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Periodic distributions on C^*-algebras

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0. Introduction.

The purpose of this paper is to apply the theory of distributions founded by L. Schwartz to C^*-algebras. The theory of distributions has made the mathematical analysis explosively develop. The starting point of this theory is to generalize the concept of ordinary functions to be continuous linear functionals on smooth functions, called distributions, and to define differentiations on such linear functionals by using the idea of integration by parts. Schwartz in [**Sch1**, **Sch2**] has studied topological and analytical property of the space of distributions. He has also discussed convolutions between distributions, Fourier transforms and applications to partial differential equations.

On the other hand, the theory of C^*-algebras can be called the theory of "non-commutative" locally compact spaces. In other words, we can say that non-commutative C^*-algebras are algebras of all continuous functions on "non-commutative" locally compact spaces because of the classical Gelfand-Naimark theorem for commutative C^*-algebras. Fourier analysis on such "non-commutative" spaces has rapidly developed by the theory of spectral subspaces induced by W. Arveson (cf. [**Ar**]).

In this paper, we will exhibit a theory of distributions on C^*-algebras. We restrict our interest to the periodic cases. Namely, we will start at a C^*-algebra A with a continuous action α of n-dimensional torus T^n. Such a triplet (A, α, T^n) is called a C^*-dynamical system. The most primitive and non-trivial example is the triplet $(C(T), \alpha, T)$ where $C(T)$ denotes the commutative C^*-algebra consisting of all continuous functions on the one dimensional torus T and α means the action induced by translation on the group T.

Let (A, α, T^n) be a C^*-dynamical system. We denote by $C_\alpha^\infty(A)$ the dense $*$-subalgebra of all elements of the C^*-algebra A whose members are differentiable infinitely many times by the derivations induced by the action α. We equip $C_\alpha^\infty(A)$ with a natural Fréchet space topology. We denote by $\mathcal{D}_\alpha(A)$ this

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Fréchet algebra. It is nothing but the "non-commutative" test function space associated to the action α. Let B be the fixed point subalgebra of A under the action α. Notice that both algebras $\mathcal{D}_\alpha(A)$ and B are non-commutative in general so that our definition for non-commutative periodic distributions as follows:

DEFINITION. A *C^*-left (resp. right) periodic distribution* is a continuous left (resp. right) B-module map from $\mathcal{D}_\alpha(A)$ to B.

We denote by $\mathcal{D}'_{\alpha l}(A)$ (resp. $\mathcal{D}'_{\alpha r}(A)$) the set of all C^*-left (resp. right) distributions. We mainly treat $\mathcal{D}'_{\alpha l}(A)$. Symmetric discussions work for $\mathcal{D}'_{\alpha r}(A)$. The object $\mathcal{D}'_{\alpha l}(A)$ becomes a topological space with left $\mathcal{D}_\alpha(A)$ and right B-module structure in a canonical way.

The differentiations can be extended to $\mathcal{D}'_{\alpha l}(A)$ in an analogous way to the classical case. Since the algebra A is contained in $\mathcal{D}'_{\alpha l}(A)$, all elements of A can be differentiable in C^*-distribution sense.

For $k \in Z^n$, let E_k be the natural projection on A to the k-th spectral subspace $A^\alpha(k)$. The k-th Fourier component $\mathcal{E}_k(\xi)$ for a C^*-left distribution ξ is defined as also a C^*-left distribution by

$$\mathcal{E}_k(\xi)(x) = \xi(E_{-k}(x)), \qquad x \in \mathcal{D}_\alpha(A).$$

Suppose that A is represented on a Hilbert space H on which the action α is spatial. Hence α is extended on the weak operator closure A'' of A as a σ-weakly continuous action of T^n, which is also denoted by α. Then we have

