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On Thom polynomials of the singularities D_k and E_k

By Yoshifumi ANDO

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Introduction.

Let A_k, D_k and E_k denote the types of the singularities of function germs studied in [**4**]. Let N and P denote smooth manifolds. When a C^∞ stable map germ $f:(N, x)\rightarrow(P, y)$ is C^∞ equivalent to a versal unfolding of a function germ with singularity A_k, D_k or E_k, we say that f has a singularity of type A_k, D_k or E_k at x respectively (see, for example, their normal forms of [**2**, Section 1]). When every singularity of a smooth map f is of type A_k or D_k (resp. A_k, D_k or E_k) with any number k, we say that f is AD-regular (resp. ADE-regular) in this paper.

Let X_k be one of A_k, D_k or E_k. We define $S_{\bar{X}_k}(f)$ to be the topological closure of the subset $S_{X_k}(f)$ consisting of all singular points of type X_k of f. We can consider the fundamental class of $S_{\bar{X}_k}(f)$ in $H_*(N; Z/2Z)$ and define the Thom polynomial of X_k for f as its Poincaré dual class denoted by $P(X_k, f)$. As usual we expect that it is represented by Stiefel-Whitney classes $w_j(TN-f^*(TP))$ (cf. [**6**]).

The purpose of this paper is to give formula calculating $P(D_k, f)$ for AD-regular maps and $P(E_k, f)$ for ADE-regular maps in a finite process ([Theorems 4.1 and 4.2]). This kind of formulas first appeared in [**9**] and [**10**] to calculate Thom polynomials of the singularities of type Σ^i and $\Sigma^{i,j}$. Their results are reviewed in Section 1. In our case of $X_k=D_k$ or E_k, we have the submanifolds $\sum X_k$ constructed in the infinite jet space $J^\infty(N, P)$ in [**2**] such that if the jet extension $j^\infty f$ of f is transverse to $\sum X_k$, then we have $S_{X_k}(f) =(j^\infty f)^{-1}(\sum X_k)$. Using the properties of $\sum X_k$ in $J^\infty(N, P)$ reviewed in Section 2, we lift $S_{\bar{X}_k}(f)$ up to a submanifold S of the total space of a certain flag bundle over N in Sections 5 and 6 so that the Poincaré dual class of S is the Euler class of some vector bundle over this total space related to the normal bundle of $\sum X_k$. This means that $P(X_k, f)$ is the image of this Euler class by the Gysin homomorphism of this flag bundle. For singularities A_k, see the similar result of [**1**].

In Section 7 we see that Theorems 4.1 and 4.2 are generalized to the situations of smooth maps into foliated manifolds or of smooth sections of fibre bundles with naturality conditions.

In the category of complex manifolds and holomorphic maps the arguments of this paper go through word for word with the exception of approximating smooth maps by transversal maps.

§ 1. Thom-Boardman manifolds and higher intrinsic derivatives.

In this section we review the necessary results about the higher intrinsic derivatives d_t and the Thom-Boardman submanifolds in [**5**] (see also [**8** and **11**]) to explain the definition and properties of $\sum D_k$ and $\sum E_k$ studied in [**2**].

Let n and p be the dimensions of manifolds N and P respectively. Throughout the paper let $n>p$ and i always denotes the number $n-p+1$. The projections of $J^\infty(N, P)$ onto N and P mapping a jet onto its source and target are written by π_N and π_P respectively. The total tangent bundle over $J^\infty(N, P)$ introduced in [**5**, Definition 1.9] is denoted by $\boldsymbol{D}$ which is isomorphic to $(\pi_N)^*(TN)$. Let $\boldsymbol{P}$ denote $(\pi_P)^*(TP)$. Then we have the homomorphism

$$d_1 : \boldsymbol{D} \longrightarrow \boldsymbol{P} \quad \text{over } J^\infty(N, P).$$

The submanifold Σ^i is the subspace of $J^\infty(N, P)$ consisting of all jets z with $\dim(\mathrm{Ker}(d_{1,z}))=i$, where $d_{1,z} : D_z \to P_z$ is the restriction of d_1 to the fibres over z (throughout the paper we use this kind of notations for fibres and restricted homomorphisms).

The symmetric product of subbundles $V_1, \cdots, V_t$ of a vector bundle V in the t-th symmetric product S^tV is denoted by $V_1 \bigcirc \cdots \bigcirc V_t$ as in [**5**]. Let $\boldsymbol{K}=\mathrm{Ker}(d_1)$ and $\boldsymbol{Q}=\mathrm{Cok}(d_1)$ over Σ^i. Notice $\dim \boldsymbol{Q}=1$. Then the second intrinsic derivative

$$d_2 : \boldsymbol{K}_1 \longrightarrow \mathrm{Hom}(\boldsymbol{K}_1, \boldsymbol{Q}) \quad \text{over } \Sigma^i$$

defines $\Sigma^{i,2}$ as the subset of all $z \in \Sigma^i$ with $\dim(\mathrm{Ker}(d_{2,z}))=2$. Note that d_2 is induced from the symmetric homomorphism of $\bigcirc^2 \boldsymbol{K}_2$ into $\boldsymbol{Q}$ denoted by d_2'. Set $\boldsymbol{K}_2=\mathrm{Ker}(d_2)$ over $\Sigma^{i,2}$. The third intrinsic derivative

$$d_3 : \boldsymbol{K}_2 \longrightarrow \mathrm{Hom}(\bigcirc^2 \boldsymbol{K}_2, \boldsymbol{Q}) \quad \text{over } \Sigma^{i,2}$$

induced from the symmetric homomorphism $d_3' : \bigcirc^3 \boldsymbol{K}_2 \to \boldsymbol{Q}$ defines $\Sigma^{i,2,j}$ as the set of all jets $z \in \Sigma^{i,2}$ with $\dim(\mathrm{Ker}(d_{3,z}))=j$. Set $\boldsymbol{K}_3=\mathrm{Ker}(d_3 | \Sigma^{i,2,j})$. If $j=1$, then $\mathrm{Cok}(d_3)$ is isomorphic to $\mathrm{Hom}(\bigcirc^2 \boldsymbol{K}_3, \boldsymbol{Q})$ over $\Sigma^{i,2,1}$. The 4-th intrinsic derivative

