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Two-cardinal versions of weak compactness: Partitions of triples

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Abstract. Let κ be a regular uncountable cardinal, and λ be a cardinal greater than κ. Our main result asserts that if $(\lambda^{<\kappa})^{<(\lambda^{<\kappa})} = \lambda^{<\kappa}$, then $(p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow \big((\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+, \mathrm{NS}_{\kappa,\lambda}s^+\big)^3$ and $(p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow (\mathrm{NS}_{\kappa,\lambda}s^+)^3$, where $\mathrm{NS}_{\kappa,\lambda}s$ (respectively, $\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}}$) denotes the smallest seminormal (respectively, strongly normal) ideal on $P_\kappa(\lambda)$, $\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}$ (respectively, $\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}$) denotes the ideal of non-ineffable (respectively, non-almost ineffable) subsets of $P_\kappa(\lambda^{<\kappa})$, and $p_{\kappa,\lambda} : P_\kappa(\lambda^{<\kappa}) \to P_\kappa(\lambda)$ is defined by $p_{\kappa,\lambda}(x) = x \cap \lambda$.

0. Introduction.

Let κ be a regular uncountable cardinal, and $\lambda > \kappa$ be a cardinal. In this paper we study $P_\kappa(\lambda)$ versions of weak compactness and associated ideals, thus continuing [**23**] which dealt with partitions of pairs. Here we are mostly concerned with partitions of triples.

This area of research has been started by Jech in a paper [**10**] published in 1973. Time has elapsed, but it remains unclear which structure we should investigate. What is the right generalization of $(\kappa, \subsetneq)$? Is it $(P_\kappa(\lambda), \subsetneq)$ or $(P_\kappa(\lambda), <)$ (where $a < b$ means that $a \in P_{|b\cap\kappa|}(b)$)? Whenever we can, we give positive results in terms of the first one, and negative results in terms of the second.

It seems to us that Johnson (see e.g. [**12**]) was right when he stressed the importance of the notion of seminormality. The point is that any κ-complete ideal J on κ is trivially seminormal (since given $A \in J^+$, $\gamma < \kappa$ and $f : A \to \gamma$, there must be $B \in J^+ \cap P(A)$ such that f is constant on B), and therefore the noncofinal ideal I_κ on κ can be seen as the smallest seminormal ideal on κ. So each time we attempt to formulate a two-cardinal version of a statement involving I_κ, we should ponder whether I_κ should be replaced by $\mathrm{I}_{\kappa,\lambda}$ (the noncofinal ideal on $P_\kappa(\lambda)$) or $\mathrm{NSS}_{\kappa,\lambda}$ (the smallest seminormal ideal on $P_\kappa(\lambda)$). Consider for example the partition property $\kappa \longrightarrow (\kappa)^2$ expressing that κ is a weakly compact cardinal. By the remarks above, it can be generalized in (at least) four different ways, namely $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$, $P_\kappa(\lambda) \longrightarrow (\mathrm{I}^+_{\kappa,\lambda})^2$, $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{NSS}^+_{\kappa,\lambda})^2$ and $P_\kappa(\lambda) \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^2$. We do not know whether these four assertions are equivalent.

We just advocated the replacement of (some occurrences of) I_κ by $\mathrm{NSS}_{\kappa,\lambda}$. Likewise we plead for the replacement of (many occurrences of) NS_κ (the nonstationary ideal on

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κ) by $\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}}$ (the smallest strongly normal ideal on $P_\kappa(\lambda)$) which seems to us more appropriate than $\mathrm{NS}_{\kappa,\lambda}$ (the nonstationary ideal on $P_\kappa(\lambda)$). Note that $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NSS}_{\kappa,\lambda}$ in case $\mathrm{cf}(\lambda) < \kappa$.

Take for instance ineffability. By work of Kunen (see [**6**]) and Baumgartner [**6**], $\mathrm{NIn}_\kappa = \{A \subseteq \kappa : A \not\longrightarrow (\mathrm{NS}_\kappa^+)^2\}$, where NIn_κ denotes the nonineffable ideal on κ. By work of Abe-Usuba [**5**], Carr [**8**], and Magidor [**14**], if $\mathrm{cf}(\lambda) \geq \kappa$, then $\mathrm{NIn}_{\kappa,\lambda} = \{A \subseteq P_\kappa(\lambda) : A \underset{<}{\not\longrightarrow} (\mathrm{NS}_{\kappa,\lambda}^+)^2\} = \big\{A \subseteq P_\kappa(\lambda) : A \not\longrightarrow \big((\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+\big)^2\big\}$. The conclusion as stated is no longer valid in case $\mathrm{cf}(\lambda) < \kappa$. In fact, it is observed in Section 3 that $\mathrm{NIn}_{\kappa,\lambda}^+ \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$ if $2^\lambda = \lambda^{<\kappa}$.

Baumgartner [**6**] also showed that $\mathrm{NIn}_\kappa = \{A \subseteq \kappa : A \not\longrightarrow (\mathrm{NS}_\kappa^+, \kappa)^3\}$. We establish the following:

THEOREM 0.1 (Theorem 2.14). *Assume* $\lambda^{<\lambda} = \lambda$. *Then* $\mathrm{NIn}_{\kappa,\lambda} = \big\{A \subseteq P_\kappa(\lambda) : A \not\longrightarrow \big((\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+, \mathrm{NSS}_{\kappa,\lambda}^+\big)^3\big\}$.

We also show that $\mathrm{NIn}_{\kappa,\lambda}^+ \underset{<}{\longrightarrow} (\mathrm{NS}_{\kappa,\lambda}^+)^3$ does not hold in case $\mathrm{cf}(\lambda) \geq \kappa$ (see Proposition 2.19).

Note the cardinality assumption in Theorem 0.1. It entails that λ is regular. In the present paper we have little to say concerning the case where $\kappa \leq \mathrm{cf}(\lambda) < \lambda$ (for some results in this case see [**23**]). Assuming λ is regular, the cardinality assumption in question is not known to be necessary. However, our guess is that there is some ideal J on $P_\kappa(\lambda)$, whose definition is similar to that of $\mathrm{NIn}_{\kappa,\lambda}$, such that $J = \big\{A \subseteq P_\kappa(\lambda) : A \underset{<}{\not\longrightarrow} \big((\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+, \mathrm{NSS}_{\kappa,\lambda}^+\big)^3\big\}$ (with $J = \mathrm{NIn}_{\kappa,\lambda}$ in case $2^{<\lambda} = \lambda$). For examples of such situations see [**23**].

Put $H = \{A \subseteq \kappa : A \not\longrightarrow (\kappa)^2\}$. If κ is weakly compact, then

(a) $H = \mathrm{I}_\kappa$, and
(b) $H^+ \longrightarrow (H^+)^3$.

In particular, $\mathrm{NS}_\kappa^* \longrightarrow (\kappa)^2$ just in case $\mathrm{NS}_\kappa^* \longrightarrow (\kappa)^3$. The $P_\kappa(\lambda)$ situation is different. Assuming $\lambda^{<\lambda} = \lambda$, $\mathrm{NS}_{\kappa,\lambda}^* \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$ if and only if κ is almost λ-ineffable (Corollary 5.11), whereas by a result of [**23**] $\mathrm{NS}_{\kappa,\lambda}^* \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$ if and only if κ is λ-Shelah.

The following provides a characterization of $\mathrm{NAIn}_{\kappa,\lambda}$ in terms of partition relations.

THEOREM 0.2 (Theorem 4.19). *Assume that* $\lambda^{<\lambda} = \lambda$, *but* λ *is not weakly compact. Then* $\mathrm{NAIn}_{\kappa,\lambda} = \{A \subseteq P_\kappa(\lambda) : A \cap C \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3\}$ *for some* $C \in \mathrm{NS}_{\kappa,\lambda}^*$.

The paper grew out of a set of notes by the second author concerning the $\underset{<}{\longrightarrow}$ partition relation. Joint work of the authors led to the present version.

The paper is organized as follows. In Section 1 we review basic material concerning the ideals on $P_\kappa(\lambda)$ considered in the paper. Sections 2 and 3 are devoted to the notion of ineffability and concerned with partitions of triples, respectively in the case $\mathrm{cf}(\lambda) = \lambda$ and $\mathrm{cf}(\lambda) < \kappa$. Sections 4–6 are also concerned with partitions of triples, but this time in connection with the notion of almost ineffability. They deal, respectively, with the following three cases: λ is regular but not weakly compact, λ is weakly compact, $\mathrm{cf}(\lambda) < \kappa$.

1. Basic material.

DEFINITION. For a set A and a cardinal μ, let $P_\mu(A) = \{a \subseteq A : |a| < \mu\}$.

DEFINITION. $\mathrm{I}_{\kappa,\lambda}$ denotes the collection of all $A \subseteq P_\kappa(\lambda)$ such that $A \cap \{a \in P_\kappa(\lambda) : b \subseteq a\} = \emptyset$ for some $b \in P_\kappa(\lambda)$.

DEFINITION. By an *ideal* on $P_\kappa(\lambda)$, we mean a collection J of subsets of $P_\kappa(\lambda)$ such that

(a) $\mathrm{I}_{\kappa,\lambda} \subseteq J$,
(b) $P(A) \subseteq J$ for all $A \in J$, and
(c) $A \cup B \in J$ for all $A, B \in J$.

J is *proper* if $P_\kappa(\lambda) \notin J$.

For an ideal J on $P_\kappa(\lambda)$, let $J^* = \{A \subseteq P_\kappa(\lambda) : P_\kappa(\lambda) \setminus A \in J\}$, $J^+ = \{A \subseteq P_\kappa(\lambda) : A \notin J\}$, and $J|X = \{A \subseteq P_\kappa(\lambda) : A \cap X \in J\}$ for every $X \in J^+$. $\mathrm{cof}(J)$ (respectively, $\overline{\mathrm{cof}}(J)$) denotes the smallest cardinality of $X \subseteq J$ with the property that for any $A \in J$, there is $Q \subseteq X$ such that $|Q| < 2$ (respectively, $|Q| < \kappa$) and $A \subseteq \bigcup Q$.

DEFINITION. Let $\xi \leq \lambda$. An ideal J on $P_\kappa(\lambda)$ is *ξ-normal* if given $A \in J^+$ and $f : A \to \xi$ with the property that $f(a) \in a$ for every $a \in A$, there is $B \in J^+ \cap P(A)$ such that f is constant on B. $\mathrm{NS}^\xi_{\kappa,\lambda}$ denotes the smallest ξ-normal ideal on $P_\kappa(\lambda)$. An ideal J on $P_\kappa(\lambda)$ is *normal* if it is λ-normal. We put $\mathrm{NS}_{\kappa,\lambda} = \mathrm{NS}^\lambda_{\kappa,\lambda}$.

Note that $\mathrm{NS}^\xi_{\kappa,\lambda} = \mathrm{I}_{\kappa,\lambda}$ for every $\xi < \kappa$.

The following is a generalization of the characterization of $\mathrm{NS}_{\kappa,\lambda}$.

LEMMA 1.1. *Let $\kappa \leq \xi \leq \lambda$ and $A \subseteq P_\kappa(\lambda)$. Then $A \in (\mathrm{NS}^\xi_{\kappa,\lambda})^*$ if and only if there is $f : \xi \times \xi \to P_\kappa(\lambda)$ such that $C^f_{\kappa,\lambda} \subseteq A$, where $C^f_{\kappa,\lambda}$ is the set of all $a \in P_\kappa(\lambda)$ such that*

(a) *$a \cap \xi \neq \emptyset$, and*
(b) *$f(\alpha, \beta) \subseteq a$ for every $(\alpha, \beta) \in (a \cap \xi) \times (a \cap \xi)$.*

DEFINITION. Given four cardinals τ, ρ, χ and σ, $\mathrm{cov}(\tau, \rho, \chi, \sigma)$ is defined as follows. If one may find $X \subseteq P_\rho(\tau)$ with the property that for any $a \in P_\chi(\tau)$, there is $Q \in P_\sigma(X)$ with $a \subseteq \bigcup Q$, let $\mathrm{cov}(\tau, \rho, \chi, \sigma) =$ the least cardinality of any such X. Otherwise let $\mathrm{cov}(\tau, \rho, \chi, \sigma) = 0$. We set $\mathrm{cov}(\tau, \rho, \chi, \sigma) = u(\tau, \chi)$ in case $\rho = \chi$ and $\sigma = 2$.

Note that $u(\kappa, \lambda) = \mathrm{cov}(\kappa, \lambda, \lambda, 2) = \min\{|X| : X \in \mathrm{I}^+_{\kappa,\lambda}\}$.

LEMMA 1.2 (Matet [**18**]). *Let μ be a cardinal with $\kappa \leq \mu < \lambda$. Then the following are equivalent*:

(i) $\mathrm{NS}^\mu_{\kappa,\lambda}|C = \mathrm{I}_{\kappa,\lambda}|C$ *for some* $C \in \mathrm{NS}^*_{\kappa,\lambda}$.

(ii) $\overline{\text{cof}}(\text{NS}_{\kappa,\mu}) \leq \lambda = \text{cov}(\lambda, \mu^+, \mu^+, \kappa)$.

DEFINITION. An ideal J on $P_\kappa(\lambda)$ is *seminormal* if it is ξ-normal for every $\xi < \lambda$. $\text{NSS}_{\kappa,\lambda}$ denotes the smallest seminormal ideal on $P_\kappa(\lambda)$.

LEMMA 1.3 (Abe [**2**]). *Suppose λ is regular. Then* $\text{NSS}_{\kappa,\lambda} = \bigcup_{\xi<\lambda} \text{NS}^{\xi}_{\kappa,\lambda}$.

LEMMA 1.4 (Matet-Shelah [**22**]). *Assuming λ is regular, the following are equivalent*:

(i) $\text{NSS}_{\kappa,\lambda}|C = \text{I}_{\kappa,\lambda}|C$ *for some* $C \in \text{NS}^*_{\kappa,\lambda}$.
(ii) $\overline{\text{cof}}(\text{NS}_{\kappa,\tau}) \leq \lambda$ *for every cardinal τ with* $\kappa \leq \tau < \lambda$.