THEOREM A (Theorem 5.13). *There exists a bijective correspondence between the set $\mathcal{D}'_{\alpha l}(A)$ of all C^*-left distributions and the set of all sequences $a_k \in (A'')^\alpha(k)$, $k \in Z^n$ satisfying the following conditions*:

(i) $A^\alpha(-k)a_k \subset B$, $k \in Z^n$,

(ii) $\{\|a_k\|\}_{k \in Z^n}$ *is slowly increasing,*

through the relation

$$\mathcal{E}_k(\xi)(x) = E_0(xa_k), \qquad x \in \mathcal{D}_\alpha(A), \ k \in Z^n .$$

Such a sequence a_k is denoted by $E_k(\xi)$ for the C^*-distribution ξ.

THEOREM B (Theorem 5.14). *Any C^*-left distribution is a finite sum of finite order derivatives of elements of A'' in the C^*-distribution sense.*

Let $\mathcal{D}(T^n)$ be the Fréchet space consisting of all smooth functions on T^n and $L^2(T^n)$ the Hilbert space consisting of all square integrable functions with respect to the Haar measure on T^n. In the pair

$$\mathcal{D}(T^n) \subset L^2(T^n)$$

and its dual pair

$$L^{2\prime}(T^n) \subset \mathscr{D}'(T^n),$$

the Hilbert space $L^2(T^n)$ is self-dual so that we have the triplet

$$\mathscr{D}(T^n) \subset L^2(T^n) \subset \mathscr{D}'(T^n),$$

which is called the Gelfand triplet. As a non-commutative version of this triplet, we take a Hilbert C^*-module in stead of the Hilbert space. We define a B-valued inner product $\langle\,,\rangle_r$ by

$$\langle a, b\rangle_r = E_0(a^*b), \qquad a, b \in A.$$

The completion of A by the norm induced by this inner product is denoted by $L^2_{\alpha r}(A)$. It has a Hilbert C^*-right B-module structure in an evident way. We similarly define Hilbert C^*-left B-module $L^2_{\alpha l}(A)$.

Let $L^2_{\alpha l}{}'(A)$ be the dual of $L^2_{\alpha l}(A)$, which consists of all continuous left B-module map from $L^2_{\alpha l}(A)$ to B. Then we naturally have the following sequence of three inclusion relations of left $\mathscr{D}_\alpha(A)$ and right B-modules

$$\mathscr{D}_\alpha(A) \subset L^2_{\alpha r}(A) \subset L^2_{\alpha l}{}'(A) \subset \mathscr{D}'_{\alpha l}(A).$$

We then introduce the notion of *left (resp. right) locally self-dual spectrum* $\Omega_l(\alpha)$ (resp. $\Omega_r(\alpha)$) for an action α. That is defined as the set of all points of Z^n in which the spectral subspaces are not self-dual as left (resp. right) B-module. We call an action α a *left locally self-dual action* if each spectral subspace $A^\alpha(k)$, $k \in Z^n$ is self-dual as left B-module, that is, $\Omega_l(\alpha)$ is empty. In this case, each element ξ of $L^2_{\alpha l}{}'(A)$ can be characterized as C^*-left distribution whose Fourier component $E_k(\xi)$ satisfies the next condition:

There exists a constant $K>0$ such that

$$\|\sum_{k\in F} E_k(\xi)^* E_k(\xi)\| < K, \qquad \text{for any finite subset } F \text{ of } Z^n.$$

In particular, ξ belongs to $L^2_{\alpha r}(A)$ if and only if $\sum_{k\in Z^n} E_k(\xi)^* E_k(\xi)$ converges in C^*-norm of B. Hence it is not generally valid that the Hilbert C^*-module is self-dual, i.e., $L^2_{\alpha r}(A) \cong L^2_{\alpha l}{}'(A)$. In fact, we have many cases in which $L^2_{\alpha r}(A)$ is properly contained in $L^2_{\alpha l}{}'(A)$. Thus we can regard the preceding quadruplet as a non-commutative version of the classical Gelfand triplet.

