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Tangential representations of one-fixed-point actions on spheres and Smith equivalence

Dedicated to Professor Anthony Bak for his 70th birthday

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Abstract. Let G be a finite Oliver group. In this paper, we discuss the relation between tangential G-representations of smooth one-fixed-point actions on spheres and the Smith equivalence of real G-representations.

1. Introduction.

Throughout this paper, let G be a finite group. Let V and W be real G-modules. If there exists a homotopy sphere (resp. standard sphere) Σ with smooth G-action such that the G-fixed-point set Σ^G consists of exactly two points, a and b say, and the tangential G-representations $T_a(\Sigma)$ and $T_b(\Sigma)$ are isomorphic to V and W, respectively, then we say that V and W are *Smith equivalent* (resp. *Smith* equivalent*). If there exists a real G-module U such that $V \oplus U$ and $W \oplus U$ are Smith equivalent (resp. Smith* equivalent), then we say that V and W are *stably Smith equivalent* (resp. *stably Smith* equivalent*). The Smith set $\mathrm{Sm}(G)$ is the subset of the real-representation ring $\mathrm{RO}(G)$ consisting of all elements $[V]-[W]$ such that V and W are Smith equivalent. If there exists a homotopy sphere Σ with smooth G-action such that the G-fixed-point set Σ^G consists of exactly one point, a say, and $T_a(\Sigma)$ is isomorphic to V, then we say that V is of *one-fixed-point type*, or *OFP type*. The Smith-equivalence problem has been studied by Atiyah-Bott [**1**], Milnor [**15**], Bredon [**4**], Sanchez [**39**], Petrie [**31**], Cappell-Shaneson [**5**], and in joint works by Cho, Dovermann, Petrie, Randall, Suh [**35**], [**6**], [**7**], [**36**], [**41**] in various contexts, and recently by Laitinen, Pawałowski, Solomon, Sumi and etc. [**13**], [**28**], [**19**], [**29**], [**20**], [**21**], while the problem of smooth one-fixed-point actions on spheres was studied by Petrie [**31**], [**32**], Laitinen-Traczyk [**14**], Morimoto [**16**], [**17**], [**18**], Laitinen-Morimoto [**11**], and Bak-Morimoto [**2**].

Theorem 1.1. *If V is a real G-module of OFP type then there exists a standard sphere S with smooth G-action such that $S^G = \{a\}$ and $T_a(S) \cong V$.*

In Section 2, we introduce two conjectures, i.e. Conjectures 2.1 and 2.2. These conjectures suggest an approach to investigate the Smith sets for Oliver groups. The

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next theorem answers to Conjecture 2.1 for gap groups G (cf. Section 2).

Theorem 1.2. *Let V and W be real G-modules of OFP type such that* $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ *for all Sylow subgroups P of G. If G is a gap group then V and W are stably Smith* equivalent.*

This will be proved in a slightly generalized form, namely as Theorem 2.1.

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2. Preliminary.

In this section, we prepare terms and notation which are necessary in the present paper.

Let X be a G-space. For a point x in X, G_x denotes the isotropy subgroup of G at x. For a subgroup H of G, we set

$$\begin{aligned} X^H &= \{x \in X \mid gx = x \text{ for all } g \in H\}, \\ X^{=H} &= \{x \in X \mid G_x = H\}, \\ X^{>H} &= X^H \smallsetminus X^{=H}. \end{aligned}$$

For a prime p, let $\mathcal{P}_p(G)$ denote the set of all subgroups P of p-power order (possibly $|P| = 1$). Let $\mathcal{P}(G)$ (resp. $\mathcal{P}_{\mathrm{odd}}(G)$) be the union of $\mathcal{P}_p(G)$, where p runs over the set of all primes (resp. all odd primes).