$$d_4 : \boldsymbol{K}_3 \longrightarrow \mathrm{Hom}(\bigcirc^2 \boldsymbol{K}_3 \bigcirc \boldsymbol{K}_2, \boldsymbol{Q}) \quad \text{over } \Sigma^{i,2,1}$$

coming from the homomorphism $d_4' : \bigcirc^3 \boldsymbol{K}_3 \bigcirc \boldsymbol{K}_2 \to \boldsymbol{Q}$ defines $\Sigma^{i,2,1,1}$ as the set of all jets $z \in \Sigma^{i,2,1}$ such that $d_{4,z}$ vanishes. Over $\Sigma^{i,2,1,1}$, $\mathrm{Ker}(d_4)$ is $\boldsymbol{K}_3$ and $\mathrm{Cok}(d_4)$ is isomorphic to $\mathrm{Hom}(\bigcirc^3 \boldsymbol{K}_3 \bigcirc \boldsymbol{K}_2, \boldsymbol{Q})$. Finally we have the 5-th intrinsic derivative

$$d_5: \boldsymbol{K}_3 \longrightarrow \mathrm{Hom}(\bigcirc^3 \boldsymbol{K}_3 \bigcirc \boldsymbol{K}_2, \boldsymbol{Q}) \quad \text{over } \Sigma^{i,2,1,1}$$

coming from $d_5': \bigcirc^4 \boldsymbol{K}_3 \bigcirc \boldsymbol{K}_2 \to \boldsymbol{Q}$. We set $\Sigma^{i,2,1,0} = \Sigma^{i,2,1} \setminus \Sigma^{i,2,1,1}$ and $\Sigma^{i,2,1,1,0}$ as the set of all jets $z \in \Sigma^{i,2,1,1}$ such that $d_{5,z}$ is injective.

§ 2. Manifolds $\sum D_k$ and $\sum E_k$.

We will briefly review the definition of $\sum D_k$. As usual $\mathrm{Hom}(\bigcirc^3 \boldsymbol{R}^2, \boldsymbol{R})$ is identified with the set of all cubic forms with variables u and v on $\boldsymbol{R}^2$. By [**4**, Lemma 5.1] it is decomposed into five orbit manifolds of the action by $GL(2)$ through $u^2v \pm v^3$, u^2v, u^3 and 0. This decomposition yields that of $\mathrm{Hom}(\bigcirc^3 \boldsymbol{K}_2, \boldsymbol{Q})$ over $\Sigma^{i,2,0}$ into five submanifolds. Let $S_4^{\pm}$ and S_5 denote the corresponding submanifolds of $\mathrm{Hom}(\bigcirc^3 \boldsymbol{K}_2, \boldsymbol{Q})$ determined by $u^2v \pm v^3$ and u^2v respectively. By identifying d_3' in Section 1 with the smooth section of $\mathrm{Hom}(\bigcirc^3 \boldsymbol{K}_2, \boldsymbol{Q})$ over $\Sigma^{i,2,0}$, we define the submanifolds $\sum D_4^{\pm}$ and $\sum \bar{D}_5$ of $\Sigma^{i,2,0}$ to be $(d_3')^{-1}(S_4^{\pm})$ and $(d_3')^{-1}(S_5)$ respectively ([**2**, Definition 3.1]). On a certain neighbourhood U of $\sum \bar{D}_5$ in $\Sigma^{i,2,0}$, there exists the line subbundle $\boldsymbol{L}$ of $\boldsymbol{K}_2$ such that for any $z \in \sum \bar{D}_5$, L_z coincides with $d_{3,z}^{-1}(H)$ where H is the set of all quadratic forms of rank 1 or 0 in $\mathrm{Hom}(\bigcirc^2 K_z, Q_z)$ and that for any $z \in U$, z lies in $\sum \bar{D}_5$ if and only if the restriction $d_{3,z}' | \bigcirc^3 L_z$ is a null homomorphism. Starting from $d_3' | \bigcirc^3 \boldsymbol{L}$ over U and $\sum \bar{D}_5$, we can successively construct the submanifolds $\sum \bar{D}_{t+1}$ and the homomorphism $r_t: \bigcirc^t \boldsymbol{L} \to \boldsymbol{Q}$ over $\sum \bar{D}_{t+1}$. In fact, by [**2**, Theorem 3.10] there exists a series of manifolds $U \supset \sum \bar{D}_5 \supset \cdots \supset \sum \bar{D}_{t+1} \supset \cdots$ with the properties

(2.1) $\sum \bar{D}_{t+1}$ is of codimension $n-p+t+1$ in $J^\infty(N, P)$,

(2.2) For $t \geqq 3$, there exists a homomorphism $r_t: \bigcirc^t \boldsymbol{L} \to \boldsymbol{Q}$ defined over $\sum \bar{D}_{t+1}$ where r_3 means $d_3' | \bigcirc^3 \boldsymbol{L}$ defined on U,

(2.3) An element z of $\sum \bar{D}_{t+1}$ belongs to $\sum \bar{D}_{t+2}$ if and only if $r_{t,z}$ vanishes,

(2.4) The intrinsic derivative of r_t

$$d(r_t): T(\sum \bar{D}_{t+1}) | \sum \bar{D}_{t+2} \longrightarrow \mathrm{Hom}(\bigcirc^t \boldsymbol{L}, \boldsymbol{Q}) | \sum \bar{D}_{t+2}$$

is surjective, that is, r_t is transverse to the zero section when considered as the section of $\mathrm{Hom}(\bigcirc^t \boldsymbol{L}, \boldsymbol{Q}) | \sum \bar{D}_{t+1}$ and

(2.5) Let $\sum D_t = \sum \bar{D}_t \setminus \sum \bar{D}_{t+1}$. If a jet extension $j^\infty f$ of a smooth map germ $f: (N, x) \to (P, y)$ is transverse to $\sum D_t$ and $j^\infty f(x) \in \sum D_t$, then f has a singularity D_t at x.

