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Mordell-Weil lattices of type D_5 and del Pezzo surfaces of degree four

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1. Introduction.

Mordell-Weil lattices of type E_8, E_7 and E_6 are closely related to del Pezzo surfaces of degree 1, 2 and 3 respectively ([**S2**], [**S3**]). In this paper, we study the relation between Mordell-Weil lattices of type D_5 ([**U**]) and del Pezzo surfaces of degree 4.

Let $f: S \to \boldsymbol{P}^1$ be a rational elliptic surface which has a section (O) and only one reducible singular fibre, of type I_4: $f^{-1}(t_0)=\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_3$. Then the (narrow) Mordell-Weil lattice of this surface is the root lattice D_5 ([**O-S**]).

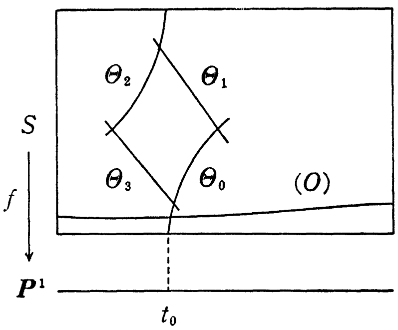

Figure 1.

Using surface theory, we can blow down (O), Θ_0 and Θ_3 in this order, and we get a smooth del Pezzo surface of degree 3, which we call S_3. By blowing down one more rational curve Θ_2, we get a smooth del Pezzo surface of degree 4, which we call S_4. In this situation, lines (exceptional curves of the first kind) on S_3 and S_4 are obtained from sections of $f: S \to \boldsymbol{P}^1$.

The contents of this paper are as follows. In section 2, starting from the elliptic curve which we have considered in [**U**] and [**S-U**] ("the excellent family of type D_5"), we describe the elliptic surface S explicitly, namely we represent S by gluing smooth surfaces defined by explicit equations. In section 3, we realize S_3 as a smooth cubic surface in $\boldsymbol{P}^3$, and S_4 as a complete intersection of two quadrics in $\boldsymbol{P}^4$ by writing down the defining equations for them. Finally in section 4, we give the equations of 27 lines on S_3 and 16 lines on S_4.

2. Description of the Kodaira-Néron model.

We consider the elliptic curve

$$E: y^2+p_5xy = x^3+p_4tx^2+(p_8t^2+p_2t^3)x+p_6t^4+t^5$$

defined over $K=k(t)$, where k is the algebraic closure of $\boldsymbol{Q}(\lambda)=\boldsymbol{Q}(p_2, p_4, p_5, p_6, p_8)$. Let

$$f: S \longrightarrow \boldsymbol{P}^1$$

denote the associated elliptic surface (the Kodaira-Néron model) of E/K. The theory of Mordell-Weil lattices says that the Mordell-Weil group $E(K)$ has a lattice structure ([**S1**]).

We assume the following two conditions on the parameter $\lambda=(p_2, \cdots, p_8)$:

($\natural$) $p_5 \neq 0$ and $p_5^2p_6-p_8^2 \neq 0$.

($\sharp$) $f: S \to \boldsymbol{P}^1$ has no reducible singular fibres other than $f^{-1}(0)$.

Then f has only one reducible singular fibre, at $t=0$, which is of type I_4. In this case we have $E(K)^0 \cong D_5$ and $E(K) \cong D_5^*$ as lattices ([**U**, Theorem 1]).

We describe the surface S explicitly. Let T^0, T^1 and T^2 be the surfaces defined as follows:

$$\begin{aligned}T^0 = \{&(x_0: y_0: z_0, s) \in \boldsymbol{P}^2 \times \boldsymbol{A}^1 \mid y_0^2z_0+p_5sx_0y_0z_0 \\ &= x_0^3+p_4sx_0^2z_0+(p_8s^2+p_2s)x_0z_0^2+(p_6s^2+s)z_0^3\}\end{aligned}$$

$$\begin{aligned}T^1 = \{&(x_1: y_1: z_1, t) \in \boldsymbol{P}^2 \times \boldsymbol{A}^1 \mid y_1^2z_1+p_5x_1y_1z_1 \\ &= tx_1^3+p_4tx_1^2z_1+(p_8t+p_2t^2)x_1z_1^2+(p_6t^2+t^3)z_1^3,\ (x_1: y_1: z_1, t) \neq (0:0:1, 0)\}\end{aligned}$$

$$\begin{aligned}T^2 = \{&(x_2: y_2: z_2, t, u) \in \boldsymbol{P}^2 \times \boldsymbol{A}^2 \mid uz_2 = tx_2,\ y_2^2+p_5x_2y_2 \\ &= tux_2^2+p_4tx_2^2+(p_8+p_2t)x_2z_2+(p_6+t)z_2^2\}.\end{aligned}$$

Let $\tilde{S}$ be the surface obtained by gluing T^0, T^1 and T^2 according to the following rules:

$$(x_1: y_1: z_1, t) = \left(sx_0: y_0: s^2z_0, \frac{1}{s}\right) \qquad \text{when } s \neq 0 \text{ and } t \neq 0,$$

$$(x_2: y_2: z_2, t, u) = \left(sx_0: y_0: sz_0, \frac{1}{s}, \frac{x_0}{sz_0}\right) \quad \text{when } sz_0 \neq 0 \text{ and } t \neq 0,$$

$$(x_2: y_2: z_2, t, u) = \left(x_1: y_1: tz_1, t, \frac{x_1}{z_1}\right) \qquad \text{when } z_1 \neq 0 \text{ and } (t, u) \neq (0, 0).$$

We define $\tilde{f}:\tilde{S}\to \boldsymbol{P}^1$ by

$$(x_0:y_0:z_0,\ s)\longrightarrow(1:s),$$

$$(x_1:y_1:z_1,\ t)\longrightarrow(t:1),$$

$$(x_2:y_2:z_2,\ t,\ u)\longrightarrow(t:1).$$

PROPOSITION 1. *$\tilde{f}:\tilde{S}\to \boldsymbol{P}^1$ is the Kodaira-Néron model of E/K.*

PROOF. By the uniqueness of the Kodaira-Néron model, we have only to show that $\tilde{S}$ is a nonsingular projective surface with generic fibre E and that no fibre has exceptional curves of the first kind. Since T^0 is obtained from E by letting $(x,\ y,\ t)=(x_0/s^2z_0,\ y_0/s^3z_0,\ 1/s)$, the generic fibre of $\tilde{f}$ is E.

