J. Math. Soc. Japan
Vol. 48, No. 3, 1996

Buekenhout geometries of rank 3 which involve the Petersen graph

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(Received Oct. 5, 1993)
(Revised Sept. 9, 1994)

1. Introduction.

Consider the diagram $(c^{*}\cdot P)$: $\circ\overset{c}{\text{——}}\circ\overset{P}{\text{——}}\circ$.

Here, the symbol $\circ\overset{c}{\text{——}}\circ$ stands for *the circle geometry with* 4 *points* and $\circ\overset{P}{\text{——}}\circ$ for the geometry of *the Petersen graph*. We will determine all simply connected geometries $\underline{G}$ with this diagram and flag-transitive automorphism group. It will turn out that there are exactly five simply connected ones. One of them is related to the alternating group of degree 6, two of them are related to the Mathieu groups M_{11} and M_{12}, all these three are finite.

One is related to the symmetric group of degree 9 and to the sporadic group He, and one to the group $SO_5(5)$, and we do not know, whether they are finite or infinite.

To be precise, we will prove the following theorem.

THEOREM. *Let $\underline{G}$ be a connected, simply connected geometry with diagram $(c^{*}\cdot P)$ and flag-transitive automorphism group G. Then $\underline{G}$ is one of the geometries $\underline{G}_5$, $\underline{G}_6$, $\underline{G}_9$, $\underline{G}_{11}$ or $\underline{G}_{12}$ defined in the next section, and G is isomorphic to one of the groups G_{6a}, G_{6b} (if $\underline{G}$ is $\underline{G}_6$), or G_{11} (if $\underline{G}$ is G_{11}), or G_{12a}, G_{12b}, G_{12c}, G_{12d}, G_{12e} (if $\underline{G}$ is $\underline{G}_{12}$), or G_{9a}, G_{9b} (if $\underline{G}$ is $\underline{G}_9$), or G_{5a}, G_{5b} (if $\underline{G}$ is $\underline{G}_5$), all defined in the last section, respectively.*

Here, $\underline{G}_{11}$ is a geometry with 66 points and automorphism group $G_{11}=M_{11}$, $\underline{G}_{12}$ is a geometry with 4752 points and automorphism group G_{12d} (resp. G_{12e})$=(A_4\times M_{12})2$ and projects onto a geometry for M_{12}, $\underline{G}_6$ is a geometry with 6480 points and automorphism group $G_{6b}=3(A_6\times A_6)2$, while we do not know, whether the geometries $\underline{G}_5$ and $\underline{G}_9$ are finite or infinite. They project onto finite geometries for $SO_5(5)$ and Σ_9 respectively and have automorphism groups G_{5b} and G_{9b} respectively.

The remark on the automorphism groups of the examples is almost trivial: the pairs $(\underline{G}, \mathrm{Aut}(\underline{G}))$ have to appear in the list, hence one has only to check in every case, which of the groups acting on the same geometry $\underline{G}$ is the "biggest one".

Presentations for the groups mentioned are given in a table at the end of the paper.

This theorem is part of an attempt to classify flag-transitive diagram geometries that have as rank 2 residues only generalized polygons defined over the field with two elements or "affine parts" (the part "outside" some hyperplane) of these, such as the affine plane of order 2 (the circle geometry on four points) or some affine part of the generalized quadrangle for $Sp_4(2)$ (the Petersen graph). Many interesting geometries and groups will have to be characterized in this program; it is somehow amazing that so many (small) sporadic simple groups often involve such geometries.

Note that the canonical examples of $(c^*\cdot P)$ geometries are truncations of geometries of type $(c^2\cdot P)$, as the circle geometry is in fact a truncation of the thin A_3-geometry, and these were investigated among others in [**IS**]. There, one also finds the corresponding examples for M_{11}, A_9, $\Omega_5(5)$ and He.

We hope that all notation is standard. For instance we use ATLAS-notation for involutions in the automorphism groups of finite simple groups (see [**At**]).

By Σ_n, we denote the symmetric group on n letters, by A_n its subgroup of even permutations. By M_{11} (resp. M_{12}) we denote the sharply 4-transitive (resp. 5-transitive) permutation groups on 11 (resp. 12) letters called the little Mathieu groups.

The Petersen graph P is the following graph on 10 vertices: to each 2-set $\{i, j\}$ of the set $\{1, 2, 3, 4, 5\}$ there corresponds a unique vertex v_{ij} of P. Vertices v_{ij} and v_{kl} are adjacent (on an edge), if $\{i, j\}\cap\{k, l\}$ is empty. The geometry of the Petersen graph has as points the vertices of P, as lines the edges of P and the natural incidence between vertices and edges. In an analogous way the circle geometry on 4 points is obtained from the complete graph on 4 letters.

If G is a group of automorphisms of some geometry $\underline{G}$ and y is some vertex of this geometry, we denote by res(y) the residue of y in $\underline{G}$, by G_y the stabilizer of y in G, which of course acts on res(y), and by K_y the kernel of the action of G_y on res(y). If y, z are two vertices of $\underline{G}$ we denote the elementwise stabilizer of y and z by G_{yz}. The (elementwise) stabilizer of a given chamber (maximal flag) is denoted by B.

The organization of the paper is as follows. In section 2, we describe the (finite) examples. In section 3, we show that any chamber-transitive automorphism group of a simply 2-connected chamber system of type $(c^*\cdot P)$ has one of twelve explicitly given presentations. Moreover, it is shown that there are precisely five (non-isomorphic) simply 2-connected chamber-transitive chamber systems with this diagram.

Let us describe the way, in which the classification is achieved.

Assume $\underline{G}$ is a connected rank-3-geometry with diagram $(c^*\cdot P)$ (in fact, for the following facts to hold, a string diagram is already sufficient), and let G be a flag-transitive group of automorphisms of $\underline{G}$. The vertices of $\underline{G}$ are called **points** (resp. **lines**, resp. **planes**), if their types correspond to the left (resp. middle, resp. right) node of the diagram.

Let $\{p, l, x\}$ be a chamber of $\underline{G}$. Then $G=\langle G_{lx}, G_{px}, G_{pl}\rangle$, and $\underline{G}$ is isomorphic to the coset geometry $\Gamma(G; G_p, G_l, G_x)$ in G, and $\underline{G}$ is also canonically isomorphic to the geometry $\Gamma(C)$ of the chamber system $C=C(G; B; G_{lx}, G_{px}, G_{pl})$.

The universal cover of $\underline{G}$ is canonically isomorphic to the coset geometry $\Gamma(G^0; G_p, G_l, G_x)$ where G^0 is the universal completion of the amalgam of the subgroups G_p, G_l, G_x in G. This group is also isomorphic to the inductive limit (amalgamated sum) of the "assemblage" $(B; G_{lx}, G_{px}, G_{pl}; G_p, G_l, G_x)$ as it arises in G (see [**T**], section 1), and the chamber system $C^0=C(G^0; B; G_{lx}, G_{px}, G_{pl})$ is the universal 2-cover of C.

Hence to obtain the universal cover of $\underline{G}$ we can apply the following well-known procedure. Choose appropriate generators $x_1, \cdots, x_n$ of the subgroups G_{lx}, G_{px}, G_{pl} and a set of relations R between the x_i that hold in G and are necessary and sufficient for the chamber system $C^1=C(G^1; B; G_{lx}, G_{px}, G_{pl})$ to have diagram $(c^*\cdot P)$, where $G^1=\langle x_1, \cdots, x_n: r=1 \text{ for } r\in R\rangle$. Then $G^1=G^0$; and the universal cover of $\underline{G}$ is isomorphic to the geometry $\Gamma(C^1)$ (see for instance [**P1**]).

Of course, this method can also be used "abstractly": if one is given some diagram (say of rank 3) and wants to determine all possible (simply 2-connected, chamber-transitive) chamber systems with this diagram, one takes a hypothetical chamber system C with this diagram and chamber-transitive group, picks some chamber c in C and determines the structure of the stabilizers $B, X_1, X_2, X_3, X_{12}, X_{13}, X_{23}$ of the chamber c, the three rank-1-residues and the three rank-2-cells on c, and the way they are amalgamated (the "assemblage" $(B; X_1, X_2, X_3; X_{12}, X_{13}, X_{23})$). This can be done in a quite abstract way without having an example. Subsequently, one determines a presentation for the inductive limit of this assemblage: this is obtained by finding generators for the groups X_{12}, X_{13}, X_{23}, and relations which force the pairwise intersections to be X_1, X_2, X_3 and the intersection of X_1, X_2, X_3 to be B—and which are necessary and sufficient for the rank-2-chamber-systems $C(X_{ij}; B; X_i, X_j)$ to be isomorphic to the corresponding prescribed rank-2-cells. Then the group G^0 with generators the union of the generators of the groups X_{ij} and relations the union of the relations in the presentations for the groups X_{ij}, $i=1, 2, 3$ is the direct limit of the assemblage $(B; X_1, X_2, X_3; X_{12}, X_{13}, X_{23})$. In G^0, the subgroups X_{ij}^0 (resp. X_j^0, resp. B^0) generated by the generators coming from X_{ij} (resp. X_j, resp. B) is isomorphic to some quotient of X_{ij} (resp. X_j, resp. B). If the corresponding chamber systems $C(X^0_{ij}; B^0; X^0_i, X^0_j)$, are isomorphic to the chamber systems $C(X_{ij}; B; X_i, X_j)$, the chamber system $C(G^0; B^0; X^0_1, X^0_2, X^0_3)$ will be a simply 2-connected chamber system of the given type with chamber-transitive group of automorphisms G^0 (compare [**P1**], Lemma 5).

If the group G^0 "collapses"—it may be equal to 1; it may also happen that G^0 is a big group, but the chamber systems $C(X^0_{ij}; B^0; X^0_i, X^0_j)$ are *not* isomorphic to the chamber systems $C(X_{ij}; B; X_i, X_j)$, one derives that the corresponding assemblage $(B; X_1, X_2, X_3; X_{12}, X_{13}, X_{23})$ is not possible for a chamber-transitive group on a chamber system of the given type.