Next we review the definition of $\sum E_t$. We define $\sum E_6$ as the set of all jets $z \in \Sigma^{i,2,1,0}$ such that $d_{4,z}' | \bigcirc^4 K_{3,z}$ does not vanish and set $\sum E_7 = \Sigma^{i,2,1,0} \setminus \sum E_6$. We can show that the restriction $d_4' | \bigcirc^4 \boldsymbol{K}_3$ is transverse to the zero section as the section of the bundle $\mathrm{Hom}(\bigcirc^4 \boldsymbol{K}_3, \boldsymbol{Q})$ over $\Sigma^{i,2,1,0}$. Hence $\sum E_7$ is a submanifold. We define $\sum E_8$ as the set of all jets $z \in \Sigma^{i,2,1,1,0}$ such that $d'_{5,z}|\bigcirc^5 K_{4,z}$ does not vanish. When we deal with only ADE-regular maps, it will be reasonable to set $\sum \bar{E}_6 = \Sigma^{i,2,1}$, $\sum \bar{E}_7 = \sum E_7 \cup \Sigma^{i,2,1,1}$ and $\sum \bar{E}_8 = \Sigma^{i,2,1,1}$. It follows that the analogous statement of (2.5) also holds for singularities E_l.

§ 3. Grassmann bundles and flag bundles.

For an n-dimensional vector space (simply n-space) V, let $G_{k,n-k}(V)$ be the grassmann manifold of all k-subspaces of V. For a vector bundle E over a space M of dimension n, let $G_{k,n-k}(E)$ over M be the grassmann bundle associated to $G_{k,n-k}(\boldsymbol{R}^n)$ whose total space consists of all pairs (m, a) where $m \in M$ and a is a k-subspace of the fibre E_m of E over m. Let $F_{i,2}(E)$ over M denote the space consisting of all triples (m, a, b) where a is an i-subspace of E_m and b is a 2-subspace of a. Similarly $F_{i,2,1}(E)$ denotes the space consisting of all quadruples (m, a, b, c) where $(m, a, b) \in F_{i,2}(E)$ and c is 1 subspace of b. Let $E_1 \to G_{i,p-1}(E)$, $E_2 \to F_{i,2}(E)$ and $E_3 \to F_{i,2,1}(E)$ denote the canonical bundles of dimensions i, 2 and 1 having fibres a, b and c respectively.

In the following commutative diagram we are applying these notations to the total tangent bundle $\boldsymbol{D}$ and $\boldsymbol{P}$ over $J^\infty(N, P)$.

$$\begin{tikzcd}
G_3(\boldsymbol{D}, \boldsymbol{P}) \arrow[r, "\phi_3"] \arrow[d, "\rho_3"] & G_2(\boldsymbol{D}, \boldsymbol{P}) \arrow[r, "\phi_2"] \arrow[d, "\rho_2"] & G_1(\boldsymbol{D}, \boldsymbol{P}) \arrow[r, "\phi_1"] \arrow[d, "\rho_1"] & G_{1,p-1}(\boldsymbol{P}) \arrow[d, "\pi"] \\
F_{i,2,1}(\boldsymbol{D}) \arrow[r] \arrow[rrr, bend right=30, "\pi_3"'] & F_{i,2}(\boldsymbol{D}) \arrow[r] \arrow[rr, bend right=20, "\pi_2"'] & G_{i,p-1}(\boldsymbol{D}) \arrow[r, "\pi_1"] & J^\infty(N, P)
\end{tikzcd} \tag{3.1}$$

where (i) π and π_i $(i=1, 2, 3)$ are the canonical projections, (ii) $G_i(\boldsymbol{D}, \boldsymbol{P})$ $(i=1, 2, 3)$ denote the fibre products of π_i and π and (iii) ρ_i and ϕ_i are also the canonical projections respectively. In the following sections we use the notations $\boldsymbol{D}' = (\pi_3 \circ \rho_3)^* \boldsymbol{D}$, $\boldsymbol{P}' = (\pi_3 \circ \rho_3)^* \boldsymbol{P}$, $\boldsymbol{D}'_1 = (\rho_1 \circ \phi_2 \circ \phi_3)^* \boldsymbol{D}_1$, $\boldsymbol{D}'_2 = (\rho_2 \circ \phi_3)^* \boldsymbol{D}_2$, $\boldsymbol{D}'_3 = (\rho_3)^* \boldsymbol{D}_3$ and $\boldsymbol{P}'_1 = (\phi_1 \circ \phi_2 \circ \phi_3)^* \boldsymbol{P}_1$.

§ 4. Results.

We can pull back the diagram (3.1) by the jet-extension $j^\infty f : N \to J^\infty(N, P)$ of an AD or ADE-regular map and obtain the similar one replacing $\boldsymbol{D}$, $\boldsymbol{P}$ and $J^\infty(N, P)$ by TN, $f^*(TP)$ and N. All of the projections in this new diagram are denoted by the same notation such as ϕ_i, ρ_i and π_i. Let $c : G_3(TN, f^*(TP)) \to G_3(\boldsymbol{D}, \boldsymbol{P})$ denote the associated map over $j^\infty f$ determined by the fact $TN = (j^\infty f)^* \boldsymbol{D}$ and $f^*(TP) = (j^\infty f)^* \boldsymbol{P}$. Then let $K_0 = c^*(\boldsymbol{D}')$, $K_j = c^*(\boldsymbol{D}'_j)$ and $Q_1 = c^*(\boldsymbol{P}'_1)$.

Now we can state the formulas to calculate the Thom polynomials $P(D_{k+1}, f)$ and $P(E_{k+1}, f)$.

THEOREM 4.1. *If $f: N \to P$ is an AD-regular map, then we have the following formulas ($k \geqq 4$).*

$$P(D_{k+1}, f) = (\pi_3 \circ \rho_3)! \Big\{ \chi\Big(\mathrm{Hom}(K_1, f^*(TP)) \\ \oplus \mathrm{Hom}\Big(K_0/K_1 \oplus K_2 \bigcirc K_1 \oplus \bigcirc^2 K_3 \bigcirc K_2 \oplus \sum_{j=4}^{k-1} \bigcirc^j K_3, Q_1\Big)\Big)\Big\}.$$