In this paper, we first study the test function space $\mathscr{D}_\alpha(A)$ and next study the Hilbert C^*-module $L^2_{\alpha r}(A)$ in focusing on Fourier expansions, which will be done in Sections 2 and 3. In Section 4, we exactly define C^*-distribution and define differentiation on them. Fourier series for C^*-distributions will be study in Section 5. The dual $L^2_{\alpha l}{}'(A)$ is characterized in terms of the Fourier series as C^*-distributions in Section 6. In Section 7, we exhibit a model of the previous quadruplet. Its C^*-dynamical system is the Cuntz algebra $\mathcal{O}_\infty$ with the canonical action α of T defined by

$$\alpha_t(S_n) = e^{itn}S_n\,, \qquad t \in R/Z = T\,,$$

where S_n, $n \in Z$ is the generators of isometries of $\mathcal{O}_\infty$ satisfying

$$\sum_{n \in Z} S_n S_n^* = 1 \qquad \text{(strong operator convergence on a Hilbert space)}.$$

The action α is left (and hence right) locally self-dual in our sense, that is, every spectral subspace $\mathcal{O}_\infty^\alpha(n)$, $n \in Z$ is self-dual as left and right B-module. But we see $L^2_{\alpha r}(\mathcal{O}_\infty)$ is properly contained in $L^2_{\alpha l}{}'(\mathcal{O}_\infty)$.

We finally, in Section 8, give an example of non locally self-dual action on a unital C^*-algebra.

In [**Ma**], C^*-distributions associated with R^n-actions are studied.

1. Preliminary.

Let A be a unital C^*-algebra and α an action of n-dimensional torus T^n on A. We identify

$$T^n = \{(e^{it_1}, e^{it_2}, \cdots, e^{it_n}) \mid t_i \in R,\ i=1, 2, \cdots, n\}.$$

For each $i=1, 2, \cdots, n$, we denote by δ_i the partial differentiation for the i-th component of T^n defined by

$$\delta_i(x) = \lim_{s \to 0} \frac{\alpha_{(0,\cdots,0,s,0,\cdots0)}(x) - x}{s}\,, \qquad x \in A$$

where the above limit is taken in the norm of A.

Set

$$Z_+^n = \{(l_1, \cdots, l_n) \in Z^n \mid l_i \geqq 0,\ 1 \leqq i \leqq n\}, \qquad N = Z_+^1\,.$$

For a multi-index $l=(l_1, \cdots, l_n) \in Z_+^n$, put the differential operator

$$D^l = \delta_1^{l_1} \cdots \delta_n^{l_n}$$

whose order is denoted by $|l| = l_1 + \cdots + l_n$.

Let $C_\alpha^\infty(A)$ be the intersection of all domains of differential operators D^l, that is,

$$C_\alpha^\infty(A) = \bigcap_{l \in Z_+^n} \text{domain of } D^l\,.$$

Let $C^\infty(T^n)$ be the set of all complex valued smooth functions on T^n. For $k \in Z^n$, the function e_k is defined on T^n by

$$e_k(e^{it_1}, \cdots, e^{it_n}) = e^{ik\cdot t} = \exp\{i(k_1t_1 + \cdots + k_nt_n)\}\,.$$

Let σ_n be the normalized Haar measure on T^n. For $x \in A$ and $k \in Z^n$, we define the k-th Fourier component of x by

$$E_k(x)=\int_{T^n}\alpha_t(x)e_{-k}(t)d\sigma_n(t) \qquad (\text{cf. [}\mathbf{Ar}\text{]}).$$

2. The space $\mathcal{D}_\alpha(A)$.

All results in this section are probably known. We will however give a proof of Corollary 2.9 for the sake of completeness.