By coset enumeration, one can try to determine the order (and structure) of G^0. Of course, if the order of G^0 equals 1, this means "collapsing". If coset enumeration does not finish, one is led to suspect that G^0 is infinite (and in particular does not collapse), but has not proved this. By finding some finite quotient of G^0, which does not "collapse", one can prove at least that the corresponding assemblage $(B; X_1, X_2, X_3; X_{12}, X_{13}, X_{23})$ exists in a chamber-transitive group on a chamber system of the given type.

Now, the discussion above applies, and we have the corresponding results on the universal (2-) covers of all flag-transitive geometries with diagram $(c^*\cdot P)$.

This (well-known) method is used in section 3.

2. The geometries.

(1) *A geometry G'_6 on 6 points.*

Let Ω be the set $\{1, 2, 3, 4, 5, 6\}$, $G=A_6$ acting in the natural way on Ω.

The geometry $\underline{G}'_6$ is defined as follows.

The points are the 6 letters of Ω, the lines are the 3-sets of Ω, and the planes are the 15 partitions of type 2/2/2 (synthemes) of Ω. The incidence between points and lines and between points and planes is defined as follows: All points p and all planes x are incident (i.e., $\underline{G}'_6$ is flat), and the point p is incident to the line l if and only if p is contained in l.

The incidence between lines l and planes x, however, is defined using the action of G as follows:

It is easily seen that Σ_6 is transitive on the set of pairs (l, x) with the property that l "cuts" x, by which we mean that the partition consisting of l and its complement, which is of type 3/3, refines x to a partition of type 1/1/1/1/1/1. But $G=A_6$ has two orbits on these pairs (interchanged by Σ_6). Take one of these two orbits to be the set of incident line-plane-pairs.

LEMMA 1. (i) *The geometry $\underline{G}'_6$ has the diagram $(c^*\cdot P)$,*
$\underline{G}'_6$ is connected and has 6 points,
(ii) *G is flag-transitive on $\underline{G}'_6$.*

PROOF. Almost by definition, $G=A_6$ is flag-transitive on $\underline{G}'_6$ and the residue of a line is a generalized digon.

Since there are exactly eight 3-sets of Ω cutting a given syntheme (plane) x, four of them are lines in res(x). By transitivity of $G_x=\Sigma_4$ on the six letters, the pairwise intersections of the four 3-sets of letters corresponding to lines in res(x) are singletons, and hence the six points in res(x) are each incident to exactly two lines in res(x). Moreover, every two lines in res(x) have a unique point in common. Hence res(x) is isomorphic to the dual affine plane on 6 points as stated.

In the residue of x, every point p is incident to exactly two lines. Therefore, res(p) is in fact a graph on the 10 lines incident to p, which correspond to the 2-sets of $\Omega-\{p\}$, and a double count yields that this is a 3-valent graph.

Since G_p is isomorphic to A_5, this graph is uniquely determined (the 10-point representation of A_5 has permutation rank 3), and is isomorphic to the Petersen graph. Clearly, $\underline{G}'_6$ has 6 points and is connected.

We denote by $\underline{G}_6$ the universal cover of $\underline{G}'_6$. It will turn out in the next section that $\underline{G}_6$ is finite but not flat any more. $\underline{G}_6$ will be seen to have 6480 points and automorphism group isomorphic to $3(A_6\times A_6)2$.

(2) *A geometry G'_{12} for M_{12}.*

Let $G=M_{12}$ act in the natural way on the 12 point set Ω.

Before describing the geometry $\underline{G}'_{12}$ itself, we have to point out some facts about the subgroups and conjugacy classes of G.

LEMMA 2. *The following hold in $G=M_{12}$.*

(i) *G has exactly two classes of involutions:*
- *$2A$ with $C(2A)=2\times\Sigma_5$*
- *$2B$ with $C(2B)=2^{1+4}\Sigma_3$.*

(ii) *Involutions of type $2A$ are fixed point free on Ω.*

(iii) *Each involution of type $2B$ has 4 fixed points on Ω.*

(iv) *G has exactly two classes of fours groups V containing involutions of type $2A$:*
$V=(2A)^2$ with $N(V)=A_4\times\Sigma_3$
$V=(2A_22B)$ with V contained in $O_2(C(2B))$.

(v) *Let U be elementary abelian of order 8 which contains a $2A$-pure fours group.*
Then U is of type $2A_62B$ and lies in $O_2(C(2B))$.

PROOF. The first three statements can be verified by a look at the character table and the information given on G in the ATLAS.

Let t be some $2A$ involution in G. The action of $C(t)$ on the 6 orbits of $\langle t\rangle$ on Ω is the transitive 6-point action of Σ_5. There, transpositions are fixed point free, and hence from transpositions in Σ_6, we get a class of $(2A)^2$ fours groups V. Assume we already know this class is unique. Then a double count gives $|N(V)|=2^3 3^2$, and in $C(2A)$ we see $N(V)\cap C(2A)=2\times 2\times \Sigma_3$. Clearly, this is precisely $C(V)$, and the involutions in V are conjugate under $N(V)$. This yields the structure of $N(V)$.

The elements of order 4 in G do not have orbit structure $4+4+4$ on Ω, since G is perfect. Therefore those involutions of G, that are squares, have fixed points on Ω, and therefore are $2B$-involutions. This implies that involutions s in $C(t)'$ are $2B$ and so there is exactly one class of $2A$-pure fours groups, and there is exactly one more class of fours groups V containing a $2A$-involution in G, and V contains a $2B$ involution. Moreover, there are no $2A$-pure eights groups in G.

The third involution in $V=\langle s, t\rangle$ is of type $2A$:

To see this consider the centralizer of s. It acts on the four fixed points of s on Ω and on the four nontrivial orbits of $\langle s\rangle$ as a Σ_4.

The kernel K of the action on nontrivial orbits is elementary abelian of order 8 and the involutions in $K-\langle s\rangle$ are all conjugate. Since they must have fixed points on Ω, they are $2B$, and therefore K is a $2B$-pure eights group.

This implies that $2A$ involutions in $C(s)$ act nontrivially on the nontrivial $\langle s\rangle$-orbits.

Since they induce fixed point free involutions on the fixed point set of s, they lie in the commutator subgroup of $C(s)$, and so induce also fixed point free involutions on the four nontrivial $\langle s\rangle$-orbits. This implies that also st is $2A$.

(v) immediately follows from (iv).

We will construct some rank 3 geometry $\underline{G}'_{12}$ consisting of subgroups of G, with incidence given by inclusion.

Consider the geometry $\underline{G}'_{12}$ of points, lines and planes where:

The points are the subgroups of order 2 of G containing $2A$-involutions, the lines are the $2A$-pure fours groups in G, and the planes are the elementary abelian subgroups of order 8 of G containing $2A$-pure fours groups.

LEMMA 3. (i) *$\underline{G}'_{12}$ has diagram $(c^*\cdot P)$,*
$\underline{G}'_{12}$ is connected and has 396 *points,*
(ii) *$G=M_{12}$ is flag-transitive on $\underline{G}'_{12}$, but also* $\mathrm{Aut}(M_{12})$ *acts on $\underline{G}'_{12}$.*

PROOF. Let W be a plane and V a line on W. Then $W\in Syl_2(C(V))$, so $N(V)$ permutes transitively the planes on V, and hence there is only one class of planes in G. Moreover, the normalizer of W in G contains a subgroup $A_4\times Z_2$ from the normalizer of V, and hence contains an element of order 3 which acts nontrivially on W. Since by the above, $N(W)$ is transitive on the four $2A$-pure fours groups contained in W, at least the alternating group A_4 is induced by $N(W)$ on these. Now, G is flag-transitive on $\underline{G}'_{12}$. A double count gives $|N(W)|=2^5 3$. This implies $N(W)=C(2B)'$.

The isomorphism type of res(W) is now as desired, and clearly, the residue of a line is a generalized digon.

Let p be a point in $\underline{G}'_{12}$. Then the stabilizer in G of p is isomorphic to $Z_2\times\Sigma_5$, and lines in res(p) are the $2A$-pure fours groups in G_p, which are in a one-to-one correspondence with the transpositions in $G_p/Z(G_p)=\Sigma_5$, and therefore to ten 2-sets of some 5-set. A double count gives that there are exactly 3 planes on each line, and also exactly 2 lines on each plane in res(p). Now, G_p acts flag-transitively on res(p), which is a 3-valent graph on the 10 lines in res(p). This graph is uniquely determined and is in fact the Petersen graph. Since G acts primitively on its 396 $2A$-involutions, the result follows.

Let $\underline{G}_{12}$ denote the universal cover of $\underline{G}'_{12}$. It will turn out in the next section that $\underline{G}_{12}$ is a 12-fold cover of $\underline{G}'_{12}$.

We point out, that truncations of $\underline{G}'_{12}$ already appear in [**B**], namely the geometries (21) and (22). There the geometry of *all* 2-sets, 3-sets and 4-sets of Ω is discussed (geometry (6)), of which $\underline{G}_{11}$ (defined in (3) below) is a subgeometry.

The geometry $\underline{G}_{11}$ is the truncation of a rank 4-geometry with diagram $(c^2\cdot P)$, described and shown to be simply connected in [**IS**], Theorem (6.1). We give some details for the convenience of the reader.

(3) *A geometry* G_{11} *for the Mathieu group* M_{11}.

Let $G=M_{11}$ act in its 3-transitive representation on a set Ω of 12 letters.

Then clearly, G is transitive on 2-sets and 3-sets of Ω. But G has two orbits on 4-sets of Ω, one of size 330, and one orbit X of size 165 with stabilizer $2\Sigma_4$. (The stabilizer of a 4-set in X is plain to see, and the fact that G has exactly 2 orbits on 4-sets can be verified comparing the permutation characters of M_{12} on cosets of "transitive" subgroups M_{11} and on 4-sets of Ω.)

Define the geometry $\underline{G}_{11}$ as follows:

The points are the 2-sets of Ω, the lines are the 3-sets of Ω, the planes are the 4-sets of Ω in X. Incidence is defined by inclusion.

LEMMA 4. (i) *$\underline{G}_{11}$ has diagram $(c^*\cdot P)$,*
$\underline{G}_{11}$ is connected and has 66 *points,*
(ii) *$G=M_{11}$ is flag-transitive on G_{11}.*

PROOF. Since the stabilizer in G of a plane x acts as Σ_4 on the four letters contained in x, we get flag-transitivity of G. A double count gives that each line is on exactly three planes. And clearly, in the residue of a point, a plane is incident to exactly two lines.