THEOREM 4.2. *If $f: N \to P$ is an ADE-regular map, then we have the following formulas.*

$$P(E_6, f) = (\pi_3 \circ \rho_3)! \{\chi(\mathrm{Hom}(K_1, f^*(TP)) \\ \oplus \mathrm{Hom}(K_0/K_1 \oplus K_2 \bigcirc K_1 \oplus K_3 \bigcirc^2 K_2, Q_1))\}$$

$$P(E_7, f) = (\pi_3 \circ \rho_3)! \{\chi(\mathrm{Hom}(K_1, f^*(TP)) \\ \oplus \mathrm{Hom}(K_0/K_1 \oplus K_2 \bigcirc K_1 \oplus K_3 \bigcirc^2 K_2 \oplus \bigcirc^4 K_3, Q_1))\}$$

$$P(E_8, f) = (\pi_3 \circ \rho_3)! \{\chi(\mathrm{Hom}(K_1, f^*(TP)) \\ \oplus \mathrm{Hom}(K_0/K_1 \oplus K_2 \bigcirc K_1 \oplus K_3 \bigcirc^2 K_2 \oplus \bigcirc^3 K_3 \bigcirc K_2, Q_1))\}.$$

REMARK 4.3. Let ν be a bundle of dimension greater than p such that $TP \oplus \nu$ is trivial. By the analogous arguments in [**1**, Section 4] we can reduce the calculation of $P(X_k, f)$ using the above formulas to that in the simpler case where $f^*(TP)$, TN and K_j are replaced by $f^*(TP \oplus \nu)$, $TN \oplus f^*(\nu)$ and the corresponding bundles K_j. However it is not necessarily easy to represent them by Stiefel-Whitney classes.

Here we give their precise formulas in the simple real case of $n = p+1$. See further calculations in complex case in Section 8. Let $W = 1 + W_1 + \cdots + W_j + \cdots$ be the Stiefel-Whitney class of $TN - f^*(TP)$ and $1 + \overline{W}_1 + \cdots + \overline{W}_j + \cdots$ be its formal inverse. For AD-regular maps $P(D_{k+1}, f)$ is equal to the part of degree $k+2$ of the polynomial

$$W(\overline{W}_1 + \overline{W}_2) \Big\{ \sum_{j=0}^{[k/2-2]} \binom{[k/2-2]}{j} \overline{W}_j \Big\} + W \Big\{ \sum_{j=0}^{[k/2-1]} \binom{[k/2-1]}{j} \overline{W}_{j+1} \Big\}$$

where [ ] means the Gauss bracket. In particular, $P(D_k, f) = 0$ for $k = 5$, 6 or 7. For ADE-regular maps, $P(E_{k+1}, f) = W_k W_2 + W_{k-1}(W_3 + W_1 W_2)$.

REMARK 4.4. The referee kindly informed the author the following. It follows from [**12**] that the Thom polynomial of D_k of ADE-regular maps vanish for $k=5$ and 7.

§ 5. Lift of the manifolds $\sum D_k$ and $\sum E_k$.

First we lift the submanifolds Σ^i and $\Sigma^{i,2}$ of $J^\infty(N, P)$ up to the diffeomorphic ones

$$(\Sigma^i)' = \{(z, K_{1,z}) \mid z \in \Sigma^i\}$$

$$(\Sigma^{i,2})' = \{(z, K_{1,2}, K_{2,z}, Q_z) \mid z \in \Sigma^{i,2}\}$$

of $G_{i,p-1}(\boldsymbol{D})$ and $G_2(\boldsymbol{D}, \boldsymbol{P})$. Note that $\boldsymbol{D}_1' \mid (\Sigma^i)' = (\pi_1 \mid (\Sigma^i)')^* \boldsymbol{K}_1$ and $\boldsymbol{D}_2' \mid (\Sigma^{i,2})' = (\pi_2 \circ \rho_2 \mid (\Sigma^{i,2})')^* \boldsymbol{K}_2$. We define

$$s_1 : G_{i,p-1}(\boldsymbol{D}) \longrightarrow \mathrm{Hom}(\boldsymbol{D}_1, \pi_1^*(\boldsymbol{P}))$$

and

$$s_2 : (\rho_1 \circ \phi_2)^{-1}((\Sigma^i)') \longrightarrow \mathrm{Hom}((\pi_2 \circ \rho_2)^* \boldsymbol{D} / (\rho_1 \circ \phi_2)^* \boldsymbol{D}_1 \oplus (\rho_2)^* \boldsymbol{D}_2 \bigcirc (\rho_1 \circ \phi_2)^* \boldsymbol{D}_1, (\phi_1 \circ \phi_2)^* \boldsymbol{P}_1)$$

to be the smooth sections of the given bundles as follows. For an element $z' = (z, a)$ of $G_{i,p-1}(\boldsymbol{D})$, set $s_1(z') = d_{1,z} \mid a$. For an element $z' = (z, K_{1,z}, b, Q_z)$ of $(\rho_1 \circ \phi_2)^{-1}((\Sigma^i)')$, define $d_{1,z}' = D_z / K_{1,z} \to Q_z$ to be the homomorphism induced from $d_{1,z}$ by $K_{1,z} = \mathrm{Ker}(d_{1,z})$. Then set $s_2(z') = d_{1,z}' \oplus d_{2,z}' \mid (b \bigcirc K_{1,z})$. The following proposition states the results of [**9** and **10**, Proposition 2.1] while Ronga's result is written in the other form due to [**1**, Lemma 3.1].

PROPOSITION 5.1. *The sections s_1 and s_2 are transverse to the zero sections and their inverse images of the zero-sections are equal to $(\Sigma^i)'$ and $(\Sigma^{i,2})'$ as sets respectively.*

We now deal with the lift of $\sum D_k$. Let $(\Sigma^{i,2,0})'$ and $U(S_5)'$ denote the subsets of $(\Sigma^{i,2})'$ such that z belongs to $\Sigma^{i,2,0}$ and $U(S_5)$ respectively.

Let $z' = (z, K_{1,z}, K_{2,z}, c, Q_z)$ be an element of $(\phi_3)^{-1}((\Sigma^{i,2,0})')$. Define the smooth section

$$s_3 : (\phi_3)^{-1}((\Sigma^{i,2,0})') \longrightarrow \mathrm{Hom}(\bigcirc^2 \boldsymbol{D}_3' \bigcirc \boldsymbol{D}_2', \boldsymbol{P}_1')$$

by $s_3(z') = d_{3,z}' \mid \bigcirc^2 c \bigcirc K_{2,z}$. Let $(\sum \bar{D}_{k+1})'$ denote the subset of $(\phi_3)^{-1}(U(S_5)')$ consisting of all elements z' with $c = L_z$ and $z \in \sum \bar{D}_{k+1}$ $(k \geqq 4)$. Note that $\boldsymbol{D}_3'$ coincides with $(\pi_3 \circ \rho_3)^* \boldsymbol{L}$ over $(\sum \bar{D}_5)'$. Then the smooth sections

$$r_k' : (\sum \bar{D}_{k+1})' \longrightarrow \mathrm{Hom}(\bigcirc^k \boldsymbol{D}_3', \boldsymbol{P}_1') \mid (\sum \bar{D}_{k+1})'$$

is defined by $r_k'(z')$ being the homomorphism $r_{k,z} : \bigcirc^k L_z \to Q_z$.