We first notice that the $*$-subalgebra $C_\alpha^\infty(A)$ is dense in A. Let $\{h_j\}_{j\in N}$ be an approximate identity on T^n which consists of a sequence of smooth functions approximating the delta function on T^n (cf. [**Yo**; p. 157]).

For a smooth function f on T^n and an element $a\in A$, put

$$a*f=\int_{T^n}\alpha_t(a)f(t)d\sigma_n(t).$$

which defines an element of A. The following lemma is routine.

LEMMA 2.1. *For $a\in A$, we have*
(i) $D^l(a*f)=a*D^l f$, $l\in Z_+^n$ *and hence* $a*f\in C_\alpha^\infty(A)$, $f\in C^\infty(T^n)$.
(ii) $\lim_{j\to\infty}\|a*h_j-a\|=0$.

Set the $*$-subalgebra of A

$$\mathcal{T}_\alpha=\{\sum_{k\in F}E_k(x)\,|\,F \text{ is a finite subset of } Z^n,\ x\in A\}.$$

Since $E_k(a)=a*e_{-k}$, $a\in A$, $k\in Z^n$ and we can take an approximate identity consisting of a sequence of finite linear combinations of e_k, $k\in Z^n$, the next lemma is clear.

LEMMA 2.2. *$\mathcal{T}_\alpha(A)$ (and hence $C_\alpha^\infty(A)$) is a dense $*$-subalgebra of A.*

We next equip the sequence of the seminorms $\{p_m\}_{m\in N}$ with $C_\alpha^\infty(A)$ by

$$p_m(x)=\max_{|k|\le m}\|D^k x\|, \qquad x\in C_\alpha^\infty(A).$$

We denote by $\mathcal{D}_\alpha(A)$ the topological $*$-algebra $C_\alpha^\infty(A)$ with the topology induced by the sequence of seminorms $\{p_m\}_{m\in N}$.

LEMMA 2.3.

$$E_k(\delta_j x)=ik_jE_k(x), \qquad x\in\mathcal{D}_\alpha(A),\ k=(k_1,\cdots,k_n)\in Z^n.$$

PROOF. By integration by parts, we see

$$E_k(\delta_j x)=\int_{T^n}\delta_j(\alpha_t(x))e^{-ik\cdot t}d\sigma_n(t)=\int_{T^n}\alpha_t(x)ik_je^{-ik\cdot t}d\sigma_n(t).$$

COROLLARY 2.4.

$$E_k(D^l x) = (ik_1)^{l_1} \cdots (ik_n)^{l_n} E_k(x), \qquad \sup_{k \in Z^n} (|k_1|^{l_1} \cdots |k_n|^{l_n}) \|E_k(x)\| < \infty,$$

where $x \in \mathcal{D}_\alpha(A)$, $k=(k_1, \cdots, k_n) \in Z^n$, $l=(l_1, \cdots, l_n) \in Z_+^n$.

By the corollary above, we can define a sequence of seminorms $\{q_m\}_{m \in N}$ on $C_\alpha^\infty(A)$ by

$$q_m(x) = \max_{|l| \leq m} \cdot \sup_{k \in Z^n} (|k_1|^{l_1} \cdots |k_n|^{l_n}) \|E_k(x)\|$$

where $l=(l_1, \cdots, l_n) \in Z_+^n$,

We will see that the topology on $C_\alpha^\infty(A)$ induced by the seminorms $\{q_m\}_{m \in N}$ coincides with that done by the seminorms $\{p_m\}_{m \in N}$.

