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On dimensions of non-Hausdorff sets for plane homeomorphisms

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§1. Introduction.

A homeomorphism is called *flowable* if there exists a topological flow whose time one map is that homeomorphism. An orientation preserving fixed point free homeomorphism of $\boldsymbol{R}^2$ which is not flowable was constructed by Kerékjártó in 1934 ([**9**]). In order to show the homeomorphism is not flowable, he defined "singular points", at which the family $\{f^n\}_{n\in Z}$ is not equicontinuous with respect to the elliptic metric.

The set of "singular points" coincides with the following non-Hausdorff set (see [**10**], [**11**]): Let f be an orientation preserving fixed point free homeomorphism of $\boldsymbol{R}^2$. Denote by $\pi: \boldsymbol{R}^2 \to \boldsymbol{R}^2/f$ the quotient map which maps each orbit of f to a point. Then $\boldsymbol{R}^2/f$ is a non-Hausdorff manifold because the non-wandering set of f is empty ([**1**], [**5**] Corollary 2.3). A point p of $\boldsymbol{R}^2$ is called *non-Hausdorff* if $\pi(p)$ is not "Hausdorff" in $\boldsymbol{R}^2/f$. We call the set of all non-Hausdorff points the *non-Hausdorff set*, denoted by $NH(f)$.

In this paper, we characterize $NH(f)$ by the limit set of continua and give the dimension of $NH(f)$.

MAIN THEOREM. *Let f be an orientation preserving fixed point free homeomorphism of $\boldsymbol{R}^2$. Then $NH(f)$ is one-dimensional unless it is empty.*

In the following, we assume that all homeomorphisms of $\boldsymbol{R}^2$ are orientation preserving and without fixed points, and the topology of $\boldsymbol{R}^2$ is given by the Euclidean metric.

In §2, we give a precise definition of non-Hausdorff points and characterize $NH(f)$ by the limit sets of continua (Theorem 1). The main theorem is proved in §3 by using Theorem 2 in §2 and Theorem 3 in §3.

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§ 2. Limit sets of continua.

First we define the non-Hausdorff points precisely. Let f be a homeomorphism of $\boldsymbol{R}^2$. Denote by $O_f(S)$ the orbit $\bigcup_{n=-\infty}^{\infty} f^n(S)$ of a subset S. A point p of $\boldsymbol{R}^2$ is called *non-Hausdorff* if there is a point $q \notin O_f(\{p\})$ contained in the closure of $O_f(U)$ for any open neighborhood U of p. We call q a *pair* of p.

If $NH(f)=\emptyset$, then $\boldsymbol{R}^2/f$ is a Hausdorff manifold, and is homeomorphic to $S^1 \times \boldsymbol{R}^1$ because the quotient map is a covering map whose covering transformations are generated by f (*i.e.*, $\pi_1(\boldsymbol{R}^2/f)$ is isomorphic to $\boldsymbol{Z}$). Thus f is topologically conjugate to the translation.

By definition, $NH(f)$ is invariant under f, and $h \cdot NH(f)=NH(hfh^{-1})$ for any homeomorphism h of $\boldsymbol{R}^2$. If a homeomorphism f of $\boldsymbol{R}^2$ is the time one map of a flow φ_t ($t \in \boldsymbol{R}$), then $NH(f)$ is invariant under φ_t for any $t \in \boldsymbol{R}$. Hence $NH(f)$ consists of 1-dimensional manifolds. Since the non-Hausdorff set of Kerékjártó's homeomorphism has branch points, it is not flowable ([**9**], [**10**] and [**11**]).

Though the orbit of any point is closed because the non-wandering set of f is empty, that of a compact set K is not always closed. The difference between $\overline{O_f(K)}$ and $O_f(K)$ consists of non-Hausdorff points as follows:

LEMMA 1. *For any compact set K of $\boldsymbol{R}^2$, $\overline{O_f(K)}-O_f(K)$ is contained in $NH(f)$.*

PROOF. Let p be a point of $\overline{O_f(K)}-O_f(K)$. Then there is a point sequence $\{z_n\}_{n=1,2,3,\dots}$ of $O_f(K)$ converging to p. For each n, we choose an integer m_n such that $z_n \in f^{m_n}(K)$. Since K is compact, we can assume that $\{f^{-m_n}(z_n)\}_{n=1,2,3,\dots}$ converges to a point q of K by taking a subsequence.

Let U and V be any open sets of $\boldsymbol{R}^2$ containing p and q, respectively. For a sufficiently large n, $f^{-m_n}(z_n) \in V$ and $z_n \in U$. Hence q is contained in $\overline{O_f(U)}$.

Since q is an element of K, p is not contained in $O_f(\{q\})$ $(\subset O_f(K))$. Thus p is a non-Hausdorff point. □

LEMMA 2. *Let $\{U_i\}_{i=1,2,3,\dots}$ be a countable base of $\boldsymbol{R}^2$ such that each $\overline{U_i}$ is compact. Then $NH(f)=\bigcup_{i=1}^{\infty}(\overline{O_f(\overline{U_i})}-O_f(\overline{U_i}))$.*

PROOF. Let p be a point of $NH(f)$, and q, a pair of p. Since the complement of $O_f(\{p\})$ is an open set containing q, there is an open ε-ball $B_\varepsilon(q)$ with center q disjoint from $O_f(\{p\})$. We choose an open set U_i from the countable base such that $q \in U_i \subset B_{\varepsilon/2}(q)$. Then p is not contained in $O_f(\overline{U_i})$ because $O_f(\{p\}) \cap \overline{U_i} \subset O_f(\{p\}) \cap B_\varepsilon(q)=\emptyset$. On the other hand, by the choice of p and q, $O_f(U_i)$ intersects any open set containing p. Hence $p \in \overline{O_f(U_i)} \subset \overline{O_f(\overline{U_i})}$. Thus $NH(f)$ is contained in $\bigcup_{i=1}^{\infty}(\overline{O_f(\overline{U_i})}-O_f(\overline{U_i}))$.

Lemma 1 implies that $NH(f)$ contains $\bigcup_{i=1}^{\infty}(\overline{O_f(\overline{U_i})}-O_f(\overline{U_i}))$. Thus Lemma 2 holds. □

We define the *limit set* $Lim_f(S)$ of a subset S by $\bigcap_{n\geq 0}\overline{\bigcup_{|i|\geq n} f^i(S)}$. Then $Lim_f(S)$ is a closed invariant set for any subset S. We consider the non-Hausdorff set in terms of this limit set in the following.

LEMMA 3. *Let K be a continuum (i.e., a compact connected set) such that $f(K)\cap K=\emptyset$. Then $Lim_f(K)=\overline{O_f(K)}-O_f(K)$.*

PROOF. Let z be an element of $\overline{O_f(K)}-O_f(K)$. Since $\overline{O_f(K)}=\overline{\bigcup_{i=-\infty}^{\infty} f^i(K)}$ $=(\bigcup_{|i|\leq n-1} f^i(K))\cup\overline{\bigcup_{|i|\geq n} f^i(K)}$ for any $n\geqq 0$, z is an element of $\overline{\bigcup_{|i|\geq n} f^i(K)}$. Thus $\overline{O_f(K)}-O_f(K)\subset Lim_f(K)$.

Next suppose that z is an element of $Lim_f(K)$ (*i.e.*, $z\in\overline{\bigcup_{|i|\geq n} f^i(K)}$ for any $n\geqq 0$). Then z is an element of $\overline{\bigcup_{i=-\infty}^{\infty} f^i(K)}=\overline{O_f(K)}$.

In order to show that z is not contained in $O_f(K)$, it suffices to prove that $f^j(K)\cap\overline{\bigcup_{i\neq j} f^i(K)}=\emptyset$ for any integer j because $z\in\overline{\bigcup_{|i|\geq |j|+1} f^i(K)}\subset\overline{\bigcup_{i\neq j} f^i(K)}$.

Suppose that $f^j(K)\cap\overline{\bigcup_{i\neq j} f^i(K)}\neq\emptyset$ for some j. Let U and V be open sets satisfying that $K\subset U$, $f(K)\subset V$ and $U\cap V=\emptyset$. Let $\varepsilon=d(f(K), \boldsymbol{R}^2-V)>0$, where $d(A, B)=\inf\{d(x, y)\,;\, x\in A,\ y\in B\}$. For any point $p\in K$, there is $\delta(p)>0$ such that $f(B_{\delta(p)}(p))\subset B_{\varepsilon/2}(f(p))$ and $B_{\delta(p)}(p)\subset U$. Since K is compact, there are finitely many points $p_1, p_2, \cdots, p_k\in K$ such that $\{B_{\delta(p_i)}(p_i)\}_{i=1,\dots,k}$ is an open covering of K. Let $W=\bigcup_{i=1}^{k} B_{\delta(p_i)}(p_i)$. Then $\overline{W}$ is also a continuum satisfying $K\subset W$ and $\overline{W}\cap f(\overline{W})\subset\overline{U}\cap\bigcup_{i=1}^{k}\overline{f(B_{\delta(p_i)}(p_i))}\subset\overline{U}\cap V=\emptyset$. Furthermore, $f^j(\overline{W})\cap\overline{\bigcup_{i\neq j} f^i(\overline{W})}\supset f^j(W)\cap\overline{\bigcup_{i\neq j} f^i(K)}\neq\emptyset$ because $f^j(W)$ is an open set containing $f^j(K)$. However this contradicts Brown's lemma ([**3**] Lemma 3.1), which implies that, if C is a continuum and $C\cap f(C)=\emptyset$, then $f^i(C)\cap f^j(C)=\emptyset$ whenever $i\neq j$. Hence $f^j(K)\cap\overline{\bigcup_{i\neq j} f^i(K)}$ is empty. Therefore $Lim_f(K)$ is contained in $\overline{O_f(K)}-O_f(K)$. □

THEOREM 1. *Let f be an orientation preserving fixed point free homeomorphism of $\boldsymbol{R}^2$. For any countable base $\{U_i\}_{i=1,2,3,\dots}$ of $\boldsymbol{R}^2$ satisfying that $\overline{U_i}\cap f(\overline{U_i})=\emptyset$ and each $\overline{U_i}$ is a continuum, the following equations hold;*

$$NH(f)=\bigcup_{i=1}^{\infty} Lim_f(\overline{U_i})=\bigcup_{i=1}^{\infty}(\overline{O_f(\overline{U_i})}-O_f(\overline{U_i})).$$

PROOF. By Lemma 2, $NH(f)=\bigcup_{i=1}^{\infty}(\overline{O_f(\overline{U_i})}-O_f(\overline{U_i}))$. Since $Lim_f(\overline{U_i})=\overline{O_f(\overline{U_i})}-O_f(\overline{U_i})$ by Lemma 3, $NH(f)=\bigcup_{i=1}^{\infty} Lim_f(\overline{U_i})$. □

REMARK. For any foliation of $\boldsymbol{R}^2$, there is a leaf preserving homeomorphism, and foliations of $\boldsymbol{R}^2$ are given by 1-dimensional non-Hausdorff manifolds ([**7**]). Since many kinds of 1-dimensional non-Hausdorff manifolds have already been given ([7]), we can make various homeomorphisms of $\boldsymbol{R}^2$. For example, we obtain a homeomorphism whose non-Hausdorff set is dense in $\boldsymbol{R}^2$.

THEOREM 2. *$NH(f)$ has no interior points.*

PROOF. By taking sufficiently small balls, we choose a countable base $\{U_i\}$ of $\boldsymbol{R}^2$ such that $\overline{U_i}\cap f(\overline{U_i})=\emptyset$ and $\overline{U_i}$ are continua. By definition, $Lim_f(\overline{U_i})$ is closed. Furthermore, $Lim_f(\overline{U_i})$ has no interior points because $Lim_f(\overline{U_i})=\overline{O_f(\overline{U_i})}-O_f(\overline{U_i})$. Since $NH(f)=\bigcup_{i=1}^{\infty} Lim_f(\overline{U_i})$ by Theorem 1, $NH(f)$ is a countable union of closed sets without interior points. By Baire's theorem, $NH(f)$ has no interior points. $\square$

§3. Proof of the main theorem.