For an integer $m \geq 0$, let $\mathfrak{S}^{(m)}$ (resp. $\mathfrak{S}^{(m)}_{\mathrm{h}}$) denote the family of all standard spheres (resp. homotopy spheres) X with smooth G-action such that $|X^G| = m$. Define the subsets $\mathrm{RO}(G, \mathfrak{S}^{(1)})$, $\mathrm{RO}(G, \mathfrak{S}^{(1)}_{\mathrm{h}})$, $\mathrm{RO}(G, \mathfrak{S}^{(2)})$, and $\mathrm{RO}(G, \mathfrak{S}^{(2)}_{\mathrm{h}})$ of $\mathrm{RO}(G)$ by

$$\mathrm{RO}(G, \mathfrak{X}) = \{[T_a(X)] \mid X \in \mathfrak{X},\ X^G = \{a\}\}$$

for $\mathfrak{X} = \mathfrak{S}^{(1)}$, $\mathfrak{S}^{(1)}_{\mathrm{h}}$, and

$$\mathrm{RO}(G, \mathfrak{X}) = \{[T_a(X)] - [T_b(X)] \mid X \in \mathfrak{X},\ X^G = \{a, b\}\}$$

for $\mathfrak{X} = \mathfrak{S}^{(2)}$, $\mathfrak{S}^{(2)}_{\mathrm{h}}$. By definition, $\mathrm{Sm}(G)$ coincides with $\mathrm{RO}(G, \mathfrak{S}^{(2)}_{\mathrm{h}})$. It is clear that

$$\mathrm{RO}(G, \mathfrak{S}^{(1)}) \subset \mathrm{RO}(G, \mathfrak{S}^{(1)}_{\mathrm{h}}) \text{ and } \mathrm{RO}(G, \mathfrak{S}^{(2)}) \subset \mathrm{RO}(G, \mathfrak{S}^{(2)}_{\mathrm{h}}).$$

By Theorem 1.1, we have

$$\mathrm{RO}(G, \mathfrak{S}^{(1)}) = \mathrm{RO}(G, \mathfrak{S}^{(1)}_{\mathrm{h}}). \tag{2.1}$$

A finite group G is called an *Oliver group* if G does not have a normal series $P \trianglelefteq H \trianglelefteq G$

such that P and G/H are of prime-power order and H/P is cyclic. By [**11**, Theorem A], $\mathrm{RO}(G, \mathfrak{S}^{(1)}_{\mathrm{h}})$ is non-empty if and only if G is an Oliver group.

For a subset A of $\mathrm{RO}(G)$ and sets $\mathcal{F}$ and $\mathcal{G}$ of subgroups of G, we set

$$\begin{aligned} A^{\mathcal{F}} &= \{[V] - [W] \in A \mid V^H = 0 = W^H \text{ for all } H \in \mathcal{F}\}, \\ A_{\mathcal{G}} &= \{[V] - [W] \in A \mid \operatorname{res}^G_H V \cong \operatorname{res}^G_H W \text{ for all } H \in \mathcal{G}\}, \\ A^{\mathcal{F}}_{\mathcal{G}} &= (A^{\mathcal{F}})_{\mathcal{G}}. \end{aligned}$$

The set $\mathrm{Sm}(G)_{\mathcal{P}(G)}$ is called the *primary Smith set*. We know that $\mathrm{Sm}(G) \smallsetminus \mathrm{Sm}(G)_{\mathcal{P}(G)}$ is a finite set (cf. [**24**, Theorem 1]).

Define the set $\mathrm{DRO}(G, \mathfrak{S}^{(1)})$ by

$$\mathrm{DRO}(G, \mathfrak{S}^{(1)}) = \{[V] - [W] \in \mathrm{RO}(G) \mid [V],\ [W] \in \mathrm{RO}(G, \mathfrak{S}^{(1)})\}.$$

We have the following two conjectures.

Conjecture 2.1. If G is an Oliver group then the inclusion

$$\mathrm{DRO}(G, \mathfrak{S}^{(1)})_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{S}^{(2)})_{\mathcal{P}(G)}$$

holds.

Let G^{nil} denote the smallest normal subgroup N of G such that G/N is nilpotent.

Conjecture 2.2. If G is an Oliver group such that a Sylow 2-subgroup of G^{nil} is not normal in G^{nil} then the coincidence

$$\mathrm{DRO}(G, \mathfrak{S}^{(1)})_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{S}^{(2)})_{\mathcal{P}(G)}$$

holds.

Let X be a smooth G-manifold and $\mathcal{F}$ a set of subgroups of G. We say that X satisfies the *$\mathcal{F}$-gap condition* (resp. *$\mathcal{F}$-weak gap condition*) if

$$\dim X^H_\alpha > 2 \dim X^K_\beta \tag{G}$$

$$\big(\text{resp.}\quad \dim X^H_\alpha \geq 2 \dim X^K_\beta\big) \tag{WG}$$

for all subgroups $H \in \mathcal{F}$ and $K > H$ of G and all connected components X^H_α and X^K_β of X^H and X^K, respectively, such that $X^H_\alpha \supset X^K_\beta$.