LEMMA 2.5. *For any* $x \in \mathcal{D}_\alpha(A)$ *and* $l \in Z_+^n$, *we have*

(i)

$$\|x\| \leqq q_0(x) + q_{2n}(x) \sum_{j=1}^n \binom{n}{j} \left(\frac{\pi^2}{3}\right)^{n-j}$$

(ii)

$$\|D^l x\| \leqq q_{|l|}(x) + q_{|l|+2n}(x) \sum_{j=1}^n \binom{n}{j} \left(\frac{\pi^2}{3}\right)^{n-j}.$$

PROOF. (i) Take an arbitrary continuous linear functional φ on A with $\|\varphi\| \leqq 1$ and put $f(t) = \varphi(\alpha_t(x))$, $t \in T^n$ so that $f \in C^\infty(T^n)$. We first notice that

$$|\varphi(x)| \leqq \sup_{t \in T^n} |\varphi(\alpha_t(x))| = \|f\|_\infty \leqq \sum_{k \in Z^n} |\hat{f}(k)|$$

where $\hat{f}(k)$ means the k-th Fourier coefficient of f.

Let Ω_j $(j=1, 2, \cdots, n)$ be the subset of Z^n consisting of all elements in which the number of nonzero components is $n-j$. Then the following inequality holds

$$\sum_{k \in \Omega_j} |\hat{f}(k)| \leqq q_{2n}(x) \binom{n}{j} \left(\frac{\pi^2}{3}\right)^{n-j}$$

because $\sum_{\substack{j \neq 0 \\ j \in Z}} 1/j^2 = \pi^2/3$. In fact, we have

$$\sum_{k_2 \neq 0, \cdots, k_n \neq 0} |\hat{f}(0, k_2, \cdots, k_n)|$$

$$\leqq \sum_{k_2 \neq 0, \cdots, k_n \neq 0} \frac{1}{k_2^2 \cdots k_n^2} \sup_{k_2, \cdots, k_n \in Z} |k_2|^2 \cdots |k_n|^2 |\hat{f}(0, k_2, \cdots, k_n)|$$

$$\leqq \sum_{k_2 \neq 0, \cdots, k_n \neq 0} \frac{1}{k_2^2 \cdots k_n^2} \cdot \sup_{k \in Z^n} (|k_1|^2 \cdots |k_n|^2 |\hat{f}(k)|).$$

As we see

$$|\hat{f}(k)| = |\varphi(E_k(x))| \leqq \|E_k(x)\|\,,$$

it follows that

$$\sum_{k_2\neq 0,\cdots,k_n\neq 0} |\hat{f}(0, k_2, \cdots, k_n)| \leqq \Big(\sum_{k_2\neq 0}\frac{1}{k_2^2}\Big)\cdots\Big(\sum_{k_n\neq 0}\frac{1}{k_n^2}\Big)q_{2n}(x)$$

so that

$$\sum_{k_2\neq 0,\cdots,k_n\neq 0} |\hat{f}(0, k_2, \cdots, k_n)| \leqq q_{2n}(x)\Big(\frac{\pi^2}{3}\Big)^{n-1}.$$

Hence we get similarly

$$\sum_{k\in\Omega_1} |\hat{f}(k)| \leqq n\cdot q_{2n}(x)\Big(\frac{\pi^2}{3}\Big)^{n-1}.$$

It is easy to see that

$$\sum_{k\in\Omega_j} |\hat{f}(k)| \leqq q_{2n}(x)\binom{n}{j}\Big(\frac{\pi^2}{3}\Big)^{n-j}$$

in a similar way. Thus we have

$$\sum_{k\in Z^n} |\hat{f}(k)| = |\hat{f}(0, \cdots, 0)| + \sum_{j=1}^{n}\sum_{k\in\Omega_j} |\hat{f}(k)| \leqq q_0(x) + q_{2n}(x)\sum_{j=1}^{n}\binom{n}{j}\Big(\frac{\pi^2}{3}\Big)^{n-j}.$$

(ii) The assertion is easily seen by the identity: $q_m(D^l x) = q_{m+|l|}(x)$.