First we prove the connectivity of $NH(f)\cup\{\infty\}$ in $\boldsymbol{R}^2\cup\{\infty\}$ in order to consider the dimension of $NH(f)$.

THEOREM 3. *$NH(f)\cup\{\infty\}$ is connected in $\boldsymbol{R}^2\cup\{\infty\}$.*

PROOF. It is enough to prove that $NH(f)$ is not contained in the union of any disjoint open sets U_1 and U_2 of $\boldsymbol{R}^2\cup\{\infty\}$ satisfying that $NH(f)\cap U_1\neq\emptyset$ and $\infty\in U_2$.

Let p be an element of $NH(f)\cap U_1$ and let q be a pair of p. Since $O_f(\{p\})$ is closed, we can choose an $\varepsilon>0$ such that $\overline{B_\varepsilon(q)}\cap O_f(\{p\})=\emptyset$ and $\overline{B_\varepsilon(q)}\cap f(\overline{B_\varepsilon(q)})=\emptyset$. Denote by K the closed ball $\overline{B_\varepsilon(q)}$. Then p is contained in $\overline{O_f(K)}$ because $O_f(B_\varepsilon(q))$ intersects any neighborhood of p. In particular, U_1 intersects $O_f(K)$.

Let Λ denote the non-empty set $\{n\in\boldsymbol{Z}; U_1\cap f^n(K)\neq\emptyset\}$. Suppose that Λ is a finite set. Then there is a positive integer N_1 such that U_1 is disjoint from $f^n(K)$ for any $|n|\geqq N_1$. Hence p is not contained in $\overline{\bigcup_{|n|\geqq N_1} f^n(K)}$. However this contradicts that $p\in\overline{O_f(K)}=(\bigcup_{|n|<N_1} f^n(K))\cup\overline{\bigcup_{|n|\geqq N_1} f^n(K)}$ and $p\notin O_f(K)$. Thus Λ is an infinite set. We denote the elements of Λ by $n_1, n_2, n_3, \cdots$ where $\lim_{i\to\infty}|n_i|=\infty$.

Let z be an element of K. Since $\lim_{n\to\pm\infty} f^n(z)=\infty$, there is a positive integer N_2 such that $f^n(z)\in U_2$ for any $|n|\geqq N_2$. In particular, U_2 intersects $f^n(K)$ for any $|n|\geqq N_2$. By taking a sufficiently large I such that $|n_i|\geqq N_2$ for any $i\geqq I$, $f^{n_i}(K)$ intersects both U_1 and U_2 for any $i\geqq I$. Since $f^{n_i}(K)$ is connected, there exists an element x_i of $f^{n_i}(K)-(U_1\cup U_2)$ for $i\geqq I$.

By taking a subsequence of $\{x_i\}_{i\geqq I}$, we can assume that $\{x_i\}$ converges to a point $z\notin U_1\cup U_2$ because $(\boldsymbol{R}^2\cup\{\infty\})-(U_1\cup U_2)$ is compact. For any integer $m\geqq 0$, there is a positive integer I_m ($I_m\geqq I$) such that $|n_i|\geqq m$ for any $i\geqq I_m$. Since $x_i\in f^{n_i}(K)\subset\bigcup_{|j|\geqq m} f^j(K)$ for any $i\geqq I_m$, z is an element of $\overline{\bigcup_{|j|\geqq m} f^j(K)}$. Thus z is a point of $Lim_f(K)$. By Lemmas 1 and 3, z is a non-Hausdorff point, which is not contained in $U_1 \cup U_2$. □

REMARK. $NH(f) \cup \{\infty\}$ is not always arcwise connected (see [**4**], Example 3).

PROOF OF THE MAIN THEOREM. Since $NH(f) \cup \{\infty\}$ is connected in $\boldsymbol{R}^2 \cup \{\infty\}$ by Theorem 3, the dimension of $NH(f) \cup \{\infty\}$ is not zero if $NH(f)$ is not empty. By [**8**], the dimension of $NH(f)$ is not zero, either. On the other hand, the dimension of $NH(f)$ is less than two by Theorem 2. Thus $NH(f)$ is one-dimensional. □

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