For a prime p, $G^{\{p\}}$ denote the intersection of all normal subgroups H of G such that $|G : H|$ is a power of p (possibly p^0). Let $\mathcal{L}(G)$ denote the set of all subgroups H of G such that $H \supset G^{\{p\}}$ for some p. For a set $\mathcal{F}$ of subgroups of G, a real G-module V is called $\mathcal{F}$-free if $V^H = 0$ for all $H \in \mathcal{F}$. A real G-module V is called a *gap real G-module* if V is $\mathcal{L}(G)$-free and V satisfies the $\mathcal{P}(G)$-gap condition. If a finite group G

not of prime-power order possesses a gap real G-module then we call G a *gap group.* A finite group G not of prime-power order is a gap group if G satisfies one of the following conditions (cf. [**11**, Theorem 2.3], [**25**, Proposition 4.3]).

(1) $G = G^{\{2\}}$.
(2) $G \neq G^{\{p\}}$ holds for at least two odd primes p.
(3) A Sylow 2-subgroup of G is normal in G.
(4) G has a normal subgroup N such that G/N is a gap group.

If G is a gap group then for homotopy spheres Σ and Ξ with smooth G-action such that $\Sigma^G = \{a\}$, $T_a(\Sigma) \cong V$, $\Xi^G = \{b\}$, $T_b(\Xi) \cong W$, there exists an $\mathcal{L}(G)$-free real G-module U such that $\Sigma \times U$ and $\Xi \times U$ both satisfy the $\mathcal{P}(G)$-gap condition.

Theorem 2.1. *Let V and W be real G-modules such that $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ for all Sylow subgroups P of G. Suppose there exist homotopy spheres Σ and Ξ with smooth G-action such that $\Sigma^G = \{a\}$, $T_a(\Sigma) \cong V$, $\Xi^G = \{b\}$, $T_b(\Xi) \cong W$, and there exists an $\mathcal{L}(G)$-free real G-module U_0 such that $\Sigma \times U_0$ and $\Xi \times U_0$ both satisfy the $\mathcal{P}(G)$-weak gap condition. Then there exists a standard sphere S with smooth G-action such that $S^G = \{a, b\}$, $T_a(S) = V \oplus U_1$, $T_b(S) = W \oplus U_1$ for some $\mathcal{L}(G)$-free real G-module U_1 and S satisfies the $\mathcal{P}(G)$-weak gap condition.*

For a natural number n, let C_n be a cyclic group of order n. For a prime p, let $G^{\cap p}$ denote the intersection of all normal subgroups H of G such that $|G : H| = 1$ or p. For various Oliver groups G, e.g.

(I) an Oliver group G such that a Sylow 2-subgroup of G^{nil} is not normal in G^{nil} and $G/G^{\mathrm{nil}} \cong C_3$,
(II) a gap Oliver group G such that a Sylow 2-subgroup of G^{nil} is not normal in G^{nil} and $G/G^{\mathrm{nil}} \cong C_6$,
(III) $G = H \times K$ such that H is a nontrivial perfect group with a dihedral subquotient D_{2pq} for distinct primes p and q and K is a finite group with $K/K^{\{2\}} = C_2 \times \cdots \times C_2$ (possibly the trivial group),

we got $\mathrm{Sm}(G)_{\mathcal{P}(G)} = \mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)}$, by essentially proving the validity of Conjecture 2.2 for those G.

3. Basic observation.

For a prime p, let $G^{\{p\}}$ denote the intersection of all normal subgroups H of G such that $|G : H|$ is a power of p.