The residue of a plane (and a line) has obviously the desired isomorphism type. Let p be a point of $\underline{G}_{11}$. Then the stabilizer of p in G is isomorphic to Σ_5, and acts flag-transitively on res(p), which is a 3-valent graph on the 10 lines in res(p). This graph is uniquely determined and is in fact the Petersen graph. Since Ω has 66 2-sets, and all 2-sets and 3-sets of Ω appear in $\underline{G}_{11}$, the result follows.

It will turn out in the next section that $\underline{G}_{11}$ is simply connected.

(4) *Graphs for* 3^7-A_9, *He and* $SO_5(5)$.

The remaining examples are—just like $\underline{G}_{11}$—truncations of rank 4 geometries with diagram

$$\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ\overset{P}{\text{——}}\circ \qquad (c^2\cdot P).$$

All of these are geometries, whose vertices are cliques in certain graphs Γ, incidence defined by inclusion, and so the same holds for the truncations. Hence we describe just the graphs Γ. We show the following.

CLAIM. *The geometries* $\underline{G}(\Gamma)$ *of all cliques of* Γ, *i-cliques having type* $i\in\{1, 2, 3, 4\}$, *incidence defined by inclusion, are flag-transitive and have the diagram* $(c^2\cdot P)$.

The first example was given in [**IS**], p. 949f in a slightly different setting. The following description is implicit in [**BCN**], p. 36 remarks (iii); it is due to Wilbrink, see the reference to [**BL**] there.

Let V_n be an n-dimensional vector space over the field $GF(5)$. Let $(\cdot,\cdot)$ be a non-degenerate symmetric bilinear form on V_n with discriminant 2. Call (+)-points in V_n the l-dimensional subspaces $\langle v\rangle$ of V_n satisfying $(v, v)\in\{1, -1\}$.

Then Γ has as vertex set the (+)-points of V_5, and two vertices are adjacent, if and only if they are perpendicular with respect to $(\cdot,\cdot)$.

Clearly maximal cliques in Γ are 4-cliques and the group $O(V_5)$ acts flag-transitively on the geometry $\underline{G}(\Gamma)$. In order to verify that $\underline{G}(\Gamma)$ has the given diagram, it suffices to consider the residue of some 2-clique, and hence to show that the corresponding graph Γ_3 on the space V_3 is isomorphic to the Petersen graph.

But Γ_3 has 10 vertices and valency 3 and admits $O_3(5)=A_5$: such graph is unique and isomorphic to the Petersen graph.

It is clear, that $SO_5(5)$ acts flag-transitively on the geometry of 2-cliques, 3-cliques and 4-cliques, of Γ, which will be denoted by $\underline{G}(\Omega_5(5))$. Its universal cover is denoted $\underline{G}_5$.

The next example is the commuting graph Γ on transpositions in Σ_9. It has maximal cliques of size 4, and the geometry $\underline{G}(\Gamma)$ is flag-transitive with diagram $(c^2\cdot P)$. It is described in [**IS**]. Its truncation to 2-cliques, 3-cliques and 4-cliques will be denoted by $\underline{G}(A_9)$. Let the universal cover of $\underline{G}(A_9)$ be denoted by $\underline{G}_9$.

The geometry $\underline{G}(A_9)$ is not simply connected, as the following example shows.

For this example a certain group Σ will play the dominant role. It has the properties:

- $O_3(\Sigma) \cong Z_3$,
- $\Sigma/O_3(\Sigma) \cong \Sigma_7$,
- $\Sigma' \cong 3\cdot A_7$ nonsplit.

The group Σ is unique up to isomorphism by these properties, and will be denoted $3\Sigma_7$ (just as in [**At**]). Involutions in $3\Sigma_7$ that correspond to transpositions in $3\Sigma_7/O_3(3\Sigma_7)$ are again called transpositions in $3\Sigma_7$. The commuting graph on transpositions in $3\Sigma_7$ is one of the four locally Petersen graphs ([**Ha**]).

Let us consider a nonsplit extension G of an elementary abelian normal subgroup N of order 3^7 with Σ_9. Using [**KP**], one could derive the existence of such group abstractly, we prefer to have a concrete example, however. It lives inside the nonsplit extension $3^7\Omega_7(3)=:X$, which itself is a subgroup of the big Fischer group Fi'_{24} (to be precise, there are two classes of them conjugate under the action of Fi_{24}).

To see this, we investigate X as a subgroup of Fi_{24} a bit more closely. It is the normalizer of a certain 3-group of order 3^7, which we again call N. For the following, we use information from [**At**], pp. 200–207.

Note that elements of order 3 in N are conjugate to their inverses, and hence subgroups of order 3 of N can be named after the two 3-elements they contain.

They split into conjugacy classes under Fi'_{24} as follows: $3A_{378}3B_{364}3C_{351}$.

Clearly, N is just the natural modulo for $X/N=\Omega_7(3)$ (with, say, bilinear form $(\cdot,\cdot)$ on N, which has discriminant 1), and nontrivial elements x of N are in the class $3A$ (resp. $3B$, resp. $3C$) if $(x, x)=2$ (resp. 0, resp. 1).

It is well-known that $\Omega_7(3)$ has (two classes of) subgroups isomorphic to Σ_9. Let G be a subgroup of X containing N such that $G/N=\Sigma_9$. Let t be an involution of G mapping onto a transposition modulo N. Then $C_G(t)$ maps onto a group of type $Z_2\times\Sigma_7$ modulo N, hence $C_N(t)$ has order 1, 3, 3^6 or 3^7, as $C_N(t)$ is certainly $C_G(t)$-invariant. But as X is perfect, the eigenvalue -1 of t must have an even multiplicity, and therefore $C_N(t)$ has precisely 3 elements. Let $C_N(t)=\langle x\rangle$. We have to determine $C_G(t)$ in some detail.

Clearly, as $P\Omega_6^+(3)$ does not have subgroups isomorphic to A_7 (see [**At**], p. 69), x is a $3C$-element. The centralizer of x in Fi'_{24} equals $C_X(x)$, and is isomorphic to $3^7 2\cdot\Omega_6^-(3)$. By a look at centralizers of 6-elements in Fi'_{24} one sees that t is of class $2B$ in Fi'_{24}. And its centralizer in Fi'_{24} looks like: $N(t)=2^{1+12}3\Omega_6^-(3)\cdot 2_2$. Moreover, the centralizer of t in Fi_{24} is of type

$2^{1+12}3\Omega_6^-(3)\cdot(2^2)_{122}$.

Assume $C_G(t)$ is isomorphic to $(Z_6\times A_7):2$, whence it has a subgroup isomorphic to $Z_2\times\Sigma_7$. Consider now the group $U=3\Sigma_7\times Z_2$ of $He:2$, which is a subgroup of Fi_{24}'. The involution in $Z(U)$ is of class $2B$, as Fi_{22} does not contain subgroups isomorphic $3A_7$ nonsplit.

We conclude that $N(t)/O_2(N(t))$ contains subgroups Σ_7 and groups of type $3\Sigma_7$. As there are just two classes of subgroups Σ_7 in the group $\Omega_6^-(3)\cdot 2_2$, and these are fused in $\Omega_6^-(3)\cdot(2^2)_{122}$, this is impossible.

Hence $C_G(t)=Z_2\times 3\Sigma_7$.

Assume now that G' has a subgroup A isomorphic to A_9. Then $N_G(A)$ is a subgroup isomorphic Σ_9, and contains subgroups $Z_2\times\Sigma_7$. This contradicts the argument above.

Hence G is "nonsplit" as claimed above.

Now it follows easily that the commuting involutions graph Γ on t^G has maximal cliques of size 4, and 2-cliques (resp. 3-cliques, resp. 4-cliques) of Γ containing t correspond to 1-cliques (resp. 2-cliques, resp. 3-cliques) of the commuting graph on transpositions in $3\Sigma_7$.

Moreover, the group G acts flag-transitively on the (connected) geometry $\underline{G}(\Gamma)$. The diagram of $\underline{G}(\Gamma)$ has diagram $\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ\overset{P}{\text{——}}\circ$.

Of course, the geometry $\underline{G}(\Gamma)$ is a 2-cover of the geometry $\underline{G}(A_9)$ of [**IS**]. And the truncation to rank 3 is a 2-cover of $\underline{G}(A_9)$.

Just like the previous example, the next one is a graph on $2B$-elements in Fi_{24}'. The automorphism group is the subgroup $He:2$ of Fi_{24}' mentioned above. This example also appears in [**IS**], but is constructed as a group geometry there. To present it in our graph theoretical way, we need some information on $\mathrm{Aut}(He)=He:2$, which we take from [**At**], p. 104f. We use the names for the conjugacy classes in $He:2$ chosen in [**At**].

Put in the following $H=He$, the simple group named after D. Held, and $G=\mathrm{Aut}(H)$, such that $H=G'$, and $G/H=Z_2$.

LEMMA 5. (i) *There are precisely two classes of involutions in H, $2A$ and $2B$, and for an involution $t\in H$ we have*: 5 *divides the order of $C_H(t)$, if and only if $t\in 2A$. (Clearly, the two classes are not fused under the action of G.)*

(ii) *There are precisely two classes of elements of order* 3 *in H, $3A$ and $3B$, and for a* 3*-element $d\in H$ we have: d is of class $3A$, if and only if $N_H(\langle d\rangle)$ is isomorphic to $3\Sigma_7$ (nonsplit, this is the group Σ defined above).*

(iii) *There are precisely two classes of elements of order* 6 *in H, $6A$ and $6B$, and proper powers of $6A$-elements (resp. $6B$-elements) are of class $2A$ and $3A$ (resp. $2B$ and $3B$).*

(iv) *There is precisely one class of involutions in* $G-H$, $2C$, *and for* $t\in 2C$ *we have* $C_G(t)$ *is isomorphic to* $Z_2\times 3\Sigma_7$.

PROOF. See [**At**], p. 104f.

LEMMA 6. *Let* t *be a* $2C$*-involution in* G, *and let* a, b, c *be three pairwise commuting transpositions in* $C_H(t)=3\Sigma_7$. *The following holds*:

(i) a *and* ab *are of class* $2A$ *in* H,

(ii) abc *is of class* $2B$ *in* H.