Let $\bar{S}$ be the surface in $\boldsymbol{P}^2\times \boldsymbol{A}^1$ defined by the equation

$$Y^2Z+p_5XYZ=X^3+p_4tX^2Z+(p_8t^2+p_2t^3)XZ^2+(p_6t^4+t^5)Z^3.$$

$\bar{S}$ is obtained from E by letting $(x,\ y)=(X/Z,\ Y/Z)$.

It is known that the only singularities of the surface obtained by gluing $\bar{S}$ and T^0 are rational double points, and that S is the minimal resolution of the surface (cf. [**K**]). So the condition ($\sharp$) implies that $S-f^{-1}(0)\cong T^0$. Then T^0 is nonsingular and when $t\neq 0$, $\tilde{f}^{-1}(t)$ has no exceptional curves of the first kind.

To show that $\tilde{S}$ is nonsingular, we have only to show that T^1 and T^2 are nonsingular at the points satisfying $t=0$.

First we show that T^1 is nonsingular at the points satisfying $t=0$. Let

$$g(x_1,\ y_1,\ z_1,\ t)=y_1^2z_1+p_5x_1y_1z_1-(tx_1^3+p_4tx_1^2z_1+(p_8t+p_2t^2)x_1z_1^2+(p_6t^2+t^3)z_1^3).$$

If $(x_1:y_1:z_1,\ 0)\in T^1$ is a singular point, then we have

$$\left.\frac{\partial g}{\partial x_1}\right|_{t=0}=p_5y_1z_1=0 \tag{1}$$

$$\left.\frac{\partial g}{\partial y_1}\right|_{t=0}=2y_1z_1+p_5x_1z_1=0 \tag{2}$$

$$\left.\frac{\partial g}{\partial z_1}\right|_{t=0}=y_1^2+p_5x_1y_1=0 \tag{3}$$

$$\left.\frac{\partial g}{\partial t}\right|_{t=0}=-x_1^3-p_4x_1^2z_1-p_8x_1z_1^2=0. \tag{4}$$

If $z_1=0$, then $x_1=0$ by (4), and $y_1=0$ by (3). If $z_1\neq 0$, then $y_1=0$ by (1) and ($\natural$), and $x_1=0$ by (2) and ($\natural$). But $(x_1:y_1:z_1,\ t)=(0:0:1,\ 0)$ is not a point on T^1. So T^1 is nonsingular.

Next we show that T^2 is nonsingular at the points satisfying $t=0$. Let

$$h_1(x_2, y_2, z_2, t, u) = uz_2 - tx_2,$$

$$h_2(x_2, y_2, z_2, t, u) = y_2^2 + p_5x_2y_2 - (tux_2^2 + p_4tx_2^2 + (p_8+p_2t)x_2z_2 + (p_6+t)z_2^2).$$

Then the Jacobian matrix is

$$\begin{pmatrix} \left.\dfrac{\partial h_1}{\partial x_2}\right|_{t=0} & \left.\dfrac{\partial h_2}{\partial x_2}\right|_{t=0} \\ \left.\dfrac{\partial h_1}{\partial y_2}\right|_{t=0} & \left.\dfrac{\partial h_2}{\partial y_2}\right|_{t=0} \\ \left.\dfrac{\partial h_1}{\partial z_2}\right|_{t=0} & \left.\dfrac{\partial h_2}{\partial z_2}\right|_{t=0} \\ \left.\dfrac{\partial h_1}{\partial t}\right|_{t=0} & \left.\dfrac{\partial h_2}{\partial t}\right|_{t=0} \\ \left.\dfrac{\partial h_1}{\partial u}\right|_{t=0} & \left.\dfrac{\partial h_2}{\partial u}\right|_{t=0} \end{pmatrix} = \begin{pmatrix} 0 & p_5y_2 - p_8z_2 \\ 0 & 2y_2 + p_5x_2 \\ u & -p_8x_2 - 2p_6z_2 \\ -x_2 & -ux_2^2 - p_4x_2^2 - p_2x_2z_2 - z_2^2 \\ z_2 & 0 \end{pmatrix}.$$

When $z_2=0$, we have $x_2\neq 0$ by $h_2=0$. If $(\partial h_2/\partial x_2)_{t=0}=0$, we have $y_2=0$ by (♮), then $(\partial h_2/\partial y_2)_{t=0}\neq 0$. Since $(\partial h_1/\partial t)_{t=0}=-x_2\neq 0$, the rank of the Jocobian matrix is 2.

When $z_2\neq 0$, if $(\partial h_2/\partial x_2)_{t=0}=(\partial h_2/\partial y_2)_{t=0}=0$, then we have

$$\begin{aligned} \left.\frac{\partial h_2}{\partial z_2}\right|_{t=0} &= -p_8\cdot\frac{-2}{p_5}y_2 - 2p_6z_2 \\ &= -p_8\cdot\frac{-2}{p_5}\cdot\frac{p_8}{p_5}z_2 - 2p_6z_2 \\ &= \frac{2}{p_5^2}(p_8^2 - p_5^2p_6)z_2. \end{aligned}$$

This is not equal to 0 by (♮). Since $(\partial h_1/\partial u)_{t=0}=z_2\neq 0$, the rank of the Jacobian matrix is 2. So T^2 is nonsingular.