We can prove the analogous result as in Proposition 5.1.

PROPOSITION 5.2. *The sections s_3 and r_k' $(k \geqq 4)$ are transverse to the zero sections and their inverse images of the zero sections coincide with $(\sum \bar{D}_5)'$ and $(\sum \bar{D}_{k+2})'$ respectively.*

PROOF. First we prove the latter statement for s_3. Let z' be any element of $(\phi_3)^{-1}((\Sigma^{i,2,0})')$ such that $d'_{3,z}$ vanishes on $\bigcirc^2 c \bigcirc K_{2,z}$. Take a metric of $K_{2,z}$ and let e be a unit vector of c and f be its orthogonal unit vector. Let u and v be the dual vectors of f and e respectively. That is, $u(f)=1$, $u(e)=0$, $v(f)=0$ and $v(e)=1$. With this notation we can write as $d'_{3,z}=a_1u^3+a_2u^2v+a_3uv^2+a_4v^3$. Then we obtain $d'_{3,z}(e\bigcirc e\bigcirc e)=6a_4$ and $d'_{3,z}(e\bigcirc e\bigcirc f)=2a_3$ by easy calculations. Since it vanishes on $\bigcirc^2 c\bigcirc K_{2,z}$, we have $a_3=a_4=0$. That is, $d'_{3,z}=a_1u^3+a_2u^2v=u^2(a_1u+a_2v)$. It is easy to see that $c=L_z$ which is the space annihilated by u. This means $z\in S_5$. On the other hand if z' belongs to $(\sum\bar{D}_5)'$, then $z\in S_5$ and $c=K_{3,z}$. So $d'_{3,z}$ vanishes on $\bigcirc^2 c\bigcirc K_{2,z}$.

Next we show the transversality of s_3. Let z_0 be any element of $(\sum\bar{D}_5)_{x,y}$ such that d'_{3,z_0} is written as u^2v under suitable coordinate systems near x and y (see [**4**, Propositions 3.5 and 3.10]). As above let e and f be the dual basis of u and v in K_{2,z_0} for the case $c=L_{z_0}$. Then for any element z' of $(\phi_3)^{-1}(U(S_5)')$ near $(\sum\bar{D}_5)'$, c is generated by $te+f$ and $d'_{3,z}=u^2v+\varepsilon v^3$ for some real numbers t and ε by Section 2. It follows that the normal coordinates of $(\sum\bar{D}_5)'$ in $(\phi_3)^{-1}(U(S_5)')$ is given by t and ε near z_0. On the other hand the normal coordinates of the zero-section of $\mathrm{Hom}(\bigcirc^2 c\bigcirc D'_{2,z}, P'_{1,z})$ are given by two numbers $d'_{3,z}((te+f)\bigcirc(te+f)\bigcirc f)$ and $d'_{3,z}((te+f)\bigcirc(te+f)\bigcirc e)$. By easy calculations we have that they are equal to $4t$ and $2t^2+6\varepsilon$ respectively. Since the mapping of (t, ε) to $(4t, 2t^2+6\varepsilon)$ is regular at $t=\varepsilon=0$, s_3 is transverse to the zero-section.

The proposition for r'_k is almost an immediate consequence of the definition of $(\sum D_{k+1})'$ using r_k in Section 2 since $\boldsymbol{D}'_3|(\sum D_{k+1})'=(\pi_3\circ\rho_3|(\sum D_{k+1})')^*\boldsymbol{L}$ $(k\geqq 4)$. Q.E.D.

We consider the lift of $\sum\bar{E}_k$. Let $z'=(z, K_{1,z}, K_{2,z}, c, Q_z)$ be any element of $(\phi_3)^{-1}((\Sigma^{i,2})')$. We define $(\sum\bar{E}_6)'$ as the set $(\Sigma^{i,2,1})'$ consisting of all elements z' with $c=K_{3,z}$ and $z\in\Sigma^{i,2}$ in $(\phi_3)^{-1}((\Sigma^{i,2})')$. We have the section

$$s_6:(\phi_3)^{-1}((\Sigma^{i,2})')\longrightarrow \mathrm{Hom}(\boldsymbol{D}'_3\bigcirc^2\boldsymbol{D}'_2, \boldsymbol{P}'_1)$$

defined by $s'_6(z')=d'_{3,z}|c\bigcirc^2 K_{2,z}$.

The set $(\sum\bar{E}_7)'$ in $(\Sigma^{i,2,1})'$ is defined as the set consisting of all elements z' with $z\in\Sigma^{i,2,1}$ such that $d'_{4,z}|\bigcirc^4 K_{3,z}$ vanishes. The set $(\sum\bar{E}_8)'$ is $(\Sigma^{i,2,1,1})'$ consisting of all elements z' with $c=K_{3,z}$ and $z\in\Sigma^{i,2,1,1}$. We have the sections

$$s'_7:(\Sigma^{i,2,1})'\longrightarrow \mathrm{Hom}(\bigcirc^4\boldsymbol{D}'_3, \boldsymbol{P}'_1)$$

$$s'_8:(\Sigma^{i,2,1})'\longrightarrow \mathrm{Hom}(\bigcirc^3\boldsymbol{D}'_3\bigcirc\boldsymbol{D}'_2, \boldsymbol{P}'_1)$$

defined by $s'_7(z')=d'_{4,z}|\bigcirc^4 K_{3,z}$ and $s'_8(z')=d'_{4,z}|\bigcirc^3 K_{3,z}\bigcirc K_{2,z}$.

Then we have the following proposition for s'_k.