PROOF. We use the information of lemma 5 without explicit mentioning. As 5 divides the order of $C_H(a)$, a is of class $2A$.

As $O_3(C_H(t))=\langle d\rangle$ for some $3A$-element $d\in H$, ab is of class $2A$. This proves (i).

As $C_H(t)$ contains some Sylow 3-subgroup S of H, there is some $3B$-element e in $C_H(t)$. It is obvious that e centralizes some involution s of $C_H(t)$ modulo $\langle d\rangle$, and so s centralizes some 3-element in $\langle d, e\rangle-\langle d\rangle$. These are all $3B$-elements, as can be seen in the nonabelian group S, and hence there is some $2B$-involution in $C_H(t)$. Now (ii) follows from (i).

We are now able to describe the graph Γ for the desired example.

Vertices of Γ are $2C$-involutions in G. Two vertices t, s are adjacent, if $[s, t]=1$ and 5 divides the order of $C_H(\langle s, t\rangle)$.

Right from the definition it is clear that G induces a group of automorphisms on Γ, which is vertex-transitive. Moreover, $\{s, t\}$ is an edge in Γ, if and only if st is a transposition in $C_H(t)$ (and also in $C_H(s)$).

LEMMA 7. (i) *Maximal cliques in* Γ *are* 4*-cliques, and* G *acts transitively on the set of* 4*-cliques of* Γ,

(ii) *If* $t\in\Gamma$, *then the* 2*-cliques* (*resp.* 3*-cliques, resp.* 4*-cliques*) *of* Γ *on* t *correspond to the* 1*-cliques* (*resp.* 2*-cliques, resp.* 3*-cliques*) *of the commuting graph on transpositions of* $3\Sigma_7$,

(iii) G *acts flag-transitively on* $\underline{G}(\Gamma)$, *and* $\underline{G}(\Gamma)$ *has diagram*

$$\circ\overset{c}{\text{——}}\circ\overset{c}{\text{——}}\circ\overset{P}{\text{——}}\circ\,.$$

PROOF. Let $t\in\Gamma$. Then if $s\in\Gamma$ is adjacent to t, the involution $st\in H$ is a transposition in $C_H(t)=3\Sigma_7$, and hence there are no i-cliques in Γ for $i>4$.

Let u, v, $w\in\Gamma$ be three vertices of Γ adjacent to t, such that they commute pairwise. Then the involutions tu, tv and tw are pairwise commuting transpositions in $3\Sigma_7$ and the group $\langle t, tu, tv, tw\rangle$ is elementary abelian of order 16. By lemma 2(i), the elements of $\langle tu, tv, tw\rangle-\langle tuvw\rangle$ are $2A$-involutions. Moreover, there is an element d of order 3 in $C_H(t)$ permuting transitively the elements u, v, w.

Claim: $\{t, u, v, w\}$ is a 4-clique in Γ.

Assume the contrary, then $\{u, v, w\}$ is a co-clique in Γ. Now, in $C_H(u)$, the element tu is a transposition, while the elements uv and uw are not. Therefore uv and uw are double transpositions in $C_H(u)$. But since tuv and tuw are $2A$-involutions, we may identify $\{tu, uv, uw\}$ in $C_H(u)/O_3(C_H(u))=\Sigma_7$ with $\{(12), (12)(34), (12)(56)\}$. Therefore $C=C_H(\langle u, v, w\rangle)$ is a $\{2, 3\}$-group with $O_3(C)=O_3(C_H(u))$, as can be seen in $C_H(u)$. And d is contained in $N_H(O_3(C))=C_H(u)$. This is clearly a contradiction.

We may conclude that vertices u, v adjacent to t form an edge, if and only if the transpositions tu and tv in $C_H(t)=3\Sigma_7$ commute. Now (ii) follows. As G (and also H) is transitive on pairs $\{t, F\}$ of vertices $t\in\Gamma$ and 4-cliques F containing t of Γ, G (resp. H) is transitive on 4-cliques of Γ, and the stabilizer G_F (resp. H_F) of a 4-clique F is transitive on vertices in F. But, given a pair $\{t, F\}$ as above, $C_H(t)$ induces a group Σ_3 on the 2-cliques of F containing t, hence G_F (resp. H_F) induces Σ_4 on vertices of F, and G (resp. H) acts flag-transitively on $\underline{G}(\Gamma)$. Together with (ii), this implies the rest of (iii).

The truncation of $\underline{G}(\Gamma)$ to 2-cliques, 3-cliques and 4-cliques will be denoted by $G(He)$. It will later on turn out, that its universal cover is $\underline{G}_9$.

3. The proof of the theorem.

Let $\underline{G}$ be a connected geometry with diagram $(c^*\cdot P)$ and flag-transitive group of automorphisms G. Let $\{p, l, x\}$ be some chamber of $\underline{G}$, where p is a point, l is a line and x is a plane.

(1) $G_p/K_p=A_5$ *or* Σ_5.

In the first case, $G_{lp}/K_p=\Sigma_3$, $G_{xp}/K_p=Z_2\times Z_2$, $B/K_p=Z_2$.

In the second case, $G_{lp}/K_p=Z_2\times\Sigma_3$, $G_{xp}/K_p=D_8$, $B/K_p=Z_2\times Z_2$.

In the first case, B/K_p does not contain a nontrivial subgroup invariant under G_{lp}, in the second case, there is a unique nontrivial subgroup of B/K_p invariant under G_{lp}, its order is 2.

PROOF. The automorphism group of the Petersen graph is Σ_5, and has only A_5 as a proper flag-transitive subgroup. The further statements can be verified on the Petersen graph.

(2) $G_x/K_x=A_4$ *or* Σ_4.

In the first case, $G_{lx}/K_x=Z_3$, $G_{xp}/K_x=Z_2$, $B=K_x$.

In the second case, $G_{lx}/K_x=\Sigma_3$, $G_{xp}/K_x=Z_2\times Z_2$, $B/K_p=Z_2$.

In any case, B/K_x does not contain a nontrivial subgroup invariant under G_{lx}.

PROOF. The automorphism group on the circle geometry on 4 points is Σ_4 and only A_4 is a proper flag-transitive subgroup. The further statements are obvious.

Note that there is no nontrivial normal subgroup of G contained in B, as G acts faithfully on the set of chambers of $\underline{G}$. But $\underline{G}$ is connected, hence $G=\langle G_i, G_j\rangle$ for $\{i, j, k\}=\{p, l, x\}$, and even $G=\langle G_i, G_{jk}\rangle$, and therefore, if K is some nontrivial normal subgroup of G_i contained in B, then K is not invariant under G_{jk}. In particular, if we can prove $K_l=K_x$ or $K_pK_l=K_x$, we already know $K_x=1$.

By (1) and (2), $G_{pl}=\langle B, d\rangle$ for some element d not in B satisfying $d^3\in B$, $G_{px}=\langle B, a\rangle$ for some element a not in B satisfying $a^2\in B$, and $G_{lx}=\langle B, e\rangle$ for some element e not in B satisfying $e^3\in B$. By connectedness of $\underline{G}$, $G=\langle B, d, a, e\rangle$. We will use the elements a and e right from the beginning and prove early statements (for instance in (3)) for all possible choices; later on, we pick specific elements d, a and e subject to further conditions. Then we can, of course, still use these early results.

(3) *$K_l\leqq K_x$, and $K_p\cap K_x\leqq K_l$; either $K_x=K_l=1$, or $|K_x:K_l|=2$. Further, $K_l\cap(K_l)^a=K_p\cap K_x$ is G_p-invariant and $K_l\cap(K_l)^a\cap(K_l)^{ae}\cap(K_l)^{aee}=1$. In particular, K_x is an elementary abelian 2-group of order at most 16.*

PROOF. By (2), $K_lK_x/K_x=1$ and we get $K_l\leqq K_x$. Since res(1) is a generalized digon, $K_p\cap K_x\leqq K_l$. And since G_l/K_l is isomorphic to a subgroup of $\Sigma_3\times\Sigma_3$, $|K_x:K_l|\leqq 2$ holds.

In case $K_l=1$, there is nothing to show.

Assume for the rest of (3) that $K_l>1$. Then K_l is not G_x-invariant, but is invariant under G_{lx}, hence has precisely the four G_x-conjugates K_l, $(K_l)^a$, $(K_l)^{ae}$ and $(K_l)^{aee}$ (which are all contained in K_x). Moreover, $K_p\cap K_x$ is a proper subgroup of K_l.

The intersection M_x of all G_x-conjugates of K_l has index at most 16 in K_x, and K_x/M_x is an elementary abelian 2-group. We have to show that $M_x=1$.

It is clear, by (1), that $K_l/K_p\cap K_l=K_lK_p/K_p$ has at most two elements.

Assume K_l is contained in K_p. Then $K_l=K_l\cap K_p=K_p\cap K_x$, contradicting the above.

Therefore $K_l/K_p\cap K_l$ has precisely 2 elements, and so $K_p\cap K_l\leqq K_p\cap K_x<K_l$ which implies $K_p\cap K_l=K_p\cap K_x$; in particular, $K_p\cap K_l$ is G_p-invariant.

Hence $K_p\cap K_l$ is contained in the group $K_l\cap(K_l)^a$. But as the group $K_l\cap(K_l)^a$ is a proper subgroup of K_l, equality holds. Now the conjugates $K_l\cap(K_l)^a$, $K_l\cap(K_l)^{ae}$ and $K_l\cap(K_l)^{aee}$ are nothing but the conjugates of

$K_l\cap(K_l)^a=K_p\cap K_l$ under the action of G_{lx}. In fact, they equal the intersections of K_l with K_q for the three points q incident to l. Clearly, the whole of G_l acts on this set, whence normalizes the intersection M_x.

Now, M_x is normalized by G_x and G_l, whence $M_x=1$.

We discuss the possible structures of G_p/K_p and G_x/K_x in the following. By (1) and (2), we consider the cases (4), (5) and (6).

(4) *Assume $G_x/K_x=A_4$. Then $|K_x|=2$ and $G_p=A_5$, or $|K_x|=8$ and $G_p=Z_2\times\Sigma_5$.*

PROOF. We have $K_p\leqq K_x=B$ by (2) and therefore $B=K_x\geqq K_l\geqq K_p$ by (3). If $K_l=1$, we get $|B|=2$, and the first possibility occurs. Assume K_l is different from 1, whence $K_x>K_l>K_p$ and $|B/K_p|=4$. Then K_l is not G_x-invariant and there are four G_x-conjugates of K_l in K_x. This clearly implies $K_p>1$. Moreover by (1), we know K_l is not contained in G'_pK_p, and $G_p/K_p=\Sigma_5$.