Lastly we show that $\tilde{f}^{-1}(0)$ has no exceptional curves of the first kind. We have

$$\tilde{f}^{-1}(0) = \Theta_0\cup\Theta_1\cup\Theta_2\cup\Theta_3,$$

where Θ_0 is the rational curve $\{z_1=0\}$, Θ_1 is the rational curve obtained by gluing $\{y_1=0,\ x_1\neq 0\}$ and $\{z_2=y_2=0\}$ by $u=x_1/z_1$, Θ_2 is the rational curve $\{u=0,\ y_2^2+p_5x_2y_2=p_8x_2z_2+p_6z_2^2\}$, Θ_3 is the rational curve obtained by gluing $\{y_1+p_5x_1=0,\ x_1\neq 0\}$ and $\{z_2=y_2+p_5x_2=0\}$ by $u=x_1/z_1$. If $\tilde{f}^{-1}(0)$ has an exceptional curve of the first kind, then we can blow it down and get a smooth model whose number of components of the fibre at $t=0$ is less than 4. On the other hand we know that the Kodaira-Néron model has a reducible singular fibre of type I_4 at $t=0$. So $\tilde{f}^{-1}(0)$ has no exceptional curves of the first kind, and $\tilde{f}:\tilde{S}\to\boldsymbol{P}^1$ is the Kodaira-Néron model of E/K. *q.e.d.*

REMARK. The surface T^1 is obtained from E by letting $(x, y)=(tx_1/z_1, ty_1/z_1)$ and removing the point $(0:0:1, 0)$. The surface T^2 is obtained from E by letting $(x, y)=(t^2x_2/z_2, t^2y_2/z_2)$ and introducing u such that $uz_2=tx_2$ (cf. [**BLR**, § 1.5]).

From now on, we identify $f:S\to\boldsymbol{P}^1$ with $\tilde{f}:\tilde{S}\to\boldsymbol{P}^1$.

3. Del Pezzo surfaces obtained from S.

First we define two surfaces S_3 and S_4. The surface S_3 is obtained from S by blowing down the zero section (O), Θ_0 and Θ_3. The surface S_4 is obtained from S by blowing down (O), Θ_0, Θ_3 and Θ_2. To be exact, S_3 and S_4 are obtained as follows.

The zero section (O), which is $(x_0:y_0:z_0, s)=(0:1:0, s)$ in T^0 and $(x_1:y_1:z_1, t)=(0:1:0, t)$ in T^1, is an exceptional curve of the first kind ([**S1**, Theorem 2.8]). When we blow it down, we have a birational morphism $\pi_1:S\to S_1$. Since $(\Theta_0^2)=-2$ and $(\Theta_0\cdot(O))=1$, $\pi_1(\Theta_0)$ is an exceptional curve of the first kind on S_1. Next we blow down $\pi_1(\Theta_0)$. Then we have a birational morphism $\pi_2:S_1\to S_2$, under which $\pi_1(\Theta_3)$ is mapped to an exceptional curve of the first kind on S_2. Then we blow down $\pi_2\circ\pi_1(\Theta_3)$ and we have a birational morphism $\pi_3:S_2\to S_3$. Under this morphism $\pi_2\circ\pi_1(\Theta_2)$ is mapped to an exceptional curve of the first kind. By blowing it down, we obtain a birational morphism $\pi_4:S_3\to S_4$.

The surfaces S_3 and S_4 are described explicitly as follows.

THEOREM 2. *Let S_3 be the surface obtained from S by blowing down (O), Θ_0 and Θ_3 as above. Then S_3 is a smooth del Pezzo surface of degree 3 and it is isomorphic to the cubic surface $\tilde{S}_3$ in $\boldsymbol{P}^3$ defined by*

$$\tilde{S}_3: \qquad Y^2Z+p_5WXY=X^3+p_4WX^2+p_8W^2X+p_2WXZ+p_6W^2Z+WZ^2.$$

THEOREM 3. *Let S_4 be the surface obtained from S by blowing down (O), Θ_0, Θ_3 and Θ_2 as above. Then S_4 is a smooth del Pezzo surface of degree 4 and it is isomorphic to the $(2, 2)$-type complete intersection $\tilde{S}_4$ in $\boldsymbol{P}^4$ defined by*

$$\tilde{S}_4: \qquad \begin{cases} V'X'=Y'^2-p_6W'^2-W'Z' \\ V'Z'=X'^2+p_4W'X'+p_8W'^2+p_2W'Z'-p_5W'Y'. \end{cases}$$

PROOF OF THEOREM 2. S is a smooth rational surface ([**S1**, (10.14)]) and S_3 is obtained from S by a sequence of blowing-down of exceptional curves of the first kind, so S_3 is a smooth rational surface. Let F be a fibre of f. The canonical divisor of S is $-F$ ([**S1**, Theorem 2.8]). Let $F_1=\pi_1(F)$, $F_2=\pi_2(F_1)$

