefore, Sobolev inequalities lead for $s < 1/2$ to

$$\begin{aligned} |C_k^{21, \text{high}}| &\lesssim \sum_{k^{2/3} \lesssim N_{12} \lesssim k} \frac{k N_{12}^{1-2s}}{N_{12}} k^{-2s} N_{12}^{-2s} \|P_k u\|_{L_t^\infty H_x^s}^2 \|P_{N_{12}} u\|_{L_T^\infty H_x^s}^2 \|P_{\lesssim N_{12}} u\|_{L_T^\infty H_x^s}^2 \\ &\lesssim k^{1-14s/3} \|P_{\lesssim k} u\|_{L_t^\infty H_x^s}^6, \end{aligned}$$

which is acceptable for $s \geq 3/14$.

Finally, we claim that $C_k^{21, \text{low}} = 0$. Indeed, performing the change of variables

$$(k_2, k_3, k_{12}, k_{13}) \mapsto (-k_{12}, -k_{13}, -k_2, -k_3) = (\hat{k}_2, \hat{k}_3, \hat{k}_{12}, \hat{k}_{13})$$

and using that u is real-valued we get

$$\begin{aligned} C_k^{21, \text{low}} &= k|\widehat{u}(k)|^2 \mathfrak{J} \int_0^t \left[\sum_{\substack{|\hat{k}_2| \vee |\hat{k}_3| \vee |\hat{k}_{12}| \vee |\hat{k}_{13}| \leq k^{2/3} \\ \hat{k}_{12} + \hat{k}_{13} = -\hat{k}_2 - \hat{k}_3 \neq 0}} \frac{1}{\hat{k}_2 + \hat{k}_3} \widehat{u}(-\hat{k}_{12}) \widehat{u}(-\hat{k}_{13}) \widehat{u}(-\hat{k}_2) \widehat{u}(-\hat{k}_3) \right] \\ &= k|\widehat{u}(k)|^2 \mathfrak{J} \int_0^t \left[\sum_{\substack{|\hat{k}_2| \vee |\hat{k}_3| \vee |\hat{k}_{12}| \vee |\hat{k}_{13}| \leq k^{2/3} \\ \hat{k}_{12} + \hat{k}_{13} = -\hat{k}_2 - \hat{k}_3 \neq 0}} \frac{1}{\hat{k}_2 + \hat{k}_3} \widehat{u}(\hat{k}_{12}) \widehat{u}(\hat{k}_{13}) \widehat{u}(\hat{k}_2) \widehat{u}(\hat{k}_3) \right] \\ &= -C_k^{21, \text{low}}, \end{aligned}$$

which ensures that $C_k^{21, \text{low}} = 0$.

• **Estimate for B_k^4 .**

By symmetry we can assume $|k_{41}| \geq |k_{42}| \geq |k_{43}|$. B_k^4 can be controlled exactly as B_k^1 and is even easier (see (4.10) and (4.15)) except for the treatment of the region ($k_{41} = -k_1$ and $|k_{42}| \vee |k_{43}| \leq k^{2/3}$) which is slightly different to the treatment of the region ($k_1 = -k_4 = k$ and $|k_{12}| \vee |k_{13}| \leq k^{2/3}$) for B_k^1 . We thus only consider the region $k_1 + k_{41} = 0$ and $|k_{42}| \vee |k_{43}| \leq k^{2/3}$. In this region, according to (4.35), we can decompose B_k^4 as

$$\begin{aligned} B_k^4 &= \Im \int_0^t \left(\sum_{\Lambda(k)} \left[\left(\frac{k^2}{(k_1 + k_2)(k_1 + k_3)} - 1 \right) + 1 \right] \right. \\ &\quad \left. \times \frac{k}{(k_2 + k_3)} \widehat{u}(k_1) \widehat{u}(k_2) \widehat{u}(k_3) \widehat{u}(-k_1) \widehat{u}(k_{42}) \widehat{u}(k_{43}) \right) \\ &= B_k^{41} + B_k^{42}, \end{aligned}$$

with

$$\begin{aligned} \Lambda(k) &= \{(k_1, k_2, k_3, k_{42}, k_{43}) \in \mathbb{Z}^5 : k_1 + k_2 + k_3 = k, \\ &\quad k_{42} + k_{43} = -k_2 - k_3 \neq 0, |k_2| \vee |k_3| \vee |k_{42}| \vee |k_{43}| \leq k^{2/3}\}. \end{aligned}$$

B_k^{41} can be easily estimated as C_k^1 (actually it is even easier) by using (4.10) and the fact that $|k_{42} + k_{43}| = |k_2 + k_3|$.

Finally, we claim that $B_k^{42} = 0$. Indeed, performing the change of variables $\varphi : (k_1, k_2, k_3, k_{42}, k_{43}) \mapsto (k_1, -k_{42}, -k_{43}, -k_2, -k_3) = (\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_{42}, \hat{k}_{43})$ and noticing that $\hat{k}_1 + \hat{k}_2 + \hat{k}_3 = k_1 + (-k_{42} - k_{43}) = k_1 + k_2 + k_3 = k$ ensures that φ is a bijection on $\Lambda(k)$, we get

$$\begin{aligned} B_k^{42} &= \Im \int_0^t \left(\sum_{\Lambda(k)} \frac{k}{\hat{k}_2 + \hat{k}_3} |\widehat{u}(\hat{k}_1)|^2 \widehat{u}(-\hat{k}_{42}) \widehat{u}(-\hat{k}_{43}) \widehat{u}(-\hat{k}_2) \widehat{u}(-\hat{k}_3) \right) \\ &= \Im \int_0^t \left(\sum_{\Lambda(k)} \frac{k}{\hat{k}_2 + \hat{k}_3} |\widehat{u}(\hat{k}_1)|^2 \overline{\widehat{u}(\hat{k}_{42}) \widehat{u}(\hat{k}_{43})} \widehat{u}(\hat{k}_2) \widehat{u}(\hat{k}_3) \right) \\ &= -B_k^{42}, \end{aligned}$$

which ensures that $B_k^{42} = 0$ and completes the proof of Proposition 4.9. \square

REMARK 4.3. For the same reasons explain in ([19, Remark 3.2]) our method of proof of the smoothing effect seems to break down for $s < 1/3$. The reason is that the term A_k^1 can neither be controlled for $s < 1/3$ nor be canceled by adding a term of order 7 in the modified energy. Indeed, it is shown in [19] that for any k large enough one can find many couples of triplets $(\vec{k}_{(3)}, \vec{k}_{1(3)})$ such that $\vec{k}_{(3)} \in D^1(k)$, $\vec{k}_{1(3)} \in D^1(k_1)$ and $|\Omega_5(\vec{k}_{1(3)}, k_2, k_3, -k)| \lesssim 1$. Therefore, a supplementary term in the modified energy will not be useful to treat this term since we would not be able to control this term for

$s < 1/3$ and the “nonlinear contribution”, of the time derivative of this term would be even worst.

On the other hand, note that even if we only give an estimate of $A^{1,0}$ for $s \geq 1/3$, we could lower the Sobolev index here by adding a supplementary term in the modified energy. This is due to the fact that on the support of $A^{1,0}$ we have $|\Omega_5(k_1, -k_1, k_1, k_2, k_3, k_4)| = |\Omega_3(k_1, k_2, k_3)| \gtrsim |k|$.