Now G_p is generated by G_p-conjugates of K_l and the subgroup K_p, therefore it centralizes K_p. Recall that $K_p=K_p\cap K_x=K_l\cap(K_l)^a$, by (3). And take some element $e'\in G_x$ of order 3 that permutes the groups K_l, $(K_l)^a$ and $(K_l)^{ae}$ among themselves. Clearly, $G_x=\langle G_{px}, e'\rangle$. Hence the group $K_l\cap(K_l)^a\cap(K_l)^{ae}$ is normal in G, and therefore it is equal to 1. Now K_l has at most 4 elements and $|K_x|=8$ follows.

We showed $|K_p|=2$; since K_x is elementary abelian, $G_p=Z_2\times\Sigma_5$.

(5) *Assume $G_x/K_x=\Sigma_4$ and $G_p/K_p=A_5$.
Then $K_p=K_x=1$, or $|K_p|=|K_x|=2$, $G_p=Z_2\times A_5$.*

PROOF. We have $|B/K_p|=|B/K_x|=2$ by (1) and (2), and $K_l\leqq K_x$ with $|K_x:K_l|\leqq 2$ by (3). Moreover, $K_p\cap K_x$ is contained in K_l by (3). Hence either $K_p=K_x=1$, or $K_pK_x=B$ and $K_p\cap K_x=K_l=1$. Then $|K_p|=|K_x|=2$. Since B contains more than one involution, the last statement holds.

(6) *Assume $G_x/K_x=\Sigma_4$ and $G_p/K_p=\Sigma_5$.
Then $|B:K_x|=2$ and $G_l/K_l=\Sigma_3\times\Sigma_3$. Moreover, $G''_p=A_5$.*

PROOF. The group B/K_x has precisely 2 elements by (2). Hence B/K_l has 4 elements by (3), and as G_l/K_l is a flag-transitive subgroup of $\Sigma_3\times\Sigma_3$, we get $G_l/K_l=\Sigma_3\times\Sigma_3$. The group G_{px}/K_x is a fours group by (2), and so G_{px} has exponent at most 4. Clearly, K_p is a 2-group of order at most 8, hence centralized by G''_p, and therefore G''_p, which is a perfect central extension of A_5, is isomorphic to A_5 or to $SL_2(5)$. Assume the second case occurs. Then some involution t from $G_p-G''_p$ normalizes some Sylow 2-subgroup S of $G''_p=SL_2(5)$, which is quaternion. By the structure of G_p/K_p, t does not centralize $S/Z(S)$. But then it is easily verified that $S\langle t\rangle\leqq G_p$ contains elements of order 8. As

G_{px} contains a Sylow 2-subgroup of G_p, this contradicts $\exp(G_{px})=4$.

For the convenience of the reader, we once more describe the procedure for the identification of G and $\underline{G}$.

We start with the flag-transitive group G on the simply 2-connected geometry $\underline{G}$. Then G also acts transitively on the chamber system $C=C(\underline{G})$, which is simply connected. Assume B (resp. X_1, X_2, X_3) are the stabilizers of some chamber c (resp. the rank 1 residues containing c) of C (the *parabolics of* G). Then C is isomorphic to $C(G; B; X_1, X_2, X_3)$ by the transitive action of G. We choose a "canonical" generator set $\{x_i : i\in I\}$ of G from the local structure of $\underline{G}$, and derive a set R of relations between these generators. We do this in a way that the parabolics of G are generated by subsets of the generator set and relations live only inside the parabolics.

Assume that for each parabolic X, the generators contained in X and the relations between these generators derived from a presentation of this parabolic. Then consider the group G^0, for which our generators and relations form a presentation. Then the corresponding subsets of generators of G^0 generate subgroups B^0, X_1^0, X_2^0 and X_3^0, again called parabolics, of G^0.

Then there is a natural projection π from G^0 onto G, where parabolics are mapped onto parabolics.

Assume we know that π induces a bijection on each parabolic of G^0. Then clearly π induces a 2-cover from $C^0=C(G^0; B^0; X_1^0, X_2^0, X_3^0)$ onto $C=C(G; B; X_1, X_2, X_3)$, which must be an isomorphism, as C is simply connected. This implies that π is an isomorphism of groups, and we derived a presentation for G. Moreover, as in our case the diagram is a string and some "fundamental condition" is satisfied, we may apply [**MT**] to see that $\underline{G}$ is isomorphic to $\underline{G}(C)=\underline{G}(C^0)$.

The mentioned knowledge that rank 2 residues of C^0 have the desired isomorphism type must be taken from some quotient, where it can be verified explicitly.

Now to the presentations. We give generators and relations for the group G. We thereby frequently use well-known presentations for A_4, Σ_4, A_5 and Σ_5.

For instance, if d, t are elements in A_5 of order 3 and 2 respectively, and t inverts d, and s is some other involution in A_5 commuting with t, then the elements ds and dst have both order 5. Conversely, $\langle d, t, s : d^3=t^2=s^2=[s, t]=(dt)^2=(ds)^5=(dst)^5=1\rangle$ is isomorphic to the group A_5.

It is also well-known that $\langle a, e : a^2=e^3=(ea)^4=1\rangle$ is isomorphic to Σ_4.

The verification that the presentations given below indeed define the groups stated, was done by coset enumeration using CAYLEY. We summarize the presentations in table at the end of the paper.

A natural division into three cases is given by the structure of G_x/K_x and G_p/K_p (see (4), (5) and (6) above).

We investigate the situation of (4) (resp. (5), resp. (6)) in sections (7) (resp. (8), resp. (9)).

(7) *Let* $G_x/K_x=A_4$. *Then* $G=G_{12a}\cong Z_3\times M_{12}$ *or* $G=G_{12b}\cong A_4\times M_{12}$.

(a) Assume $K_p=1$.

Then by (4), $G_p=A_5$, and it is easily seen that $G_{px}=Z_2\times Z_2$, $G_x=Z_2\times A_4$, $G_{lx}=Z_6$, $G_{pl}=\Sigma_3$, $G_l=Z_3\times\Sigma_3$.

Pick $d\in G_{pl}$ of order 3, $e\in G_{xl}$ of order 3, $s\in G_{xl}$ of order 2, $a\in G_{px}\cap G'_x$, not in B, of order 2. Then $B=\langle s\rangle$, $G_{pl}=\langle s, d\rangle$, $G_{px}=\langle B, a\rangle$ and $G_{lx}=\langle B, e\rangle$, and the relations $[d, e]=(ds)^2=(da)^5=(dsa)^5=[e, s]=(ae)^3=1$ hold.

The group $G_{12a}=\langle a, d, e, s: a^2=d^3=e^3=s^2=[a, s]=[d, e]=[e, s]=(ds)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle$ *is isomorphic to* $Z_3\times M_{12}$.

(b) Assume next $G_x/K_x=A_4$, $|K_p|>1$.

Then by (1), $G_p=Z_2\times\Sigma_5$, $G_{px}=Z_2\times D_8$, $G_{pl}=Z_2\times Z_2\times\Sigma_3$, $K_x=(Z_2\times Z_2\times Z_2)$, $G_l=A_4\times\Sigma_3$, $G_{lx}=A_4\times Z_2$.

Moreover, $1<K_p<K_l<B=K_x$, and so we may pick elements t in K_p of order 2, $d\in G_{pl}$ of order 3, $e\in G_{xl}$ of order 3, u in K_l with $u=t^e$, $s\in Z(G_{xl})$ of order 2, $a\in G_{px}\cap G'_p$, not in B, of order 2.

Then $B=\langle t, u, s\rangle$, $G_{pl}=\langle B, d\rangle$, $G_{xl}=\langle B, e\rangle$, $G_{px}=\langle B, a\rangle$ and $G_{px}\cap G'_p=\langle s, a\rangle$, and the relations $[t, u]=[t, s]=[t, d]=[t, a]=t^eu=u^eut=[u, s]=[u, a]s=[u, d]=[d, e]=(ds)^2=(da)^5=(dsa)^5=[e, s]=1$ hold. Moreover $(ae)^3\in B$.

The possible relations $(ae)^3=u, t, ut, us, st, uts$ make the presented group collapse to a group with at most 3 elements, the relations $(ae)^3=1$ and $(ae)^3=s$ are equivalent (replace a by as).

The group $G_{12b}=\langle a, d, e, s, t, u: a^2=d^3=e^3=s^2=t^2=u^2=[t, u]=[s, t]=[d, t]=[a, t]=t^eu=u^eut=[s, u]=[a, u]s=[d, u]=[a, s]=[e, s]=[d, e]=(ds)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle$ *is isomorphic to* $A_4\times M_{12}$.

(8) *Let* $G_x/K_x=\Sigma_4$ *and* $G_p/K_p=A_5$. *Then* $G=G_{6a}\cong 3(A_6\times A_6)$ *or* $G=G_{6b}\cong Z_3(A_6\times A_6)Z_2$ *or* $G=G_{12c}\cong(Z_3\times M_{12})Z_2$.

(a) Assume $K_p=K_x=1$.

Then $G_p=A_5$, $G_{pl}=\Sigma_3$, $G_{px}=Z_2\times Z_2$, $B=Z_2$, $G_l=(Z_3\times Z_3)Z_2$.

Hence we may pick elements a, b, d, e in the following way:

s is the involution in B, d is an element of order 3 in G_{pl}, e is an element of order 3 in G_{xl}, and a is the involution in $G_{px}\cap G'_x$.

Then the relations $[a, s]=(ds)^2=(es)^2=[d, e]=(ae)^3=1$, and $(ad)^5=(asd)^5=1$ hold in G.

The group $G_{6a}=\langle a, d, e, s: a^2=d^3=e^3=s^2=[a, s]=(ds)^2=(es)^2=[d, e]=(da)^5=(dsa)^5=(ae)^3=1\rangle$ *is isomorphic to the central product of two perfect central extensions* Z_3A_6.

(b) Assume $|K_p|=|K_x|=2$.