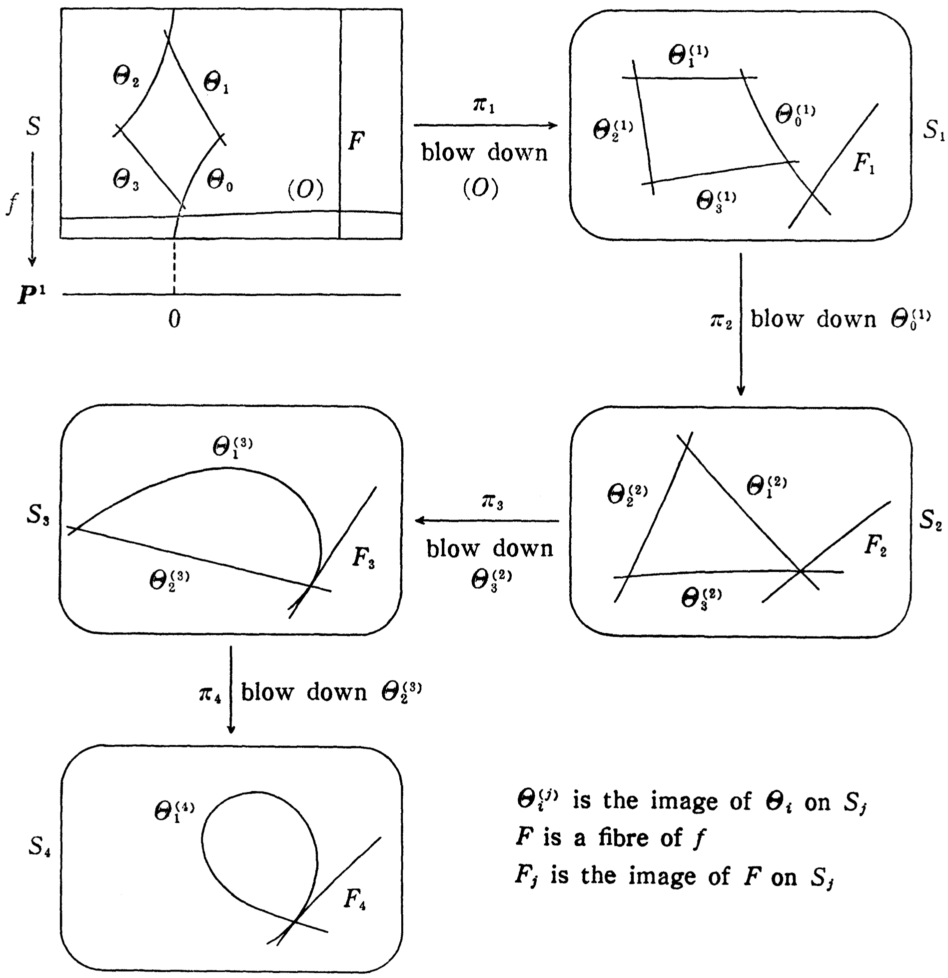

Figure 2.

and $F_3=\pi_3(F_2)$. The canonical divisor of S_3 is $-F_3$ and $(F_3^2)=3$. If C is an irreducible curve on S_3, then we have $(C\cdot F_3)\geqq 0$ (we may assume that F_3 is an irreducible curve). Now we assume $(C\cdot F_3)=0$. Then $C_2=\pi_3^*C$ is an irreducible curve on S_2 and $(C_2\cdot F_2)=0$, so $C_1=\pi_2^*C_2$ is an irreducible curve on S_1 and $(C_1\cdot F_1)=0$, hence $C_0=\pi_1^*C_1$ is an irreducible curve on S and $(C_0\cdot F)=0$. So C_0 is an irreducible component of a fibre of $f: S\to \boldsymbol{P}^1$. Since C is a curve, $C_0\neq\Theta_0$ and $C_0\neq\Theta_3$. If $C_0=F$ then $(C\cdot F_3)=3$, if $C_0=\Theta_1$ then $(C\cdot F_3)=2$, and if $C_0=\Theta_2$ then $(C\cdot F_3)=1$. This contradicts the assumption that $(C\cdot F_3)=0$, so we have $(C\cdot F_3)>0$. This shows that the anti-canonical divisor F_3 on S_3 is an ample divisor, so S_3 is a del Pezzo surface of degree 3 $((F_3^2)=3)$.

Next we define a morphism $\varphi: S \to \tilde{S}_3$ as follows.

$$\varphi|_{T^0}: (x_0 : y_0 : z_0,\ s) \longrightarrow (W : X : Y : Z) = (sz_0 : x_0 : y_0 : z_0),$$

$$\varphi|_{T^1}: (x_1 : y_1 : z_1,\ t) \longrightarrow (W : X : Y : Z) = (tz_1 : tx_1 : y_1 : t^2 z_1),$$

$$\varphi|_{T^2}: (x_2 : y_2 : z_2,\ t,\ u) \longrightarrow (W : X : Y : Z) = (z_2 : tx_2 : y_2 : tz_2).$$

This definition is compatible with the gluing, so the morphism is well-defined. Under this morphism, (O), Θ_0 and Θ_3 are mapped to one point $P_0=(0:0:1:0)$. Let us show the isomorphism $S' := S-((O)\cup\Theta_0\cup\Theta_3) \cong \tilde{S}_3 - \{P_0\}$. By the defining equation of $\tilde{S}_3$, for the point of $\tilde{S}_3-\{P_0\}$, we have $W\neq 0$ or $Z\neq 0$. When $Z\neq 0$, let $\alpha_1 : \{Z\neq 0\} \to T^0-(O)$ be the morphism defined by

$$(x_0 : y_0 : z_0,\ s) = \left(X : Y : Z,\ \frac{W}{Z}\right).$$

The morphism $\varphi\circ\alpha_1$ is the identity morphism on $\{Z\neq 0\}$. When $W\neq 0$, let $\alpha_2 : \{W\neq 0\} \to T^2-\Theta_3$ be the morphism defined by

$$(x_2 : y_2 : z_2,\ t,\ u) = \left(X : \frac{YZ}{W} : Z,\ \frac{Z}{W},\ \frac{X}{W}\right).$$