The following corollary of Theorem 4.1 will be crucial for the local well-posedness result.

COROLLARY 4.1. *Assume that $s \geq 1/3$, $0 < T \leq 1$ and $u, v \in Z_T^s$ are two solutions to (2.1) defined in the time interval $[0, T]$. Then, for all integer number $k \geq 2^9$ such that $|\widehat{u}(0, k)| = |\widehat{v}(0, k)|$ and all $0 < s' < s$, it holds*

$$\sup_{t \in]0, T[} k^{1+s'-s} \left| |\widehat{u}(t, k)|^2 - |\widehat{v}(t, k)|^2 \right| \lesssim \|u - v\|_{Z_T^{s'}} (\|u\|_{Z_T^s} + \|v\|_{Z_T^s})^3 (1 + \|u\|_{Z_T^s} + \|v\|_{Z_T^s})^4, \quad (4.37)$$

where the implicit constant is independent of k .

PROOF. Proceeding as in the proof of Proposition 4.9 we obtain (4.30) for u and for v . Taking the difference of these two identities and estimating the right-hand side member as in Proposition 4.9 and estimating the non quadratic terms of the modified energy as in Lemma 4.8, the triangular inequality leads for any $k \geq 1$ to

$$\begin{aligned} \sup_{t \in]0, T[} k \left| |\widehat{u}(t, k)|^2 - |\widehat{v}(t, k)|^2 \right| &\lesssim \sup_{N \geq k} \left(\frac{k}{N} \right)^{s-} \|P_{\leq N}(u - v)\|_{Z_T^s} \\ &\quad \times (\|P_{\leq N}u\|_{Z_T^s} + \|P_{\leq N}v\|_{Z_T^s})^3 (1 + \|P_{\leq N}u\|_{Z_T^s} + \|P_{\leq N}v\|_{Z_T^s})^4. \end{aligned}$$

This last inequality clearly yields (4.37). \square

5. Estimates for the difference.

We will need the following multilinear estimates of order three and five.

PROPOSITION 5.1. *Assume that $0 < T \leq 1$, η_1, η_2 are bounded functions and u_i are functions in $Z^0 := X^{-11/10, 1} \cap L_T^\infty L_x^2$. Assume also that $N \gg 1$, $M \geq 1$ and $j' \in \{1, 2, 3\}$. Then*

$$\left| G_{\eta_1, M}^T (\Pi_{\eta_2, M'}^{j'}(u_1, u_2, u_3), u_4, u_5, u_6) \right| \lesssim T M M' \prod_{i=1}^6 \|u_i\|_{L_T^\infty L_x^2}, \quad (5.1)$$

where $G_{\eta, M}^T$ is defined in (3.10). Let also $N_1, N_2, N_3 \geq 1$ be dyadic integers and $(K_1, K_2) \in]0, +\infty[^2$ such that $K_2 \gg K_1$. Then it holds

$$\left| G_{\eta_1 \mathbb{1}_{D^1} \mathbb{1}_{|\Omega_3| \sim K_1}, M}^T \left(\Pi_{\eta_2 \mathbb{1}_{|\Omega_3| \geq K_2}, M'}^{j'}(P_{N_1}u_1, P_{N_2}u_2, P_{N_3}u_3), u_4, u_5, P_N u_6 \right) \right|$$

$$\lesssim \frac{T^{1/8}}{K_2} MM' \max(N_1, N_2, N_3)^{11/10} \prod_{i=1}^6 \|u_i\|_{Z_T^0}, \quad (5.2)$$

where D^1 and $\Pi_{\eta, M}^{j'}$ are defined in respectively (3.30) and (3.2). Moreover, the implicit constant in estimates (5.1) and (5.2) only depends on the L^∞ -norm of the functions η_1 and η_2 .

PROOF. (5.1) follows by using twice (3.8). To prove (5.2), we first notice that $K_2 \gg K_1$ and (4.6) ensure that $|\Omega_5(\vec{k}_{(5)})| \sim |\Omega_3(k_1, k_2, k_3)| \geq K_2$. Then the result follows by proceeding as in the proof of (3.11) with $R = [K_2/\max(N_1, N_2, N_3)^{11/10}]^{8/7}$. \square

5.1. Definition of the modified energy for the difference.

Let $N_0 \geq 2^9$, N be a nonhomogeneous dyadic number and $(u, v) \in H^s(\mathbb{T})^2$ with $s \in \mathbb{R}$. We define the modified energy of the difference at the dyadic frequency N by

$$\mathcal{E}_N[u, v, N_0] = \begin{cases} \frac{1}{2} \|P_N(u - v)\|_{L_x^2}^2 & \text{for } N \leq N_0 \\ \frac{1}{2} \|P_N(u - v)\|_{L_x^2}^2 + \mathcal{E}_N^3[u, v] & \text{for } N > N_0, \end{cases} \quad (5.3)$$

where

$$\begin{aligned} \mathcal{E}_N^3[u, v] &= \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \frac{k}{\Omega(\vec{k}_{(3)})} \varphi_N^2(k) \\ &\times \Re \left[\left(\hat{u}(k_1)\hat{u}(k_2) + \hat{u}(k_1)\hat{v}(k_2) + \hat{v}(k_1)\hat{v}(k_2) \right) (\hat{u} - \hat{v})(k_3)(\hat{u} - \hat{v})(-k) \right] \end{aligned}$$

where $\vec{k}_{(3)} = (k_1, k_2, k_3)$, $m_{\min} = \min_{1 \leq i \neq j \leq 3} (|k_i + k_j|)$ and where the set A_3 is defined in (3.32). The modified energy $E^{s'}[u, v, N_0]$ of the difference $u - v$ is defined by

$$E^{s'}[u, v, N_0] = \sum_{N \geq 1} N^{2s'} \mathcal{E}_N[u, v, N_0].$$

The following lemma ensures that $E^{s'}[u, v, N_0]$ is well-defined as soon as $(u, v) \in H^s(\mathbb{T})^2$ with $s > 0$. Moreover, for $N_0 > 2^9$ large enough we have $E^{s'}[u, v, N_0] \sim \|u - v\|_{H^{s'}}^2$.

LEMMA 5.2. *Let $(u, v) \in H^s(\mathbb{T})^2$ with $s > 0$. Then, for any $s' \in \mathbb{R}$ and any $N_0 \gg (\|u\|_{H^s} + \|v\|_{H^s})^{1/s}$, it holds*

$$C^{-1} \|u - v\|_{H^{s'}}^2 \leq E^{s'}[u, v, N_0] \leq C \|u - v\|_{H^{s'}}^2 \quad (5.4)$$

for some constant $C > 1$.