Lemma 3.1. *Let G be an Oliver group and let $\Sigma \in \mathfrak{S}^{(1)}_{\mathrm{h}}$ with $\Sigma^G = \{a\}$ and $V = T_a(\Sigma)$. Then the following properties hold.*

(1) *In general, the equality $V^{G^{\cap 2}} = 0$ holds.*
(2) *If a Sylow 2-subgroup of G is normal in G then $V^{G^{\cap p}} = 0$ for all primes p.*
(3) *For each prime p and $P \in \mathcal{P}_p(G)$, Σ^P is a mod-p homology sphere.*
(4) *For each prime p and a Sylow p-subgroup P, $\Sigma^P \cap \Sigma^{G^{\{p\}}} = \{a\}$.*

(5) *Let p be a prime with $P \in \mathcal{P}_p(G)$ such that* $\dim V^P = 0$. *Then* $\Sigma^{G^{\{q\}}} = \{a\}$ *for all primes $q \neq p$.*
(6) *Let p be a prime with $P \in \mathcal{P}_p(G)$ such that* $\dim V^P > 0$. *Then* $\dim V^P > \dim \Sigma^{G^{\{q\}}}$ *for all primes $q \neq p$.*
(7) *For any $P \in \mathcal{P}(G)$, $\Sigma^P \smallsetminus \bigcup_q \Sigma^{G^{\{q\}}} \neq \emptyset$, where q runs over the set of all primes.*

We remark that there exist Oliver groups G with $[V] \in \mathrm{RO}(G, \mathfrak{S}_{\mathrm{h}}^{(1)})$ and odd primes p such that $V^{G^{\cap p}} \neq 0$, e.g. $G = \mathrm{P\Sigma L}(2, 27)$, $A_n \times C_p$ with $n \geq 6$ (cf. [**20**]).

Proof of Lemma 3.1. (1): Let N be a normal subgroup of G with $|G : N| = 2$. By Lemma 2.1 of [**19**], there never exists a connected closed manifold M of dimension ≥ 1 with smooth C_2-action such that $|M^{C_2}| = 1$. Thus we get $V^N = 0$.

Since $G/G^{\cap 2} \cong C_2 \times \cdots \times C_2$, the result above implies $V^{G^{\cap 2}} = 0$.

(2): Let p be an odd prime. Let N be a normal subgroup of G with $|G : N| = p$. Since a Sylow 2-subgroup of N is normal, Σ^N is orientable (cf. [**8**]). By the same argument as the proof of Lemma 2.1 of [**19**] for its G $(= C_2)$ replaced by $G = C_p$ (and in the category of orientation-preserving actions), there never exists a connected closed orientable manifold M of dimension ≥ 1 with smooth C_p-action such that $|M^{C_p}| = 1$. Thus we get $V^N = 0$.

Since $G/G^{\cap p} \cong C_p \times \cdots \times C_p$, the result above implies $V^{G^{\cap p}} = 0$.

(3): This follows from the Smith theory.

(4): Since $PG^{\{p\}} = G$, $\Sigma^P \cap \Sigma^{G^{\{p\}}} = \Sigma^G = \{a\}$.

(5): We have

$$\{a\} \subset \Sigma^{G^{\{q\}}} \subset \Sigma^P = S^0.$$

Since

$$\left|\Sigma^{G^{\{q\}}}\right| = \chi\left(\Sigma^{G^{\{q\}}}\right) \equiv 1 \mod q,$$

we get $\Sigma^{G^{\{q\}}} = \{a\}$.

(6): Note that Σ^P is a mod-p homology sphere of dimension ≥ 1 and $\chi(\Sigma^P) = 0$ or 2. For any prime q, we have

$$\chi\left(\Sigma^{G^{\{q\}}}\right) \equiv 1 \mod q.$$

Thus $\Sigma^{G^{\{q\}}} \subsetneq \Sigma^P$, which implies $\dim \Sigma^{G^{\{q\}}} < \dim \Sigma^P = \dim V^P$.

(7): This follows from (4)–(6). □

Lemma 3.2. *Let G be an Oliver group and let $\Sigma \in \mathfrak{S}_{\mathrm{h}}^{(2)}$ with $\Sigma^G = \{a, b\}$, $V = T_a(\Sigma)$, and $W = T_b(\Sigma)$. Then the following properties hold.*

(1) *If N is a subgroup of G with $|G : N| = 2$ then $V^N = 0 = W^N$ or* $\mathrm{res}^G_N V \cong \mathrm{res}^G_N W$, *and hence* $\dim V^N = \dim W^N$. *Thus* $V^{G^{\cap 2}} \cong W^{G^{\cap 2}}$ *as $G/G^{\cap 2}$-modules.*