LEMMA 1.5 (Abe [**2**]). *Suppose* $\kappa \leq \text{cf}(\lambda) < \lambda$. *Then* $\text{NS}_{\kappa,\lambda} = \text{NSS}_{\kappa,\lambda}|C$ *for some* $C \in \text{NS}^*_{\kappa,\lambda}$.

DEFINITION. Let $\delta \leq \lambda$. An ideal J on $P_\kappa(\lambda)$ is $[\delta]^{<\kappa}$*-normal* if given $A \in J^+$ and $f : A \to P_\kappa(\delta)$ with the property that $f(a) \in P_{|a\cap\kappa|}(a \cap \delta)$ for all $a \in A$, there is $B \in J^+ \cap P(A)$ such that f is constant on B. J is *strongly normal* if it is $[\lambda]^{<\kappa}$-normal.

The following is a generalization of a result of Carr-Levinski-Pelletier [**9**].

LEMMA 1.6. *Suppose κ is a limit cardinal, and let $\kappa \leq \delta \leq \lambda$. Then there exists a $[\delta]^{<\kappa}$-normal ideal if and only if κ is Mahlo.*

Assuming there exists a $[\delta]^{<\kappa}$-normal ideal on $P_\kappa(\lambda)$, $\text{NS}^{[\delta]^{<\kappa}}_{\kappa,\lambda}$ denotes the smallest such ideal.

LEMMA 1.7 (Carr-Levinski-Pelletier [**9**], Matet [**15**]). *Suppose κ is Mahlo and* $\lambda^{<\kappa} = \lambda$. *Then there is* $E \in \big(\text{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}\big)^*$ *such that* $\text{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = \text{NS}_{\kappa,\lambda}|E$.

LEMMA 1.8 (Matet-Péan-Shelah [**20**]). (i) *Suppose κ is Mahlo and* $\overline{\text{cof}}(\text{NS}_{\kappa,\lambda}) \leq \lambda^{<\kappa}$. *Then there is* $E \in \big(\text{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}\big)^*$ *such that* $\text{NS}_{\kappa,\lambda}|E = \text{I}_{\kappa,\lambda}|E$.
(ii) *Suppose* $\text{cf}(\lambda) < \kappa$. *Then* $\overline{\text{cof}}(\text{NS}_{\kappa,\lambda}) \leq \bigcup_{\kappa\leq\tau<\lambda} \overline{\text{cof}}(\text{NS}_{\kappa,\tau})$.

Thus if $\text{cf}(\lambda) < \kappa$, then $\overline{\text{cof}}(\text{NS}_{\kappa,\lambda}) \leq 2^{<\lambda}$.

DEFINITION. For $a, b \in P_\kappa(\lambda)$, $a < b$ means that $a \in P_{|b\cap\kappa|}(b)$.

DEFINITION. Let $n \in \omega \setminus 2$. For $A \subseteq P_\kappa(\lambda)$, let $[A]^n_< = \{(a_1, \ldots, a_n) \in A^n : a_1 < \cdots < a_n\}$ and $[A]^n = \{(a_1, \ldots, a_n) \in A^n : a_1 \subsetneq \cdots \subsetneq a_n\}$. Given $\mathcal{A}, \mathcal{B} \subseteq P(P_\kappa(\lambda))$ and $\eta \in \text{On}$, $\mathcal{A} \underset{<}{\longrightarrow} (\mathcal{B})^n_\eta$ (respectively, $\mathcal{A} \longrightarrow (\mathcal{B})^n_\eta$) asserts that for any $A \in \mathcal{A}$ and any $F : [A]^n \to \eta$, there is $B \in \mathcal{B} \cap P(A)$ such that F is constant on $[B]^n_<$ (respectively, $[B]^n$). For $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq P(P_\kappa(\lambda))$, $\mathcal{A} \underset{<}{\longrightarrow} (\mathcal{B}, \mathcal{C})^n$ (respectively, $\mathcal{A} \longrightarrow (\mathcal{B}, \mathcal{C})^n$) asserts that for any $A \in \mathcal{A}$ and any $F : [A]^n \to 2$, there is either $B \in \mathcal{B} \cap P(A)$ such that F takes the constant value 0 on $[B]^n_<$ (respectively, $[B]^n$), or $C \in \mathcal{C} \cap P(A)$ such that F takes the constant value 1 on $[C]^n_<$ (respectively, $[C]^n$). $\mathcal{A} \underset{<}{\longrightarrow} (\mathcal{B})^n$ (respectively, $\mathcal{A} \longrightarrow (\mathcal{B})^n$)

means that $\mathcal{A} \underset{<}{\longrightarrow} (\mathcal{B},\mathcal{B})^n$ (respectively, $\mathcal{A} \longrightarrow (\mathcal{B},\mathcal{B})^n$). For $A \subseteq P_\kappa(\lambda)$, $A \underset{<}{\longrightarrow} (\mathcal{B},\mathcal{C})^n$ (respectively, $A \underset{<}{\longrightarrow} (\mathcal{B},\mathcal{C})^n$) means that $\{A\} \underset{<}{\longrightarrow} (\mathcal{B},\mathcal{C})^n$ (respectively, $\{A\} \longrightarrow (\mathcal{B},\mathcal{C})^n$). Similarly, $A \underset{<}{\longrightarrow} (\mathcal{B})^n_\eta$ (respectively, $A \longrightarrow (\mathcal{B})^n_\eta$) means that $\{A\} \underset{<}{\longrightarrow} (\mathcal{B})^n_\eta$ (respectively, $\{A\} \longrightarrow (\mathcal{B})^n_\eta$). Each of the above partition relations is negated by crossing the arrow.

LEMMA 1.9 (Jech [**10**]). *Suppose* $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. *Then* κ *is weakly compact.*

DEFINITION. κ is *mildly* λ*-ineffable* if given $f_a : a \to 2$ for $a \in P_\kappa(\lambda)$, there is $g : \lambda \to 2$ such that for any $a \in P_\kappa(\lambda)$, we may find $b \in P_\kappa(\lambda)$ such that $a \subseteq b$ and $f_b|a = g|a$.

LEMMA 1.10 (Carr [**8**], Matet [**17**]). *If* $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$, *then* κ *is mildly* $\lambda^{<\kappa}$*-ineffable.*

LEMMA 1.11 (Usuba [**27**]). *Suppose* $\mathrm{cf}(\lambda) \geq \kappa$ *and* κ *is mildly* λ*-ineffable. Then* $\lambda^{<\kappa} = \lambda$.

DEFINITION. $\mathrm{NSJ}_{\kappa,\lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ for which one can find $f_a : a \to 2$ for $a \in A$ so that for every $g : \lambda \to 2$, there is $\xi \in \lambda$ such that $\{a \in A : \forall\gamma \in a \cap \xi(f_a(\gamma) = g(\gamma))\} \in \mathrm{NS}^\xi_{\kappa,\lambda}$.

It was observed in [**23**] that if $\mathrm{cf}(\lambda) \geq \kappa$ and $P_\kappa(\lambda) \notin \mathrm{NSJ}_{\kappa,\lambda}$, then κ is mildly λ-ineffable.

DEFINITION. NSh_κ is the set of all $B \subseteq \kappa$ for which one may find $k_\beta : \beta \to \beta$ for $\beta \in B$ such that for any $t : \kappa \to \kappa$, there is $\delta < \kappa$ with the property that $k_\beta|\delta \neq t|\delta$ for all $\beta \in B$ with $\beta \geq \delta$.

$\mathrm{NSh}_{\kappa,\lambda}$ is the set of all $A \subseteq P_\kappa(\lambda)$ with the property that we may find $f_a : a \to a$ for $a \in A$ such that for every $g : \lambda \to \lambda$, there is $b \in P_\kappa(\lambda)$ with $\{a \in A : b \subseteq a$ and $g|b = f_a|b\} = \emptyset$. κ is λ*-Shelah* if $P_\kappa(\lambda) \notin \mathrm{NSh}_{\kappa,\lambda}$.

DEFINITION. $\mathrm{NAIn}_{\kappa,\lambda}$ (respectively, $\mathrm{NIn}_{\kappa,\lambda}$) is the set of all $A \subseteq P_\kappa(\lambda)$ with the property that one may find $f_a : a \to 2$ for $a \in A$ such that there does not exist $g : \lambda \to 2$ and B in $\mathrm{I}^+_{\kappa,\lambda} \cap P(A)$ (respectively, $\mathrm{NS}^+_{\kappa,\lambda} \cap P(A)$) such that $g|a = f_a$ for any $a \in B$. κ is λ*-ineffable* (respectively, *almost* λ*-ineffable*) if $P_\kappa(\lambda)$ does not lie in $\mathrm{NIn}_{\kappa,\lambda}$ (respectively, $\mathrm{NAIn}_{\kappa,\lambda}$).

LEMMA 1.12. (i) (Matet-Usuba [**23**]) $\mathrm{NSJ}_{\kappa,\lambda}$ *is a* (*possibly improper*) *seminormal ideal on* $P_\kappa(\lambda)$.
(ii) (Carr [**7**]) *Each of* $\mathrm{NSh}_{\kappa,\lambda}$, $\mathrm{NAIn}_{\kappa,\lambda}$, $\mathrm{NIn}_{\kappa,\lambda}$ *is a* (*possibly improper*) *normal ideal on* $P_\kappa(\lambda)$. *Moreover* $\mathrm{NSh}_{\kappa,\lambda} \subseteq \mathrm{NAIn}_{\kappa,\lambda} \subseteq \mathrm{NIn}_{\kappa,\lambda}$.

It is simple to see that if μ is a cardinal with $\kappa < \mu < \lambda$, and $P_\kappa(\lambda) \notin \mathrm{NSJ}_{\kappa,\lambda}$ (respectively, κ is λ-Shelah, κ is almost λ-ineffable, κ is λ-ineffable), then $P_\kappa(\mu) \notin \mathrm{NSJ}_{\kappa,\mu}$ (respectively, κ is μ-Shelah, κ is almost μ-ineffable, κ is μ-ineffable).

LEMMA 1.13 (Carr [**8**], Magidor [**14**]). *Let $A \subseteq P_\kappa(\lambda)$ be such that $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^2$. Then $A \in \mathrm{NIn}^+_{\kappa,\lambda}$.*

DEFINITION. $\mathrm{NAIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}$ (respectively, $\mathrm{NIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}$) is the set of all $A \subseteq P_\kappa(\lambda)$ with the property that one can find $f_a : P_{|a\cap\kappa|}(a) \to 2$ for $a \in A$ so that there does not exist $g : P_\kappa(\lambda) \to 2$ and B in $\mathrm{I}^+_{\kappa,\lambda} \cap P(A)$ (respectively, $(\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+ \cap P(A)$) such that $g|P_{|a\cap\kappa|}(a) = f_a$ whenever $a \in B$.

DEFINITION. $\mathrm{NIn}_{\kappa,\lambda,2}$ is the set of all $A \subseteq P_\kappa(\lambda)$ with the property that one can find $f_{a_0a_1} : a_0 \to 2$ for $(a_0, a_1) \in [A]^2_<$ so that there does not exist $g : \lambda \to 2$ and B in $\mathrm{NS}^+_{\kappa,\lambda} \cap P(A)$ such that $g|a_0 = f_{a_0a_1}$ for every $(a_0, a_1) \in [B]^2_<$.

LEMMA 1.14 (Kamo [**13**], Matet [**16**]). *Each of* $\mathrm{NAIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}$, $\mathrm{NIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}$, $\mathrm{NIn}_{\kappa,\lambda,2}$ *is a* (*possibly improper*) *normal ideal on* $P_\kappa(\lambda)$.

DEFINITION. We define $p_{\kappa,\lambda} : P_\kappa(\lambda^{<\kappa}) \to P_\kappa(\lambda)$ by $p_{\kappa,\lambda}(x) = x \cap \lambda$.

DEFINITION. For a regular uncountable cardinal μ, a *μ-Aronszajn tree* is a tree of height μ with every level of size less than μ and no cofinal branch.

Specker [**26**] established that for every infinite cardinal ν such that $\nu^{<\nu} = \nu$, there exists a ν^+-Aronszajn tree.

2. Ineffability 1.

We first show that if $\lambda^{<\lambda} = \lambda$, then $\mathrm{NIn}^+_{\kappa,\lambda} \longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+, \mathrm{NSS}^+_{\kappa,\lambda})^3$. We need to recall a few facts.

LEMMA 2.1 (Matet-Usuba [**23**]). *Suppose $\lambda^{<\lambda} = \lambda$, and let $A \in \mathrm{NSh}^+_{\kappa,\lambda}$ and $F : [A]^2_< \to \eta$, where $2 \le \eta < \kappa$. Then there is $Q \subseteq A$ such that either $Q \in \mathrm{NS}^+_{\kappa,\lambda}$ and F takes the constant value 0 on $[Q]^2_<$, or $Q \in \mathrm{I}^+_{\kappa,\lambda}$ and F takes the constant value i on $[Q]^2_<$ for some i with $0 < i < \eta$.*

LEMMA 2.2 (Folklore). *Suppose κ is Mahlo. Then $\{a \in P_\kappa(\lambda) : a \cap \kappa$ is an inaccessible cardinal$\} \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^*$.*

LEMMA 2.3 (Usuba [**27**]). *Suppose κ is λ-Shelah. Then $\mathrm{NSh}_{\kappa,\lambda}$ is a strongly normal ideal.*

LEMMA 2.4. *Suppose κ is λ-Shelah. Then the following hold*:

- (i) (Johnson [**12**]) $\{a \in P_\kappa(\lambda) : \mathrm{o.t.}(a)$ *is a cardinal*$\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.
- (ii) (Abe [**3**]) *If λ is regular, then $\{a \in P_\kappa(\lambda) : |a|$ is regular$\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.*
- (iii) (Abe [**3**]) *If λ is a strong limit cardinal, then $\{a \in P_\kappa(\lambda) : |a|$ is a strong limit cardinal$\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.*
- (iv) (Abe [**3**]) *Let μ be a cardinal such that $\lambda = 2^\mu$. Then $\{a \in P_\kappa(\lambda) : |a| = 2^{|a\cap\mu|}\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.*

Lemma 2.5. *Suppose κ is λ-Shelah, $\lambda^{<\lambda} = \lambda$ and λ is not inaccessible. Then* $\{a \in P_\kappa(\lambda) : 2^{<|a|} = |a|\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.