PROPOSITION 5.3. *The section s_k' is transverse to the zero-section and its inverse image of the zero-section is $\sum\bar{E}_k$ ($k=6$, 7 or 8).*

PROOF. The latter half is almost an immediate consequence of the definition of s_t. The transversality of s_7' and s_8' also follows from that of s_7 and s_8 reviewed in Section 2. So we prove that of s_6. Let z_0 be any jet of $\Sigma^{i,2,1}$ such that $d_{3,z}'$ is written as u^3 under suitable coordinate systems near x and y. Let e and f be the dual basis of u and v in K_{2,z_0} such that $K_{3,z_0}=\mathrm{Ker}(u)=\{f\}$. Then for any element z' of $(\phi_3)^{-1}((\Sigma^{i,2})')$ near z_0, c is generated by $te+f$ and that $d_{3,z}'=u^3+auv^2+bv^3$ for some real numbers t, a and b. Then the normal coordinates of $(\Sigma^{i,2,1})'$ in $(\phi_3)^{-1}((\Sigma^{i,2})')$ is given by t, a and b near z_0. On the other hand the normal coordinates of the zero-section of $\mathrm{Hom}(c\bigcirc^2 D_{2,z}', P_{1,z}')$ over z are given by the three numbers $d_{3,z}'((te+f)\bigcirc e\bigcirc e)$, $d_{3,z}'((te+f)\bigcirc e\bigcirc f)$ and $d_{3,z}'((te+f)\bigcirc f\bigcirc f)$. By easy calculations we know that they are equal to $6t$, $2a$ and $2at+6b$ respectively. Since the mapping of (t, a, b) to $(6t, 2a, 2at+6b)$ is regular at the origin, we obtain the transversality of s_3 at z_0. Q.E.D.

§ 6. Proof of Theorems.

In this section we prove Theorems 4.1 and 4.2 using the results in the previous section together with the following well known facts about algebraic topology.

(6.1) Let s be a smooth section of a vector bundle E over M transverse to the zero-section. Then the Poincaré dual class of its inverse image of the zero-section is congruent modulo 2 to the Euler class $\chi(E)$.

(6.2) Let M_1 and M_2 be locally closed submanifolds of M with $M_1\supset M_2$. Let m_1 be the Poincaré dual of $[M_1]$ in M and m_2 be that of $[M_2]$ in M_1 where brackets mean fundamental classes. If there exists a class m_2' of $H^*(M; \boldsymbol{Z}/2\boldsymbol{Z})$ such that $i^*(m_2')=m_2$ where i is an inclusion of M_1 into M, then the Poincaré dual class of M_2 in M is equal to m_1m_2'.

PROOF OF THEOREM 4.1. We use the notations in Section 4. Let S be the submanifold $c^{-1}((\sum\bar{D}_{k+1})')$ of $G_3(TN, f^*(TP))$. Since f is AD-regular and $j^\infty f$ is transverse to $\sum\bar{D}_{k+1}$, S is mapped diffeomorphically onto $S_{\bar{D}_{k+1}}(f)$ by $\pi_3\circ\rho_3$. Hence by definition of the Gysin homomorphism, $(\pi_3\circ\rho_3)!$ maps the Poincaré dual class $[S]^c$ in $G_3(TN, f^*(TP))$ onto $[S_{\bar{D}_{k+1}}(f)]^c$. Therefore we need to show that $[S]^c$ is equal to the Euler class of the given vector bundle in the formula of Theorem 4.1. If necessary, we slightly deform f by homotopy and obtain a series of submanifolds of $G_3(TN, f^*(TP))$; $c^{-1}((\rho_1\circ\phi_2\circ\phi_3)^{-1}((\Sigma^i)'))\supset c^{-1}(\phi_3^{-1}((\Sigma^{i,2})'))\supset c^{-1}((\sum\bar{D}_5)')\supset\cdots\supset c^{-1}((\sum\bar{D}_{k+1})')$. It follows from the definition of c that every submanifold coincides with $c^*(s_t)$'s or $c^*(r_k')$'s inverse image of the zero-section of some vector bundle induced from one appeared in Propositions 5.1 and 5.2 by c. These bundles are extended to ones over $G_3(TN, f^*(TP))$.

First we prove Theorem 4.1 for $P(D_5, f)$. The manifold $c^{-1}((\sum\bar{D}_5)')$ is $c^*(s_3)$'s inverse image of the zero-section of $\mathrm{Hom}(\bigcirc^2 K_3 \bigcirc K_2, Q_1)$ by Proposition 5.2. Therefore $[c^{-1}((\sum\bar{D}_5)')]^c$ is equal to $[c^{-1}\circ\phi_3^{-1}((\Sigma^{i,2})')]^c \chi(\mathrm{Hom}(\bigcirc^2 K_3 \bigcirc K_2, Q_1))$ by (6.1) and (6.2). By [**1**, Proposition 3.1] or the similar arguments above using Proposition 5.1 and the naturality of Gysin homomorphisms we know that $[c^{-1}\circ\phi_3^{-1}((\Sigma^{i,2})')]^c$ is equal to the Euler class of $\mathrm{Hom}(K_1, f^*(TP)) \oplus \mathrm{Hom}(K_0/K_1 \oplus K_2 \bigcirc K_1, Q_1)$. This shows the formula of $P(D_5, f)$.

By combining the arguments above and Proposition 5.2 we can prove the general case. Q. E. D.

We can also prove Theorem 4.2 by applying the analogous discussion using Propositions 5.1, 5.2, (6.1) and (6.2) to the case of E_t. So the details are left to the readers.

§ 7. Foliated manifolds and bundles with naturality.

In this section we explain that the results about Thom polynomials for smooth maps in the previous sections also hold in more general settings of smooth maps into foliated manifolds (cf. [**3**]) or sections of smooth bundles with naturality (cf. [**7**]) where $J^\infty(N, P)$ and $\boldsymbol{P}$ in the diagram (3.1) and also $f^*(TP)$ and Q_1 in Theorems 4.1 and 4.2 should be replaced by appropriate other jet spaces and bundles respectively. Their proofs are very like that of the case of smooth maps and so are left to the readers.