(2) *Suppose a Sylow* 2*-subgroup of* G *is normal in* G*. Let* p *be an odd prime. If* N *is a normal subgroup of* G *with* $|G:N|=p$ *then* $V^N=0=W^N$ *or* $\mathrm{res}^G_N V \cong \mathrm{res}^G_N W$*, and hence* $\dim V^N = \dim W^N$*. Thus* $V^{G^{\cap 3}} \cong W^{G^{\cap 3}}$ *as* $G/G^{\cap 3}$*-modules.*

(3) *Let* H *be a subgroup of* G *such that* $V^H \neq 0$ *or* $W^H \neq 0$*. If there exists* $P \in \mathcal{P}(H)$ *satisfying* $V^P = V^H$ *or* $W^P = W^H$*, then* $\mathrm{res}^G_H V \cong \mathrm{res}^G_H W$*.*

Proof. (1): Suppose $V^N \neq 0$ or $W^N \neq 0$. Let X_a and X_b be the connected components of Σ^N containing a and b, respectively. Then X_a or X_b has positive dimension. The group $C_2 = G/N$ smoothly acts on X_a and X_b. Suppose $\dim X_a > 0$. Since there never exists a connected closed smooth C_2-manifold Y with $\dim Y > 0$ and $|Y^{C_2}| = 1$, X_a contains b, i.e. $X_a = X_b$. If $\dim X_b > 0$ then by the same argument we get $X_b = X_a$. Thus $X_a = X_b$ holds in the both cases. This implies $\mathrm{res}^G_N V \cong \mathrm{res}^G_N W$.

The equality $\dim V^N = \dim W^N$ holds in any case where $\dim V^N = 0$ or not. This implies $V^{G^{\cap 2}} \cong W^{G^{\cap 2}}$ as real $G/G^{\cap 2}$-modules.

(2): Suppose $V^N \neq 0$ or $W^N \neq 0$. Since a Sylow 2-subgroup G_2 of G is normal and $G_2 \subset N$, Σ^N is orientable. Note that there never exists a connected closed orientable smooth C_p-manifold Y such that $\dim Y > 0$ and $|Y^{C_p}| = 1$. By the argument same as the proof of (1), we get $\mathrm{res}^G_N V \cong \mathrm{res}^G_N W$.

The equality $\dim V^N = \dim W^N$ holds in any case where $\dim V^N = 0$ or not. This implies $V^{G^{\cap 3}} \cong W^{G^{\cap 3}}$ as real $G/G^{\cap 3}$-modules.

(3): By the Smith theory, Σ^P is a mod-p homology sphere, where $|P| = p^k$. By the assumption that $V^P = V^H \neq 0$ or $W^P = W^H \neq 0$ holds, $\Sigma^H = \Sigma^P$ is a connected manifold containing a and b. Thus $\mathrm{res}^G_H V \cong \mathrm{res}^G_H W$ as real H-modules. □

Real G-modules V and W are $\mathcal{P}(G)$-matched Smith-equivalent if V and W are Smith-equivalent and $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$ for a Sylow 2-subgroup P of G. If G does not contain an element of order 8 then Smith-equivalent V and W are $\mathcal{P}(G)$-matched Smith-equivalent. The next proposition immediately follows from Lemma 3.2.

Proposition 3.3. *$\mathcal{P}(G)$-matched Smith-equivalent real G-modules V and W are isomorphic as real G-modules if for each cyclic subgroup C of G there exists $P \in \mathcal{P}(C)$ such that $V^P = V^C \neq 0$ or $W^P = W^C \neq 0$.*

This is available to show that $G = \mathrm{SL}(2,5)$ does not have a pair (V,W) of Smith-equivalent non-isomorphic real G-modules V and W of dimension ≤ 17. For the convenience of readers, we give an outline of the proof. Let $G = \mathrm{SL}(2,5)$. A. Borowiecka [**3**] tabulated the character of U and the dimension of U^H for irreducible real G-modules U and subgroups H of G. We can also obtain the data by using the computer software GAP. The order of an element of G is 1, 2, 3, 4, 5, 6 or 10. Let (V,W) be a pair of Smith-equivalent real G-modules of dimension ≤ 17. Since G does not contain elements of order 8, V and W are $\mathcal{P}(G)$-matched. By using this with $\dim V, \dim W \leq 17$, we can see that each irreducible component of V and W is of dimension 3, 4 or 5, and moreover that $V^{C_p} = V^{C_{2p}}$ and $W^{C_p} = W^{C_{2p}}$ for $p = 3$ and 5, and that $\dim V^{C_n} = \dim W^{C_n} > 0$ for $n = 1, 2, 3, 4, 5$. Thus, by Proposition 3.3 we get $V \cong W$.