Proof. Suppose otherwise. Then by Lemmas 2.3 and 2.4(i), we may find $A \in \mathrm{NSh}^+_{\kappa,\lambda}$ and $\alpha \in \lambda$ such that $2^{|a\cap\alpha|} > |a|$ for every $a \in A$. Put $C = \{a \in P_\kappa(\lambda) : |a \cap \alpha| = |a \cap |\alpha||\}$. Note that $C \in \mathrm{NS}^*_{\kappa,\lambda}$. Pick a cardinal $\mu \geq |\alpha|$ with $2^\mu = \lambda$. Then for any $a \in A \cap C$, $2^{|a\cap\mu|} \geq 2^{|a\cap|\alpha||} > |a|$, which contradicts Lemma 2.4(iv). $\square$

Definition. Let $\mathcal{A}_{\kappa,\lambda}$ be the set of all $a \in P_\kappa(\lambda)$ such that

(a) $a \cap \kappa$ is an uncountable inaccessible cardinal, and
(b) $\mathrm{o.t.}(a)$ is a cardinal greater than $a \cap \kappa$.

Lemma 2.6. *Suppose κ is λ-Shelah. Then $\mathcal{A}_{\kappa,\lambda} \in \mathrm{NSh}^*_{\kappa,\lambda}$.*

Proof. By Lemmas 2.2, 2.3 and 2.4(i). $\square$

Lemma 2.7. *Suppose κ is λ-Shelah and $\lambda^{<\lambda} = \lambda$. Then* $\{a \in \mathcal{A}_{\kappa,\lambda} : \mathrm{o.t.}(a)^{<\mathrm{o.t.}(a)} = \mathrm{o.t.}(a)\} \in \mathrm{NSh}^*_{\kappa,\lambda}$.

Proof. By Lemmas 2.4 ((ii) and (iii)), 2.5 and 2.6. $\square$

Lemma 2.8 (Abe [**4**]). *Suppose* $\mathrm{cf}(\lambda) \geq \kappa$, *$A \in \mathrm{NAIn}^+_{\kappa,\lambda} \cap P(\mathcal{A}_{\kappa,\lambda})$, and $s_a \subseteq P_{a\cap\kappa}(a)$ for $a \in A$. Then the set of all $a \in A$ such that $\{b \in A \cap P_{a\cap\kappa}(a) : s_b = s_a \cap P_{b\cap\kappa}(b)\} \in \mathrm{NSh}_{a\cap\kappa,a}$ lies in $\mathrm{NAIn}_{\kappa,\lambda}$.*

Proof. This is immediate from Proposition 3.6, Fact 3.7 and Lemma 3.8 of [**4**]. $\square$

Lemma 2.9 (Kamo [**13**]). $\mathrm{NIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}})$.

Lemma 2.10 (Abe-Usuba [**5**]). *Suppose $A \in \mathrm{NIn}^+_{\kappa,\lambda}$. Then there is $H \in \mathrm{NIn}^+_{\kappa,\lambda} \cap P(A)$ and $t_a : a \to a$ for $a \in H$ such that $a < b$ for every $(a,b) \in [H]^2$ with $t_a = t_b|a$.*

Proposition 2.11. *Suppose $\lambda^{<\lambda} = \lambda$, and let $A \in \mathrm{NIn}^+_{\kappa,\lambda}$ and $F : [A]^3 \to \eta$, where $2 \leq \eta < \kappa$. Then there is $Q \subseteq A$ such that either $Q \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+$ and F takes the constant value 0 on $[Q]^3$, or $Q \in \mathrm{NSS}^+_{\kappa,\lambda}$ and F takes the constant value i on $[Q]^3$ for some i with $0 < i < \eta$.*

Proof. By Lemmas 1.4 and 1.7, we may find $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\mathrm{NSS}_{\kappa,\lambda}|C = \mathrm{I}_{\kappa,\lambda}|C$, and $E \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^*$ such that $\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = \mathrm{NS}_{\kappa,\lambda}|E$. By Lemma 2.10, there is $H \in \mathrm{NIn}^+_{\kappa,\lambda} \cap P(A)$ and $t_a : a \to a$ for $a \in H$ such that $a < b$ for every $(a,b) \in [H]^2$ with $t_a = t_b|a$. Select a bijection $j : P_\kappa(\lambda) \times P_\kappa(\lambda) \times (1+\eta) \to P_\kappa(\lambda)$. Let B be the set of all $d \in C \cap E \cap H \cap \mathcal{A}_{\kappa,\lambda}$ such that

(a) $d \cap \kappa \geq 1 + \eta$,
(b) $\mathrm{o.t.}(d)^{<\mathrm{o.t.}(d)} = \mathrm{o.t.}(d)$, and
(c) $j(a,b,i) < d$ for any $(a,b) \in [P_{d\cap\kappa}(d)]^2_<$ and any $i < 1 + \eta$.

Then $B \in \mathrm{NIn}^+_{\kappa,\lambda}$ by Lemmas 2.3 and 2.7. For $d \in B$, define $f_d : [B \cap P_{d\cap\kappa}(d)]^2_< \to \eta$ by $f_d(a,b) = F(a,b,d)$, and put

- $v_d = \{j(a,b,1+i) : (a,b) \in [P_{d\cap\kappa}(d)]^2_<$ and $f_d(a,b) = i\}$,
- $w_d = \{j(\{\gamma\},\{\delta\},0) : (\gamma,\delta) \in d \times d$ and $t_d(\gamma) = \delta\}$,
- $s_d = v_d \cup w_d$, and
- $z_d = \{c \in B \cap P_{d\cap\kappa}(d) : s_c = s_d \cap P_{c\cap\kappa}(c)\}$.

Set $W = \{d \in B : z_d \in \mathrm{NSh}^+_{d\cap\kappa,d}\}$. Then $W \in \mathrm{NIn}^+_{\kappa,\lambda}$ by Lemma 2.8. For $d \in W$, we may find by Lemma 2.1 $Q_d \subseteq z_d$ and $i_d < \eta$ such that

(α) f_d takes the constant value i_d on $[Q_d]^2_<$, and
(β) Q_d lies in $\mathrm{NS}^+_{d\cap\kappa,d}$ if $i_d = 0$, and in $\mathrm{I}^+_{d\cap\kappa,d}$ otherwise.

There must be $i < \eta$ such that $\{d \in W : i_d = i\} \in \mathrm{NIn}^+_{\kappa,\lambda}$. By Lemma 2.9, $\mathrm{NIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = \mathrm{NIn}_{\kappa,\lambda}$. Hence we may find $Q \subseteq P_\kappa(\lambda)$ and $R \in \mathrm{NS}^+_{\kappa,\lambda}$ with $R \subseteq \{d \in W : i_d = i\}$ such that $Q \cap P_{d\cap\kappa}(d) = Q_d$ for every $d \in R$. If $i > 0$, then clearly $Q \in \mathrm{I}^+_{\kappa,\lambda}$, and in fact $Q \in \mathrm{NSS}^+_{\kappa,\lambda}$ since $Q \subseteq C$.

Claim 1. *Suppose $i = 0$. Then $Q \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+$.*

Proof of Claim 1. Since $Q \subseteq E$, it suffices to show that $Q \in \mathrm{NS}^+_{\kappa,\lambda}$. Fix $D \in \mathrm{NS}^*_{\kappa,\lambda}$. Select $G : \lambda \times \lambda \to P_\kappa(\lambda)$ so that $\{a \in P_\kappa(\lambda) : \forall(\zeta,\xi) \in a \times a(G(\zeta,\xi) \subseteq a)\} \subseteq D$. Since $R \in \mathrm{NS}^+_{\kappa,\lambda}$, we may find $e \in R$ such that $G(\zeta,\xi) < e$ for every $(\zeta,\xi) \in e \times e$. Now $Q_e \in \mathrm{NS}^+_{e\cap\kappa,e}$, so we may find $a \in Q_e$ such that $G(\zeta,\xi) \subseteq a$ for every $(\zeta,\xi) \in a \times a$. Then clearly $a \in Q \cap D$, which completes the proof of the claim. □

Finally, let us show that F takes the constant value i on $[Q]^3$. Thus let $(a_0,a_1,a_2) \in [Q]^3$. Pick $d \in R$ with $a_2 < d$. Then $\{a_0,a_1,a_2\} \subseteq Q_d \subseteq z_d$.

Claim 2. *Let $l < 3$. Then $t_{a_l} = t_d | a_l$.*

Proof of Claim 2. Fix $\gamma \in a_l$. Then $j(\{\gamma\},\{t_{a_l}(\gamma)\},0) \in s_d$ since $s_{a_l} = s_d \cap P_{a_l\cap\kappa}(a_l)$, and therefore $t_{a_l}(\gamma) = t_d(\gamma)$, which completes the proof of Claim 2. □

It follows from Claim 2 that $a_0 < a_1 < a_2$. Then $f_d(a_0,a_1) = i$, so $j(a_0,a_1,1+i) \in s_d$. Now $s_{a_2} = s_d \cap P_{a_2\cap\kappa}(a_2)$, and therefore $j(a_0,a_1,1+i) \in s_{a_2}$. Hence $i = f_{a_2}(a_0,a_1) = F(a_0,a_1,a_2)$. □

Our next result asserts that $\{A \subseteq P_\kappa(\lambda) : A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, [P_\kappa(\lambda)]^4_<)^3\} \subseteq \mathrm{NIn}^+_{\kappa,\lambda}$.

Lemma 2.12. *Let J be an ideal on $P_\kappa(\lambda)$, and $A \subseteq P_\kappa(\lambda)$ such that for any $g : [A]^3_< \to 2$, there is either $B \in J^+ \cap P(A)$ such that g takes the constant value 0 on $[B]^3_<$, or $(a_0,a_1,a_2,a_3) \in [A]^4_<$ such that $g(a_0,a_1,a_2) = g(a_1,a_2,a_3) = 1$. Then $A \underset{<}{\longrightarrow} (J^+)^2$.*

Proof. Fix $f : [A]^2_< \to 2$. Define $g : [A]^3_< \to 2$ by: $g(b_0,b_1,b_2) = 1$ just in case $f(b_0,b_1) = 0$ and $f(b_1,b_2) = 1$. Then clearly there must be $B \in J^+ \cap P(A)$ such that g

takes the constant value 0 on $[B]^3_<$. Now suppose there is $(c,d)\in[B]^2_<$ with $f(c,d)=0$. Put $C=\{a\in B: d<a\}$. We claim that f takes the constant value 0 on $[C]^2_<$. Suppose otherwise, and pick $(v,w)\in[C]^2_<$ with $f(v,w)=1$. Then $f(d,v)=1$ since $g(c,d,v)=0$, and hence $g(c,d,v)=1$. Contradiction. $\square$

PROPOSITION 2.13. *Let $A\subseteq P_\kappa(\lambda)$ be such that $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},[P_\kappa(\lambda)]^4_<)^3$. Then $A\in\mathrm{NIn}^+_{\kappa,\lambda}$.*

PROOF. By Lemmas 1.13 and 2.12. $\square$

If $\lambda^{<\lambda}=\lambda$, then by a result of [**23**], for any $A\subseteq P_\kappa(\lambda)$, $A\in\mathrm{NSh}^+_{\kappa,\lambda}$ if and only if $A \underset{<}{\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NSS}^+_{\kappa,\lambda})^2$ if and only if $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^2$. Replacing pairs by triples, we obtain the following:

THEOREM 2.14. *Suppose $\lambda^{<\lambda}=\lambda$. Then for any $A\subseteq P_\kappa(\lambda)$, the following are equivalent*:

(i) $A\in\mathrm{NIn}^+_{\kappa,\lambda}$.
(ii) $A\longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NSS}^+_{\kappa,\lambda})^3$.
(iii) $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^3$.

PROOF. By Propositions 2.11 and 2.13. $\square$

To conclude this section, let us observe that $\mathrm{NIn}^+_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^3$ does not hold in case $\mathrm{cf}(\lambda)\geq\kappa$.