When $X=Z=0$, $(X:Z)=(Y^2-p_6W^2 : p_8W^2-p_5WY)$. By the condition ($\natural$), we have $(Y^2-p_6W^2,\ p_8W^2-p_5WY)\neq(0,0)$, so α_2 is a well-defined morphism on $\{W\neq 0\}$. The morphism $\varphi\circ\alpha_2$ is the identity morphism on $\{W\neq 0\}$. We can check that $\alpha_1=\alpha_2$ on $\{W\neq 0\}\cap\{Z\neq 0\}$, so by gluing them we get a morphism $\alpha : \tilde{S}_3-\{P_0\} \to S'$. The morphism $\alpha\circ\varphi|_{S'}$ is the identity morphism on S', so $\varphi|_{S'} : S' \to \tilde{S}_3-\{P_0\}$ is the isomorphism. This shows the isomorphism $S_3-\{\pi_3\circ\pi_2\circ\pi_1((O)\cup\Theta_0\cup\Theta_3)\}\cong\tilde{S}_3-\{P_0\}$. If we let

$$\begin{aligned} &m(W, X, Y, Z) \\ &= Y^2Z+p_5WXY-(X^3+p_4WX^2+p_8W^2X+p_2WXZ+p_6W^2Z+WZ^2), \end{aligned}$$

then

$$\left.\frac{\partial m}{\partial Z}\right|_{P_0} \neq 0.$$

This shows the non-singularity of $\tilde{S}_3$ at P_0, and we get $S_3\cong\tilde{S}_3$. *q.e.d.*

In T^2, the curve Θ_1 is $\{(x_2 : y_2 : z_2,\ t,\ u)=(1:0:0,\ 0,\ u)\}$. By the defining equation of T^2, we have

$$\frac{y_2}{x_2}\frac{y_2}{z_2}+p_5\frac{y_2}{z_2} = u^2+p_4u+p_8+p_2t+(p_6+t)\frac{z_2}{x_2}.$$

When $(x_2 : y_2 : z_2,\ t,\ u)=(1:0:0,\ 0,\ u)$, we have

$$p_5\frac{y_2}{z_2}=u^2+p_4u+p_8.$$

Since

$$\varphi(x_2:y_2:z_2,\ t,\ u)=(z_2:tx_2:y_2:tz_2)=\left(1:u:\frac{y_2}{z_2}:t\right),$$

the image of Θ_1 is in the curve $\{p_5WY=X^2+p_4WX+p_8W^2,\ Z=0\}$. When $W\neq 0$, α_2 is the inverse morphism of φ. When $W=0$, the curve $\{p_5WY=X^2+p_4WX+p_8W^2,\ Z=0\}$ has only one point $P_0=(0:0:1:0)$, and this is the image of the point $\Theta_1-\Theta_1\cap T^2$. So Θ_1 is mapped to the curve $\{p_5WY=X^2+p_4WX+p_8W^2,\ Z=0\}$.

The curve Θ_2 is $\{y_2^2+p_5x_2y_2=p_8x_2z_2+p_6z_2^2,\ t=u=0\}$. Since

$$\varphi(x_2:y_2:z_2,\ t,\ u)=(z_2:tx_2:y_2:tz_2),$$

the image is in the curve $\{X=Z=0\}$. When $W\neq 0$, α_2 is the inverse morphism of φ. When $W=0$, the curve $\{X=Z=0\}$ has only one point $P_0=(0:0:1:0)$, and this is the image of the point $(1:-p_5:0,\ 0,\ 0)$. So Θ_2 is mapped to the curve $\{X=Z=0\}$.

Proof of Theorem 3. In the same way as in the proof of Theorem 2, we can show that the surface S_4 is a smooth del Pezzo surface of degree 4.

We define a morphism $\psi:\tilde{S}_3\to\tilde{S}_4$ by

$$(V':W':X':Y':Z')=(Y^2-p_6W^2-WZ:WX:X^2:XY:XZ).$$

When $X=0$, by the defining equation of $\tilde{S}_3$, we have $(Y^2-p_6W^2-WZ)Z=0$. If $Z\neq 0$ then

$$\begin{aligned}&(Y^2-p_6W^2-WZ:WX:X^2:XY:XZ)\\&=(X^2+p_4WX+p_8W^2+p_2WZ-p_5WY:WZ:XZ:YZ:Z^2).\end{aligned}$$

If $X=Z=0$, by the condition ($\natural$), we have

$$(Y^2-p_6W^2-WZ,\ X^2+p_4WX+p_8W^2+p_2WZ-p_5WY)\neq(0,\ 0),$$

so the line $\{X=Z=0\}$ is mapped to the point $Q_0=(1:0:0:0:0)$. When $(V':W':X':Y':Z')\neq Q_0$, $(W:X:Y:Z)=(W':X':Y':Z')$ defines the inverse morphism of $\psi|_{\tilde{S}_3-\{X=Z=0\}}$, so $\tilde{S}_3-\{X=Z=0\}\cong\tilde{S}_4-\{Q_0\}$. Since $\{X=Z=0\}=\varphi(\Theta_2)$, we have the isomorphism $S_4-\{\pi_4\circ\pi_3\circ\pi_2\circ\pi_1((O)\cup\Theta_0\cup\Theta_3\cup\Theta_2)\}\cong\tilde{S}_4-\{Q_0\}$. If we let