Let $\mathcal{F}$ be a nonsingular foliation of codimension p on a smooth manifold E. For $\mathcal{F}$ we take a local coordinate system $\{U_\lambda, \psi_\lambda\}$ of E with submersion $\psi_\lambda : U \to \boldsymbol{R}^p$ having the well known required properties of foliations. For a smooth map $f : N \to E$ and $\mathcal{F}$, a point x of N is called a singular point of type A_k, D_k or E_k with respect to $\mathcal{F}$ when x is that of a smooth map $\psi_\lambda\circ(f|U_\lambda)$ for some λ respectively. We also define an AD (resp. ADE)-regular smooth map $f : N \to E$ with respect to $\mathcal{F}$ similarly. Let $S_{X_k}(f, \mathcal{F})$ denote the set of all singular points of type X_k with respect to $\mathcal{F}$ of f and $S_{\bar{X}_k}(f, \mathcal{F})$ denote its topological closure. Our purpose is to see that $[S_{\bar{X}_k}(f, \mathcal{F})]^c$ is calculated by the similar formulas in Theorems 4.1 and 4.2.

Let $\psi'_\lambda : J^\infty(N, U_\lambda) \to J^\infty(N, \boldsymbol{R}^p)$ be the induced submersion of ψ_λ mapping a jet $j_x^\infty f$ onto $j_x^\infty(\psi_\lambda\circ f)$ and identify $J^\infty(N, U)$ canonically with a subspace of $J^\infty(N, E)$ by the inclusion of U into E. Then we can define the submanifold $\sum X_k(\mathcal{F})$ in $J^\infty(N, E)$ as the union of all submanifolds $(\psi'_\lambda)^{-1}(\sum X_k(N, \boldsymbol{R}^p))$ for all λ. Since $\sum X_k$ is defined by using the kernel ranks of the higher intrinsic derivatives and related homomorphisms such as r_k, it follows that $\sum X_k(\mathcal{F})$ does not depend on a choice of $\{U_\lambda, \psi_\lambda\}$. It will be easy to see that $S_{\bar{X}_k}(f, \mathcal{F})=(j^\infty f)^{-1}(\sum\bar{X}_k(\mathcal{F}))$. As in Section 4 we write its Poincaré dual class as $P(X_k, f; \mathcal{F})$. In this situation we must replace $J^\infty(N, P)$ and $\boldsymbol{P}$ by $J^\infty(N, E)$ and the induced bundle from the normal bundle $n(\mathcal{F})$ of $\mathcal{F}$ by the projection of $J^\infty(N, E)$ onto E in (3.1) respectively. Then $P(D_k, f; \mathcal{F})$ and $P(E_k, f; \mathcal{F})$ are calculated by the same formula of Theorems 4.1 and 4.2 respectively, while $f^*(TP)$ must be changed by $f^*(n(\mathcal{F}))$ together with its associated bundles K_i and Q_1. For A-regular maps, $P(A_k, f; \mathcal{F})$ is also dealt with similarly (cf. [**1**, Theorem 3.2]).

For example consider an immersion f of N into E with $\dim N=n$, $\dim E=n+1$ and $\operatorname{codim}\mathcal{F}=n-1$ for $n=7$ or 8. Since $\operatorname{codim}\Sigma^{2,2,2}(n, n-1)=9$, f becomes an ADE-regular map with respect to $\mathcal{F}$. It follows from Remark 4.3 that $P(E_{k+1}, f; \mathcal{F})=W_kW_2+W_{k-1}(W_3+W_1W_2)$ for $k=5$ or 6 and $P(E_8, f; \mathcal{F})=0$ where $W_j=W_j(TN-f^*(n(\mathcal{F})))$.

Let $\pi: E\to N$ be a smooth fibre bundle having a fibre P with naturality condition (see [**7**]). Let $\{U_\lambda\}$ be its covering of N with trivialization $\psi_\lambda: E|U_\lambda\to P$. For a section s of E, we define its A_k, D_k or E_k singular point by considering that of $\psi_\lambda\circ(s|U_\lambda)$ and AD (resp. ADE)-regular sections similarly as above. Let $J^\infty E$ be its infinite jet space consisting of all jets of local sections of E. Then we have the identification $\psi'_\lambda: J^\infty(E|U_\lambda)\to J^\infty(U_\lambda, P)$. Let $\sum X_k(E)$ denote the union of all spaces $(\psi'_\lambda)^{-1}(\sum X_k(U_\lambda, P))$ for all λ in $J^\infty(E)$. Again $\sum X_k(E)$ is well defined. Thus we can define the Thom polynomial $P(X_k, s; \pi)$ similarly. If we replace $J^\infty(N, P)$ by $J^\infty(E)$ and $\boldsymbol{P}$ by the induced bundle of the tangent bundle along the fibre $T(P_E)$ of E by the projection of $J^\infty(E)$ onto E in (3.1), then we can calculate $P(X_k, s; \pi)$ by the same formulas of Theorems 4.1 and 4.2 for D_k and E_k and of [**1**, Theorem 3.2] for A_k, while $f^*(TP)$ must be replaced by $f^*(T(P_E))$ together with its associated bundles K_i and Q_1.

The homotopy principle for AD or ADE-regular maps is valid (see [**3**]) and therefore their existence problem is reduced to a homotopy theoretic problem. The primary obstructions of this problem modulo two become the Thom polynomials studied in this paper.

§ 8. Calculation.

We sketch a method to calculate the polynomial $P(D_{k+1}, f)$ for the case $n=p+1$ stated in Section 4 (the case of $P(E_{k+1}, f)$ is similar and omitted). The calculation of the Thom polynomials in [**1**, Section 4] will be helpful to understand its details. By the analogous argument to that in [**1**] we may reduce its calculation to the situation of $P(D_{k+1}, f')$ for $f': N'\to P'$ where TN' is stably equivalent to $TN-f^*(TP)$, TP' is trivial and $\dim N'-\dim P'=n-p$.

For simplicity we may set $\dim N'=n$, $\dim P'=p$ and use the same notation for bundles which are induced from one bundle over any space in the pull-backed diagram of (3.1) by c in the following.