4. Proofs of Theorems 1.1 and 2.1.

Let X and Y be connected closed oriented smooth manifolds with smooth G-action such that for each $g \in G$, g preserves the orientation of X if and only if g preserves the orientation of Y. Let x_0 and y_0 be points of X and Y, respectively, such that $G_{x_0} \subset G_{y_0}$ and there exists an orientation-reversing linear G_{x_0}-isomorphism $\varphi : T_{x_0}(X) \to T_{y_0}(Y)$. Clearly, an element $g \in G_{x_0}$ preserves the orientation of $T_{x_0}(X)$ if and only if it preserves the orientation of $T_{y_0}(Y)$. Consider the G-manifold $G \times_{G_{x_0}} Y$. Forgetting the G-actions, the connected component $Y' = \{[e, y] \in G \times_{G_{x_0}} Y \mid y \in Y\}$ of $G \times_{G_{x_0}} Y$ is canonically identified with Y and hence oriented, where e is the unit of G. We can choose an orientation of $G \times_{G_{x_0}} Y$ such that $g\varphi g^{-1} : T_{gx_0}(X) \to gT_{[e,y_0]}(\{e\} \times Y)$ is orientation-reversing for arbitrary $g \in G$. Thus we can obtain the G-connected sum $X \#_{G,(G_{x_0})} (G \times_{G_{x_0}} Y)$ at points gx_0 and $[g, y_0]$, $g \in G$. If we choose the other orientation of X then the resulting G-manifold is denoted by $-X$. The canonical identification map from X to $-X$ is orientation-reversing. Hence for arbitrary $x_0 \in X$, we obtain the G-connected sum

$$X(\#, x_0) = X \#_{G,(G_{x_0})} \big(G \times_{G_{x_0}} -X\big)$$

at points gx_0 and $[g, x_0]$, $g \in G$.

Let $\Sigma \in \mathfrak{S}_{\mathrm{h}}^{(1)}$, $\Sigma^G = \{a\}$ and $V = T_a(\Sigma)$. For a point $b \in \Sigma$ with $b \neq a$, the resulting space $\Sigma(\#, b)$ belongs to $\mathfrak{S}_{\mathrm{h}}^{(1)}$ and possesses a specific point $a' = [e, a]$. There is a canonical orientation-reversing linear G_b-isomorphism $T_a(\Sigma) \to T_{a'}(\Sigma(\#, b))$. We set

$$A_\Sigma = \bigcup_p \Sigma^{G\{p\}}$$

where p ranges over the set of all primes dividing $|G|$. By Lemma 3.1, $\Sigma^P \smallsetminus A_\Sigma \neq \emptyset$ for all $P \in \mathcal{P}(G)$. Let $M_\Sigma = M(A_\Sigma, \Sigma)$ be the G-regular (manifold) neighborhood of A_Σ in Σ such that $\Sigma^P \smallsetminus M_\Sigma \neq \emptyset$ for all $P \in \mathcal{P}(G)$. Let p be a prime and P a Sylow p-subgroup of G. Take a point x_p in $\Sigma^P \smallsetminus M_\Sigma$. Then the isotropy subgroup G_{x_p} satisfies

$$P \subset G_{x_p} \notin \mathcal{L}(G).$$

Thus $|G : G_{x_p}|$ divides $|G : P|$ and $|G : G_{x_p}| \neq 1$.

Proof of Theorem 1.1. There exist points $y_1, \ldots, y_m$ in Σ with the following properties.

(1) The conjugacy classes $(G_{y_1}), \ldots, (G_{y_m})$ of isotropy subgroups $G_{y_1}, \ldots, G_{y_m}$ are all distinct.
(2) $G_{y_i} \notin \mathcal{L}(G)$ for every $i = 1, \ldots, m$.
(3) G_{y_i} contains a Sylow subgroup of G for every $i = 1, \ldots, m$.
(4) There exist positive integers $k(1), \ldots, k(m)$ such that

$$\sum_{i=1}^{m} k(i)|G : G_{y_i}| \equiv -1 \mod |\Theta_n|,$$

where Θ_n is the group of homotopy spheres of dimension $n = \dim \Sigma$.