$$n_1(V',\ W',\ X',\ Y',\ Z')=V'X'-(Y'^2-p_6W'^2-W'Z'),$$

$$n_2(V',\ W',\ X',\ Y',\ Z')=V'Z'-(X'^2+p_4W'X'+p_8W'^2+p_2W'Z'-p_5W'Y'),$$

then the Jacobian matrix at Q_0 is

$$\begin{pmatrix} \left.\frac{\partial n_1}{\partial V'}\right|_{Q_0} & \left.\frac{\partial n_2}{\partial V'}\right|_{Q_0} \\ \left.\frac{\partial n_1}{\partial W'}\right|_{Q_0} & \left.\frac{\partial n_2}{\partial W'}\right|_{Q_0} \\ \left.\frac{\partial n_1}{\partial X'}\right|_{Q_0} & \left.\frac{\partial n_2}{\partial X'}\right|_{Q_0} \\ \left.\frac{\partial n_1}{\partial Y'}\right|_{Q_0} & \left.\frac{\partial n_2}{\partial Y'}\right|_{Q_0} \\ \left.\frac{\partial n_1}{\partial Z'}\right|_{Q_0} & \left.\frac{\partial n_2}{\partial Z'}\right|_{Q_0} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows the non-singularity of $\tilde{S}_4$ at Q_0, and we have $S_4 \cong \tilde{S}_4$. *q.e.d.*

4. Lines on S_3 and S_4.

There are 27 lines on a del Pezzo surface of degree 3, and they are exceptional curves of the first kind. A section of $f: S \to \boldsymbol{P}^1$ is an exceptional curve of the first kind on S ([**S1**]). If a section (P) does not meet (O), Θ_0 nor Θ_3, by $\pi_3 \circ \pi_2 \circ \pi_1$, (P) is mapped to an exceptional curve of the first kind on S_3. Such a section is one of the following two types:

i) (P) such that $((P)\cdot(O))=0$ and $((P)\cdot\Theta_2)=1$. It is of the form

$$\begin{cases} x = gt^2 \\ y = ht^3 + ct^2 \quad g, h, c \in k. \end{cases}$$

ii) (P) such that $((P)\cdot(O))=0$ and $((P)\cdot\Theta_1)=1$. It is of the form

$$\begin{cases} x = gt^2 + at \\ y = ht^3 + ct^2 \quad g, a, h, c \in k, \ a \neq 0. \end{cases}$$

There are 10 sections of type i) and 16 sections of type ii) ([**U**]).

The curve Θ_2 is also mapped to an exceptional curve of the first kind on S_3. So a line on $\tilde{S}_3$ is one of the following three types:

i) $\varphi((P))$ for (P) of type i)
ii) $\varphi((P))$ for (P) of type ii)
iii) $\varphi(\Theta_2)$.

If the section (P) of type i) is

$$\begin{cases} x = gt^2 \\ y = ht^3 + ct^2 \quad g, h, c \in k, \end{cases}$$

then $\varphi((P))$ is the line

$$\begin{cases} X = gZ \\ Y = hZ+cW. \end{cases}$$

If the section (P) of type ii) is

$$\begin{cases} x = gt^2+at \\ y = ht^3+ct^2 \quad g,\ a,\ h,\ c \in k, \end{cases}$$

then $\varphi((P))$ is the line

$$\begin{cases} X = gZ+aW \\ Y = hZ+cW. \end{cases}$$

$\varphi(\Theta_2)$ is the line

$$\begin{cases} X = 0 \\ Z = 0. \end{cases}$$

There are 16 lines on a del Pezzo surface of degree 4, and they are exceptional curves of the first kind. If (P) is a section of type ii), by $\pi_4\circ\pi_3\circ\pi_2\circ\pi_1$, (P) is mapped to an exceptional curve of the first kind on S_4. So a line on $\tilde{S}_4$ is $\phi\circ\varphi((P))$ for a section (P) of type ii).

If (P) is

$$\begin{cases} x = gt^2+at \\ y = ht^3+ct^2 \quad g,\ a,\ h,\ c \in k, \end{cases}$$

then $\phi\circ\varphi((P))$ is the line

$$\begin{cases} X' = gZ'+aW' \\ Y' = hZ'+cW' \\ V' = g^2Z'+(2ag+p_4g+p_2-p_5h)W'. \end{cases}$$

Now we obtain the following corollary by [**U**, Theorem 4].

COROLLARY 4. *Take $u_1, \cdots, u_5 \neq 0$ such that $u_1^2, \cdots, u_5^2$ are mutually distinct and for any choice of signs,*

$$\pm u_1 \pm \cdots \pm u_5 \neq 0.$$

Let

$$\begin{cases} p_2 = -\dfrac{1}{2}\varepsilon_2 \\ p_4 = \dfrac{1}{2}\varepsilon_4 - \dfrac{1}{2}p_2^2 \end{cases}$$

$$\begin{cases} p_6 = \frac{1}{4}\varepsilon_6 + \frac{1}{2}p_2p_4 \\ p_8 = -\frac{1}{4}\varepsilon_8 + \frac{1}{4}p_4^2 \\ p_5 = u_1u_2u_3u_4u_5 . \end{cases}$$

Here $\varepsilon_{2\nu}$ is the ν-th elementary symmetric function of $u_1^2, \cdots, u_5^2$.