LEMMA 2.15. *Let μ be a cardinal with $\kappa\leq\mu\leq\lambda$, and let J be an ideal on $P_\kappa(\lambda)$ that is ξ-normal for every $\xi<\mu$. Further let $A\in J^+$ be such that $A \underset{<}{\longrightarrow} (J^+)^3$, and let $f_{a_0a_1}: a_0\cap\mu\to 2$ for $(a_0,a_1)\in[A]^2_<$. Then we may find $B\in J^+\cap P(A)$, $h:\mu\to 2$, and $Q_\xi\in J$ for $\xi<\mu$ such that for any $\xi<\mu$ and any $(a_0,a_1)\in[B\setminus Q_\xi]^2_<$, $h|(a_0\cap\xi)=f_{a_0a_1}|(a_0\cap\xi)$.*

PROOF. Define $F:[A]^3_<\to 2$ by: $F(a_0,a_1,a_2)=1$ just in case there is $\alpha\in a_0$ such that $f_{a_0a_1}|(a_0\cap\alpha)=f_{a_1a_2}|(a_0\cap\alpha)$, $f_{a_0a_1}(\alpha)=0$, and $f_{a_1a_2}(\alpha)=1$. We may find $B\in J^+\cap P(A)$ and $i<2$ such that F takes the constant value i on $[B]^3_<$. We inductively construct $h_\xi:\xi\to 2$ and $Q_\xi\in J$ for $\xi<\mu$ so that for any $\xi<\mu$ and any $(a_0,a_1)\in[B\setminus Q_\xi]^2_<$, $h_\xi|(a_0\cap\xi)=f_{a_0a_1}|(a_0\cap\xi)$. For $\xi=0$, put $h_\xi=\emptyset=Q_\xi$. Now suppose $\xi>0$, and h_η and Q_η have already been defined for all $\eta<\xi$. In case ξ is a limit ordinal, put $h_\xi=\bigcup_{\eta<\xi}h_\eta$ and $Q_\xi=S\cup T$, where $S=\{a\in P_\kappa(\lambda):\exists\eta\in a\cap\xi(a\in Q_\eta)\}$ and $T=\{a\in P_\kappa(\lambda):\exists\eta\in a\cap\xi(\eta+1\notin a)\}$. Next suppose ξ is a successor ordinal, say $\xi=\zeta+1$. Put $R=\{a\in P_\kappa(\lambda):\zeta\notin a\}$. If $f_{c_0c_1}(\zeta)=1-i$ for every $(c_0,c_1)\in[B\setminus(Q_\zeta\cup R)]^2_<$, set $h_\xi=h_\zeta\cup\{(\zeta,1-i)\}$ and $Q_\xi=Q_\zeta\cup R$. Now assume there is $(c_0,c_1)\in[B\setminus(Q_\zeta\cup R)]^2_<$ such that $f_{c_0c_1}(\zeta)=i$. Let Z be the set of all $a\in P_\kappa(\lambda)$ such that $c_1<a$ does not hold. Then clearly for any $(a_0,a_1)\in[B\setminus(Q_\zeta\cup R\cup Z)]^2_<$, $f_{c_1a_0}(\zeta)=i$ (since $F(c_0,c_1,a_0)=i$), and therefore $f_{a_0a_1}(\zeta)=i$ (since $F(c_1,a_0,a_1)=i$). Put $h_\xi=h_\zeta\cup\{(\zeta,i)\}$ and $Q_\xi=Q_\zeta\cup R\cup Z$. Finally, set $h=\bigcup_{\xi<\mu}h_\xi$. $\square$

Lemma 2.16. *Let $A \subseteq P_\kappa(\lambda)$ be such that $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^3$. Then $A \in \mathrm{NIn}^+_{\kappa,\lambda,2}$.*

Proof. This easily follows from Lemma 2.15. $\square$

Lemma 2.17 (Abe [**4**]). *Let $A \in \mathrm{NIn}^+_{\kappa,\lambda}$. Then $\{d \in \mathcal{A}_{\kappa,\lambda} : A \cap P_{d\cap\kappa}(d) \in \mathrm{NIn}_{d\cap\kappa,d}\} \in \mathrm{NIn}^+_{\kappa,\lambda}$.*

Lemma 2.18. *$\{d \in \mathcal{A}_{\kappa,\lambda} : d \cap \kappa$ is not d-ineffable$\} \in \mathrm{NIn}_{\kappa,\lambda,2}$.*

Proof. Suppose otherwise. Pick a bijection $j : \lambda \times \lambda \times \lambda \times \lambda \to \lambda$. Let A be the set of all $d \in \mathcal{A}_{\kappa,\lambda}$ such that $j``(d \times d \times d \times d) = d$ and $d \cap \kappa$ is not d-ineffable. Then by Lemmas 2.3 and 2.6, $A \in \mathrm{NIn}^+_{\kappa,\lambda,2}$. For $d \in A$, select $s^d_b \subseteq b$ for $b \in P_{d\cap\kappa}(d)$ so that for any $s \subseteq d$, $\{b \in P_{d\cap\kappa}(d) : s^d_b = s \cap b\} \in \mathrm{NS}_{d\cap\kappa,d}$. For $(d,e) \in [A]^2_<$, put $x_{de} = \{b \in A \cap P_{d\cap\kappa}(d) : s^d_b = s^e_d \cap b\}$. Note that $x_{de} \in \mathrm{NS}_{d\cap\kappa,d}$. Pick $f_{de} : d \times d \to d$ so that $x_{de} \cap \{b \in P_{d\cap\kappa}(d) : f_{de}``(b \times b) \subseteq b\} = \emptyset$. Set $t_{de} = v_{de} \cup w_{de}$, where $v_{de} = \{j(0,0,0,\xi) : \xi \in s^e_d\}$ and $w_{de} = \{j(1,\alpha,\beta,\gamma) : \alpha,\beta,\gamma \in d$ and $f_{de}(\alpha,\beta) = \gamma\}$.

We may find $B \in \mathrm{NS}^+_{\kappa,\lambda} \cap P(A)$ and $t \subseteq \lambda$ such that $t_{de} = t \cap d$, for all $(d,e) \in [B]^2_<$. Set $S = \{\xi < \lambda : j(0,0,0,\xi) \in t\}$, and define $f : \lambda \times \lambda \to \lambda$ by $f(\alpha,\beta) =$ the unique γ such that $j(1,\alpha,\beta,\gamma) \in t$. Let C be the set of all $a \in P_\kappa(\lambda)$ such that $f``(a \times a) \subseteq a$. Now let $(b,d,e) \in [B \cap C]^3_<$. Then $b \in x_{de}$ since $s^d_b = S \cap b = s^e_b \cap b$. Moreover $f_{de} = f|(d \times d)$, so $f_{de}``(b \times b) \subseteq b$. Contradiction. $\square$

Proposition 2.19. *Assume $\mathrm{cf}(\lambda) \geq \kappa$. Then $\mathrm{NIn}^+_{\kappa,\lambda} \underset{<}{\not\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^3$.*

Proof. By Lemmas 2.16, 2.17 and 2.18. $\square$

Note that by Lemmas 1.5 and 2.3 and Proposition 2.19, $\mathrm{NIn}^+_{\kappa,\lambda} \underset{<}{\not\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, \mathrm{NSS}^+_{\kappa,\lambda})^3$ in case $\kappa \leq \mathrm{cf}(\lambda) < \lambda$.

Question 1. *Does $\mathrm{NIn}^+_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, \mathrm{I}^+_{\kappa,\lambda})^3$ hold in case $\kappa \leq \mathrm{cf}(\lambda) < \lambda = 2^{<\lambda}$?*

3. Ineffability 2.

In this section we are concerned with the case $\mathrm{cf}(\lambda) < \kappa$. We show that if $2^\lambda = \lambda^{<\kappa}$, then $(p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+, \mathrm{NS}^+_{\kappa,\lambda})^3$. Furthermore, we establish that $\{A \subseteq P_\kappa(\lambda) : A \underset{<}{\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+, [P_\kappa(\lambda)]^4_<)^3\} \subseteq (p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+$ in case $\mathrm{cf}(\lambda) < \kappa$.

The reason we work with $p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}})$ is that the ideal $\mathrm{NIn}_{\kappa,\lambda}$ is not large enough. In fact, if $2^\lambda = \lambda^{<\kappa}$, then by results of [**23**] and [**27**], for any $A \in \mathrm{NIn}^+_{\kappa,\lambda}$, there is $B \in \mathrm{NIn}^+_{\kappa,\lambda} \cap P(A)$ with $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$ (we can take $B = \{a \in A \cap \mathcal{A}_{\kappa,\lambda} \cap E : A \cap \mathcal{A}_{\kappa,\lambda} \cap E \cap P_{a\cap\kappa}(a) \in \mathrm{NIn}_{a\cap\kappa,a}\}$, where $E \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^*$ is such that $\mathrm{NS}_{\kappa,\lambda}|E = \mathrm{I}_{\kappa,\lambda}|E$). On the other hand it can be shown that if $\mathrm{cf}(\lambda) < \kappa$, then $p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}) = \{A \subseteq P_\kappa(\lambda) : A \not\longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+)^2\} = \{A \subseteq P_\kappa(\lambda) : A \underset{<}{\not\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+)^2\}$.

Definition. Suppose κ is inaccessible and $\mathrm{cf}(\lambda) < \kappa$. Let $\langle y_\alpha : \lambda \leq \alpha < \lambda^{<\kappa}\rangle$ be a one-to-one enumeration of the elements of $P_\kappa(\lambda)$. Define $q_{\kappa,\lambda} : P_\kappa(\lambda) \to P_\kappa(\lambda^{<\kappa})$ by

$q_{\kappa,\lambda}(a) = a \cup \{\alpha \in \lambda^{<\kappa} \setminus \lambda : y_\alpha < a\}$, and set $\mathcal{X}_{\kappa,\lambda} = \{x \in P_\kappa(\lambda^{<\kappa}) : x = q_{\kappa,\lambda}(x \cap \lambda)\}$.

LEMMA 3.1 (Abe [**1**]). *Suppose κ is Mahlo and* $\mathrm{cf}(\lambda) < \kappa$. *Then the following hold*:

(i) $\mathcal{X}_{\kappa,\lambda} \in (\mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}})^*$.
(ii) $q_{\kappa,\lambda}$ *is an isomorphism from* $(P_\kappa(\lambda), \subsetneq)$ *onto* $(\mathcal{X}_{\kappa,\lambda}, \subsetneq)$.
(iii) $q_{\kappa,\lambda}(\mathrm{I}_{\kappa,\lambda}) = \mathrm{I}_{\kappa,\lambda^{<\kappa}} | \mathcal{X}_{\kappa,\lambda}$.
(iv) $q_{\kappa,\lambda}(\mathrm{NS}_{\kappa,\lambda}) = \mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{\lambda} | \mathcal{X}_{\kappa,\lambda}$.
(v) $q_{\kappa,\lambda}(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}}) = \mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}} = \mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{[\lambda]^{<\kappa}} | \mathcal{X}_{\kappa,\lambda}$.

LEMMA 3.2. *Suppose κ is inaccessible and* $\mathrm{cf}(\lambda) < \kappa$, *and let* $Q \subseteq \mathcal{X}_{\kappa,\lambda}$. *Then* $q_{\kappa,\lambda}^{-1}(Q) = \{x \cap \lambda : x \in Q\}$.

PROOF. $\subseteq$: Let $a \in P_\kappa(\lambda)$ be such that $q_{\kappa,\lambda}(a) \in Q$. Then $a = \lambda \cap q_{\kappa,\lambda}(a)$.
$\supseteq$: Let $x \in Q$. Then $x = q_{\kappa,\lambda}(x) \cap \lambda$, so $x \cap \lambda \in q_{\kappa,\lambda}^{-1}(Q)$ □

LEMMA 3.3 (Usuba [**27**]). *Suppose κ is λ-Shelah and* $\mathrm{cf}(\lambda) < \kappa$. *Then* $\lambda^{<\kappa} = \lambda^+$.

The above lemmas and Proposition 2.11 give:

PROPOSITION 3.4. *Suppose* $\mathrm{cf}(\lambda) < \kappa$ *and* $2^\lambda = \lambda^{<\kappa}$ *and let* $A \in (p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+$. *Further let* $F : [A]^3 \to \eta$, *where* $2 \le \eta < \kappa$. *Then there is* $B \subseteq A$ *such that either* $B \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+$ *and F takes the constant value* 0 *on* $[B]^3$, *or* $B \in \mathrm{NS}_{\kappa,\lambda}^+$ *and F takes the constant value i on* $[B]^3$ *for some i with* $0 < i < \eta$.

PROOF. Let $X = \{x \in P_\kappa(\lambda^{<\kappa}) : x \cap \lambda \in A\}$. Then by Lemmas 2.3 and 3.1, $X \cap \mathcal{X}_{\kappa,\lambda} \in \mathrm{NIn}_{\kappa,\lambda^{<\kappa}}^+$. Define $G : [X \cap \mathcal{X}_{\kappa,\lambda}]^3 \to \eta$ by $G(x_0, x_1, x_2) = F(x_0 \cap \lambda, x_1 \cap \lambda, x_2 \cap \lambda)$. Since $(\lambda^{<\kappa})^{<(\lambda^{<\kappa})} = \lambda^{<\kappa}$ by Lemma 3.3, we may find by Proposition 2.11 $Q \subseteq X \cap \mathcal{X}_{\kappa,\lambda}$ and $i < \eta$ such that

(a) G takes the constant value i on $[Q]^3$, and
(b) $Q \in (\mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}})^+$ if $i = 0$, and $Q \in \mathrm{NSS}_{\kappa,\lambda^{<\kappa}}^+$ otherwise.

Put $B = \{x \cap \lambda : x \in Q\}$. Note that $B \subseteq A$. By Lemma 3.2, $B = q_{\kappa,\lambda}^{-1}(Q)$, so by Lemma 3.1 $Q \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+$ if $i = 0$, and $Q \in \mathrm{NS}_{\kappa,\lambda}^+$ otherwise.

Let us show that F takes the constant value i on $[B]^3$. Thus let $(a_0, a_1, a_2) \in [B]^3$. For $j < 3$, set $x_j = q_{\kappa,\lambda}(a_j)$. Note that $x_j \in Q$ and $x_j \cap \lambda = a_j$. By Lemma 3.1 $(x_0, x_1, x_2) \in [Q]^3$, so $i = G(x_0, x_1, x_2) = F(a_0, a_1, a_2)$. □

PROPOSITION 3.5. *Suppose* $\mathrm{cf}(\lambda) < \kappa$, *and let* $A \subseteq P_\kappa(\lambda)$ *be such that* $A \underset{<}{\longrightarrow} ((\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+, [P_\kappa(\lambda)]_<^4)^3$. *Then* $A \in (p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+$.

PROOF. Set $Z = \{x \in P_\kappa(\lambda^{<\kappa}) : x \cap \lambda \in A\}$. By Lemma 1.13 it suffices to show that $Z \underset{<}{\longrightarrow} ((\mathrm{NS}_{\kappa,\lambda^{<\kappa}})^+)^2$. Fix $F : Z \times Z \to 2$. Define $G : [A]_<^2 \to 2$ by $G(a_0, a_1) = F(q_{\kappa,\lambda}(a_0), q_{\kappa,\lambda}(a_1))$. By Lemma 2.12 we may find $B \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^+ \cap P(A)$ and $i < 2$ such that G takes the constant value i on $[B]_<^2$. Set $X = q_{\kappa,\lambda}``B$. Then clearly $X \subseteq Z$.