Then $B=K_pK_x$ is elementary abelian of order 4, and $G_p=Z_2\times A_5$.

Now $G_{pl}=Z_2\times\Sigma_3$, $G_{px}=Z_2\times Z_2\times Z_2$ and $G_x=Z_2\times\Sigma_4$. Moreover, $K_l=K_x\cap K_p=1$ and hence $G_l=\Sigma_3\times\Sigma_3$.

Pick elements k in K_p of order 2, s in K_x of order 2, d in G_{pl} of order 3 and e in G_{lx} of order 3. Moreover, take the involution a in $(G_{px}\cap G'_x)-B$.

Then $B=\langle k, s\rangle$, $G_{pl}=\langle B, d\rangle$, $G_{px}=\langle B, a\rangle$, $G_{lx}=\langle B, e\rangle$, and, if $\langle u, v\rangle=G_{px}\cap G'_p$, where u inverts d, the following relations hold:

$[k, s]=[k, d]=[k, a]=[e, s]=[a, s]=[d, e]=(ke)^2=(sd)^2=(dv)^5=(dvu)^5=1$, and $(ae)^3=1$.

As possibilities for $\langle u, v\rangle$ we get all fours groups in G_{px} that do not contain the element k. The two possibilities $\langle u, v\rangle=\langle s, ka\rangle$, $\langle sk, ak\rangle$ give a collapsing of the corresponding group, whereas the choices $\langle u, v\rangle=\langle sk, a\rangle$, $\langle s, a\rangle$ do not force the group to collapse. So we end up with the following groups:

$G_{6b}=\langle a, d, e, k, s: a^2=d^3=e^3=k^2=s^2=[a, k]=[a, s]=[k, s]=[d, k]=[e, s]=[d, e]=(ds)^2=(ek)^2=(da)^5=(dska)^5=(ae)^3=1\rangle$, *which is isomorphic to a group* $Z_3(A_6\times A_6)Z_2$, $G_{12c}=\langle a, d, e, k, s: a^2=d^3=e^3=k^2=s^2=[a, k]=[d, k]=[e, s]=[k, s]=[a, s]=[d, e]=(ds)^2=(ke)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle$, *which is isomorphic to a group* $(Z_3\times M_{12})Z_2$.

(9) *Let* $G_x/K_x=\Sigma_4$ *and* $G_p/K_p=\Sigma_5$. *Then* $G=G_{11}\cong M_{11}$ *or* $G=G_{12d}$ *(resp.* G_{12e}*)* $\cong(A_4\times M_{12})Z_2$ *or* $G=G_{5a}$ *or* G_{5b} *or* $G=G_{9a}$ *or* G_{9b}, *in which cases we do not know the isomorphism type.*

Recall that $K_l\leqq K_x$; and as res(1) is a digon, $K_x\cap K_p\leqq K_l$.

(a) $K_x>1$.

Assume $K_x=1$. Then $|B|=2$, which contradicts $|B/K_x|=4$.

(b) Assume $|K_x|=2$. Then $G=G_{11}$.

We have $|B|=4$, and $K_p=1$. Then $G_p=\Sigma_5$, and $G_{px}=D_8$, $G_{pl}=Z_2\times\Sigma_3$. Now $|B|=Z_2\times Z_2$, and $G_l=\Sigma_3\times\Sigma_3$. Clearly, $G_{lx}=Z_2\times\Sigma_3$, and $K_x=Z(G_{lx})=Z(G_x)$. Pick an element z in K_x of order 2, and an element t in $Z(G_{pl})$ of order 2, an element d in G_{pl} of order 3, and an element e in G_{lx} of order 3. Pick further an involution a in $G_{px}\cap G'_p-B$. Then $B=\langle z, t\rangle$, $G_{pl}=\langle B, d\rangle$, $G_{px}=\langle B, a\rangle$ and $G_{lx}=\langle B, e\rangle$. As G_l is 3-closed, the elements d and e commute. K_x centralizes G_x, hence G_{px}, and hence z is contained in G'_{px}.

This implies $K_x\leqq G'_p$ and so $G_{px}\cap G'_p=\langle a, z\rangle$, and the relations $[z, t]=[z, e]=[z, a]=(zd)^2=[t, d]=(te)^2=[t, a]z=(da)^5=(dza)^5=1$ hold. Moreover, as $G_{px}=D_8$, z lies in the commutator subgroup of Sylow 2-subgroups of G_x, whence $O_2(G_x)=Q_8$. Now the involution a is not contained in $O_2(G_x)$, but the 4-element ta is. Hence $(ate)^3$ is contained in K_x. Up to replacing a by az, this gives a unique presentation $G_{11}=\langle a, d, e, z, t: a^2=d^3=e^3=z^2=t^2=[e, z]=$

$[a, z]=[z, t]=(dz)^2=[d, t]=(et)^2=[a, t]z=(da)^5=(dza)^5=(ate)^3=1\rangle$.

A coset enumeration shows that G_{11} *is isomorphic to* M_{11}.

REMARK. The identification could be achieved also as follows. Consider the chamber system $C=C(G; K_x; B, \langle K_x, te\rangle, \langle K_x, a\rangle, \langle K_x, d\rangle)$. Then C has diagram $(c^2\cdot P)$, and by [**MT**], the corresponding geometry $\Gamma(C)$ has the same diagram and flag-transitive group of automorphisms G. As the stabilizer of a point-line-flag F in G induces only A_5 on $\boldsymbol{F}$, which has type P, by [**IS**], Theorem 6.1, we get $\Gamma(C)=\underline{G}_{11}$ and $G=M_{11}$.

(c) The case $K_x=Z_2\times Z_2$ does not occur.

Assume $K_x=Z_2\times Z_2$. Then $|K_l|=|K_p|=2$. As K_x can not contain four different conjugates of K_l, but G_{lx} normalizes K_l, the whole of G_x leaves invariant K_l, a contradiction.

(d) Assume $K_x=Z_2\times Z_2\times Z_2$. Then G_{12d}, G_{12e}, G_{9a} or G_{6a}.

Then $K_l=Z_2\times Z_2$, and K_l is contained in K_x; $|K_p|=4$, and K_p is not contained in K_x, as it is not equal to K_l; furthermore K_l is not G_x-invariant and hence has precisely four G_x-conjugates in K_x. This is only possible, if these are precisely the four 2-spaces in K_x missing a certain 1-space Z_x, which of course equals $Z(G_x)$.

Again $B=K_xK_p$, and K_p acts nontrivially on K_x. Hence $[K_x, K_p]=K_x\cap K_p=K_l\cap K_p$. As $K_p\leqq G_x$ centralizes Z_x, we get $[K_l, K_p]=K_l\cap K_p$.

Recall that $G_x/K_x=\Sigma_4$ acts faithfully on K_x. Hence $[G_{px}, K_x]=(K_l\cap K_p)\cdot Z_x$. Therefore, $(K_l\cap K_p)\cdot Z_x$ is contained in G'_{px}.

Now $G_p=(K_p\times G''_p):2$, with $K_p=Z_2\times Z_2$ or Z_4; moreover, K_l is not contained in $(K_p\times G''_p)$, and $\langle K_p, K_l\rangle=D_8$, whence K_p is not central in G_p, and $Z(G_p)=[K_p, K_l]$.

We get $G_{pl}=D_8\times\Sigma_3$, and $G_l=\Sigma_4\times\Sigma_3$.

Now we may pick elements k in $Z(G_p)$ of order 2, d in G_{pl} of order 3, m in $K_p-\langle k\rangle$ with $m^2=k$ or 1, u in $K_l-\langle k\rangle$ of order 2, e in G_{lx} of order 3 such that $\langle e\rangle$ is inverted by some element t in the coset K_xm, and $z\in Z(G_x)$ of order 2, $a\in G_{px}\cap G''_p$, not in B, of order 2. As G''_p centralizes K_p, we get $[a, m]=[a, k]=1$. As a is contained in G_{px}, we get $[a, z]=1$.

The element t must be an involution, since k is the only nontrivial square in $\langle K_x, m\rangle$, but does not centralize e. Involutions in K_xm are the elements m, mk, mz, mkz if $m^2=1$, and the elements mu, muk, muz, $mukz$ if $m^2=k$. Hence up to replacing m by mk, we may assume $t=m$ in the first case, and $t=mu$ in the second case.

This gives the relations

(*) $\qquad m^2=1$ and $(em)^2=1$, or $m^2=k$ and $(emu)^2=1$.

Then $K_p=\langle m, k\rangle$, $K_l=\langle k, u\rangle$, $K_x=\langle k, u, z\rangle$, $B=\langle m, k, u, z\rangle$, $G_{lx}=\langle B, e\rangle$, $G_{px}=\langle B, a\rangle$, $G_{pl}=\langle B, d\rangle$.

Now, the element a does not normalize K_l, but centralizes k and z, hence maps K_l onto some conjugate of K_l different from K_l, containing k and not containing z. Thus $K_l^a=\langle k, uz\rangle$, and we get

(**) $$[u, a]=z \text{ or } kz.$$

In the first case, a induces a fixed point free involution on the four conjugates of K_l in K_x, while in the second case, a fixes the conjugates $\langle u, kz\rangle$ and $\langle uk, kz\rangle$. As m has two fixed points $\langle K_l\rangle$ and $\langle k, uz\rangle$, am induces a fixed point free involution in this case.

But e acts fixed point freely on $O_2(G_x)/Z_x$, hence elements of K_x centralized by an element of order 3 in G_x are contained in Z_x, and we get the relations

(***) $$(ae)^3=1 \text{ or } z, \text{ or } (ame)^3=1 \text{ or } z.$$

Moreover, there are the trivial relations $[k, m]=[k, u]=[k, z]=[k, d]=[k, a]=[u, m]k=[u, z]=[u, k]=[u, d]=[m, z]=[m, d]=[e, z]=[e, d]=1$, and $(zd)^2=(da)^5=1$. Up to replacing u by ku, we may assume $k^eu=u^eku=1$.

Now checking (using again CAYLEY) the group generated by k, u, z, a, d, e and one set of the relations indicated, we get a contradiction, (the parabolic subgroups of G are smaller than we assumed) as soon as the resulting group has fewer than 60 elements. This happens, whenever the relation $m^2=k$ holds.

Hence we know that always $m^2=1$.