In the right hand term of the formulas of Theorems 4.1 and 4.2, let V denote the vector bundle whose Euler class is considered and V' denote the vector bundle so that V is written as $\mathrm{Hom}(K_1, f^*(TP'))\oplus\mathrm{Hom}(V', Q_1)$ over $G_3(TN', f^*(TP'))$. For D_{k+1}, as an example, V' is $K_0/K_1\oplus K_2\bigcirc K_1\oplus K_3\bigcirc K_3\bigcirc K_2\oplus\sum_{j=4}^{k-1}\bigcirc^j K_3$. Let $C(V')$ be written as $\prod_{i=1}^{n+k-1}(1-u_i)$ and $C(Q_1)=1+y$. Then we have

$$C(V)=C(K_1^*)^p\prod_{i=1}^{n+k-1}(1+u_i+y).$$

Note that $\chi(V)=C_{2p+n+k-1}(V)$ and that its coefficient of y^{n-1} turns out to be $(-1)^{k+1}C_2(K_1^*)^pC_{k+1}(V')$. Since TP is trivial, we have $(\rho_3)_!(y^{n-2})=1$ and $(\rho_3)_!(y^j)=0$ when $j\neq n-2$ (see, for example, [**1**, Proposition 4.1(b)]). Therefore

$$(\rho_3)_!(\chi(V))=(-1)^{k+1}C_2(K_1^*)^pC_{k+1}(V').$$

Consider the following decomposition of π_3 to compute $(\pi_3)_!$.

$$F_{2,2,1}(TN')=G_{1,n-2}(\tau^*(TN')/(TN')_1)\xrightarrow{\tau_1}G_{1,n-1}(TN')\xrightarrow{\tau}N'.$$

Let $C(K_1/K_2)=1+d$ and $C(K_3)=1+l$. Then we have

$$C(K_1)=(1+d)(1+l),\quad C(K_1^*)=(1-d)(1-l)$$
$$C(K_0/K_1)=C(K_0)(1+d)^{-1}(1+l)^{-1}$$
$$C(K_2\bigcirc K_1)=(1+2d)(1+d+l)(1+2l)$$
$$C(K_3\bigcirc K_3\bigcirc K_2)=(1+3l)(1+d+2l)$$
$$C(\bigcirc^j K_3)=(1+jl)$$
$$C(K_3\bigcirc K_2\bigcirc K_2)=(1+d+2l)(1+3l)(1+2d+l)$$

and

$$C(\bigcirc^3K_3\bigcirc K_2)=(1+4l)(1+d+3l).$$

So $C_2(K^*)=dl$ and we can represent $C_{k+1}(V')$ as a polynomial with respect to $C_i(K_0)$, d and l. Suppose that $C_2(K_1^*)C_{k+1}(V')$ is written as a polynomial

$$d^pl^p\Big(\sum_{i=0}^{k+1}C_i(K_0)\Big(\sum_{s+t=k+1-i}a_{st}d^sl^t\Big)\Big)$$

where a_{st} are integers. We note here that by [**1**, Proposition 4.1(b)]

$$\begin{aligned}(\tau_1)_!(d^{n+s-1})&=(-1)^{s+1}\overline{C}_{s+1}(K_0/K_3)\\&=(-1)^{s+1}(\overline{C}_{s+1}(K_0)+\overline{C}_s(K_0)l)\end{aligned}$$

and

$$(\tau)!(l^{n+s-1}) = (-1)^s \bar{C}_s(K_0).$$

By applying these formulas to the polynomial above we obtain the following by Theorems 4.1 and 4.2.

$$P(X_{k+1}, f) = (-1)^{k+1} \sum_{i=0}^{k+1} (-1)^{k-i+1} C_i \Big(\sum_{s+t=k-i+1} a_{st}(-\bar{C}_{s+1}\bar{C}_t + \bar{C}_s\bar{C}_{t+1}) \Big)$$

$$= \sum_{i=0}^{k+1} (-1)^i C_i \Big(\sum_{s+t=k-i+1} a_{st}(-\bar{C}_{s+1}\bar{C}_t + \bar{C}_s\bar{C}_{t+1}) \Big)$$

where $C_i = C_i(K_0) = C_i(TN - f^*(TP))$. Hence $P(X_{k+1}, f)$ can be written as follows.

$$\sum_{i=1}^{k+1} (-1)^i C_i \Big(\sum_{s=0}^{k+2-i} (a_{s,\, k+1-i-s} - a_{s-1,\, k+2-i-s}) \bar{C}_s \bar{C}_{k+2-i-s} \Big)$$

where $a_{-1,\, k+2-i} = a_{k+2-i,\, -1} = 0$.

Let $p(d, l)$ be the polynomial

$$C(K_0) d^p l^p (1+d)^{-1} (1+l)^{-1} (1+2d)(1+2l)(1+3l)(1+d+l)(1+2l+d).$$

For D_{k+1}, $C_2(k_1^*)^P C_{k+1}(V')$ becomes the part of the degree $2p+k+1$ of the polynomial

$$p(d, l) \prod_{j=4}^{k-1} (1+jl).$$

Similarly for E_{k+1} (k=5, 6 or 7), $C_2(K_1^*) C_{k+1}(V')$ is the part of degree $2p+k+1$ of the polynomial

$$p(d, l)(1+2d+l),$$

$$p(d, l)(1+2d+l)(1+4l)$$

or

$$p(d, l)(1+2d+l)(1+4l)(1+3d+3l)$$

respectively and we give two tables of a_{st} for D_5 and E_6.

a_{st} for D_5

t \ s	0	1	2	3	4	5
5	0					
4	0	−2				
3	12	14	−12			
2	16	32	2	2		
1	7	18	8	0	0	
0	1	3	2	0	0	0

a_{st} for E_6

$t \backslash s$	0	1	2	3	4	5	6
6	0						
5	0	0					
4	12	12	−16				
3	28	70	18	−14			
2	23	82	74	6	2		
1	8	35	46	16	0	0	
0	1	5	8	4	0	0	0

The precise formula of $P(D_6, f)$ for $n=p+1$ is as follows.

$$\begin{aligned}
&-2\bar{C}_1\bar{C}_5-12\bar{C}_2\bar{C}_4+14\bar{C}_3^2\\
&\quad -C_1(14\bar{C}_1\bar{C}_4-14\bar{C}_2\bar{C}_3)\\
&\qquad +C_2(12\bar{C}_4+12\bar{C}_1\bar{C}_3-24\bar{C}_2^2)\\
&\qquad\quad -C_3(14\bar{C}_3-14\bar{C}_1\bar{C}_2)\\
&\qquad\qquad +C_4(4\bar{C}_2-4\bar{C}_1^2).
\end{aligned}$$

The real version of the arguments above shows the formulas stated in Section 4.

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Yoshifumi ANDO
Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan