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Invariant Kohn-Rossi cohomology and obstruction to embedding of compact real $(2n-1)$-dimensional CR manifolds in C^N

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§ 0. Introduction.

Let X be a compact connected CR manifold of real dimension $2n-1$. One of the most important invariants in CR-geometry is the Kohn-Rossi cohomology $H_{KR}^{p,q}(X)$ introduced by Kohn and Rossi in 1965 [**Ko-Ro**]. Throughout this paper we shall assume that $n\geqq 3$ and X is strongly pseudoconvex. As a consequence of Kohn's solution to the $\bar{\partial}$-Neumann problem, Kohn-Rossi showed that $H_{KR}^{p,q}(X)$ is finite dimensional if $1\leqq q\leqq n-2$. In 1974, Boutet de Monvel [**Bo**] (see also Kohn [**Ko$_3$**]) proved that X is CR-embeddable in some C^N. There are two fundamental questions raised by the theorem of Boutet de Monvel. The first one is the complex Plateau problem. Specifically the problem asks whether such X is a boundary of complex submanifold in C^N. The celebrated theorem of Harvey and Lawson [**Ha-La**] in 1975 asserts that X is the boundary of a subvariety V in C^N. In [**Ya**], the second named author related the Kohn-Rossi cohomology to the local cohomology at the singularities of the subvariety. In case $N=n+1$, the Kohn-Rossi cohomology was computed explicitly. This allows him to conclude that X is the boundary of a complex submanifold in C^N if and only if $H_{KR}^{p,q}(X)=0$ for $1\leqq q\leqq n-2$. The second fundamental question is to find the minimal embedding dimension. The key contribution of [**Ko$_3$**] is the study of a pseudo-differential operator on functions in $X=\partial V$ which is the transfer, via the Dirichlet problem, of the operator of the form $\sum a_j \partial/\partial\bar{z}_j$ normal to ∂V. As was pointed out by Kohn [**Ko$_3$**], there are indications that the study of this operator will provide us with obstructions to embedding in a space of given dimension. The only example of such obstructions is in the second named author's previous work [**Ya**] which in particular implies that for $n\geqq 3$, certain $(2n-1)$-dimensional CR manifolds which are embedded in C^{n+1}, cannot be embedded in C^n. In fact, as shown in [**Lu-Ya**] there are obstructions for embedding $(2n-1)$-dimensional CR manifolds in C^{n+1} as well, by the work of [**Ya**].

In this paper we shall consider a strongly pseudoconvex CR manifold X

which admits a transversal holomorphic S^1 action in the sense of Lawson-Yau [**La-Ya**]. Following Tanaka [**Ta**], this S^1-action defines naturally a differential operator N which acts on the Kohn-Rossi cohomology. Denote by $\tilde{H}_{KR}^{p,q}(X)$ those parts of $H_{KR}^{p,q}(X)$ on which N acts trivially. The purpose of this paper is to show the following result.

THEOREM A. *Let X be a strongly pseudoconvex compact CR manifold of dimension $2n-1$, $n\geqq 3$, which admits a transversal holomorphic S^1-action. Suppose that X is CR-embeddable in $\boldsymbol{C}^N$. Then the invariant Kohn-Rossi cohomology $\tilde{H}_{KR}^{p,q}(X)=0$ for all $1\leqq p+q\leqq 2n-N-1$.*

COROLLARY B. *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1$ with a transversal holomorphic S^1-action. Suppose that $\tilde{H}_{KR}^{p,q}(X)\neq 0$ for some (p, q) such that $1\leqq p+q\leqq 2n-N-1$. Then X is not CR-embeddable in $\boldsymbol{C}^N$.*

REMARK. The above Corollary gives obstructions to embedding a strongly pseudoconvex CR-manifold of dimension $2n-1$ with a transversal holomorphic S^1-action in $\boldsymbol{C}^N$ where $N\leqq 2n-2$.

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In § 1 we shall recall the definition of the invariant Kohn-Rossi cohomology $\tilde{H}_{KR}^{p,q}(X)$ on a CR-manifold with a transversal holomorphic S^1-action. In § 2, we shall recall Kohn's beautiful harmonic theory and Tanaka's differential geometric study on a strongly pseudoconvex CR manifold. We follow Tanaka's approach (cf. [**Ta**]). It is included here partly for the sake of clarity of the paper and partly for the sake of convenience to the readers because it seems to us that the beautiful work of Tanaka [**Ta**] is not easily accessible to many mathematicians. In § 3, we prove Theorem A.

§ 1. Invariant Kohn-Rossi cohomology.

In [**Ko$_2$**], Kohn considered the $\bar{\partial}_b$ complex intrinsically on a compact CR manifold of real dimension $2n-1$. Unfortunately, his definition of the $\bar{\partial}_b$ complex is different from Kohn-Rossi's $\bar{\partial}_b$ complex which was originally considered by Kohn-Rossi in [**Ko-Ro**]. Following Tanaka [**Ta**], we reformulate the $\bar{\partial}_b$ complex in a way independent of the interior manifold.

DEFINITION 1.1. Let X be a connected real manifold of dimension $2n-1$ and S be an $(n-1)$-dimensional subbundle S of CTX such that

(1) $S\cap\bar{S}=\{0\}$

(2) If L, L' are local sections of S, then so is $[L, L']$.

The manifold X, together with the structure S, is called a CR manifold.

REMARK 1.2. There exists a unique subbundle $\mathcal{H}$ of $T(X)$ such that $\boldsymbol{C}\mathcal{H}=S\oplus\bar{S}$. Furthermore, there exists a unique homomorphism $J:\mathcal{H}\to\mathcal{H}$ such that $J^2=-$identity and $S=\{W-\sqrt{-1}JW: W\in\mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of S.

There is a natural filtration of the De Rham complex of X, as follows: Let $A^k(X)=\wedge^k(\boldsymbol{C}T(X))^*$. Denote by $A^{p,q}(X)$ the subbundle of $A^{p+q}(X)$ consisting of all $\varphi\in A^{p+q}(X)$ such that $\varphi(Y_1, \cdots, Y_{p-1}, \bar{Z}_1, \cdots, \bar{Z}_{q+1})=0$ for all $Y_1, \cdots, Y_{p-1}\in\boldsymbol{C}T(X)_x$ and $Z_1, \cdots, Z_{q+1}\in S_x$, x being the origin of φ. Let $\mathcal{A}^k(X)=\Gamma(A^k(X))$ and $\mathcal{A}^{p,q}(X)=\Gamma(A^{p,q}(X))$. Then $A^k(X)=A^{0,k}(X)\supset A^{1,k-1}(X)\supset\cdots\supset A^{k,0}(X)\supset 0$ and $d\mathcal{A}^{p,q}(X)\subset\mathcal{A}^{p,q+1}(X)$, giving a filtration of the De Rham complex $\{\mathcal{A}^k(X), d\}$.

DEFINITION 1.3. Let $C^{p,q}(X)=A^{p,q}(X)/A^{p+1,q-1}(X)$ and $\mathcal{C}^{p,q}(X)=\Gamma(C^{p,q}(X))$. The exterior differentiation d induces an operator $d'': \mathcal{C}^{p,q}(X)\to\mathcal{C}^{p,q+1}(X)$. The cohomology groups of the resulting complex $\{\mathcal{C}^{p,q}(X), d''\}$ will be denoted by $H^{p,q}_{KR}(X)$.

REMARK 1.4. Consider the case where X is the boundary of a complex manifold M. The complex $\{\underline{C}^{p,q}(X), d''\}$ in the sheaf category, associated with the complex $\{\mathcal{C}^{p,q}(X), d''\}$, coincides with the boundary complex $\{\mathcal{B}^{p,q}, \bar{\partial}_b\}$ introduced by Kohn-Rossi (cf. [**Ko-Ro**], p. 465). As a consequence, $H^{p,q}_{KR}(X)$ is called the Kohn-Rossi cohomology group of type (p, q).

Let $L_1, \cdots, L_{n-1}$ be a local frame of S. Choose a purely imaginary local section N of $\boldsymbol{C}T(X)$ such that $L_1, \cdots, L_{n-1}, \bar{L}_1, \cdots, \bar{L}_{n-1}, N$ span $\boldsymbol{C}T(X)$. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j]=\sum a_{ij}^k L_k+\sum b_{ij}^k \bar{L}_k+c_{ij}N$$

is Hermitian, and is called the Levi form. The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \cdots, L_{n-1}, N$.

DEFINITION 1.5. X is said to be strongly pseudoconvex if the Levi form is definite at each point of X.

DEFINITION 1.6. (Cf. [**La-Ya**] p. 558.) A smooth S^1-action on X is said to be holomorphic if it preserves the subbundle $\mathcal{H}\subset T(X)$ and commutes with J. It is said to be transversal if, in addition, the vector field V which generates the action, is transversal to $\mathcal{H}$ at all points of X.

The condition of being holomorphic can be written in terms of the Lie derivation $\mathcal{L}_V$ as: $\mathcal{L}_V(\Gamma(\mathcal{H}))\subset\Gamma(\mathcal{H})$ and $\mathcal{L}_V(J)=0$, or equivalently as: $\mathcal{L}_V(\Gamma(S))\subset\Gamma(S)$.

DEFINITION 1.7. Let X be a strongly pseudoconvex CR manifold with transversal holomorphic S^1-action. Let V be the generating vector field. Following Tanaka [**Ta**], we define, for every k, a differential operator $N: \mathcal{A}^k(X) \to \mathcal{A}^k(X)$ by $N\varphi = \sqrt{-1}\mathcal{L}_V\varphi$, $\varphi \in \mathcal{A}^k(X)$.

The operator N leaves invariant the spaces $\mathcal{A}^{p,q}(X)$ and $\mathcal{C}^{p,q}(X)$, and commutes with the operators d, d''. It follows that the operator N acts on the cohomology groups $H_{KR}^{p,q}(X)$.

DEFINITION 1.8. Let X be a strongly pseudoconvex CR manifold with transversal holomorphic S^1-action. The invariant Kohn-Rossi cohomology $\tilde{H}_{KR}^{p,q}(X)$ is defined to be $\tilde{H}_{KR}^{p,q}(X) = \{c \in H_{KR}^{p,q}(X): Nc = 0\}$, where N is the operator in Definition 1.7.

§ 2. Kohn's harmonic theory and Tanaka's differential geometric study on strongly pseudoconvex *CR* manifolds.

In this section, we shall follow Tanaka's treatment [**Ta**] of Kohn's harmonic theory and the Kohn-Rossi cohomology $H_{KR}^{p,q}(X)$.

DEFINITION 2.1. Let X be a CR manifold with structure S. A complex vector bundle E over X is said to be holomorphic if there is a differential operator $\bar{\partial}_E: \Gamma(E) \to \Gamma(E \otimes \bar{S}^*)$ satisfying the conditions:

(1) $\bar{L}_1(fu) = (\bar{L}_1 f)u + f(\bar{L}_1 u)$

(2) $[\bar{L}_1, \bar{L}_2]u = \bar{L}_1\bar{L}_2 u - \bar{L}_2\bar{L}_1 u$,

where $u \in \Gamma(E)$, f is a complex valued function on X, $L_1, L_2 \in \Gamma(S)$ and $(\bar{\partial}_E u)(\bar{Z})$ is denoted by $\bar{Z}u$ for $Z \in \Gamma(S)$.

For a holomorphic vector bundle E over X, set

$$C^q(X, E) = E \otimes \wedge^q \bar{S}^*, \qquad \mathcal{C}^q(X, E) = \Gamma(C^q(X, E)).$$

The differential operators $\bar{\partial}_E^q: \mathcal{C}^q(X, E) \to \mathcal{C}^{q+1}(X, E)$ defined by

$$(\bar{\partial}_E^q \varphi)(\bar{L}_1, \cdots, \bar{L}_{q+1}) = \sum_i (-1)^{i+1} \bar{L}_i(\varphi(\bar{L}_1, \cdots, \hat{\bar{L}}_i, \cdots, \bar{L}_{q+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \varphi([\bar{L}_i, \bar{L}_j], \bar{L}_1, \cdots, \hat{\bar{L}}_i, \cdots, \hat{\bar{L}}_j, \cdots, \bar{L}_{q+1})$$

for all $\varphi \in \mathcal{C}^q(X, E)$ and $L_1, \cdots, L_{q+1} \in \Gamma(S)$, satisfy $\bar{\partial}_E^{q+1} \circ \bar{\partial}_E^q = 0$.

DEFINITION 2.2. The cohomology groups of the complex $\{\mathcal{C}^q(X, E), \bar{\partial}_E^q\}$ will be denoted by $H^q(X, E)$.

In case X is compact, strongly pseudoconvex and E is equipped with a Hermitian inner product $\langle\, ,\rangle$, a Laplacian operator of $\bar{\partial}_E$ can be defined as follows: Let $(\mathcal{H}, J)$ be the real expression of S. Then there exists a nowhere vanishing 1-form $\theta \in \Gamma(T(X)^*)$ such that at each $x \in X$, θ annihilates $\mathcal{H}_x$ and $-d\theta(JY, Y)$, $Y \in \mathcal{H}_x$, is a positive definite Hermitian form. Corresponding to θ, there exists a unique vector field $\xi \in \Gamma(T(X))$ such that $[\xi, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$, $\theta(\xi)=1$ and $\xi \rfloor d\theta=0$. In the following, we fix one such θ. Extend J to a tensor field of type (1, 1) by setting $J\xi=0$ and define a tensor field g of type (0, 2) by $g(Y_1, Y_2)=-d\theta(JY_1, Y_2)$. Then g induces a Hermitian inner product on S which, together with $\langle\ ,\ \rangle$ on E, induces a Hermitian inner product on $C^q(X, E)$ in the usual way. Integration with respect to the volume element $dv=\theta\wedge(d\theta)^{n-1}$ provides $C^q(X, E)$ with a Hermitian inner product $(\ ,\ )$. Let ϑ_E be the formal adjoint operator of $\bar{\partial}_E$ with respect to $(\ ,\ )$.

DEFINITION 2.3. $\square_E=\vartheta_E\bar{\partial}_E+\bar{\partial}_E\vartheta_E$ is called the Laplacian of $\bar{\partial}_E$ with respect to θ and $\langle\ ,\ \rangle$. The space of harmonic forms $\{\varphi \in C^q(X, E) : \square_E\varphi=0\}$ will be denoted by $\mathcal{H}^q(X, E)$.

Tanaka [**Ta**] defined canonical connections on X and in E with respect to θ and $\langle\ ,\ \rangle$ to describe $\square_E$ and prove the subelliptic 1/2-estimate of $\square_E$. Working through Kohn's harmonic theory in this differential geometric setting yields in particular:

THEOREM 2.4. (Cf. Kohn [**Ko$_2$**].) *Assume $n \geqq 3$. Every cohomology class in $H^q(X, E)$ is represented by a unique harmonic form. Hence $H^q(X, E) \cong \mathcal{H}^q(X, E)$.*

Theorem 2.4 can be applied to the Kohn-Rossi cohomology $H^{p,q}_{KR}(X)$ by identifying $C^{p,q}(X)$ with $C^q(X, E^p)$ for a holomorphic vector bundle E^p as follows:

DEFINITION 2.5. Let h be the Riemannian metric on X defined by $h=g+\theta^2$. Denote by $\langle\ ,\ \rangle$ Hermitian inner product on $A^k(X)$ induced by h in the usual way. For $\varphi, \psi \in \mathcal{A}^k(X)$, set $(\varphi, \psi)=\int_X \langle\varphi, \psi\rangle dv$.

Let $\mathcal{H}'$ be the subbundle of $T(X)$ spanned by the vector field ξ. By the decomposition $\boldsymbol{C}T(X)=\boldsymbol{C}\mathcal{H}' \oplus S \oplus \bar{S}$, $C^{p,q}(X)$ can be identified with the subbundle $\wedge^p(\boldsymbol{C}\mathcal{H}' \oplus S)^* \otimes \wedge^q \bar{S}^*$ of $A^{p+q}(X)$.

DEFINITION 2.6. With $C^{p,q}(X)$ identified as a subspace of $\mathcal{A}^{p+q}(X)$, let δ'' be the formal adjoint operator of d'' with respect to $(\ ,\ )$ in Definition 2.5 and $\Delta''=\delta''d''+d''\delta''$ be the corresponding Laplacian. The harmonic space $\{\varphi \in C^{p,q}(X) : \Delta''\varphi=0\}$ will be denoted by $\mathcal{H}^{p,q}_{KR}(X)$.

Let $E^p=\wedge^p(\boldsymbol{C}\mathcal{H}' \oplus S)^*$. For $\varphi \in \Gamma(E^p)$, $Y \in S$, $Z_1, \cdots, Z_p \in \Gamma(\boldsymbol{C}\mathcal{H}' \oplus S)$, define $(\bar{\partial}_{E^p}\varphi)(\bar{Y})=\bar{Y}\varphi$ by $(\bar{Y}\varphi)(Z_1, \cdots, Z_p)=\bar{Y}(\varphi(Z_1, \cdots, Z_p))+\sum(-1)^i\varphi(\omega[\bar{Y}, Z_i], Z_1, \cdots, \hat{Z}_i, \cdots, Z_p)$, where ω denotes the projection of $\boldsymbol{C}T(X)$ onto $\boldsymbol{C}\mathcal{H}' \oplus S$.

Then E^p is holomorphic with respect to $\bar{\partial}_{E^p}$. Furthermore, with $C^{p,q}(X)$ identified as $C^q(X, E^p)$, $d''=(-1)^p\bar{\partial}_{E^p}$. Hence, $H^{p,q}_{KR}(X)$ may be identified with $H^q(X, E^p)$. On the other hand, E^p as a subbundle of $A^p(X)$ can also be equipped with the Hermitian inner product $\langle\,,\rangle$ induced by h. Then $\vartheta_{E^p}=(-1)^p\delta''$, $\square_{E^p}=\Delta''$ and $\mathcal{H}^q(X, E^p)$ coincides with $\mathcal{H}^{p,q}_{KR}(X)$. Thus, Theorem 2.4 yields:

THEOREM 2.7. (Kohn [**Ko**$_2$].) $H^{p,q}_{KR}(X)\cong\mathcal{H}^{p,q}_{KR}(X)$.

REMARK 2.8. The $*$ operator with respect to h and the orientation by dv defines a duality between $A^k(X)$ and $A^{2n-1-k}(X)$. Putting $\#\varphi=\overline{*\varphi}$, $\varphi\in A^k(X)$, defines an anti-isomorphism $\#: C^{p,q}(X)\to C^{n-p,n-1-q}(X)$ for $p+q=k$. Then $\delta''=(-1)^{p+q}\#d''\#$ on $C^{p,q}(X)$ and $\#\Delta''=\Delta''\#$. Hence $\#\mathcal{H}^{p,q}_{KR}(X)=\mathcal{H}^{n-p,n-1-q}_{KR}(X)$.

From now on, we shall assume that X is a strongly pseudoconvex compact CR manifold with transversal holomorphic S^1 action (cf. Definition 1.6). We choose θ such that the corresponding ξ is the vector field V which generates the action.

DEFINITION 2.9. Let $B^{p,q}(X)=\wedge^p S^*\otimes\wedge^q\bar{S}^*\subset A^{p+q}(X)$ and $\mathcal{B}^{p,q}(X)=\Gamma(B^{p,q}(X))$. For $\varphi\in\mathcal{B}^{p,q}(X)$, define $\partial\varphi\in\mathcal{B}^{p+1,q}(X)$ and $\bar{\partial}\varphi\in\mathcal{B}^{p,q+1}(X)$ by the decomposition $d\varphi\equiv\partial\varphi+\bar{\partial}\varphi \pmod{\theta}$.

By the holomorphic condition satisfied by the generating vector field, $\partial^2=0$ and $\bar{\partial}^2=0$. Theorem 2.4 can again be applied to the cohomology groups of the resulting complexes. For our purpose, we consider the following operators:

DEFINITION 2.10. Let $\square$ be the Laplacian of $\bar{\partial}$ with respect to the Hermitian inner product $(\,,)$ in Definition 2.5. Let $L: B^{p,q}(X)\to B^{p+1,q+1}(X)$ be the operator defined by $L\varphi=-d\theta\wedge\varphi$ and Λ be the adjoint of L with respect to the Hermitian inner product $\langle\,,\rangle$ induced by h.

The operator $N=\sqrt{-1}\mathcal{L}_\xi$ introduced in Definition 1.7 leaves invariant the subspaces $C^{p,q}(X)$ and $\mathcal{B}^{p,q}(X)$ of $\mathcal{A}^{p+q}(X)$. N commutes not only with d, d'', $\bar{\partial}$, but also their formal adjoint operators with respect to $(\,,)$, because $\mathcal{L}_\xi h=0$. In particular, it operates on $\mathcal{H}^{p,q}_{KR}(X)$.

DEFINITION 2.11. The invariant harmonic space $\{\varphi\in\mathcal{H}^{p,q}_{KR}(X): N\varphi=0\}$ will be denoted by $\tilde{\mathcal{H}}^{p,q}_{KR}(X)$.

REMARK 2.12. Since $N\#=-\#N$, Remark 2.8 implies $\#\tilde{\mathcal{H}}^{p,q}_{KR}(X)=\tilde{\mathcal{H}}^{n-p,n-1-q}_{KR}(X)$.

§3. Proof of Theorem A.

We are now ready to prove Theorem A. Let us first recall some terminology. Suppose that Y is a complex analytic space, and let $\mathcal{O}_Y$ denote the sheaf of germs of holomorphic functions on Y. Denote by $\mathcal{O}_Y^w$ the sheaf of germs of weakly holomorphic functions on Y, that is, the sheaf of germs of bounded functions which are holomorphic on the regular set of Y (cf. [**Gu-Ro**]). Of course we have $\mathcal{O}_Y \subset \mathcal{O}_Y^w$, and the space Y is called normal if $\mathcal{O}_Y = \mathcal{O}_Y^w$.

DEFINITION 3.1. Let φ_t be a continuous action of S^1 on a complex analytic space Y, and suppose it preserves the set of regular points of Y. Then this action is called holomorphic if