Then 27 lines on the cubic surface

$$\tilde{S}_3: \qquad Y^2Z + p_5WXY = X^3 + p_4WX^2 + p_8W^2X + p_2WXZ + p_6W^2Z + WZ^2$$

are given as follows:

i) 5 *lines*

$$\begin{cases} X = u_i^{-2}Z \\ Y = u_i^{-3}Z + c_iW \quad (i = 1, 2, 3, 4, 5), \end{cases}$$

where

$$c_i = \frac{1}{2}(p_4u_i^{-1} + p_2u_i + u_i^3 - p_5u_i^{-2}).$$

5 *lines*

$$\begin{cases} X = u_i^{-2}Z \\ Y = -u_i^{-3}Z - (p_5u_i^{-2} + c_i)W \quad (i = 1, 2, 3, 4, 5), \end{cases}$$

where

$$c_i = \frac{1}{2}(p_4u_i^{-1} + p_2u_i + u_i^3 - p_5u_i^{-2}).$$

ii) 16 *lines*

$$\begin{cases} X = u^{-2}Z + aW \\ Y = u^{-3}Z + cW. \end{cases}$$

Here $u = \sigma(u_0)$, $a = \sigma(a_0)$, $c = \sigma(c_0)$ are the transforms of u_0, a_0, c_0 below under the sign change σ of even number of $u_1, \cdots, u_5$.

$$u_0 = \frac{1}{2}(u_1 + \cdots + u_5),$$

$$a_0 = u_0^{-1}\prod_{i=1}^{5}(u_i - u_0),$$

$$c_0 = \frac{1}{2}(3a_0u_0^{-1} + p_4u_0^{-1} + p_2u_0 + u_0^3 - p_5u_0^{-2}).$$

iii) 1 *line*

$$\begin{cases} X = 0 \\ Z = 0. \end{cases}$$

COROLLARY 5. *Under the same assumption as Corollary* 4, 16 *lines on the del Pezzo surface of degree* 4

$$\tilde{S}_4: \quad \begin{cases} V'X' = Y'^2 - p_6W'^2 - W'Z' \\ V'Z' = X'^2 + p_4W'X' + p_8W'^2 + p_2W'Z' - p_5W'Y' \end{cases}$$

are given as follows:

$$\begin{cases} X' = u^{-2}Z' + aW' \\ Y' = u^{-3}Z' + cW' \\ V' = u^{-4}Z' + (2au^{-2} + p_4u^{-2} + p_2 - p_5u^{-3})W', \end{cases}$$

where u, a *and* c *are the same as in Corollary* 4.

If we take $u_1, \cdots, u_5 \in \boldsymbol{Q}$, then we get a del Pezzo surface of degree 3 and 27 lines on it defined over $\boldsymbol{Q}$, and a del Pezzo surface of degree 4 and 16 lines on it defined over $\boldsymbol{Q}$.

EXAMPLE 1. If we take $(u_1, \cdots, u_5) = (1, 2, 3, 4, 5)$, then $\tilde{S}_3$ and the 27 lines on it are as follows:

$$\tilde{S}_3: Y^2Z + 120WXY = X^3 + \frac{1067}{8}WX^2 - \frac{210375}{256}W^2X - \frac{55}{2}WXZ + \frac{2475}{32}W^2Z + WZ^2.$$

$$X = Z, \qquad Y = -\frac{105}{16}W + Z$$

$$X = \frac{1}{4}Z, \qquad Y = -\frac{165}{32}W + \frac{1}{8}Z$$

$$X = \frac{1}{9}Z, \qquad Y = -\frac{195}{16}W + \frac{1}{27}Z$$

$$X = \frac{1}{16}Z, \qquad Y = -\frac{645}{64}W + \frac{1}{64}Z$$

$$X = \frac{1}{25}Z, \qquad Y = \frac{75}{16}W + \frac{1}{125}Z$$

$$X = Z, \qquad Y = \frac{1815}{16}W - Z$$

$$X = \frac{1}{4}Z, \qquad Y = -\frac{795}{32}W - \frac{1}{8}Z$$

$$X = \frac{1}{9}Z, \qquad Y = -\frac{55}{48}W - \frac{1}{27}Z$$

$$X = \frac{1}{16}Z, \qquad Y = \frac{165}{64}W - \frac{1}{64}Z$$

$$X = \frac{1}{25}Z, \qquad Y = -\frac{759}{80}W - \frac{1}{125}Z$$

$$X=-\frac{3003}{16}W+\frac{4}{225}Z, \qquad Y=\frac{781}{10}W+\frac{8}{3375}Z$$

$$X=-\frac{3675}{16}W+\frac{4}{9}Z, \qquad Y=\frac{355}{2}W-\frac{8}{27}Z$$

$$X=-\frac{6075}{16}W+4Z, \qquad Y=\frac{1545}{2}W-8Z$$

$$X=\frac{1701}{16}W+4Z, \qquad Y=\frac{411}{2}W+8Z$$

$$X=\frac{1925}{16}W+4Z, \qquad Y=\frac{495}{2}W+8Z$$

$$X=-\frac{539}{16}W+\frac{4}{9}Z, \qquad Y=-\frac{209}{6}W+\frac{8}{27}Z$$

$$X=-\frac{891}{16}W+\frac{4}{25}Z, \qquad Y=-\frac{429}{10}W+\frac{8}{125}Z$$

$$X=-\frac{14175}{208}W+\frac{4}{169}Z, \qquad Y=-\frac{14835}{338}W-\frac{8}{2197}Z$$

$$X=\frac{325}{16}W+\frac{4}{9}Z, \qquad Y=\frac{115}{6}W+\frac{8}{27}Z$$

$$X=-\frac{91}{16}W+\frac{4}{25}Z, \qquad Y=-\frac{129}{10}W+\frac{8}{125}Z$$

$$X=-\frac{1053}{112}W+\frac{4}{49}Z, \qquad Y=-\frac{1623}{98}W+\frac{8}{343}Z$$

$$X=-\frac{2025}{176}W+\frac{4}{121}Z, \qquad Y=-\frac{4485}{242}W-\frac{8}{1331}Z$$