Moreover by Lemma 3.1, $X \in (\mathrm{NS}^{[\lambda^{<\kappa}]^{<\kappa}}_{\kappa,\lambda^{<\kappa}})^+$. We claim that F takes the constant value i on $[X]^2_<$. Fix $a_0, a_1 \in B$ with $q_{\kappa,\lambda}(a_0) < q_{\kappa,\lambda}(a_1)$. Then $a_0 \subseteq a_1$ since $q_{\kappa,\lambda}(a_0) \subseteq q_{\kappa,\lambda}(a_1)$. Furthermore, $|a_0| \leq |q_{\kappa,\lambda}(a_0)| < |q_{\kappa,\lambda}(a_1) \cap \kappa| = |a_1 \cap \kappa|$. Thus $a_0 < a_1$, and consequently $F(q_{\kappa,\lambda}(a_0), q_{\kappa,\lambda}(a_1)) = G(a_0, a_1) = i$. □

4. Almost ineffability 1.

We start this section by showing that if $\lambda^{<\lambda} = \lambda$, then $\mathrm{NAIn}^+_{\kappa,\lambda} \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3$. The following easily follows from Lemma 3.1:

Lemma 4.1. $\mathrm{NAIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}})$.

Lemma 4.2. *Suppose* $\lambda^{<\lambda} = \lambda$. *Then* $\mathrm{NAIn}^+_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3_\eta$ *for every* η *with* $2 \leq \eta < \kappa$.

Proof. Fix $A \in \mathrm{NAIn}^+_{\kappa,\lambda}$ and $F : [A]^3_< \to \eta$, where $2 \leq \eta < \kappa$. Select a bijection $j : P_\kappa(\lambda) \times P_\kappa(\lambda) \times \eta \to P_\kappa(\lambda)$. Let B be the set of all $d \in A \cap \mathcal{A}_{\kappa,\lambda}$ such that

(a) $d \cap \kappa \geq \eta$,
(b) $\mathrm{o.t.}(d)^{<\mathrm{o.t.}(d)} = \mathrm{o.t.}(d)$, and
(c) $j(a, b, i) < d$ for any $(a, b) \in [P_{d\cap\kappa}(d)]^2_<$ and any $i \in \eta$.

Then $B \in \mathrm{NAIn}^+_{\kappa,\lambda}$ by Lemmas 2.3 and 2.7. For $d \in B$, define $f_d : [B \cap P_{d\cap\kappa}(d)]^2_< \to \eta$ by $f_d(a, b) = F(a, b, d)$, and put

- $s_d = \{j(a, b, i) : (a, b) \in [P_{d\cap\kappa}(d)]^2_< \text{ and } f_d(a, b) = i\}$ and
- $z_d = \{c \in B \cap P_{d\cap\kappa}(d) : s_c = s_d \cap P_{c\cap\kappa}(c)\}$.

Set $W = \{d \in B : z_d \in \mathrm{NSh}^+_{d\cap\kappa,d}\}$. Then $W \in \mathrm{NAIn}^+_{\kappa,\lambda}$ by Lemma 2.8. For $d \in W$, we may find by Lemma 2.1 $Q_d \in \mathrm{I}^+_{d\cap\kappa,d} \cap P(z_d)$ and $i_d < \eta$ such that f_d takes the constant value i_d on $[Q_d]^2_<$. There must be $i < \eta$ such that $\{d \in W : i_d = i\} \in \mathrm{NAIn}^+_{\kappa,\lambda}$. By Lemma 4.1, $\mathrm{NAIn}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} = \mathrm{NAIn}_{\kappa,\lambda}$. Hence we may find $Q \subseteq P_\kappa(\lambda)$ and $R \in \mathrm{I}^+_{\kappa,\lambda}$ with $R \subseteq \{d \in W : i_d = i\}$ such that $Q \cap P_{d\cap\kappa}(d) = Q_d$ for every $d \in R$. It is simple to see that $Q \in \mathrm{I}^+_{\kappa,\lambda}$.

We claim that F takes the constant value i on $[Q]^3_<$. Thus let $(a, b, c) \in [Q]^3_<$. Pick $d \in R$ with $c < d$. Then $(a, b, c) \in [Q_d]^3_<$ since $Q \cap P_{d\cap\kappa}(d) = Q_d$. Hence $f_d(a, b) = i$, so $j(a, b, i) \in s_d$. Now $s_c = s_d \cap P_{c\cap\kappa}(c)$ since $c \in z_d$, and consequently $j(a, b, i) \in s_c$. Thus $i = f_c(a, b) = F(a, b, c)$. □

Lemma 4.3. *Suppose* $u(\kappa, \lambda) = \lambda$ *and there is* $C \in \mathrm{NS}^*_{\kappa,\lambda}$ *such that* $\mathrm{NSS}_{\kappa,\lambda}|C = \mathrm{I}_{\kappa,\lambda}|C$. *Then for any* $A \in \mathrm{I}^+_{\kappa,\lambda} \cap P(C)$, *there is* $B \in \mathrm{I}^+_{\kappa,\lambda} \cap P(A)$ *with* $[B]^2_< = [B]^2$.

Proof. Select $e_\alpha \in P_\kappa(\lambda)$ for $\alpha < \lambda$ so that $\{e_\alpha : \alpha < \lambda\} \in \mathrm{I}^+_{\kappa,\lambda}$. Now given $A \in \mathrm{I}^+_{\kappa,\lambda} \cap P(C)$, define inductively $a_\alpha \in A$ for $\alpha < \lambda$ so that

(a) $\alpha \in a_\alpha$ and $e_\alpha \subseteq a_\alpha$,
(b) $a_\beta < a_\alpha$ for every $\beta \in a_\alpha \cap \alpha$, and
(c) $a_\alpha \setminus a_\beta \neq \emptyset$ for every $\beta < \alpha$.

Then $B = \{a_\alpha : \alpha < \lambda\}$ is as desired. □

Proposition 4.4. *Suppose* $\lambda^{<\lambda} = \lambda$. *Then* $\mathrm{NAIn}^+_{\kappa,\lambda} \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3_\eta$ *for every* η *with* $2 \leq \eta < \kappa$.

Proof. By Lemmas 1.4, 2.3, 4.2 and 4.3. □

In the remainder of this section we establish that if $\lambda^{<\lambda} = \lambda$ but λ is not weakly compact, then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that

(a) $\{A \subseteq C : A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3\} \subseteq \mathrm{NAIn}^+_{\kappa,\lambda}$, and
(b) for any $A \subseteq C$ such that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$, there is $B \subseteq A$ such that $B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$ but $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

Lemma 4.5 (Johnson [**12**]). *Suppose* $\mathrm{cf}(\lambda) \geq \kappa$. *Then for any* $A \subseteq P_\kappa(\lambda)$, *the following are equivalent*:

(i) $A \in \mathrm{NAIn}^+_{\kappa,\lambda}$.
(ii) *Given* $g : [A]^2_< \to \lambda$ *such that* $g(a_0, a_1) \in a_0$ *for every* $(a_0, a_1) \in [A]^2_<$, *there is* $B \in \mathrm{I}^+_{\kappa,\lambda} \cap P(A)$ *such that* g *is constant on* $[B]^2_<$.

Lemma 4.6. *Assume* λ *is regular and there is a* λ*-Aronszajn tree, and let* J *be a seminormal ideal on* $P_\kappa(\lambda)$. *Then there is* $C \in \mathrm{NS}^*_{\kappa,\lambda}$ *with the following property*: *Suppose* $A \in J^+ \cap P(C)$ *is such that given* $f_{a_0a_1} : a_0 \to 2$ *for* $(a_0, a_1) \in [A]^2_<$, *there is* $B \in J^+ \cap P(A)$, $h : \lambda \to 2$, *and* $Q_\xi \in J$ *for* $\xi < \lambda$ *such that for any* $\xi < \lambda$ *and any* $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = f_{a_0a_1}|(a_0 \cap \xi)$. *Suppose further that* $g : [A]^2_< \to \lambda$ *is such that* $g(a_0, a_1) \in a_0$ *for every* $(a_0, a_1) \in [A]^2_<$. *Then there is* $D \in J^+ \cap P(A)$ *such that* g *is constant on* $[D]^2_<$.

Proof. Select a λ-Aronszajn tree $T = \langle \lambda, <_T \rangle$. For $\alpha < \lambda$, let T_α denote the α-th level of T. Let C be the set of all $a \in P_\kappa(\lambda)$ such that

(a) $\beta + 1 \in a$ for every $\beta \in a$,
(b) $a \cap T_\alpha \neq \emptyset$ for every $\alpha \in a$, and
(c) $\{\gamma <_T \xi : \gamma \in \bigcup_{\delta \in a \cap \alpha} T_\delta\} \subseteq a$ for any $\alpha \in \lambda$ and any $\xi \in a \cap T_\alpha$.

Let us check that C is as desired. It is immediate that $C \in \mathrm{NS}^*_{\kappa,\lambda}$. Now fix $A \in J^+ \cap P(C)$ with the property that given $f_{a_0a_1} : a_0 \to 2$ for $(a_0, a_1) \in [A]^2_<$, we may find $B \in J^+ \cap P(A)$, $h : \lambda \to 2$, and $Q_\xi \in J$ for $\xi < \lambda$ such that for any $\xi < \lambda$ and any $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = f_{a_0a_1}|(a_0 \cap \xi)$. Let $g : [A]^2_< \to \lambda$ be such that $g(a_0, a_1) \in a_0$ for every $(a_0, a_1) \in [A]^2_<$. For $(a_0, a_1) \in [A]^2_<$, pick $\xi_{a_0a_1} \in a_0 \cap T_{g(a_0,a_1)}$, and define $f_{a_0a_1} : a_0 \to 2$ by: $f_{a_0a_1}(\gamma) = 1$ just in case $\gamma <_T \xi_{a_0a_1}$. There must be $B \in J^+ \cap P(A)$, $h : \lambda \to 2$ and $Q_\xi \in J$ for $\xi < \lambda$ such that for any $\xi < \lambda$ and any $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = f_{a_0a_1}|(a_0 \cap \xi)$. It is simple to see that

(i) if γ and γ' are any two distinct members of $h^{-1}(\{1\})$, then either $\gamma <_T \gamma'$, or $\gamma' <_T \gamma$, and
(ii) $\{\gamma' \in \lambda : \gamma' <_T \gamma\} \subseteq h^{-1}(\{1\})$ for every $\gamma \in h^{-1}(\{1\})$.

Set $\delta =$ the least $\alpha < \lambda$ such that $T_\alpha \cap h^{-1}(\{1\}) = \emptyset$. Define $k : \delta \to \lambda$ by $k(\alpha) =$ the unique element of $T_\alpha \cap h^{-1}(\{1\})$. Pick a limit ordinal $\sigma < \lambda$ with $T_\delta \cup \mathrm{ran}(k) \subseteq \sigma$. Let D be the set of all $a \in B$ such that

(α) $\delta \in a$,
(β) for any $\zeta \in a \cap \sigma$, $a \notin Q_\zeta$, and
(γ) for any $\alpha \in a \cap \delta$, $k(\alpha) \in a$.

Then clearly $D \in J^+$.

We claim that $g(a_0, a_1) = \delta$ for each $(a_0, a_1) \in [D]^2_<$. Suppose otherwise, and select $(a_0, a_1) \in [D]^2_<$ with $g(a_0, a_1) \neq \delta$. If $g(a_0, a_1) < \delta$, then $h(k(g(a_0, a_1))) = 1$ and $f_{a_0 a_1}(k(g(a_0, a_1))) = 0$, which yields a contradiction. Thus $g(a_0, a_1) > \delta$. Put $\gamma =$ the unique element η of T_δ such that $\eta <_T \xi_{a_0 a_1}$. Then $h(\gamma) = 0$ and $f_{a_0 a_1}(\gamma) = 1$. Contradiction. □

Proposition 4.7. *Suppose that λ is regular, there is a λ-Aronszajn tree, and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$. Then there is $D \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\{A \subseteq D : A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3\} \subseteq \mathrm{NAIn}^+_{\kappa,\lambda}$.*

Proof. By Lemmas 1.4, 4.5, 4.6, and Theorem 2.14. □

Lemma 4.8 (Matet-Usuba [**23**]). *Let $A \subseteq P_\kappa(\lambda)$ be such that $A \underset{<}{\longrightarrow} ((\bigcup_{\xi<\lambda} \mathrm{NS}^\xi_{\kappa,\lambda})^+)^2$. Then $A \in \mathrm{NSJ}^+_{\kappa,\lambda}$.*

Lemma 4.9 (Matet-Usuba [**23**]). *Suppose λ is regular, there is a λ-Aronszajn tree, and $P_\kappa(\lambda) \notin \mathrm{NSJ}_{\kappa,\lambda}$. Then $\mathrm{NSh}_{\kappa,\lambda} \subseteq \mathrm{NSJ}_{\kappa,\lambda}|C$ for some $C \in \mathrm{NSJ}^+_{\kappa,\lambda} \cap \mathrm{NS}^*_{\kappa,\lambda}$.*

Lemma 4.10 (Usuba [**27**]). *Let $A \in \mathrm{NSh}^+_{\kappa,\lambda} \cap P(\mathcal{A}_{\kappa,\lambda})$. Then $\{a \in A : A \cap P_{a\cap\kappa}(a) \in \mathrm{NSh}_{a\cap\kappa,a}\} \in \mathrm{NSh}^+_{\kappa,\lambda}$.*

Lemma 4.11. *Suppose $\mathrm{cf}(\lambda) \geq \kappa$, and let $A \in \mathrm{NSh}^+_{\kappa,\lambda}$. Then there is $B \subseteq A$ with $B \in \mathrm{NSh}^+_{\kappa,\lambda} \cap \mathrm{NAIn}_{\kappa,\lambda}$.*

Proof. We can assume that $A \in \mathrm{NAIn}^+_{\kappa,\lambda}$ since otherwise the result is trivial. Set $T = A \cap \mathcal{A}_{\kappa,\lambda}$ and $B = \{a \in T : T \cap P_{a\cap\kappa}(a) \in \mathrm{NSh}_{a\cap\kappa,a}\}$. Then by Lemmas 2.6, 2.8 and 4.10, B is as desired. □

Proposition 4.12. *Suppose that λ is regular, there is a λ-Aronszajn tree, and $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\tau}) \leq \lambda$ for every cardinal τ with $\kappa \leq \tau < \lambda$. Then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ with the following property: for any $A \subseteq C$ such that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$, there is $B \in \mathrm{NSh}^+_{\kappa,\lambda} \cap P(A)$ such that $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.*

Proof. Use Lemma 1.4 to get $C_0 \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\mathrm{NSS}_{\kappa,\lambda}|C_0 = \mathrm{I}_{\kappa,\lambda}|C_0$, and Proposition 4.7 to get $C_1 \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\{B \subseteq C_1 : B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3\} \subseteq \mathrm{NAIn}^+_{\kappa,\lambda}$. We define C_2 as follows. If $P_\kappa(\lambda) \in \mathrm{NSJ}_{\kappa,\lambda}$, we set $C_2 = P_\kappa(\lambda)$. Otherwise we appeal to Lemma 4.9 and choose C_2 so that $C_2 \in \mathrm{NSJ}^+_{\kappa,\lambda} \cap \mathrm{NS}^*_{\kappa,\lambda}$ and $\mathrm{NSh}_{\kappa,\lambda} \subseteq \mathrm{NSJ}_{\kappa,\lambda}|C_2$. Put

$C = C_0 \cap C_1 \cap C_2$. Now fix $A \subseteq C$ with the property that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. By Lemmas 1.3 and 4.8, $A \in \mathrm{NSh}^+_{\kappa,\lambda}$. Hence by Lemma 4.11, there is $B \in \mathrm{NSh}^+_{\kappa,\lambda} \cap P(A)$ such that $B \in \mathrm{NAIn}_{\kappa,\lambda}$. Then clearly $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$. $\square$

The following lemma shows that the existence of a λ-Aronszajn tree in Lemma 4.6 can be replaced by a certain cardinal arithmetic assumption.