Now the relations $[u, a]=z$ and $(ae)^3=1$ or z give a presentation for the group $(A_4\times M_{12})Z_2$. The so presented groups are named G_{12d} and G_{12e}; it can be checked that factoring out the normal subgroup A_4 yields a projection of chamber systems, hence geometries, onto $\underline{G}'_{12}$. Hence we get $\underline{G}=\underline{G}_{12}$ in both cases.

The relations $[u, a]=kz$ and $(ame)^3=1$ and $[u, a]=kz$ and $(ame)^3=z$ do not force the respective groups (named G_{9a} and G_{5a}) to collapse either; in contrast, it can be verified, by letting act G_{9a} on cosets of a subgroup of index 126 or G_{5a} on cosets of a subgroup of index 300, that A_9 is a quotient of G_{9a} and $\Omega_5(5)$ is a quotient of G_{5a}, and that the projection maps are injective on parabolic subgroups of the chamber systems. This can most effectively be achieved using the cosact homomorphism implemented in CAYLEY. Now $\underline{G}$ is the universal 2-cover of $\underline{G}(A_9)$ respectively the universal 2-cover of $\underline{G}(\Omega_5(5))$ in the two cases.

Let us summarize the result of case (d):

The group G is hence generated by elements k, u, z, m, d, e, a and satisfies the relations $k^2=u^2=z^2=m^2=d^3=e^3=a^2=1$, $[k, m]=[k, u]=[k, z]=[k, d]=[k, a]=[u, m]k=[u, z]=[u, k]=[u, d]=[m, z]=[m, d]=[e, z]=[e, d]=1$,

$k^e u = u^e k u = 1$, $(em)^2 = 1$, $(zd)^2 = (da)^5 = 1$, and further relations $[u, a] = z$, and $(ae)^3 = 1$ or z, or $[u, a] = kz$, and $(ame)^3 = 1$ or z.

As already stated, these are presentations for G in the four respective case.

What about the universal 2-cover of $\underline{G}(He)$? It must appear in case (d) by the structure of its parabolic subgroups. Certainly, it is not covered by the geometry $\underline{G}_{12}$, but which of $\underline{G}_5$ and $\underline{G}_9$ is the universal 2-cover of $\underline{G} = \underline{G}(He)$?

Analogously as in the remark at the end of case (b), we can switch to the chamber system $C(G; K_x; B, \langle K_x, me\rangle, \langle K_x, a\rangle, \langle K_x, d\rangle)$ which has diagram $(c^2 \cdot P)$ and to the geometry $\Gamma(C)$. By ([**P2**], Theorem 1), the universal 2-cover of $\underline{G}$ is the appropriate truncation of the universal 2-cover of $\Gamma(C)$, and by ([**P2**], Lemma 4), all rank 3 residues in the universal 2-cover of $\underline{G}$ (viewed as a rank 4 geometry) and in $\Gamma(C)$ are isomorphic. The same holds also for the geometries $\underline{G}(A_9)$ and $\underline{G}(\Omega_5(5))$ and their universal 2-covers respectively.

Now point residues in the rank 4 geometries $\underline{G}(A_9)$ and $\underline{G}(He)$ are isomorphic to the $(c \cdot P)$-geometry for the group $3\Sigma_7$, point residues in the rank 4 geometry $\underline{G}(\Omega_5(5))$ are isomorphic to the $(c \cdot P)$-geometry for the group $\Omega_4^-(5)$ in the other case.

Hence the universal 2-covers for the Held group example and the universal 2-cover for $\underline{G}(A_9)$ are isomorphic, but are not isomorphic to the universal 2-cover of $\underline{G}(\Omega_5(5))$.

(e) Assume $K_x = Z_2 \times Z_2 \times Z_2 \times Z_2$. Then $G = G_{5b}$ or $G = G_{9b}$.

We will see that G has a flag-transitive subgroup with $|K_x| = 8$ in this case, whence the geometry must already occur in case (d). The flag-transitive subgroup can be identified to be G_{5a} (resp. G_{9a}).

Now, $K_l = Z_2 \times Z_2 \times Z_2 \leqq K_x$, and we claim that K_x is the (dual) permutation module for $G_x/K_x = \Sigma_4$. To see this, assume by way of contradiction, that the four G_x-conjugates of K_l in K_x intersect in a 1-space Z. Then Z is certainly contained in $K_p \cap K_l = K_l \cap (K_l)^a$ and therefore centralized by G_p'. But then, as G is generated by G_x and G_p', Z is G-invariant, which is a contradiction. Hence the intersection of the four G_x-conjugates of K_l in K_x is 1, and the claim is proved.

Now G_x fixes precisely one 1-space Z_x, no 2-space and precisely one 3-space H_x in K_x and Z_x is contained in H_x, but not in K_l.

Hence Z_x equals $Z(G_x)$ and is a complement to K_l in K_x. As $K_p K_x = B$, K_p acts nontrivially on K_x, and as K_p normalizes K_l and Z_x, we get $[K_x, K_p] = [K_l, K_p]$ is a group of order 2 in $K_l \cap K_p$, which itself is a fours group.

By (3), $K_l \cap K_p = K_p \cap K_x$ is G_p-invariant, and hence not invariant under G_l. Therefore, there are three conjugates of $K_l \cap K_p$ in K_l under G_{xl}.

Assume by way of contradiction that they intersect in a group Z of order 2. Then Z is central under G_{xl}, and contained in K_p. But K_p is centralized by G'_p, and $G_p=G'_p\cdot B$; hence Z is centralized by G_p and is G-invariant, which is a contradiction (the same contradiction proof was performed ten lines above). Therefore, K_l is the dual permutation module for the group $\Sigma_3=G_{lx}/K_x$. But clearly also G_l induces Σ_3 on K_l. This implies that there is a G_l-invariant 2-space M_{xl} and a G_l-invariant 1-space Z_l in K_l, such that Z_l is a complement to M_{xl} and to $K_l\cap K_p$ and G_{lx}/K_x acts faithfully on M_{xl}. Clearly, $H_x=Z_xM_{xl}$.

As K_p acts nontrivially on K_x, it has to act nontrivially on $K_l\cap K_p$ and K_p (being nonabelian of order 8 and containing the fours group $K_p\cap K_x$) is dihedral of order 8.

By (6), $G_p=(D_8\times A_5)2$. But Z_l is not contained in K_p, it is not even contained in $K_pG'_p$ by (1), and it centralizes K_p. Hence, $G_p=D_8\times\Sigma_5$. In particular, $G_{px}=D_8\times D_8$. Moreover, $G'_{px}=Z(G_{px})=Z_2\times Z_2$. And this group is contained in K_x, as G_{px}/K_x is abelian.

The action of G_x on the permutation module K_x shows that Z_x is contained in G'_{px}. Hence, $G'_{px}=[K_x, G_{px}]=Z_x[K_p, K_x]$.

Let $\langle k\rangle=Z(K_p)$, $\langle z\rangle=Z_x$, $\langle l\rangle=Z_l$. Clearly, these three involutions commute. To generate B, we pick an element $r\in K_p\cap K_l-\langle k\rangle$ and a further involution m in K_p-K_x. Then $[m, z]=[m, l]=[m, k]=[r, k]=[r, z]=[r, l]=1$, and $[m, r]=k$.

For the convenience of the reader, we list some consequences. $K_p\cap K_l=\langle k, r\rangle$, $K_l=\langle k, r, l\rangle$, $K_x=\langle k, z, l, r\rangle$ and $B=\langle K_x, m\rangle$. Moreover, $K_p=\langle k, r, m\rangle$.

The group $G''_p=A_5$ acts still flag-transitively on res(p), whence $G''_p\cap B=Z_2$ is properly contained in $G''_p\cap G_{px}=Z_2\times Z_2$.

Hence, we may pick an involution $a\in G_{px}-B$ such that a is contained in G''_p, and hence centralizes K_p. This gives relations $[a, z]=[a, m]=[a, k]=[r, a]=1$, and $[a, l]$ is contained in $Z(G_{px})=\langle k, z\rangle$. Assume $[a, l]$ is contained in $\langle k\rangle$. Then K_l is normalized by the element a, hence invariant under $G_{px}=\langle B, a\rangle$ and G_l, and therefore G-invariant, a contradiction. Hence we get $[a, l]=z$ or kz.

Assume $[a, l]=z$. Then a induces some transvection with center Z_x on K_x, which contradicts the action of $G_x/K_x=\Sigma_4$ on K_x. Hence

$$(*)\qquad [a, l]=kz.$$

In particular, $K_l^a=\langle k, r, lz\rangle$. Moreover, a induces a transposition on the four G_x-conjugates of K_l. Let these be denoted by K_l, K_l^a, Q and R. Clearly, $K_l\cap K_l^a=\langle r, k\rangle$ is centralized by a, and $\langle k, r, z\rangle$, the centralizer of a in K_x, is the sum of $K_l\cap K_l^a$ and $[K_x, a]=\langle kz\rangle$.

Also the element m acts as a transvection on K_x, hence as a transposition on $\{K_l, K_l^a, Q, R\}$. As $[m, a]=1$, m fixes K_l and K_l^a, and interchanges Q and R. Therefore, $Q\cap R$ is centralized by m, hence lies in $\langle k, l, z\rangle$. By the above argument, $Q\cap R$ is a complement to $\langle k\rangle=[K_x, m]$ in $\langle k, l, z\rangle$.

Moreover, a centralizes K_x/Q and K_x/R, hence $Q\cap R$ contains $[a, K_x]=\langle kz\rangle$.

Now, $Q\cap R=\langle kz, l\rangle$ or $\langle kz, lk\rangle$.

Therefore, $Q\cap R\cap K_l=\langle l\rangle$ or $\langle lk\rangle$. This implies $Q\cap R\cap(K_l^a)=\langle lkz\rangle$ or $\langle lz\rangle$. As this group is invariant under the action of G_{lx}, which centralizes $\langle l, z\rangle$ but not k, we get $Q\cap R\cap(K_l)^a=\langle lz\rangle$, $Q\cap R\cap K_l=\langle lk\rangle$, and $Q\cap R=\langle lz, lk\rangle$.