$$X=\frac{143}{48}W+\frac{4}{81}Z, \qquad Y=-\frac{187}{54}W+\frac{8}{729}Z$$

$$X=\frac{175}{48}W+\frac{4}{81}Z, \qquad Y=-\frac{145}{54}W-\frac{8}{729}Z$$

$$X=\frac{675}{112}W+\frac{4}{49}Z, \qquad Y=\frac{15}{98}W-\frac{8}{343}Z$$

$$X=-\frac{27}{16}W+\frac{4}{25}Z, \qquad Y=-\frac{87}{10}W-\frac{8}{125}Z$$

$$X=0, \qquad Z=0.$$

EXAMPLE 2. If we take $(u_1, \cdots, u_5)=(1, 2, 3, 4, 5)$, $\tilde{S}_4$ and 16 lines on it are as follows:

$$\tilde{S}_4: \quad \begin{cases} V'X' = Y'^2 - \dfrac{2475}{32}W'^2 - W'Z' \\ V'Z' = X'^2 + \dfrac{1067}{8}W'X' - \dfrac{210375}{256}W'^2 - \dfrac{55}{2}W'Z' - 120W'Y' \end{cases}$$

$$X' = -\frac{3003}{16}W' + \frac{4}{225}Z', \quad Y' = \frac{781}{10}W' + \frac{8}{3375}Z', \quad V' = -\frac{4813}{150}W' + \frac{16}{50625}Z'$$

$$X' = -\frac{3675}{16}W' + \frac{4}{9}Z', \quad Y' = \frac{355}{2}W' - \frac{8}{27}Z', \quad V' = -\frac{821}{6}W' + \frac{16}{81}Z'$$

$$X' = -\frac{6075}{16}W' + 4Z', \quad Y' = \frac{1545}{2}W' - 8Z', \quad V' = -\frac{3143}{2}W' + 16Z'$$

$$X' = \frac{1701}{16}W' + 4Z', \quad Y' = \frac{411}{2}W' + 8Z', \quad V' = \frac{793}{2}W' + 16Z'$$

$$X' = \frac{1925}{16}W' + 4Z', \quad Y' = \frac{495}{2}W' + 8Z', \quad V' = \frac{1017}{2}W' + 16Z'$$

$$X' = -\frac{539}{16}W' + \frac{4}{9}Z', \quad Y' = -\frac{209}{6}W' + \frac{8}{27}Z', \quad V' = -\frac{607}{18}W' + \frac{16}{81}Z'$$

$$X' = -\frac{891}{16}W' + \frac{4}{25}Z', \quad Y' = -\frac{429}{10}W' + \frac{8}{125}Z', \quad V' = -\frac{1583}{50}W' + \frac{16}{625}Z'$$

$$X' = -\frac{14175}{208}W' + \frac{4}{169}Z', \quad Y' = -\frac{14835}{338}W' - \frac{8}{2197}Z', \quad V' = -\frac{119219}{4394}W' + \frac{16}{28561}Z'$$

$$X' = \frac{325}{16}W' + \frac{4}{9}Z', \quad Y' = \frac{115}{6}W' + \frac{8}{27}Z', \quad V' = \frac{257}{18}W' + \frac{16}{81}Z'$$

$$X' = -\frac{91}{16}W' + \frac{4}{25}Z', \quad Y' = -\frac{129}{10}W' + \frac{8}{125}Z', \quad V' = -\frac{783}{50}W' + \frac{16}{625}Z'$$

$$X' = -\frac{1053}{112}W' + \frac{4}{49}Z', \quad Y' = -\frac{1623}{98}W' + \frac{8}{343}Z', \quad V' = -\frac{14369}{686}W' + \frac{16}{2401}Z'$$

$$X' = -\frac{2025}{176}W' + \frac{4}{121}Z', \quad Y' = -\frac{4485}{242}W' - \frac{8}{1331}Z', \quad V' = -\frac{61573}{2662}W' + \frac{16}{14641}Z'$$

$$X' = \frac{143}{48}W' + \frac{4}{81}Z', \quad Y' = -\frac{187}{54}W' + \frac{8}{729}Z', \quad V' = -\frac{10661}{486}W' + \frac{16}{6561}Z'$$

$$X' = \frac{175}{48}W' + \frac{4}{81}Z', \quad Y' = -\frac{145}{54}W' - \frac{8}{729}Z', \quad V' = -\frac{9349}{486}W' + \frac{16}{6561}Z'$$

$$X' = \frac{675}{112}W' + \frac{4}{49}Z', \quad Y' = \frac{15}{98}W' - \frac{8}{343}Z', \quad V' = -\frac{8801}{686}W' + \frac{16}{2401}Z'$$

$$X' = -\frac{27}{16}W' + \frac{4}{25}Z', \quad Y' = -\frac{87}{10}W' - \frac{8}{125}Z', \quad V' = \frac{49}{50}W' + \frac{16}{625}Z'$$

References

[BLR] S. Bosch, W. Lütkebohmert and M. Raynaud, Néron Models, Springer-Verlag, 1990.
[K] A. Kas, Weierstrass normal forms and invariants of elliptic surfaces, Trans. Amer. Math. Soc., **225** (1977), 259-266.
[O-S] K. Oguiso and T. Shioda, The Mordell-Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Paul., **40**, No. **1** (1991), 83-99.
[S1] T. Shioda, On the Mordell-Weil lattices, Comment. Math. Univ. St. Paul., **39**, No. **2** (1990), 211-240.
[S2] T. Shioda, Theory of Mordell-Weil lattices, Proc. ICM Kyoto 1990, vol. **I**, 1991, pp. 473-489.
[S3] T. Shioda, Plane quartics and Mordell-Weil lattices of type E_7, Comment. Math. Univ. St. Paul., **42**, No. **1** (1993), 61-79.
[S-U] T. Shioda and H. Usui, Fundamental invariants of Weyl groups and excellent families of elliptic curves, Comment. Math. Univ. St. Paul., **41**, No. **2** (1992), 169-217.
[U] H. Usui, On Mordell-Weil lattices of type D_5, Math. Nachr., **161** (1993), 219-232.

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