PROOF. Let us recall that Lemma 3.10 ensures that $|k_1| \sim |k_2| \sim |k_3| \sim |k|$ on D^1 . Therefore, a direct application of (3.8) leads to

$$N^{2s'} |\mathcal{E}_N^3[u, v]| \lesssim \sum_{M_{\min} \geq 1} \frac{N^{2s'} N M_{\min}}{M_{\min}^2 N} N^{-2s'} N^{-2s} \\ \times (\|P_{\sim N} u\|_{H^s}^2 + \|P_{\sim N} v\|_{H^s}^2) \|P_{\sim N}(u-v)\|_{H^{s'}}^2.$$

Summing over M_{\min} and $N \geq N_0$, we obtain

$$\sum_{N \geq N_0} N^{2s'} |\mathcal{E}_N^3[u, v]| \lesssim N_0^{-2s} (\|u\|_{H^s}^2 + \|v\|_{H^s}^2) \|u-v\|_{H^{s'}}^2,$$

that clearly implies (5.4) for $N_0 \gg (\|u\|_{H^s} + \|v\|_{H^s})^{1/s}$. \square

Let now (u, v) be a couple of solutions to the renormalized mKdV equation on $]0, T[$. The following proposition enables to control $E^{s'}[u, v, N_0]$ on $]0, T[$.

PROPOSITION 5.3. *Let $0 < T < 1$. Let u and v be two solutions of the renormalized mKdV (2.1) belonging to $L^\infty(0, T; H^s(\mathbb{T}))$ with $1/3 \leq s < 1/2$ and associated with the same initial data $u_0 \in H^s(\mathbb{T})$. Then, for $s/2 < s' < s - 1/10$ and any $N_0 \geq 2^{10}$, it holds*

$$\sup_{t \in [0, T]} E^{s'}[u(t), v(t), N_0] \lesssim T^{1/8} N_0^{3/2} (1 + \|u\|_{Z_T^s} + \|v\|_{Z_T^s})^8 \|w\|_{Z_T^{s'}}^2, \quad (5.5)$$

where we set $w = u - v$.

PROOF. To simplify the notation, we denote $\mathcal{E}_N[u(t), v(t), N_0]$ simply by $\mathcal{E}_N(t)$. Note that $u(t)$ and $v(t)$ are well defined for any $t \in [0, T]$ since, by the equation, $(u, v) \in (C([0, T]; H^{s-3}))^2$ and that, for any $N \in 2^{\mathbb{N}}$, $E_N(0) = 0$ since $u(0) = v(0) = u_0$. For $N \leq N_0$, the definition of $\mathcal{E}_N(t)$ easily leads to

$$\frac{d}{dt} \mathcal{E}_N(t) = \int_{\mathbb{T}} P_N \left(w(u^2 + uv + v^2) - 3P_0(w(u+v))u - 3P_0(v^2)w \right) \partial_x P_N w$$

which yields after applying Bernstein inequalities, integrating on $]0, t[$ and summing over $N \leq N_0$,

$$\sum_{N \leq N_0} \mathcal{E}_N(t) \lesssim N_0^{1+2s'} \left(\|w\|_{L_T^\infty L_x^3}^2 (\|u\|_{L_T^\infty L_x^6}^2 + \|v\|_{L_T^\infty L_x^6}^2) \right. \\ \left. + \|w\|_{L_T^\infty L_x^2}^2 (\|u\|_{L_T^\infty L_x^2}^2 + \|v\|_{L_T^\infty L_x^2}^2) \right) \\ \lesssim T N_0^{1+2s'} \|w\|_{L_T^\infty H_x^{1/6}}^2 (\|u\|_{L_T^\infty H_x^{1/3}}^2 + \|v\|_{L_T^\infty H_x^{1/3}}^2) \\ \lesssim T N_0^{3/2} \|w\|_{L_T^\infty H_x^{s'}}^2 (\|u\|_{L_T^\infty H_x^s}^2 + \|v\|_{L_T^\infty H_x^s}^2),$$

since, by hypotheses, $1/6 \leq s' < 1/4$ and $s \geq 1/3$.

Now for $N > N_0$, we first notice that the difference $w = u - v$ satisfies

$$w_t + \partial_x^3 w = -\partial_x A(u, u, w) - \partial_x A(u, v, w) - \partial_x A(v, v, w) \\ + \partial_x \left(B(u, u, w) + (B(u, u, v) - B(v, v, v)) \right), \quad (5.6)$$

where A and B are defined in (3.40). Therefore, differentiating \mathcal{E}_N with respect to time and integrating between 0 and t yields

$$\begin{aligned} N^{2s'} \mathcal{E}_N(t) &= -N^{2s'} \int_0^t \mathfrak{R} \left(\int_{\mathbb{T}} \partial_x P_N [A(u, u, w) + A(u, v, w) + A(v, v, w)] P_N w \, d\tau \right) \\ &\quad + N^{2s'} \int_0^t \mathfrak{R} \left(\int_{\mathbb{T}} \partial_x P_N B(u, u, w) P_N w \, d\tau \right) \\ &\quad + N^{2s'} \int_0^t \mathfrak{R} \left(\int_{\mathbb{T}} \partial_x P_N (B(u, u, v) - B(v, v, v)) P_N w \, d\tau \right) \\ &\quad + N^{2s'} \int_0^t \mathfrak{R} \frac{d}{dt} \mathcal{E}_N^3(\tau) \, d\tau \\ &= C_N(t) + D_N(t) + F_N(t) + G_N(t). \end{aligned}$$

As in (3.41) we notice that, since u and v are real-valued,

$$\int_{\mathbb{T}} \partial_x P_N B(u, u, w) P_N w = ik \sum_{k \in \mathbb{Z}} |\hat{u}(k)|^2 |\varphi_N(k) \hat{w}(k)|^2 \in i\mathbb{R}$$

and thus $D_N(t) = 0$. On the other hand, the smoothing effect (4.37) leads to

$$\begin{aligned} |F_N(t)| &\lesssim N^{2s'} \left| \int_0^t k \sum_{k \in \mathbb{Z}} (|\hat{u}(\tau, k)|^2 - |\hat{v}(\tau, k)|^2) \varphi_N(k)^2 \hat{v}(\tau, k) \hat{w}(\tau, -k) \, d\tau \right| \\ &\lesssim \sup_{\tau \in [0, T]} \sup_{\substack{k \in \mathbb{Z} \\ N/2 \leq |k| \leq 2N}} \left(|k|^{1+(s'-s)} \left| |\hat{u}(\tau, k)|^2 - |\hat{v}(\tau, k)|^2 \right| \right) \\ &\quad \times \int_0^t \|v_N(\tau)\|_{H^s} \|w_N(\tau)\|_{H^{s'}} \, d\tau \\ &\lesssim \delta_N T (1 + \|u\|_{Z_T^s}^7 + \|v\|_{Z_T^s}^7) \|w\|_{Z_T^{s'}} \|v\|_{L_T^\infty H^s} \|w\|_{L_T^\infty H^{s'}} \end{aligned}$$

with $\|(\delta_{2^j})_j\|_{l^1(\mathbb{N})} \lesssim 1$.

It thus remains to control $C_N(t) + G_N(t)$. We notice that $C_N(t)$ can be decomposed as

$$\begin{aligned} C_N(t) &= \sum_{j=1}^3 N^{2s'} \mathfrak{R} \int_{[0, t] \times \mathbb{T}} \partial_x P_N \left(\Pi_{\perp_{D^1}} + \Pi_{\perp_{D^2}} \right) (z_1^j, z_2^j, w) P_N w \\ &= C_N^1(t) + C_N^2(t), \end{aligned}$$

with $(z_1^1, z_2^1) = (u, u)$, $(z_1^2, z_2^2) = (u, v)$ and $(z_1^3, z_2^3) = (v, v)$. We then decompose further $C_N^1(t)$ and $C_N^2(t)$ as

$$\begin{aligned} C_N^1(t) &= \sum_{j=1}^3 N^{2s'} \mathfrak{R} \int_{[0, t] \times \mathbb{T}} \partial_x P_N \left(\Pi_{\perp_{D^1 \cap \{m_{\min} > N^{1/2}\}}} + \Pi_{\perp_{D^1 \cap \{m_{\min} \leq N^{1/2}\}}} \right) (z_1^j, z_2^j, w) P_N w \\ &= C_N^{1, \text{low}}(t) + C_N^{1, \text{high}}(t) \end{aligned}$$

and

$$\begin{aligned} C_N^2(t) &= \sum_{j=1}^3 N^{2s'} \mathfrak{R} \int_{[0,t] \times \mathbb{T}} \partial_x P_N \left(\Pi_{\mathbb{1}_{D^2 \cap \{|k_1| \vee |k_2| \ll N\}}} + \Pi_{\mathbb{1}_{D^2 \cap \{|k_1| \vee |k_2| \gtrsim N\}}} \right) (z_1^j, z_2^j, w) P_N w \\ &= C_N^{2,\text{low}}(t) + C_N^{2,\text{high}}(t). \end{aligned}$$

• **Estimate for $C_N^{2,\text{low}}$.**

From Lemma 3.10, we infer that for any $N \geq 1$,

$$D^2 \cap \{|k_1| \vee |k_2| \ll N\} \cap \left\{ \sum_{i=1}^3 k_i \in \text{supp } \Phi_N \right\} = D \cap \{|k_1| \vee |k_2| \ll N\} \cap \left\{ \sum_{i=1}^3 k_i \in \text{supp } \Phi_N \right\}.$$

Since $|k_1| \vee |k_2| \ll N \sim |k_3|$ on the support of $C_N^{2,\text{low}}$, it can thus be rewritten as

$$C_N^{2,\text{low}}(t) = \sum_{j=1}^3 N^{2s'} \sum_{1 \leq M \ll N} \mathfrak{R} \int_{[0,t] \times \mathbb{T}} \partial_x P_N \Pi_{\mathbb{1}_{D \cap \{|k_1| \vee |k_2| \ll N\}}, M}^3 (z_1^j, z_2^j, w) P_N w.$$

Since $\mathbb{1}_{D \cap \{|k_1| \vee |k_2| \ll N\}}$ satisfies the symmetry hypotheses of Lemma 3.2, it can further be rewritten as

$$C_N^{2,\text{low}}(t) = \sum_{j=1}^3 N^{2s'} \sum_{1 \leq M \ll N} M \mathfrak{R} \int_{[0,t] \times \mathbb{T}} P_N \Pi_{\eta \mathbb{1}_{D \cap \{|k_1| \vee |k_2| \ll N\}}, M}^3 (z_1^j, z_2^j, w) P_N w$$

with $\|\eta\|_\infty \lesssim 1$. Using Lemma 3.10, Proposition 3.4 and (3.9) it thus can be estimated by

$$\begin{aligned} |C_N^{2,\text{low}}(t)| &\lesssim T^{1/8} \sum_{1 \leq M \ll N} \sum_{N_1 \vee N_2 \gtrsim M} N^{2s'} \frac{M^2 N_{\max}^{11/10}}{M N_{\max}^2} N^{-2s'} N_1^{-s} N_2^{-s} \\ &\quad \times \|P_{N_1} z_1\|_{Z_T^s} \|P_{N_2} z_2\|_{Z_T^s} \|P_{\sim N} w\|_{Z_T^{s'}}^2 \\ &\lesssim T^{1/8} N^{1/10-s} (\|u\|_{Z_T^s}^2 + \|v\|_{Z_T^s}^2) \|w\|_{Z_T^{s'}}^2, \end{aligned}$$

which is acceptable for $s > 1/10$.

• **Estimate for $C_N^{2,\text{high}}$.**

On account of Proposition 3.4 we have

$$\begin{aligned} |C_N^{2,\text{high}}(t)| &\lesssim T^{1/8} \sum_{N_3 \geq 1} \sum_{N_1 \vee N_2 \gtrsim N} \sum_{1 \leq M \leq N_{\max}} N^{2s'} \frac{M N N_{\max}^{11/10}}{M N_{\max}^2} N^{-s'} N_1^{-s} N_2^{-s} N_3^{-s'} \\ &\quad \times \|P_{N_1} z_1\|_{Z_T^s} \|P_{N_2} z_2\|_{Z_T^s} \|P_{N_3} w\|_{Z_T^{s'}} \|P_N w\|_{Z_T^{s'}} \\ &\lesssim T^{1/8} N^{-s+s'+1/10+} (\|u\|_{Z_T^s}^2 + \|v\|_{Z_T^s}^2) \|w\|_{Z_T^{s'}}^2, \end{aligned}$$

which is acceptable since $s' < s - 1/10$.

• **Estimate for $C_N^{1,\text{high}}$.**

Performing a dyadic decomposition in $m_{\min} \sim M$, Lemma 3.10, Proposition 3.4 and (3.9) lead to

$$\begin{aligned} |C_N^{1,\text{high}}| &\lesssim T^{1/8} \sum_{M \gtrsim N^{1/2}} N^{2s'} \frac{MNN^{11/10}}{M^2N} N^{-2s'} N^{-2s} \\ &\quad \times (\|P_{\sim N}u\|_{Z_T^s}^2 + \|P_{\sim N}v\|_{Z_T^s}^2) \|P_{\sim N}w\|_{Z_T^{s'}}^2 \\ &\lesssim T^{1/8} N^{-2s+3/5+} (\|u\|_{Z_T^s}^2 + \|v\|_{Z_T^s}^2) \|w\|_{Z_T^{s'}}^2, \end{aligned}$$

which is acceptable for $s > 3/10$.

• **Estimate for $C_N^{1,\text{low}} + G_N$.**

We have

$$G_N(t) = N^{2s'} \Re \int_0^t \frac{d}{dt} \mathcal{E}_N^3(t) =: -C_N^{1,\text{low}} + A_{N,1} + A_{N,2} + A_{N,3},$$

where

$$\begin{aligned} A_{N,1} &= N^{2s'} \int_0^t \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \sum_{1 \leq q \neq q' \leq 2} \sum_{i=1}^3 \frac{k \varphi_N^2(k) k_q}{\Omega^3(\vec{k}_{(3)})} \\ &\quad \times \mathfrak{I} \left[\hat{z}_{q',i}(k_{q'}) \hat{w}(k_3) \hat{w}(-k) \left(3 |\hat{z}_{q,i}(k_q)|^2 \hat{z}_{q,i}(k_q) + \sum_{\vec{k}_{q(3)} \in D(k_q)} \prod_{j=1}^3 \hat{z}_{q,i}(k_{q,j}) \right) \right], \\ A_{N,2} &= N^{2s'} \int_0^t \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \sum_{i=1}^3 \frac{k \varphi_N^2(k) k_3}{\Omega^3(\vec{k}_{(3)})} \\ &\quad \times \mathfrak{I} \left[\hat{w}(-k) \hat{z}_{1,i}(k_1) \hat{z}_{2,i}(k_2) \left((|\hat{u}(k_3)|^2 + |\hat{v}(k_3)|^2) \hat{w}(k_3) + \hat{w}(-k_3) \hat{u}(k_3) \hat{v}(k_3) \right) \right. \\ &\quad \left. + \sum_{\vec{k}_{3(3)} \in D(k_3)} [\hat{u}(k_{3,1}) \hat{u}(k_{3,2}) + \hat{u}(k_{3,1}) \hat{v}(k_{3,2}) + \hat{v}(k_{3,1}) \hat{v}(k_{3,2})] \hat{w}(k_{3,3}) \right] \end{aligned}$$

and

$$\begin{aligned} A_{N,3} &= N^{2s'} \int_0^t \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \sum_{i=1}^3 \frac{k^2 \varphi_N^2(k)}{\Omega^3(\vec{k}_{(3)})} \\ &\quad \times \mathfrak{I} \left[\hat{z}_{1,i}(k_1) \hat{z}_{2,i}(k_2) \hat{w}(k_3) \left((|\hat{u}(k)|^2 + |\hat{v}(k)|^2) \hat{w}(-k) + \hat{w}(k) \hat{u}(-k) \hat{v}(-k) \right) \right] \end{aligned}$$

$$+ \sum_{\vec{k}_{4(3)} \in D(-k)} \left[\widehat{u}(k_{4,1})\widehat{u}(k_{4,2}) + \widehat{u}(k_{4,1})\widehat{v}(k_{4,2}) + \widehat{v}(k_{4,1})\widehat{v}(k_{4,2}) \right] \widehat{w}(k_{4,3}) \Bigg],$$

where we set $(z_{1,1}, z_{2,1}) = (u, u)$, $(z_{1,2}, z_{2,2}) = (u, v)$ and $(z_{1,3}, z_{2,3}) = (v, v)$. Hence,

$$C_N^{1,\text{low}} + G_N = \sum_{j=1}^3 A_{N,j}.$$

For any sextuplet $\vec{N} = (N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N) \in (2^{\mathbb{N}})^6$, any $\vec{z} = (z_{1,1}, z_{1,2}, z_{1,3}, z_2, z_3, z_4) \in (Z_T^s)^6$ and any function $\eta \in L^\infty(\mathbb{R})$, we set

$$R_{\vec{N},\eta}(\vec{z}) = N^{2s'+2} \left| \int_0^t \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \frac{\eta(k_1, k_2, k_3)}{\Omega^3(\vec{k}_{(3)})} \widehat{P_{N_2 z_2}(k_2)} \widehat{P_{N_3 z_3}(k_3)} \widehat{P_{N z_4}(-k)} \right. \\ \left. \times \widehat{P_{N_{1,1} z_{1,1}}(k_1)} \widehat{P_{N_{1,2} z_{1,2}}(-k_1)} \widehat{P_{N_{1,3} z_{1,3}}(k_1)} \right|$$

and

$$S_{\vec{N},\eta}(\vec{z}) = N^{2s'+2} \left| \int_0^t \sum_{k \in \mathbb{Z}} \sum_{\substack{\vec{k}_{(3)} \in D^1(k) \\ m_{\min} \leq N^{1/2}}} \frac{\eta(k_1, k_2, k_3)}{\Omega^3(\vec{k}_{(3)})} \widehat{P_{N_2 z_2}(k_2)} \widehat{P_{N_3 z_3}(k_3)} \widehat{P_{N z_4}(-k)} \right. \\ \left. \times \sum_{\vec{k}_{1(3)} \in D(k_1)} \widehat{P_{N_{1,1} z_{1,1}}(k_{1,1})} \widehat{P_{N_{1,2} z_{1,2}}(k_{1,2})} \widehat{P_{N_{1,3} z_{1,3}}(k_{1,3})} \right|.$$

We observe that to get the desired estimates for the $A_{N,j}$, it suffices to prove that for any sextuplet $\vec{N} = (N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N) \in (2^{\mathbb{N}})^6$, any $\vec{z} = (z_{1,1}, z_{1,2}, z_{1,3}, z_2, z_3, z_4) \in (Z^s)^6$ and any $\eta \in L^\infty$ with $\|\eta\|_\infty \lesssim 1$,

$$R_{\vec{N},\eta}(\vec{z}) + S_{\vec{N},\eta}(\vec{z}) \lesssim T^{1/8} \widetilde{N}_{\max}^{-2(s-s') +} \|P_{N z_4}\|_{Z_T^s} \prod_{i=2}^3 \|P_{N_i z_i}\|_{Z_T^s} \prod_{j=1}^3 \|P_{N_{1,j} z_{1,j}}\|_{Z_T^s} \quad (5.7)$$

where $\widetilde{N}_{\max} = \max(N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N)$.

Indeed, the modulus of the $A_{N,j}$ are controlled by sums of terms of this form with

$$\eta(k_1, k_2, k_3) = \frac{k_4 k_j}{N^2} \mathbb{1}_{D^1} \varphi_N(k_4), \quad j = 1, 2, 3, 4, \quad -k_4 = k_1 + k_2 + k_3,$$

and where w appears two times in the components of \vec{z} and all the other components are u or v . Therefore (5.7) leads to

$$\sum_{j=1}^3 |A_{j,N}| \lesssim T^{1/8} \widetilde{N}_{\max}^{0-} \|P_{\lesssim \widetilde{N}_{\max}} w\|_{Z_T^{s'}}^2 \left(\|P_{\lesssim \widetilde{N}_{\max}} u\|_{Z_T^s} + \|P_{\lesssim \widetilde{N}_{\max}} v\|_{Z_T^s} \right)^4 \quad (5.8)$$

and (5.5) then follows by summing over $(N_{1,1}, N_{1,2}, N_{1,3}, N_2, N_3, N)$ thanks to the factor \tilde{N}_{\max}^{0-} .

To simplify the notation, we denote $P_{N_i} z_i$, $P_{N_{1,j}} z_{1,j}$ and $P_N z_4$ by respectively z_i and $z_{1,j}$ and z_4 in the sequel.

• **Estimate for $R_{\tilde{N},\eta}(\vec{z})$.**

We recall that on D^1 we must have $|k_1| \sim |k_2| \sim |k_3| \sim |k|$ and thus $R_{\tilde{N},\eta}(\vec{z})$ vanishes except if $N_{1,1} \sim N_{1,2} \sim N_{1,3} \sim N_2 \sim N_3 \sim N$. In particular, $\tilde{N}_{\max} \sim N$. Then, estimate (3.8) leads to

$$\begin{aligned} R_{\tilde{N},\eta}(\vec{z}) &\lesssim \sum_{M \geq 1} T \frac{N^{2s'+2} M}{M^2 N} N^{-6s} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s} \\ &\lesssim T \tilde{N}_{\max}^{-2(s-s')} \tilde{N}_{\max}^{1-4s} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s}, \end{aligned}$$

which is acceptable since for $s > 1/4$.

• **Estimate for $S_{\tilde{N},\eta}(\vec{z})$.**

We set $\vec{k}_{1(3)} = (k_{1,1}, k_{1,2}, k_{1,3})$. By symmetry, we may assume that $N_{1,1} \geq N_{1,2} \geq N_{1,3}$ so that $N_{1,1} \sim \tilde{N}_{\max}$. We separate different contributions.

Case 1: $M_{1,\text{med}} \geq 2^{-9} N$.

Case 1-1: $|\Omega_3(\vec{k}_{1(3)})| \gg |\Omega_3(\vec{k}_{(3)})|$. Noticing that $|\Omega_3(\vec{k}_{(3)})| \sim M_{\min} M_{\text{med}} N$ and $|\Omega_3(\vec{k}_{1(3)})| \gtrsim 2^{-9} M_{1,\text{min}} N N_{1,1}$ in this region, (5.2) leads to

$$\begin{aligned} S_{\tilde{N},\eta}(\vec{z}) &\lesssim \sum_{1 \leq M_{1,\text{min}} \lesssim N_{1,2}} \sum_{M_{\text{med}} \gtrsim M_{\text{min}} \geq 1} T^{1/8} \frac{N^{2s'+2} M_{\min}}{M_{\min} M_{\text{med}} N^2} \tilde{N}_{\max}^{1/10} \tilde{N}_{\max}^{-2s} N^{-2s} \\ &\quad \times \prod_{i=2}^4 \|z_i\|_{Z_T^s} \prod_{j=1}^3 \|z_{1,j}\|_{Z_T^s} \\ &\lesssim T^{1/8} N_{\max}^{-2(s-s')} \tilde{N}_{\max}^{-2s'+1/10} N^{2(s'-s)} \prod_{i=2}^4 \|z_i\|_{Z_T^s} \prod_{j=1}^3 \|z_{1,j}\|_{Z_T^s}, \end{aligned}$$

which is acceptable since for $s' > s/2$ and $s \geq 1/3$ force $s' > 1/6$.

Case 1-2: $|\Omega_3(\vec{k}_{1(3)})| \lesssim |\Omega_3(\vec{k}_{(3)})|$. Then $m_{\min} \leq N^{1/2}$ yields

$$M_{1,\text{min}} \lesssim \frac{M_{\min} M_{\text{med}} N}{M_{1,\text{med}} N_{1,1}} \lesssim \frac{M_{\text{med}} N^{3/2}}{N N_{1,1}} = \frac{M_{\text{med}} N^{1/2}}{N_{1,1}}.$$

Therefore (5.1) leads to

$$S_{\tilde{N},\eta}(\vec{z}) \lesssim T \sum_{1 \leq M_{\min} \leq M_{\text{med}} \lesssim N} \frac{N^{2s'+2} M_{\min} M_{\text{med}} N^{1/2}}{M_{\min} M_{\text{med}} N N_{1,1}} N_{1,1}^{-s} N^{-3s} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s}$$

$$\lesssim T \tilde{N}_{\max}^{-2(s-s')} \tilde{N}_{\max}^{s-1-2s'} N^{2s'-3s+3/2+} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s}$$

which is acceptable since $s' > s/2$ ensures that $s - 1 - 2s' < 0$ and since

$$(s - 1 - 2s') + (2s' - 3s + \frac{3}{2}+) = -2s + \frac{1}{2}+ < 0 \quad \Leftrightarrow \quad s > \frac{1}{4}.$$

Case 2: $M_{1,\text{med}} < 2^{-9}N$. Then $N_{1,1} \sim N_{1,2} \sim N_{1,3} \sim N_2 \sim N_3 \sim N$.

Case 2-1: $M_{1,\text{min}} \leq 2^9 M_{\text{med}}$. Then (5.1) leads to

$$\begin{aligned} S_{\tilde{N},\eta}(\vec{z}) &\lesssim T \sum_{1 \leq M_{\text{min}} \leq M_{\text{med}} \lesssim N} \frac{N^{2s'+2} M_{\text{min}} M_{\text{med}}}{M_{\text{min}} M_{\text{med}} N} N^{-6s} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s} \\ &\lesssim T \tilde{N}_{\max}^{-2(s-s')} \tilde{N}_{\max}^{-4s+1+} \prod_{i=2}^4 \|z_i\|_{L_T^\infty H_x^s} \prod_{j=1}^3 \|z_{1,j}\|_{L_T^\infty H_x^s}, \end{aligned}$$

which acceptable for $s > 1/4$.

Case 2-2: $M_{1,\text{min}} > 2^9 M_{\text{med}}$. Then,

$$|\Omega_3(\vec{k}_{1(3)})| \sim M_{1,\text{min}} M_{1,\text{med}} M_{1,\text{max}} \geq 2^{18} M_{\text{min}} M_{\text{med}} N \gg |\Omega_3(\vec{k}_{(3)})|,$$

and thus (5.2) leads to

$$\begin{aligned} S_{\tilde{N},\eta}(\vec{z}) &\lesssim T^{1/8} \sum_{M_{\text{min}} \geq 1} \sum_{1 \leq M_{1,\text{med}} \lesssim N} \frac{N^{2s'+2} M_{\text{min}}}{M_{\text{min}}^2 N M_{1,\text{med}}} N^{-6s} N^{1/10} \\ &\quad \times \prod_{i=2}^4 \|z_i\|_{Z_T^s} \prod_{j=1}^3 \|z_{1,j}\|_{Z_T^s} \\ &\lesssim T^{1/8} \tilde{N}_{\max}^{-2(s-s')} \tilde{N}_{\max}^{-4s+11/10} \prod_{i=2}^4 \|z_i\|_{Z_T^s} \prod_{j=1}^3 \|z_{1,j}\|_{Z_T^s}, \end{aligned}$$

which is acceptable for $s > 11/40$. \square

LEMMA 5.4. *Assume that $0 < T \leq 1$, $s \geq 1/3$ and $(u, v) \in L^\infty(0, T; H^s(\mathbb{T}))^2$ are two solution to (2.1) associated to the initial data $(u_0, v_0) \in H^s(\mathbb{T})^2$. Then, setting $s' = 1/3 - 1/8 = 5/24$, it holds*

$$\|u - v\|_{Z_T^{s'}} \lesssim (1 + \|u\|_{L_T^\infty H_x^s}^2 + \|v\|_{L_T^\infty H_x^s}^2)^3 \|u - v\|_{L_T^\infty H_x^{s'}}. \quad (5.9)$$