Now consider $K_l\cap(K_l)^a\cap Q$ and $K_l\cap(K_l)^a\cap R$. These 1-spaces of K_x are interchanged by the element m. Hence they equal $\langle r\rangle$ and $\langle rk\rangle$, and we found a "canonical" basis of the 4-space K_x in the vectors

$$lz,\ lk,\ r,\ rk,$$

which are permuted "naturally" via conjugation by $G_x/K_x=\Sigma_4$. Note that $H_x=\langle k, rl, z\rangle$ is the unique G_x-invariant 3-space in K_x, and $M_{xl}=K_l\cap H_x=\langle k, rl\rangle$ is the unique G_{xl}-irreducible 2-space in K_x.

We go on with the presentation for G.

Clearly, G_{pl} is 3-closed and we take $\langle d\rangle=O_3(G_{pl})$.

Then $[d, k]=[d, m]=[d, r]=[d, l]=1$, and so d is inverted by z, as $G_{pl}/K_p=Z_2\times\Sigma_3$. In G''_p we see $(da)^5=1$. It is easily seen that $\langle d\rangle=O_3(G_l)$, as the centralizer of K_l in G_l is 3-closed.

Consider the group G_{lx}. We know $G_{lx}/K_x=\Sigma_3$, and $\langle Z_x, Z_l\rangle\leqq K_x$ is central in G_{lx}. Therefore, the four Sylow 3-subgroups of G_{lx} are permuted transitively by the fours group $K_x/\langle Z_x, Z_l\rangle$ and $G_{xl}/\langle Z_x, Z_l\rangle$ induces Σ_4 on $\mathrm{Syl}_3(G_{xl})$. The involution m in $B-K_x$ must therefore induce a transposition there. Let $\langle e\rangle$ be a Sylow 3-subgroup of G_{lx}, which is normalized by m. Up to replacing e by its inverse, we get

$$(lz)^e=lz,\ (lk)^e=r,\ r^e=rk,\ (rk)^e=lk.$$

Now, the element ema acts as an element of order 3 on K_x, with centralizer $\langle r, z\rangle$. Hence we get the condition

$$(**)\qquad (ema)^3\in\langle r, z\rangle.$$

The element ea induces a 4-cycle on $\{lz, lk, r, rk\}$, hence $\langle e, a\rangle K_x=G_x$.

Assume $(ema)^3$ is not contained in $\langle z\rangle$. Then H_x is contained in $\langle e, m, a\rangle$, and since the involution m normalizes $\langle e, a\rangle$, also $\langle e, a\rangle$ contains H_x. But $(ea)^4$ is contained in $\langle z\rangle$, as it is centralized by ea. Therefore, $\langle e, a, z\rangle/\langle z\rangle$ is generated by an element of order 3 and an involution, such that their product is of order 4; such a group is isomorphic to Σ_4, a contradiction.

Hence $(ema)^3=1$ or z.

Together with all other relations derived so far, in particular $[d, e]=1$, the first relation defines a group G_{5b}, the second a group G_{9b}, and we find again subgroups of indices 126 (resp. 300), such that in the permutation groups induced on the cosets of these subgroups no residue collapses. Let $K_x^1=\langle k, z, rl\rangle$, $B^1=$

$\langle K_x^1, m\rangle$; then the group $G^1=\langle k, z, rl, e, a, d\rangle$ is obviously a flag-transitive subgroup of G. The chamber system $C(G^1; K_x^1; \langle K_x^1, m\rangle, \langle K_x^1, em\rangle, \langle K_x^1, a\rangle, \langle K_x^1, d\rangle)$ has diagram $(c^2\cdot P)$, the chamber system $C(G^1; B^1; \langle b^1, em\rangle, \langle B^1, a\rangle, \langle B^1, d\rangle)$ has diagram $(c^*\cdot P)$. Hence we are back in case (d) and the result follows.

We omit the identification of the group G^1 as G_{5a} respectively G_{9a} in the two cases.

As we already know that there must occur the two non-isomorphic geometries $\underline{G}_5$ and $\underline{G}_9$ in case (e), this follows anyway.

TABLE OF PRESENTATIONS.

$$\begin{aligned}G_{6a} &= \langle a, d, e, s : a^2=d^3=e^3=s^2=[a, s]=(ds)^2=(es)^2 \\ &\quad =[d, e]=(da)^5=(dsa)^5=(ae)^3=1\rangle \\ &\cong Z_3(A_6\times A_6).\end{aligned}$$

$$\begin{aligned}G_{6b} &= \langle a, d, e, k, s : a^2=d^3=e^3=k^2=s^2=[a, k]=[a, s]=[k, s] \\ &\quad =[d, k]=[e, s]=[d, e]=(ds)^2=(ek)^2=(da)^5=(dska)^5=(ae)^3=1\rangle \\ &\cong Z_3(A_6\times A_6)Z_2.\end{aligned}$$

$$\begin{aligned}G_{11} &= \langle a, d, e, z, t : a^2=d^3=e^3=z^2=t^2=[e, z]=[a, z]=[z, t]=(dz)^2 \\ &\quad =[d, t]=(et)^2=[d, e]=[a, t]z=(da)^5=(dza)^5=(ate)^3=1\rangle \\ &\cong M_{11}.\end{aligned}$$

$$\begin{aligned}G_{12a} &= \langle a, d, e, s : a^2=d^3=e^3=s^2=[a, s]=[d, e]=[e, s] \\ &\quad =(ds)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle \\ &\cong Z_3\times M_{12}.\end{aligned}$$

$$\begin{aligned}G_{12b} &= \langle a, d, e, s, t, u : a^2=d^3=e^3=s^2=t^2=u^2=[t, u]=[s, t] \\ &\quad =[d, t]=[a, t]=t^e u=u^e ut=[s, u]=[a, u]s=[d, u] \\ &\quad =[a, s]=[e, s]=[d, e]=(ds)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle \\ &\cong A_4\times M_{12}.\end{aligned}$$

$$\begin{aligned}G_{12c} &= \langle a, d, e, k, s : a^2=d^3=e^3=k^2=s^2=[a, k]=[d, k]=[e, s]=[k, s] \\ &\quad =[a, s]=[d, e]=(ds)^2=(ke)^2=(da)^5=(dsa)^5=(ae)^3=1\rangle \\ &\cong (Z_3\times M_{12})Z_2.\end{aligned}$$

$$\begin{aligned}G_{12d} &= \langle k, u, z, m, d, e, a : k^2=u^2=z^2=m^2=d^3=e^3=a^2=[k, m] \\ &\quad =[k, u]=[k, z]=[k, d]=[k, a]=[u, m]k=[u, z]=[u, d] \\ &\quad =[m, z]=[m, d]=[e, z]=[e, d]=k^e u=u^e ku=[m, a] \\ &\quad =[z, a]=(em)^2=(zd)^2=(da)^5=1,\ [u, a]=z,\ (ae)^3=1\rangle \\ &\cong (A_4\times M_{12})Z_2.\end{aligned}$$

$$G_{12e}=\langle k,\ u,\ z,\ m,\ d,\ e,\ a:k^2=u^2=z^2=m^2=d^3=e^3=a^2=[k,\ m]$$
$$=[k,\ u]=[k,\ z]=[k,\ d]=[k,\ a]=[u,\ m]k=[u,\ z]=[u,\ d]$$
$$=[m,\ z]=[m,\ d]=[e,\ z]=[e,\ d]=[m,\ a]=[z,\ a]=k^e u$$
$$=u^e ku=(em)^2=(zd)^2=(da)^5=1,\ [u,\ a]=z,\ (ae)^3=z\rangle$$
$$\cong (A_4\times M_{12})Z_2.$$

$$G_{5a}=\langle k,\ u,\ z,\ m,\ d,\ e,\ a:k^2=u^2=z^2=m^2=d^3=e^3=a^2=[k,\ m]$$
$$=[k,\ u]=[k,\ z]=[k,\ d]=[k,\ a]=[u,\ m]k=[u,\ z]=[u,\ d]$$
$$=[m,\ z]=[m,\ d]=[e,\ z]=[e,\ d]=[m,\ a]=[z,\ a]=k^e u=u^e ku$$
$$=(em)^2=(zd)^2=(da)^5=1,\ [u,\ a]=kz,\ (ame)^3=z\rangle.$$

$$G_{5b}=\langle k,\ z,\ l,\ r,\ m,\ a,\ d,\ e:k^2=z^2=l^2=r^2=m^2=a^2=d^3=e^3$$
$$=(dz)^2=(da)^5=[k,\ z]=[k,\ l]=[l,\ z]=[m,\ z]=[m,\ l]$$
$$=[m,\ k]=[r,\ k]=[r,\ z]=[r,\ l]=[m,\ r]k=(em)^2=[d,\ e]$$
$$=[l,\ e]=[z,\ e]=k^e rl=r^e rk=[a,\ z]=[a,\ m]=[a,\ k]$$
$$=[r,\ a]=[a,\ l]kz=[d,\ k]=[d,\ r]=[d,\ l]=(ema)^3=1\rangle.$$

$$G_{9a}=\langle k,\ u,\ z,\ m,\ d,\ e,\ a:k^2=u^2=z^2=m^2=d^3=e^3=a^2=[k,\ m]$$
$$=[k,\ u]=[k,\ z]=[k,\ d]=[k,\ a]=[u,\ m]k=[u,\ z]=[u,\ d]$$
$$=[m,\ z]=[m,\ d]=[e,\ z]=[e,\ d]=[m,\ a]=[z,\ a]=k^e u$$
$$=u^e ku=(em)^2=(zd)^2=(da)^5=1,\ [u,\ a]=kz,\ (ame)^3=1\rangle.$$

$$G_{9b}=\langle k,\ z,\ l,\ r,\ m,\ a,\ d,\ e:k^2=z^2=l^2=r^2=m^2=a^2=d^3=e^3=(dz)^2$$
$$=(da)^5=[k,\ z]=[k,\ l]=[l,\ z]=[m,\ z]=[m,\ l]=[m,\ k]$$
$$=[r,\ k]=[r,\ z]=[r,\ l]=[m,\ r]k=(em)^2=[d,\ e]=[l,\ e]$$
$$=[z,\ e]=k^e rl=r^e rk=[a,\ z]=[a,\ m]=[a,\ k]=[r,\ a]$$
$$=[a,\ l]kz=[d,\ k]=[d,\ r]=[d,\ l]=(ema)^3 z=1\rangle.$$

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