Lemma 4.13. *Assume $\mu < \lambda$ is a cardinal with $2^\mu = \lambda$, and let J be a μ-normal ideal on $P_\kappa(\lambda)$. Then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ with the following property: Suppose $A \in J^+ \cap P(C)$ is such that given $f_{a_0 a_1} : a_0 \cap \mu \to 2$ for $(a_0, a_1) \in [A]^2_<$, there is $B \in J^+ \cap P(A)$, $h : \mu \to 2$, and $Q_\xi \in J$ for $\xi < \mu$ such that for any $\xi < \mu$ and any $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = f_{a_0 a_1}|(a_0 \cap \xi)$. Suppose further that $g : [A]^2_< \to \lambda$ is such that $g(a_0, a_1) \in a_0$ for every $(a_0, a_1) \in [A]^2_<$. Then there is $D \in J^+ \cap P(A)$ such that g is constant on $[D]^2_<$.*

Proof. Let $\langle e_\eta : \eta < \lambda \rangle$ be a one-to-one enumeration of the subsets of μ. Let C be the set of all $a \in P_\kappa(\lambda)$ such that $a \cap e_\zeta \neq a \cap e_\eta$ for any two distinct members ζ, η of a. Let us verify that C is as desired. Clearly, $C \in \mathrm{NS}^*_{\kappa,\lambda}$. Now fix $A \in J^+ \cap P(C)$ with the property that given $f_{a_0 a_1} : a_0 \cap \mu \to 2$ for $(a_0, a_1) \in [A]^2_<$, we may find $B \in J^+ \cap P(A)$, $h : \mu \to 2$ and $Q_\xi \in J$ for $\xi < \mu$ such that for any $\xi < \mu$ and any $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = g_{a_0 a_1}|(a_0 \cap \xi)$. Let $g : [A]^2_< \to \lambda$ be such that $g(a_0, a_1) \in a_0$ for every $(a_0, a_1) \in [A]^2_<$. For $(a_0, a_1) \in [A]^2_<$, define $f_{a_0 a_1} : a_0 \cap \mu \to 2$ by: $f_{a_0 a_1}(\alpha) = 1$ if and only if $\alpha \in e_{g(a_0, a_1)}$. There must be $B \in J^+ \cap P(A)$, $h : \mu \to 2$ and $Q_\xi \in J$ for $\xi < \mu$ such that for any $\xi < \mu$ and any $(a_0, a_1) \in [B \setminus Q_\xi]^2_<$, $h|(a_0 \cap \xi) = f_{a_0 a_1}|(a_0 \cap \xi)$. Let $h^{-1}(\{1\}) = e_\delta$. Now let D be the set of all $a \in B$ such that

(a) $\delta \in a$,
(b) $\alpha + 1 \in a$ for every $\alpha \in a \cap \mu$, and
(c) $a \notin Q_\xi$ for every $\xi \in a \cap \mu$.

Then clearly, $D \in J^+$. We claim that g takes the constant value δ on $[D]^2_<$. Suppose otherwise, and pick $(a_0, a_1) \in [D]^2_<$ with $g(a_0, a_1) \neq \delta$. There must be $\alpha \in a_0 \cap \mu$ such that $\alpha \in e_\delta \triangle e_{g(a_0, a_1)}$. Then $h(\alpha) \neq f_{a_0 a_1}(\alpha)$. Contradiction. $\square$

Proposition 4.14. *Suppose $\lambda = 2^\mu$ for some cardinal $\mu < \lambda$. Then there is $D \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\{A \subseteq D : A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3\} \subseteq \mathrm{NAIn}^+_{\kappa,\lambda}$.*

Proof. We can assume that $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$ since otherwise the result is trivial. Pick a cardinal $\mu < \lambda$ such that $2^\mu = \lambda$. Then by Lemma 1.9, $\mu \geq \kappa$, and consequently $\mathrm{cf}(\lambda) \geq \kappa$. Set $J = \mathrm{NS}^\mu_{\kappa,\lambda}$. Let $C \in \mathrm{NS}^*_{\kappa,\lambda}$ be as in the statement of Lemma 4.13. By Lemma 1.2, there is $Z \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $J|Z = \mathrm{I}_{\kappa,\lambda}|Z$. Then by Theorem 2.14 and Lemma 4.5, $D = C \cap Z$ is as desired. $\square$

Lemma 4.15 (Matet-Usuba [**23**]). *Suppose λ is regular, and let $2 \leq \eta < \kappa$. Then $\mathrm{NSJ}^+_{\kappa,\lambda} \longrightarrow (\mathrm{I}^+_{\kappa,\lambda})^2_\eta$.*

Lemma 4.16 (Matet-Usuba [**23**]). *Suppose $2^{<\lambda} = \lambda$. Then $\mathrm{NSJ}_{\kappa,\lambda} \subseteq \mathrm{NSh}_{\kappa,\lambda}$.*

Lemma 4.17 (Matet-Usuba [**23**]). *Suppose $\lambda = 2^\mu$ for some cardinal μ. Then* $\mathrm{NSh}_{\kappa,\lambda} \cap P(C) \subseteq \mathrm{NSJ}_{\kappa,\lambda}$ *for some* $C \in \mathrm{NS}^*_{\kappa,\lambda}$.

Proposition 4.18. *Suppose that $\lambda^{<\lambda} = \lambda$, but λ is not a strong limit cardinal. Then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ with the following property: For any $A \subseteq C$ such that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$, there is $B \subseteq A$ such that $B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$ but $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.*

Proof. We proceed as in the proof of Proposition 4.12. There must be a cardinal $\mu < \lambda$ such that $\lambda = 2^\mu$. By Proposition 4.14, and Lemmas 1.4 and 4.17, we may find $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\{B \subseteq C : B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3\} \subseteq \mathrm{NAIn}^+_{\kappa,\lambda}$, $\mathrm{NSS}_{\kappa,\lambda}|C = \mathrm{I}_{\kappa,\lambda}|C$, and $\mathrm{NSh}_{\kappa,\lambda} \cap P(C) \subseteq \mathrm{NSJ}_{\kappa,\lambda}$. Now fix $A \subseteq C$ with $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. By Lemma 4.8 $A \in \mathrm{NSh}^+_{\kappa,\lambda}$, so by Lemma 4.11 there is $B \in \mathrm{NSh}^+_{\kappa,\lambda} \cap P(A)$ with $B \in \mathrm{NAIn}_{\kappa,\lambda}$. Then clearly $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$. On the other hand $B \longrightarrow (\mathrm{I}^+_{\kappa,\lambda})^2$ by Lemmas 4.15 and 4.16. □

Let us observe that by a result of Neeman [**24**], it is consistent relative to infinitely many supercompact cardinals that there is a cardinal ν such that

(a) there is no ν^+-Aronszajn tree, and
(b) ν is a strong limit cardinal of cofinality ω, and $2^\nu = \nu^{++}$ (and therefore $2^\mu \neq \nu^+$ for every cardinal $\mu < \nu^+$).

If $\lambda^{<\lambda} = \lambda$ but λ is not weakly compact, then by a result of [**23**] and Lemma 4.3, for any $A \in \mathrm{NS}^+_{\kappa,\lambda}$, $A \in \mathrm{NSh}^+_{\kappa,\lambda}$ if and only if $(\mathrm{NS}_{\kappa,\lambda}|A)^* \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^2$ if and only if $(\mathrm{NS}_{\kappa,\lambda}|A)^* \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. For triples, the following holds.

Theorem 4.19. *Suppose that $\lambda^{<\lambda} = \lambda$ and λ is not weakly compact. Then for any $A \in \mathrm{NS}^+_{\kappa,\lambda}$, the following are equivalent:*

(i) $A \in \mathrm{NAIn}^+_{\kappa,\lambda}$.
(ii) $(\mathrm{NS}_{\kappa,\lambda}|A)^* \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3$.
(iii) $(\mathrm{NS}_{\kappa,\lambda}|A)^* \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

Proof. By Propositions 4.4, 4.7 and 4.14. □

5. Almost ineffability 2.

This section is concerned with the case when λ is weakly compact. We show that if ($\lambda^{<\lambda} = \lambda$ and) λ is weakly compact, then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that

(a) for every $A \subseteq C$ with $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$, there is $B \subseteq A$ with $B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$ and $B \in \mathrm{NAIn}_{\kappa,\lambda}$, and
(b) for any $A \subseteq C$, $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$ if and only if $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

These results contrast with those in Section 4 concerning the case when $\lambda^{<\lambda} = \lambda$ and λ is not weakly compact.

LEMMA 5.1 (Shelah [**25**]). *Suppose κ is weakly compact. Then* NSh_κ *is a normal ideal on* κ. *Moreover,* $\{\mu \in \kappa : \mu$ *is a Mahlo cardinal*$\} \in \mathrm{NSh}_\kappa^*$.

LEMMA 5.2 (Johnson [**11**]). *Let* $A \in \mathrm{NSh}_\kappa^+$ *and* $h_a : \alpha \to \alpha$ *for* $\alpha \in A$. *Then there is* $h : \kappa \to \kappa$ *such that for any* $\eta < \kappa$, $\{\alpha \in A : h_\alpha | \eta = h | \eta\} \in \mathrm{NSh}_\kappa^+$.

LEMMA 5.3 (Carr [**7**]). *If* κ *is* $2^{(\lambda^{<\kappa})}$*-Shelah, then* κ *is* λ*-supercompact.*

PROPOSITION 5.4. *Suppose* λ *is weakly compact and* κ *is* λ*-Shelah. Then* κ *is almost* λ*-ineffable.*

PROOF. We use Lemma 4.5. Let $g : [P_\kappa(\lambda)]^2_< \to \lambda$ be such that $g(a_0, a_1) \in a_0$ for every $(a_0, a_1) \in [P_\kappa(\lambda)]^2_<$. Let W be the set of Mahlo cardinals μ with $\kappa \le \mu < \lambda$. By Lemma 5.3, κ is almost μ-ineffable for every $\mu \in W$. For each $\mu \in W$, we may find $B_\mu \in \mathrm{I}^+_{\kappa,\mu}$ and $\xi_\mu \in \mu$ such that g takes the constant value ξ_μ on $[B_\mu]^2_<$. By Lemma 5.1, there must be $A \in \mathrm{NSh}^+_\lambda \cap P(W)$ and $\xi \in \lambda$ such that $\xi_\mu = \xi$ for every $\mu \in A$. Let $P_\kappa(\lambda) = \{e_\beta : \beta < \lambda\}$. Let D be the set of all $\mu \in A$ such that

(a) $e_\beta \subseteq \mu$ for every $\beta \in \mu$, and
(b) for every cardinal ν with $\kappa \le \nu < \mu$, $P_\kappa(\nu) \subseteq \{e_\beta : \beta < \mu\}$.

It is simple to see that $D \in \mathrm{NSh}^+_\lambda$. Note that for any $\mu \in D$, $\{e_\beta : \beta < \mu\} = P_\kappa(\mu)$. For $\mu \in D$, define $h_\mu : \mu \to \mu$ by $h_\mu(\beta) =$ the least γ such that $e_\beta \subseteq e_\gamma$ and $e_\gamma \in B_\mu$. By Lemma 5.2, there is $h : \lambda \to \lambda$ such that for any $\eta < \lambda$, $\{\mu \in D : h_\mu | \eta = h | \eta\} \in \mathrm{NSh}^+_\lambda$. Set $H = \{e_\delta : \delta \in \mathrm{ran}(h)\}$. Then clearly $e_\beta \subseteq e_{h(\beta)}$ for every $\beta < \lambda$, so $H \in \mathrm{I}^+_{\kappa,\lambda}$. Let us show that g is constant on $[H]^2_<$. Thus let $\gamma, \sigma < \lambda$ with $e_{h(\gamma)} < e_{h(\sigma)}$. Pick $\eta < \lambda$ with $\{\gamma, \sigma\} \subseteq \eta$. We may find $\mu \in D$ such that $h_\mu | \eta = h | \eta$. Then $(e_{h(\gamma)}, e_{h(\sigma)}) \in [B_\mu]^2_<$, and therefore $g(e_{h(\gamma)}, e_{h(\sigma)}) = \xi$. □

LEMMA 5.5 (Matet-Usuba [**23**]). *Suppose* λ *is weakly compact, and* $2 \le \eta < \kappa$. *Then* $\mathrm{NSJ}^+_{\kappa,\lambda} \longrightarrow (\mathrm{NSJ}^+_{\kappa,\lambda})^2_\eta$.