PROOF. We proceed as in Lemma 3.9 so that we are reduced to estimate $\|u - v\|_{X_T^{s'-11/10,1}}$. Setting $w = u - v$, the Duhamel formula associated to (1.1), the standard linear estimates in Bourgain's spaces and the fractional Leibniz rule lead to

$$\|w\|_{X_T^{s'-11/10,1}}$$

$$\begin{aligned}
 &\lesssim \|u_0 - v_0\|_{H^{s'}} + \|\partial_x(w(u^2 + uv + v^2))\|_{X_T^{s'-11/10,0}} + \|P_0(u^2)w_x\|_{X_T^{s'-11/10,0}} \\
 &\quad + \|P_0(u^2 - v^2)v_x\|_{X_T^{s'-11/10,0}} \\
 &\lesssim \|u - v\|_{L_T^\infty H_x^{s'}} + \|J_x^{s'-1/10}(w(u^2 + uv + v^2))\|_{L_T^2 L_x^2} + \|u\|_{L_T^\infty L_x^2}^2 \|w\|_{L_T^2 H_x^{s'-1/10}} \\
 &\quad + (\|u\|_{L_T^\infty L_x^2} + \|v\|_{L_T^\infty L_x^2}) \|v\|_{L_T^2 H_x^{s'-1/10}} \|w\|_{L_T^\infty L_x^2}. \tag{5.10}
 \end{aligned}$$

Then, we notice that

$$\begin{aligned}
 &\|J_x^{s'-1/10}(w(u^2 + uv + v^2))\|_{L_T^2 L_x^2} \\
 &\lesssim \|J_x^{s'}(w(u^2 + uv + v^2))\|_{L_T^2 L_x^{5/3}} \\
 &\lesssim (\|u\|_{L_T^4 L_x^{20}}^2 + \|v\|_{L_T^4 L_x^{20}}^2) \|J_x^{s'} w\|_{L_T^\infty L_x^2} \\
 &\quad + (\|u\|_{L_T^4 L_x^{20}} + \|v\|_{L_T^4 L_x^{20}}) \|w\|_{L_T^\infty L_x^{24/7}} (\|J_x^{5/24} u\|_{L_T^4 L_x^{120/31}} + \|J_x^{5/24} v\|_{L_T^4 L_x^{120/31}})
 \end{aligned}$$

which leads to (5.9) thanks to (3.20)–(3.21) and Sobolev inequalities since $H^{5/24}(\mathbb{T}) \hookrightarrow L^{24/7}(\mathbb{T})$ and for $s \geq 1/3$, it holds $s' = s - 1/8 \geq 5/24$ and $120/31 < 4$. \square

6. Proof of Theorem 2.3.

6.1. Unconditional uniqueness for the renormalized mKdV equation.

Let us start by proving the unconditional uniqueness of (2.1). Let $T > 0$ and $(v_1, v_2) \in L^\infty(0, T; H^{1/3})^2$ be a couple of functions that satisfies (2.1) in the distributional sense with $v_1(0) = v_2(0) = u_0 \in H^s(\mathbb{T})$. We first notice that Lemma 3.9 ensures that $(u, v) \in Z_{\tilde{T}}^{1/3}$ with $\tilde{T} = \min(1, T)$ and, from Proposition 3.11, we infer that

$$\|v_1\|_{Z_{\tilde{T}}^{1/3}} + \|v_2\|_{Z_{\tilde{T}}^{1/3}} \lesssim \|u_0\|_{H^{1/3}} + \tilde{T}^{1/8} \left(1 + \|v_1\|_{L_T^\infty H_x^{1/3}}^3 + \|v_2\|_{L_T^\infty H_x^{1/3}}^3\right)^4.$$

Hence, taking $\tilde{T} \leq \min(1, T, (1 + \|u_0\|_{H^{1/3}})^{-16})$, we get

$$\|v_1\|_{Z_{\tilde{T}}^{1/3}} + \|v_2\|_{Z_{\tilde{T}}^{1/3}} \lesssim \|u_0\|_{H^{1/3}}.$$

Then, noticing that $1/6 < 5/24 < 1/3 - 1/9$, (5.4)–(5.5) and (5.9) lead to

$$\|v_1 - v_2\|_{L_T^\infty H_x^{5/24}}^2 \lesssim T^{1/8} N_0^{3/2} (1 + \|u_0\|_{H^{1/2}})^{25} \|v_1 - v_2\|_{L_T^\infty H_x^{5/24}}^2$$

with $N_0 \gg \|u_0\|_{H^{1/3}}^{-3}$. This forces

$$\|v_1 - v_2\|_{L_T^\infty H_x^{5/24}} = 0$$

with $T' \sim \min(\tilde{T}, (1 + \|u_0\|_{H^{1/3}})^{-300})$. Hence $v_1 = v_2$ a.e. on $[0, T']$. Therefore there exists $t_1 \in [T'/2, T']$ such that $v_1(t_1) = v_2(t_1)$ and $\|v_1(t_1)\|_{H^{1/3}} \leq \|v_1\|_{L_T^\infty H_x^{1/3}}$. Using this bound we can repeat this argument a finite number of times to extend the uniqueness result on $[0, T]$.

6.2. Local well-posedness of the renormalized mKdV equation.

It is known from the classical well-posedness theory that an initial data $u_0 \in H^\infty(\mathbb{T})$ gives rise to a global solution $u \in C(\mathbb{R}; H^\infty(\mathbb{T}))$ to the Cauchy problem (1.1). Then combining Lemma 3.9 and Proposition 3.11 we infer that u verifies

$$\|u\|_{L_T^\infty H_x^s}^2 \lesssim \|u_0\|_{H^s}^2 + T^{1/8} \left(\|u\|_{L_T^\infty H_x^s} + \|u\|_{L_T^\infty H_x^s}^3 \right)^4 \tag{6.1}$$

for any $0 < T < 1$. Taking $T = T(\|u_0\|_{H^s}) \sim \min(1, (1 + \|u_0\|_{H^s})^{-10})$, the continuity of: $T \mapsto \|u\|_{L_T^\infty H_x^s}$ ensures that

$$\|u\|_{L_T^\infty H^s} \lesssim \|u_0\|_{H^s}$$

and Lemma 3.9 then leads to

$$\|u\|_{Z_T^s} \lesssim \|u_0\|_{H^s} (1 + \|u_0\|_{H^s}^2). \tag{6.2}$$

Moreover, we infer from Theorem 4.1 that for any $K \in 2^{\mathbb{N}}$ it holds

$$\begin{aligned} \|P_{\geq K} u(t)\|_{L_T^\infty H_x^s}^2 &\leq \sum_{k \geq K} \sup_{t \in [0, T]} |k|^{2s} |\widehat{u}(t, k)|^2 \\ &\lesssim \sum_{k \geq K} \left(|k|^{2s} |\widehat{u_0}(k)|^2 + k^{2s-1} \|u\|_{Z^s}^4 (1 + \|u\|_{Z^s})^4 \right) \\ &\leq \|P_{\geq K} u_0\|_{H^s}^2 + K^{2s-1} (1 + \|u_0\|_{H^s})^{24}. \end{aligned} \tag{6.3}$$

Now let us fix $1/3 \leq s < 1/2$. For $u_0 \in H^s(\mathbb{T})$ we set $u_{0,n} = P_{\leq n} u_0$ and we denote by $u_n \in C(\mathbb{R}; H^\infty(\mathbb{T}))$ the solutions to (2.1) emanating from $u_{0,n}$. In view of (6.2) we infer that for any $n \in \mathbb{N}$,

$$\|u_n\|_{Z_T^s} \lesssim \|u_0\|_{H^s} (1 + \|u_0\|_{H^s}^2),$$

with $T = T(\|u_0\|_{H^s})$, and (6.3) ensures that

$$\lim_{K \rightarrow +\infty} \sup_{n \in \mathbb{N}} \|P_{\geq K} u_n(t)\|_{L_T^\infty H_x^s} = 0. \tag{6.4}$$