By iterated replacements of Σ by $\Sigma(\#, y_i)$, where i ranges from 1 to m, we may assume that there exist orientation-reversing linear G_{y_i}-isomorphisms $T_{y_i}(\Sigma) \to T_a(\Sigma)$. Then the resulting space Y of iterated G-connected sums of copies of Σ,

$$\begin{aligned} Y = \Sigma & \underbrace{\#_{G,(G_{y_1})}(G \times_{G_{y_1}} \Sigma)\#_{G,(G_{y_1})} \cdots \#_{G,(G_{y_1})}(G \times_{G_{y_1}} \Sigma)}_{k(1) \text{ fold}} \\ & \underbrace{\#_{G,(G_{y_2})}(G \times_{G_{y_2}} \Sigma)\#_{G,(G_{y_2})} \cdots \#_{G,(G_{y_2})}(G \times_{G_{y_2}} \Sigma)}_{k(2) \text{ fold}} \\ & \cdots\cdots\cdots \\ & \underbrace{\#_{G,(G_{y_m})}(G \times_{G_{y_m}} \Sigma)\#_{G,(G_{y_m})} \cdots \#_{G,(G_{y_m})}(G \times_{G_{y_m}} \Sigma)}_{k(m) \text{ fold}}, \end{aligned} \tag{4.1}$$

is a standard sphere with smooth G-action such that $Y^G = \{a\}$ and $T_a(Y) = V$ (cf. [**12**, Proposition 1.3 and Example 1.2]). $\square$

For a real G-module V, we define $V^{\mathcal{L}(G)}$ to be the smallest G-submodule of V containing $V^{G\{q\}}$ for all primes q. With respect to some G-invariant inner product on V, we have the orthogonal decomposition

$$V = V^{\mathcal{L}(G)} \oplus V_{\mathcal{L}(G)}.$$

Proof of Theorem 2.1. First fix a prime p and a Sylow p-subgroup P of G. Then take a point $x_p \in \Sigma^P$ as above. Let $D(x_p, \varepsilon_p)$ be a small closed disk P-neighborhood of x_p in Σ such that $D(x_p, \varepsilon_p) \cap M_\Sigma = \emptyset$. Consider the contractible P-manifold $Y_p = \Sigma \smallsetminus \mathrm{Int}(D(x_p, \varepsilon_p))$. Then $T(Y_p)$ is a (non-equivariantly) trivial real vector bundle over Y_p and $[T(Y_p)] = 0$ in $\widetilde{KO}_P(Y_p)_{(p)}$. Thus we get the following properties.

(1) $T(M_\Sigma)$ is a (non-equivariantly) trivial real vector bundle.
(2) $[T(M_\Sigma)] = 0$ in $\widetilde{KO}_Q(\mathrm{res}^G_Q M_\Sigma)_{(q)}$ for all primes q and $Q \in \mathcal{P}_q(G)$.

We obtain the G-space A_Ξ and the G-manifold M_Ξ similarly to A_Σ and M_Σ, respectively. We set $M = (M_\Sigma \amalg M_\Xi) \times D(U_0)$. Then the following properties are obtained.

(1) $T(M)$ is (non-equivariantly) a product bundle.
(2) $[T(M)] = 0$ in $\widetilde{KO}_Q(\mathrm{res}^G_Q M)_{(q)}$ for all primes q and $Q \in \mathcal{P}_q(G)$.
(3) $M^G = \{a, b\}$.
(4) $M^L = A^L_\Sigma \amalg A^L_\Xi$ for all $L \in \mathcal{L}(G)$.
(5) M satisfies the $\mathcal{P}(G)$-weak gap condition.

There may be words for (5) above. By the hypothesis on U_0 in Theorem 2.1, the G-space

$N = (\Sigma \amalg \Xi) \times U_0$ satisfies the $\mathcal{P}(G)$-weak gap condition. Note $\dim \Sigma = \dim \Xi$. By the definition of M, the dimension of an arbitrary connected component of M is equal to $\dim N$. For a subgroup H of G and a point x in M^H, the equality $\dim T_x(M)^H = \dim T_x(N)^H$ obviously holds. Thus M satisfies the $\mathcal{P}(G)$-weak gap condition as well as N.