LEMMA 5.6 (Matet-Usuba [**23**]). *Suppose* λ *is weakly compact. Then for any* $A \in \mathrm{NSJ}^+_{\kappa,\lambda}$, *there is* $B \in \mathrm{NSJ}^+_{\kappa,\lambda} \cap P(A)$ *such that* $\mathrm{NSJ}_{\kappa,\lambda} | B = \mathrm{NSS}_{\kappa,\lambda} | B$.

PROPOSITION 5.7. *Suppose* λ *is weakly compact, and let* $2 \le \eta < \kappa$. *Then* $\mathrm{NSJ}^+_{\kappa,\lambda} \longrightarrow (\mathrm{NSJ}^+_{\kappa,\lambda})^3_\eta$.

PROOF. By Lemmas 1.1 and 1.3 we may find $Q_\zeta \in \mathrm{NSS}_{\kappa,\lambda}$ for $\zeta < \lambda$ such that $\mathrm{NSS}_{\kappa,\lambda} = \bigcup_{\zeta < \lambda} P(Q_\zeta)$.

Let $A \in \mathrm{NSJ}^+_{\kappa,\lambda}$ and $F : [P_\kappa(\lambda)]^3_< \to \eta$, where $2 \le \eta < \kappa$. By Lemma 5.6 there is $B \in \mathrm{NSJ}^+_{\kappa,\lambda} \cap P(A)$ such that $\mathrm{NSJ}_{\kappa,\lambda} | B = \mathrm{NSS}_{\kappa,\lambda} | B$.

Select bijections $\pi : [P_\kappa(\lambda)]^2 \to \lambda$ and $\sigma : \lambda \times \eta \to \lambda$. For $c \in B$, define $f_c : c \to 2$ by: $f_c(\beta) = 1$ if and only if one can find a, b such that $a \subsetneq b \subsetneq c$ and $\beta = \sigma(\pi(a, b), F(a, b, c))$. Pick $g : \lambda \to 2$ so that for any $\alpha < \lambda$, $\{c \in B : \forall \gamma \in c \cap \alpha (f_c(\gamma) = g(\gamma))\} \in \mathrm{NSS}^+_{\kappa,\lambda}$.

Put $h = \{((a, b), j) \in [B]^2 \times \eta : g(\sigma(\pi(a, b), j)) = 1\}$. Let us show that h is a function with domain $[B]^2$. Thus let $(a, b) \in [B]^2$. Set $z = \{\sigma(\pi(a, b), j) : j < \eta\}$. Pick $\alpha \in \lambda$ with $z \subseteq \alpha$. There must be $d \in B$ such that

- $b \subseteq d$.
- $z \subseteq d$.
- $f_d|(d \cap \alpha) = g|(d \cap \alpha)$.

Then for each $j < \eta$, $f_d(\sigma(\pi(a, b), j)) = g(\sigma(\pi(a, b), j))$.
By induction on $\xi < \lambda$, we define $a_\xi \in B$ so that

- $\eta \subseteq a_\xi$.
- $\xi \in a_\xi$.
- $a_\xi \setminus a_\delta \neq \emptyset$ for all $\delta < \xi$.
- $a_\xi \notin Q_\xi$.
- $F(a_\gamma, a_\delta, a_\xi) = h(a_\gamma, a_\delta)$ whenever $\gamma < \delta < \xi$ and $a_\gamma \subsetneq a_\delta \subsetneq a_\xi$.

Suppose a_ζ has been constructed for each $\zeta < \xi$. Pick $e \in P_\kappa(\lambda)$ so that $\eta \subseteq e$ and $e \setminus a_\zeta \neq \emptyset$ for every $\zeta < \xi$. Now select $\theta < \lambda$ so that $\theta = \sigma``(\theta \times \eta)$, and $\pi(a_\gamma, a_\delta) \in \theta$ whenever $\gamma < \delta < \xi$ and $a_\gamma \subsetneq a_\delta$. Select $t \in B$ so that

- $\{\xi\} \cup e \subseteq t$.
- $\sigma(\pi(a_\gamma, a_\delta), j) \in t$ whenever $j < \eta$, $\{\gamma, \delta\} \subseteq t \cap \xi$ and $a_\gamma \subsetneq a_\delta$.
- $t \notin Q_\xi$.
- $f_t|(t \cap \theta) = g|(t \cap \theta)$.

Note that if γ, δ are such that $\gamma < \delta < \xi$ and $a_\gamma \subsetneq a_\delta \subsetneq t$, then $F(a_\gamma, a_\delta, t) = h(a_\gamma, a_\delta)$, since $f_t(\sigma(\pi(a_\gamma, a_\delta), F(a_\gamma, a_\delta, t))) = 1 = g(\sigma(\pi(a_\gamma, a_\delta), F(a_\gamma, a_\delta, t)))$. We set $a_\xi = t$.

Now put $D = \{a_\zeta : \zeta < \lambda\}$. Then clearly $D \in \mathrm{NSS}^+_{\kappa,\lambda}$. Hence by Lemma 5.5, we may find $E \in \mathrm{NSJ}^+_{\kappa,\lambda} \cap P(D)$ and $i < \eta$ such that h takes the constant value i on $[E]^2$. It is simple to see that F takes the constant value i on $[E]^3$. $\square$

By (the proof of) Theorem 6.2 in [**12**], it follows that if λ is weakly compact, then $\mathrm{NSJ}^+_{\kappa,\lambda} \longrightarrow (\mathrm{NSJ}^+_{\kappa,\lambda})^n_\eta$ whenever $2 \leq n < \omega$ and $2 \leq \eta < \kappa$.

Corollary 5.8. *Suppose λ is weakly compact. Then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ with the property that for any $A \subseteq C$ such that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$, we may find $B \subseteq A$ such that $B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$ and $B \in \mathrm{NAIn}_{\kappa,\lambda}$.*

Proof. Let $A \in \mathrm{NAIn}^+_{\kappa,\lambda}$ be such that $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$. By Lemma 4.11, there is $B \subseteq A$ with $B \in \mathrm{NSh}^+_{\kappa,\lambda} \cap \mathrm{NAIn}_{\kappa,\lambda}$. Then by Lemma 4.16 and Proposition 5.7, $B \longrightarrow (\mathrm{NSJ}^+_{\kappa,\lambda})^3$. $\square$

Lemma 5.9 (Matet-Usuba [**23**]). *Suppose λ is weakly compact. Then κ is λ-Shelah just in case $P_\kappa(\lambda) \notin \mathrm{NSJ}_{\kappa,\lambda}$.*

Proposition 5.10. *Suppose λ is weakly compact. Then the following are equivalent:*

(i) *κ is almost λ-ineffable.*
(ii) $\mathrm{NS}^*_{\kappa,\lambda} \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3$.

(iii) $\mathrm{NS}^*_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$.

Proof. (i) → (ii): By Proposition 5.7 and Lemmas 2.3 and 4.16.

(ii) → (iii): Trivial.

(iii) → (i): By Proposition 5.4 and Lemmas 1.4, 4.8 and 5.9. □

It was shown in [**23**] that if $\lambda^{<\lambda} = \lambda$, then κ is λ-Shelah if and only if $(\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^* \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^2$ if and only if $\mathrm{NS}^*_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. Here is the corresponding result for triples:

Corollary 5.11. *Suppose $\lambda^{<\lambda} = \lambda$. Then the following are equivalent*:

(i) *κ is almost λ-ineffable.*

(ii) $(\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^* \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3$.

(iii) $\mathrm{NS}^*_{\kappa,\lambda} \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

Proof. (i) → (ii): By Lemmas 1.12 and 2.3, we have $\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} \subseteq \mathrm{NSh}_{\kappa,\lambda} \subseteq \mathrm{NAIn}_{\kappa,\lambda}$. Now apply Proposition 4.4.

(ii) → (iii): Trivial.

(iii) → (i): By Theorem 4.19 and Proposition 5.10. □

Proposition 5.12. *Suppose λ is weakly compact. Then there is $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that for any $A \subseteq C$, the following are equivalent*:

(i) $A \longrightarrow (\mathrm{I}^+_{\kappa,\lambda})^2$.

(ii) $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$.

(iii) $A \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

(iv) $A \longrightarrow (\mathrm{I}^+_{\kappa,\lambda})^3$.

Proof. By Lemma 1.4, we may find $C \in \mathrm{NS}^*_{\kappa,\lambda}$ such that $\mathrm{NSS}_{\kappa,\lambda}|C = \mathrm{I}_{\kappa,\lambda}|C$. Then by Lemma 4.8 and Proposition 5.7, C is as desired. □

6. Almost ineffability 3.

This section is concerned with the case $2^\lambda = \lambda^{<\kappa}$.

Lemma 6.1. *Suppose $2^\lambda = \lambda^{<\kappa}$ and $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$. Then $\mathrm{cf}(\lambda) < \kappa$, and moreover $\lambda^{<\kappa} = \lambda^+$.*

Proof. By Lemmas 1.10 and 1.11. □

Let us show that if $2^\lambda = \lambda^{<\kappa}$, then $(p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^3$.

Proposition 6.2. *Suppose $2^\lambda = \lambda^{<\kappa}$. Then $(p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^3_\eta$ for every η with $2 \leq \eta < \kappa$.*

Proof. The proof is a straightforward modification of that of Proposition 3.4. □

Next we prove that if κ is Mahlo and $2^\lambda = \lambda^{<\kappa}$, then there is $C \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^*$ such that

(a) $\{A \subseteq C : A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3\} \subseteq (p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+$, and

(b) for any $A \subseteq C$ with $A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$, there is $B \subseteq A$ with $B \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$ and $B \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$.

Lemma 6.3. *Suppose that κ is inaccessible and* $\mathrm{cf}(\lambda) < \kappa$, *and let $2 \leq n < \omega$ and $2 \leq \eta < \kappa$. Then the following hold*:

(i) *Let $X \subseteq \{x \in \mathcal{X}_{\kappa,\lambda} : |x \cap \kappa|$ is an inaccessible cardinal$\}$ be such that $X \underset{<}{\longrightarrow} (I_{\kappa,\lambda^{<\kappa}}^+)_\eta^n$. Then $q_{\kappa,\lambda}^{-1}(X) \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)_\eta^n$.*

(ii) *Let $A \subseteq P_\kappa(\lambda)$ be such that $A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)_\eta^n$. Then $q_{\kappa,\lambda}``A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)_\eta^n$.*

Proof. Proceed as in the proofs of Propositions 3.4 and 3.5. □

Proposition 6.4. *Suppose that κ is Mahlo and $2^\lambda = \lambda^{<\kappa}$. Then there is $C \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^*$ such that $\{A \subseteq C : A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3\} \subseteq (p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+$.*

Proof. We can assume that $(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^* \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$ since otherwise the result is trivial. Then by Lemma 6.1, $\mathrm{cf}(\lambda) < \kappa$ and $\lambda^{<\kappa} = \lambda^+$. Now Proposition 4.14 tells us that there is $E \in \mathrm{NS}_{\kappa,\lambda^{<\kappa}}^*$ such that $\{X \subseteq E : X \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^3\} \subseteq \mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}^+$. Set $C = q_{\kappa,\lambda}^{-1}(E)$. Note that $C \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^*$ by Lemma 3.1.

Given $A \subseteq C$ with $A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$, put $X = q_{\kappa,\lambda}``A$. Then clearly $X \subseteq E$. Moreover by Lemma 6.3, $X \underset{<}{\longrightarrow} (I_{\kappa,\lambda^{<\kappa}}^+)^3$, and therefore $X \in \mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}^+$. It follows that $A \notin p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}})$, since by Lemma 3.2 $A = \{x \cap \lambda : x \in X\}$. □

Proposition 6.5. *Suppose that κ is Mahlo and $2^\lambda = \lambda^{<\kappa}$. Then there is $C \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^*$ with the following property*: *For any $A \subseteq C$ such that $A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$, there is $B \subseteq A$ such that $B \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$ but $B \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$.*

Proof. Exactly as in the proof of the preceding proposition, we can assume that $(\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^* \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^3$, which entails that $\mathrm{cf}(\lambda) < \kappa$ and $\lambda^{<\kappa} = \lambda^+$. By Proposition 4.18 and Lemmas 2.2 and 3.1 we may find $Z \in (\mathrm{NS}_{\kappa,\lambda^{<\kappa}}^{[\lambda^{<\kappa}]^{<\kappa}})^*$ such that

(a) $Z \subseteq \{x \in \mathcal{X}_{\kappa,\lambda} : x \cap \kappa$ is an inaccessible cardinal$\}$, and

(b) for any $X \subseteq Z$ with $X \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^2$, there is $Y \subseteq X$ with $Y \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^2$ and $Y \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^3$.

Put $C = q_{\kappa,\lambda}^{-1}(Z)$. Note that $C \in (\mathrm{NS}_{\kappa,\lambda}^{[\lambda]^{<\kappa}})^*$ by Lemma 3.1. Now let $A \subseteq C$ be such that $A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda}^+)^2$. Since by Lemma 6.3, $q_{\kappa,\lambda}``A \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^2$, we may find $Y \subseteq q_{\kappa,\lambda}``A$ with $Y \underset{<}{\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^2$ and $Y \underset{<}{\not\longrightarrow} (\mathrm{I}_{\kappa,\lambda^{<\kappa}}^+)^3$. Set $B = q_{\kappa,\lambda}^{-1}(Y)$. Then clearly, $B \subseteq A$. Moreover by Lemma 6.3, $B \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$ but $B \underset{<}{\not\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$. □

Lemma 6.6. *Suppose that κ is Mahlo, $\mathrm{cf}(\lambda) < \kappa$, and there is $Z \in \mathrm{NS}^*_{\kappa,\lambda^{<\kappa}}$ such that $\mathrm{NSS}_{\kappa,\lambda^{<\kappa}}|Z = \mathrm{I}_{\kappa,\lambda^{<\kappa}}|Z$. Then there is $C \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^*$ with the property that for any $A \in \mathrm{I}^+_{\kappa,\lambda} \cap P(C)$, there is $B \in \mathrm{I}^+_{\kappa,\lambda} \cap P(A)$ with $[B]^2_< = [B]^2$.*

Proof. Put $C = q^{-1}_{\kappa,\lambda}(Z \cap \mathcal{X}_{\kappa,\lambda})$. Then by Lemma 3.1, $C \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^*$. Let $A \in \mathrm{I}^+_{\kappa,\lambda} \cap P(C)$. Set $X = q_{\kappa,\lambda}``A$. Then by Lemma 3.1, $X \in I^+_{\kappa,\lambda^{<\kappa}} \cap P(Z \cap \mathcal{X}_{\kappa,\lambda})$. Hence by Lemma 4.3, we may find $Y \in \mathrm{I}^+_{\kappa,\lambda^{<\kappa}} \cap P(X)$ with $[Y]^2_< = [Y]^2$. Put $B = q^{-1}_{\kappa,\lambda}(Y)$. It is simple to see that $B \in \mathrm{I}^+_{\kappa,\lambda} \cap P(A)$, and $[B]^2_< = [B]^2$. □

By a result of [**23**] and Lemma 6.6, if $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) \le \lambda^{<\kappa}$, then for any $A \subseteq P_\kappa(\lambda)$, $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, \mathrm{I}^+_{\kappa,\lambda})^2$ just in case $A \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^2$. We now show that the result remains valid when 3 is substituted for 2.