This proves that the sequence $\{u_n\}$ is bounded in $L^\infty(]0, T[; H^s(\mathbb{T}))$ and thus u_n^3 is bounded in $L^\infty(]0, T[; L^2(\mathbb{T}))$. Moreover, in view of the equation (2.1), the sequence $\{\partial_t u_n\}$ is bounded in $L^\infty(]0, T[; H^{-3}(\mathbb{T}))$. By Aubin–Lions compactness theorem we infer that, for any $0 < T \leq T(\|u_0\|_{H^s})$, $\{u_n\}$ is relatively compact in $L^2(]0, T[\times \mathbb{T})$. Therefore, using a diagonal extraction argument, we obtain the existence of an increasing sequence $\{n_k\} \subset \mathbb{N}$ and $u \in L^\infty(]0, T[; H^s(\mathbb{T}))$ such that

$$u_{n_k} \rightharpoonup u \text{ weak star in } L^\infty(]0, T[; H^s(\mathbb{T})) \tag{6.5}$$

$$u_{n_k} \rightarrow u \text{ in } L^2(]0, T[; L^2(\mathbb{T})) \cap L^3(]0, T[; L^3(\mathbb{T})) \tag{6.6}$$

$$u_{n_k} \rightarrow u \text{ a.e. in }]0, T[\times \mathbb{T} \tag{6.7}$$

$$u_{n_k}^3 \rightarrow u^3 \text{ in } L^1(]0, T[; L^1(\mathbb{T})). \tag{6.8}$$

These convergences results enable us to pass to the limit on the equation and to obtain that the limit function u satisfies (2.2) with $F(u) = u^3 - 3P_0(u^2)$. Therefore the unconditional uniqueness result ensures that u is the only accumulation point of $\{u_n\}$ and thus $\{u_n\}$ converges to u in the sense (6.5)–(6.8). Now, using the bounds on $\{u_n\}$ and $\{\partial_t u_n\}$, it is clear that for any $\phi \in C^\infty(\mathbb{T})$ and any $T > 0$, the sequence $\{t \mapsto (u_n, \phi)_{H^s}\}$ is uniformly equi-continuous on $[0, T]$. By Ascoli’s theorem it follows that

$$(u_n, \phi)_{H^s} \rightarrow (u, \phi) \text{ in } C([0, T]).$$

In particular, for any fixed $N \geq 1$, it holds

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \|P_{\leq N}(u_n - u)(t)\|_{H^s} = 0.$$

This last limit combined with (6.4) ensures that

$$u_n \rightarrow u \text{ in } C([0, T]; H^s(\mathbb{T}))$$

and thus $u \in C([0, T]; H^s(\mathbb{T}))$.

Finally, to prove the continuity with respect to initial data, we take a sequence $\{u_0^m\} \subset B_{H^s}(0, 2\|u_0\|_{H^s})$ that converges to u_0 in $H^s(\mathbb{T})$. Denoting by u^m the associated solutions to (2.1) that we have constructed above, we obtain in exactly the same way as above that for $T \sim \min(1, (1 + \|u_0\|_{H^s})^{-10})$ it holds

$$\|u_m\|_{Z_T^s} \lesssim \|u_0\|_{H^s}(1 + \|u_0\|_{H^s}^2), \quad \lim_{K \rightarrow +\infty} \sup_{m \in \mathbb{N}} \|P_{\geq K} u_m(t)\|_{L_T^\infty H^s} = 0$$

and

$$(u_m, \phi)_{H^s} \rightarrow (u, \phi) \text{ in } C([0, T]).$$

This ensures that $u_m \rightarrow u$ in $C([0, T]; H^s(\mathbb{T}))$ and completes the proof of the unconditional well-posedness of (2.1).

6.3. Back to the mKdV equation.

For $s \geq 0$ we define the mapping

$$\begin{aligned} L_T^\infty H_x^s &\longrightarrow L_T^\infty H_x^s \\ \Psi : u = u(t, x) &\longmapsto \Psi(u) = \Psi(u)(t, x) = u\left(t, x + \int_0^t P_0(u^2(\tau)) d\tau\right). \end{aligned}$$

It is easy to check that Ψ is a bijection from $L_T^\infty H_x^s$ into itself and also from $C([0, T]; H^s(\mathbb{T}))$ into itself with inverse bijection defined by

$$\Psi^{-1}(u) = u\left(t, x - \int_0^t P_0(u^2(\tau)) d\tau\right).$$

Moreover, for $s \geq 1/3$, it is not too hard to check that $u \in L_T^\infty H_x^s$ is a solution of (2.2) with $F(u) = u^3$ if and only if $\Psi(u) \in L_T^\infty H_x^s$ is a solution to (2.2) with $F(u) = u^3 - 3P_0(u^2)$.

Finally, we claim that Ψ and Ψ^{-1} are continuous from $C([0, T]; H^s(\mathbb{T}))$ into itself. Indeed, let $\{v_n\}_{n \geq 1} \subset C([0, T]; H^s(\mathbb{T}))$ that converges to v in $C([0, T]; H^s(\mathbb{T}))$. Then denoting $\int_0^t P_0(v_n^2)(s) ds$ by $\alpha_n(t)$ and $\int_0^t P_0(v^2)(s) ds$ by $\alpha(t)$, it is easy to check that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} (\alpha_n(t) - \alpha(t)) = 0 \quad (6.9)$$

and

$$\begin{aligned} \sup_{t \in [0, 1]} \|\Psi(v_n)(t) - \Psi(v)(t)\|_{H^s} &\leq \sup_{t \in [0, 1]} \left\| v_n(t, \cdot + \alpha_n(t)) - v(t, \cdot + \alpha_n(t)) \right\|_{H^s} \\ &\quad + \sup_{t \in [0, 1]} \left\| v(t, \cdot + \alpha_n(t)) - v(t, \cdot + \alpha(t)) \right\|_{H^s}. \end{aligned}$$

It is clear that the first term of the right-hand side of the above estimate converges to 0. Now, the second term can be rewritten as

$$I_n = \sup_{t \in [0, 1]} \left(\sum_{k \in \mathbb{Z}} \left| k^s (e^{ik\alpha_n(t)} - e^{ik\alpha(t)}) \widehat{v}(t, k) \right|^2 \right)^{1/2}.$$

Since $v \in C([0, T]; H^s(\mathbb{T}))$, $\{v(t) : t \in [0, T]\}$ is a compact set of $H^s(\mathbb{T})$ and thus

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} \sum_{|k| \geq N} |k|^{2s} |\widehat{v}(t, k)|^2 = 0,$$

which combined with (6.9) ensures that $\lim_{n \rightarrow \infty} I_n = 0$ and completes the proof of the desired continuity result.

These properties of Ψ combined with the unconditional local well-posedness of the renormalized mKdV equation in $H^s(\mathbb{T})$, clearly leads to Theorem 2.3.

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