Now regarding $x_0 = a$, $\xi_M = T(M)$, $\nu_M = \varepsilon_M(0)$ and $U = T_a(M)$, we use Lemmas 4.2 and 4.3 of [**20**]. There exists a disk $D \supset M$ with smooth G-action satisfying the following conditions.

(1) $D^G = \{a, b\}$.
(2) For $L \in \mathcal{L}(G)$, the connected components D^L_a and D^L_b of D^L containing a and b coincide with those of $M^L_\Sigma = A^L_\Sigma$ and $M^L_\Xi = A^L_\Xi$, respectively.
(3) $T_a(D) = V \oplus U_0 \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus k}$ and $T_b(D) = W \oplus U_0 \oplus \mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus k}$ for some $k \geq 3$.
(4) For any prime q and $Q \in \mathcal{P}_q(G)$, $\pi_1(D^Q)$ is a finite abelian group of order prime to q.
(5) For each $x \in D$, there exists $y \in M$ such that $G_y \supset G_x$ and $T_x(D) \cong \operatorname{res}^{G_y}_{G_x} T_y(M) \oplus \operatorname{res}^G_{G_x}(\mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus k})$. Hence D satisfies the $\mathcal{P}(G)$-weak gap condition.

Since the integer k appearing in (3) above is greater than or equal to 3, the following properties are obtained.

(6) For $Q \in \mathcal{P}(G)$, $\dim D^Q \geq 6$.
(7) For H with $Q \in \mathcal{P}(G)$ such that $Q \lhd H$ and H/Q is cyclic, $\dim D^{=H} \geq 3$.

If necessary, we replace D by $D \times D(\mathbb{R}[G]_{\mathcal{L}(G)}{}^{\oplus 2})$ so that

(8) If $\dim D^Q = 2 \dim D^H$ holds for $Q \in \mathcal{P}(G)$ and $H > Q$ then
 (a) $|H : Q| = 2$, $|HG^{\{2\}} : QG^{\{2\}}| = 2$,
 (b) $QG^{\{r\}} = G$ for all odd primes r, and
 (c) $\dim D^{>H} \leq \dim D^H - 2$.

By the proof of [**21**, Theorem 5.1], there exists a G-framed map $\boldsymbol{f} = (f, b_X)$, where $f : (X, \partial X) \to (D, \partial D)$ is a degree-one G-map and $b_X : T(X) \oplus \varepsilon_X(\mathbb{R}^u) \to f^*T(D) \oplus \varepsilon_X(\mathbb{R}^u)$ is a G-vector bundle isomorphism for some non-negative integer u, such that $X^G = \emptyset$, $\partial f = f|_{\partial X} : \partial X \to \partial D$ is the identity map, and $f : X \to D$ is a homotopy equivalence. Hence, X is a contractible smooth G-manifold with $\partial X = \partial D$. By virtue of the bundle datum b_X, it holds that $\dim X^H \leq \dim D^H$ for all subgroups H of G. Since D and X are contractible, D^P and X^P are non-empty and connected for all $P \in \mathcal{P}(G)$, and hence the equality $\dim X^H = \dim D^H$ holds for $H \in \mathcal{P}(G)$. As D satisfies the $\mathcal{P}(G)$-weak gap condition, X also satisfies the $\mathcal{P}(G)$-weak gap condition. The glued space $Y = D \bigcup_{\partial D} X$ along the boundary is a homotopy sphere and satisfies the $\mathcal{P}(G)$-weak gap condition. For each prime p and a Sylow p-subgroup P of G, since $\dim Y^P \geq 6$ and $\dim Y^P > \dim Y^H$ if $P < H \leq G$, there exists a point $y_p \in Y$ with $G_{y_p} = P$. By taking G-equivariant connected sum of copies of Y (similarly to (4.1)), we obtain a standard sphere S with smooth G-action such that $S^G = \{a, b\}$, $T_a(S) = V \oplus U_1$, $T_b(S) = W \oplus U_1$ for some $\mathcal{L}(G)$-free real G-module U_1, and S satisfies the $\mathcal{P}(G)$-weak gap condition. Hence V and W are stably Smith* equivalent. □

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