$$\varphi_t^* \mathcal{O}_Y = \mathcal{O}_Y \qquad \text{for all } t.$$

It is called weakly holomorphic if

$$\varphi_t^* \mathcal{O}_Y^w = \mathcal{O}_Y^w \qquad \text{for all } t.$$

Suppose that X is a strongly pseudoconvex CR manifold of dimension $2n-1$ with a transversal holomorphic S^1-action. In order to prove Theorem A, we are going to prove that if X is CR-embeddable in $\boldsymbol{C}^N$, then the invariant Kohn-Rossi cohomology group $\tilde{H}_{KR}^{p,q}(X)=0$ for all $1 \leqq p+q \leqq 2n-N-1$. In what follows we shall assume that X is already in $\boldsymbol{C}^N$. By the theorem of Harvey and Lawson [**Ha-La**], X forms the boundary of a compact complex analytic subvariety Y with boundary in $\boldsymbol{C}^N$. In view of the results of Lawson-Yau [**La-Ya**], the transversal holomorphic S^1-action on X extends to a weakly holomorphic action on Y. In fact the extended action on Y has exactly one fixed point. This point, say p, may be a singular point of Y. However $Y-\{p\}$ is a smooth manifold with boundary. Let $\Delta^* = \{z \in \boldsymbol{C} : 0 < |z| \leqq 1\}$ be the punctured disk. Then Theorem 1.17 of [**La-Ya**] states that the transversal holomorphic action of S^1 on X extends to a weakly holomorphic representation of the analytic semigroup Δ^* as a semi-group of analytic embeddings of Y into itself, which we shall denote by $\Phi : \Delta^* \times Y \to Y$. This action has a single fixed-point, and given any neighborhood $\mathcal{U}$ of p in Y, there is an $\varepsilon > 0$ so that $\Phi(z, Y) \subset \mathcal{U}$ for all z with $|z| < \varepsilon$. It follows that X is homeomorphic to the link of Y at p (i.e., intersection of Y with sufficiently small sphere in C^N centered at p). By the result of Hamm [**Ha**], $\pi_k(X)=0$ for all $1 \leqq k \leqq 2n-N-1$. In particular we have $H^k(X)=0$ for all $1 \leqq k \leqq 2n-N-1$ in view of Hurewicz theorem.

Consider the Laplacian of d with respect to the Hermitian inner product $(\,,\,)$ in Definition 2.5. Let $\mathcal{H}^k(X)$ be the corresponding space of harmonic k-forms. For $k \leqq n-1$, Tanaka [**Ta**] proved that for any $\alpha \in \mathcal{A}^k(X)$, $\alpha \in \mathcal{H}^k(X)$ if and only if $\alpha \in \bigoplus_{p+q=k} \mathcal{B}^{p,q}(X)$, $\square \alpha = 0$, $N\alpha = 0$ and $\Lambda \alpha = 0$, while for any $\varphi \in$

$C^{p,q}(X)$, $\varphi\in\tilde{\mathcal{H}}^{p,q}_{KR}(X)$ if and only if $\varphi\in\mathcal{B}^{p,q}(X)$, $\square\alpha=0$, $N\alpha=0$ and $\Lambda\alpha=0$. This yields $\mathcal{H}^k(X)=\bigoplus_{p+q=k}\tilde{\mathcal{H}}^{p,q}_{KR}(X)$ for $k\leqq n-1$. By the dualities $\#\mathcal{H}^k(X)=\mathcal{H}^{2n-1-k}(X)$ and $\#\tilde{\mathcal{H}}^{p,q}_{KR}(X)=\tilde{\mathcal{H}}^{n-p,n-1-q}_{KR}(X)$, $\mathcal{H}^k(X)=\bigoplus_{p+q=k}\tilde{\mathcal{H}}^{p,q}_{KR}(X)$ holds for $k\geqq n$ as well. Now $\mathcal{H}^k(X)=0$ for all $1\leqq k\leqq 2n-N-1$. Hence $\tilde{\mathcal{H}}^{p,q}_{KR}(X)=0$ for all $1\leqq p+q\leqq 2n-N-1$. Since $\tilde{H}^{p,q}_{KR}(X)\cong\tilde{\mathcal{H}}^{p,q}_{KR}(X)$, we have $\tilde{H}^{p,q}_{KR}(X)=0$ for $1\leqq p+q\leqq 2n-N-1$.

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