Lemma 6.7. *$\{A \subseteq P_\kappa(\lambda) : A \underset{<}{\not\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, \mathrm{I}^+_{\kappa,\lambda})^3\}$ is a (possibly improper) strongly normal ideal on $P_\kappa(\lambda)$ extending $\mathrm{NIn}_{\kappa,\lambda}$.*

Proof. Set $J = \{A \subseteq P_\kappa(\lambda) : A \underset{<}{\not\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda}, \mathrm{I}^+_{\kappa,\lambda})^3\}$. Clearly, $P(A) \subseteq J$ for all $A \in J$. It is also simple to see that if A_1, A_2 are any two disjoint members of J, then $A_1 \cup A_2 \in J$. It follows that J is a (possibly improper) ideal on $P_\kappa(\lambda)$. If $A \in P(P_\kappa(\lambda)) \setminus J$, then by Lemma 2.12 $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^2$, so by Lemma 1.13 $A \in \mathrm{NIn}^+_{\kappa,\lambda}$. Thus $\mathrm{NIn}_{\kappa,\lambda} \subseteq J$. By Lemma 2.3, it follows that $\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda} \subseteq J$.

Now let $A \in J^+$ and $f : A \to P_\kappa(\lambda)$ with the property that $f(a) < a$ for every $a \in A$. Let B be the set of all $a \in A$ such that $a \cap \kappa$ is an inaccessible cardinal. Note that by Lemma 2.2, $B \in J^+$. Further note that for any $a \in B$, $\mathrm{o.t.}(f(a)) \in a \cap \kappa$. For $a \in B$, let $h_a : \mathrm{o.t.}(f(a)) \to f(a)$ be the increasing enumeration of $f(a)$. For $e \in P_\kappa(\lambda)$, put $B_e = \{a \in B : f(a) = e\}$. Suppose toward a contradiction that $\{B_e : e \in P_\kappa(\lambda)\} \subseteq J$. For $e \in P_\kappa(\lambda)$, pick $F_e : [B]^3_< \to 2$ with the property that

(a) there is no $H \in \mathrm{NS}^+_{\kappa,\lambda} \cap P(B_e)$ such that F_e takes the constant value 0 on $[H]^3_<$, and
(b) there is no $Q \in \mathrm{I}^+_{\kappa,\lambda} \cap P(B_e)$ such that F_e takes the constant value 1 on $[Q]^3_<$.

Now define $F : [B]^3 \to 2$ by: $F(a_0, a_1, a_2) = 0$ if and only if either $f(a_0) = f(a_1)$ and $F_{f(a_0)}(a_0, a_1, a_2) = 0$, or $\mathrm{o.t.}(f(a_0)) < \mathrm{o.t.}(f(a_1))$, or $f(a_0) \ne f(a_1)$, $\mathrm{o.t.}(f(a_0)) = \mathrm{o.t.}(f(a_1))$ and $h_{a_0}(\sigma) < h_{a_1}(\sigma)$, where $\sigma =$ the least ζ such that $h_{a_0}(\zeta) \ne h_{a_1}(\zeta)$.

We may find $C \subseteq B$ and $i < 2$ such that

(α) F takes the constant value i on $[C]^3_<$, and
(β) $C \in \mathrm{NS}^+_{\kappa,\lambda}$, if $i = 0$, and $C \in \mathrm{I}^+_{\kappa,\lambda}$ otherwise.

Case I: $i = 0$. There must be $D \in \mathrm{NS}^+_{\kappa,\lambda} \cap P(C)$ and $\alpha \in \kappa$ such that $\mathrm{o.t.}(f(a)) = \alpha$ for every $a \in D$. We inductively define $\delta_\sigma \in \lambda$ and $W_\sigma \in \mathrm{NS}^*_{\kappa,\lambda}$ for $\sigma < \alpha$ so that $h_a(\sigma) = \delta_\sigma$ for each $a \in D \cap W_\sigma$. Suppose δ_ξ and W_ξ have already been constructed for each $\xi < \sigma$. Set $S = \bigcap_{\xi<\sigma} W_\xi$ and $y = \{h_a(\sigma) : a \in D \cap S\}$. For $\beta \in y$, pick $d_\beta \in D \cap S$ with $h_{d_\beta}(\sigma) = \beta$.

Claim 1. *y has a largest element.*

Proof of Claim 1. Suppose otherwise. Let T be the set of all $a \in D \cap S$ such that

(1) for every $\beta \in a \cap y$, $d_\beta < a$, and
(2) $\mathrm{o.t.}(a \cap y)$ is an infinite limit ordinal.

Then clearly $T \in \mathrm{NS}^+_{\kappa,\lambda}$. Pick $w \in T$. There must be $\beta \in w \cap y$ with $h_w(\sigma) < \beta$. Now select $x \in D$ with $w < x$. Then $F(d_\beta, w, x) = 1$. This contradiction completes the proof of Claim 1. □

Claim 2. *$h_a(\sigma) = \max(y)$ for every $a \in D \cap S$ with $d_{\max(y)} < a$.*

Proof of Claim 2. Suppose otherwise, and pick $s \in D \cap S$ with $d_{\max(y)} < s$ and $h_s(\sigma) \neq \max(y)$. Select $s' \in D$ with $s < s'$. Then $F(d_{\max(y)}, s, s') = 1$. This contradiction completes the proof of Claim 2. □

Now set $\delta_\sigma = \max(y)$ and $W_\sigma = \{a \in S : d_{\max(y)} < a\}$.

Finally, put $e = \{\delta_\xi : \xi < \alpha\}$ and $W = \bigcap_{\xi<\alpha} W_\xi$. Then $D \cap W \in \mathrm{NS}^+_{\kappa,\lambda} \cap P(B_e)$, and moreover F_e takes the constant value 0 on $[D \cap W]^3_<$. Contradiction.

Case II: $i = 1$.

Claim 3. *There is $z \in C$ such that $\mathrm{o.t.}(f(a)) = \mathrm{o.t.}(f(z))$ for every $a \in C$ with $z < a$.*

Proof of Claim 3. Suppose otherwise. Inductively pick $b_n \in C$ for $n < \omega$ so that $b_n < b_{n+1}$ and $\mathrm{o.t.}(f(b_n)) \neq \mathrm{o.t.}(f(b_{n+1}))$. For each $n < \omega$, $F(b_n, b_{n+1}, b_{n+2}) = 1$, so $\mathrm{o.t.}(f(b_n)) > \mathrm{o.t.}(f(b_{n+1}))$. Thus $\mathrm{o.t.}(f(b_0)) > \mathrm{o.t.}(f(b_1)) > \mathrm{o.t.}(f(b_2)) > \cdots$. This contradiction completes the proof of Claim 3. □

Put $\beta = \mathrm{o.t.}(f(z))$ and $C' = \{a \in C : z < a\}$. We define inductively $\eta_\sigma \in \lambda$ and $t_\sigma \in C'$ for $\sigma < \beta$ so that $h_a(\sigma) = \eta_\sigma$ for each $a \in C'$ with $t_\sigma < a$. Suppose η_ξ and t_ξ have already been constructed for each $\xi < \sigma$. Set $u = \bigcup_{\xi<\sigma} t_\xi$ and $R = \{a \in C' : u < a\}$.

Claim 4. *There is $v \in R$ such that $h_a(\sigma) = h_v(\sigma)$ for every $a \in R$ with $v < a$.*

Proof of Claim 4. Suppose otherwise. Inductively select $c_n \in R$ for $n < \omega$ so that $c_n < c_{n+1}$ and $h_{c_n}(\sigma) \neq h_{c_{n+1}}(\sigma)$. For each $n < \omega$, $F(c_n, c_{n+1}, c_{n+2}) = 1$, so $h_{c_n}(\sigma) > h_{c_{n+1}}(\sigma)$. Thus $h_{c_0}(\sigma) > h_{c_1}(\sigma) > h_{c_2}(\sigma) > \cdots$. This contradiction completes the proof of Claim 4. □

Now put $\eta_\sigma = h_v(\sigma)$ and $t_\sigma = v$.

Finally, set $e = \{\eta_\xi : \xi < \beta\}$ and $t = \bigcup_{\xi<\beta} t_\xi$. Then clearly $\{a \in C' : t < a\} \in \mathrm{I}^+_{\kappa,\lambda} \cap P(B_e)$. Moreover F_e takes the constant value 1 on $[\{a \in C' : t < a\}]^3_<$. Contradiction. □

Proposition 6.8. *Suppose $\overline{\mathrm{cof}}(\mathrm{NS}_{\kappa,\lambda}) \leq \lambda^{<\kappa}$, and let $A \subseteq P_\kappa(\lambda)$ with $A \underset{<}{\longrightarrow}$*

$(\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^3$. *Then* $A \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^3$.

Proof. By Lemmas 1.8, 6.6 and 6.7. $\square$

If κ is Mahlo and $2^\lambda = \lambda^{<\kappa}$, then by a result of [**23**] and Lemma 6.6, for any $A \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+$, $A \in (p_{\kappa,\lambda}(\mathrm{NSh}_{\kappa,\lambda^{<\kappa}}))^+$ if and only if $A \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^2$ if and only if $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^2$ if and only if $(\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}|A)^* \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^2$. The corresponding result for triples reads as follows:

Proposition 6.9. *Suppose κ is Mahlo and $2^\lambda = \lambda^{<\kappa}$. Then for any $A \in (\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+$, the following are equivalent*:

(i) $A \in (p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+$.
(ii) $A \longrightarrow (\mathrm{NS}^+_{\kappa,\lambda})^3$.
(iii) $A \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^3$.
(iv) $(\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda}|A)^* \underset{<}{\longrightarrow} (\mathrm{I}^+_{\kappa,\lambda})^3$.

Proof. By Lemma 6.7 and Propositions 6.2 and 6.4. $\square$

If $\lambda^{<\lambda} = \lambda$, then by Lemma 1.13 and Propositions 2.11 and 2.13, $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda})^2$ just in case $P_\kappa(\lambda) \underset{<}{\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^3$. In contrast to this, if $2^\lambda = \lambda^{<\kappa}$ and κ is $\lambda^{<\kappa}$-Shelah but not almost $\lambda^{<\kappa}$-ineffable, then (by a result of [**23**]) $P_\kappa(\lambda) \underset{<}{\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NS}^+_{\kappa,\lambda})^2$ but (by Lemma 6.7 and Propositions 6.4 and 6.8), $P_\kappa(\lambda) \underset{<}{\not\longrightarrow} (\mathrm{NS}^+_{\kappa,\lambda},\mathrm{I}^+_{\kappa,\lambda})^3$. (Note that it can be shown that if $2^\lambda = \lambda^{<\kappa}$ and κ is almost $\lambda^{<\kappa}$-ineffable, then the set of all $a \in \mathcal{A}_{\kappa,\lambda}$ such that $2^{\mathrm{o.t.}(a)} = \mathrm{o.t.}(a)^{<(a\cap\kappa)}$ and $a\cap\kappa$ is $\mathrm{o.t.}(a)^{<(a\cap\kappa)}$-Shelah but not almost $\mathrm{o.t.}(a)^{<(a\cap\kappa)}$-ineffable lies in $(p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}})^+)$. On the other hand, if $2^\lambda = \lambda^{<\kappa}$, then by Proposition 3.4 and Lemma 2.12, $P_\kappa(\lambda) \underset{<}{\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+)^2$ just in case $P_\kappa(\lambda) \underset{<}{\longrightarrow} ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{I}^+_{\kappa,\lambda})^3$.

Finally, we combine Propositions 2.11 and 3.4 on the one hand, and Propositions 4.4 and 6.2 on the other hand, thus showing that the two cases $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^{<\kappa}$ can be (at least to some extent) handled simultaneously.

Proposition 6.10. *Suppose $(\lambda^{<\kappa})^{<(\lambda^{<\kappa})} = \lambda^{<\kappa}$. Then $(p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NSS}^+_{\kappa,\lambda})^3$ and $(p_{\kappa,\lambda}(\mathrm{NAIn}_{\kappa,\lambda^{<\kappa}}))^+ \longrightarrow (\mathrm{NSS}^+_{\kappa,\lambda})^3$.*

Proof. We prove the first assertion and leave the proof of the second to the reader. Thus let $A \in (p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}))^+$. If $\mathrm{cf}(\lambda) \geq \kappa$, then by Lemma 1.11 $\lambda^{<\lambda} = \lambda$ and $p_{\kappa,\lambda}(\mathrm{NIn}_{\kappa,\lambda^{<\kappa}}) = \mathrm{NIn}_{\kappa,\lambda}$, so by Proposition 2.11, $A \longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NSS}^+_{\kappa,\lambda})^3$. If $\mathrm{cf}(\lambda) < \kappa$, then by Lemma 3.3 $2^\lambda = \lambda^{<\kappa} = \lambda^+$ and therefore by Proposition 3.4, $A \longrightarrow ((\mathrm{NS}^{[\lambda]^{<\kappa}}_{\kappa,\lambda})^+,\mathrm{NSS}^+_{\kappa,\lambda})^3$. $\square$

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