COROLLARY 2.6. *The two seminorms $p_m(\cdot)_{m\in N}$ and $q_l(\cdot)_{l\in N}$ on $C_\alpha^\infty(A)$ yield the same topology on it.*

PROOF. For any $l\in Z_+^n$, we have

$$\sup_{k\in Z_n}(|k_1|^{l_1}\cdots|k_n|^{l_n})\|E_k(x)\| = \sup_{k\in Z_n}\|E_k(D^l x)\| \leqq \|D^l x\|$$

which shows $q_m(x)\leqq p_m(x)$. By combining Lemma 2.5 with the inequality above, we conclude the assertion.

We call the topology on $C_\alpha^\infty(A)$ induced by these seminorms $\mathcal{D}_\alpha(A)$-topology.

Since the differential operators D^l for $l\in Z_+^n$ are closed on $\mathcal{D}_\alpha(A)$, the next proposition is clear.

PROPOSITION 2.7. *The topological vector space $\mathcal{D}_\alpha(A)$ is complete. Namely, it becomes a Fréchet space.*

DEFINITION. Let x_k, $k\in Z^n$ be a sequence of A which satisfies $E_k(x_k)=x_k$, $k\in Z^n$. Such a sequence is said to be *rapidly decreasing* if for any multi-index $l=(l_1, \cdots, l_n)\in Z_+^n$, there exists a positive constant c_l such that

$$|k_1|^{l_1}\cdots|k_n|^{l_n}\|x_k\| \leqq c_l\,, \qquad \text{for all } k=(k_1, \cdots, k_n)\in Z^n.$$

LEMMA 2.8. *Let $\{x_k\}_{k\in Z^n}$ be a sequence in A with $E_k(x_k)=x_k$. Then the* *following two conditions are equivalent*:

(i) *There exists* $x\in\mathcal{D}_\alpha(A)$ *such that* $E_k(x)=x_k$.

(ii) $\{x_k\}$ *is rapidly decreasing.*

PROOF. The implication (i)$\Rightarrow$(ii) is clear by the inequality

$$|k_1|^{l_1}\cdots|k_n|^{l_n}\|E_k(x)\|=\|E_k(D^l x)\|\leqq\|D^l x\|.$$

To prove the other implication (ii)$\Rightarrow$(i) we set for $j\in N$

$$Y_j=\{(k_1,\cdots,k_n)\in Z^n \mid \ |k_1|+\cdots+|k_n|\leqq j\}.$$

Hence we have

$$Y_0\subset Y_1\subset Y_2\subset\cdots,\ \bigcup_{j=0}^{\infty}Y_j=Z^n.$$

Put

$$x(j)=\sum_{k\in Y_j}x_k\in A.$$

Since $\{x_k\}_{k\in Z^n}$ is rapidly decreasing, for the multi-index $(2,\cdots,2)\in Z_+^n$, we take a positive constant $c_2>0$ satisfying

$$|k_1|^2\cdots|k_n|^2\|x_k\|\leqq c_2\qquad\text{for all } k=(k_1,\cdots,k_n)\in Z^n.$$

As in the proof of Lemma 2.5, we get

$$\sum_{k\in Z^n}\|x_k\|\leqq\|x_{(0,\dots,0)}\|+c_2\sum_{j=1}^{n}\binom{n}{j}\left(\frac{\pi^2}{3}\right)^{n-j}.$$

This implies that the sum $\sum_{k\in Z^n}x_k$ converges absolutely in the norm of A so that it defines an element of A. We write it as x. Namely, $x=\sum_{k\in Z^n}x_k=\lim_{j\to\infty}x(j)\in A$. For $l=(l_1,\cdots,l_n)\in Z_+^n$, we have

$$D^l x(j)=\sum_{k\in Y_j}D^l x_k=\sum_{k\in Y_j}(ik_1)^{l_1}\cdots(ik_n)^{l_n}x_k.$$

It is routine to show that the $\lim_{j\to\infty}D^l x(j)$ converges to an element of A and it is nothing but $D^l x$. Thus we conclude that the summation $\sum_{k\in Z^n}x_k$ (or the $\lim_{j\to\infty}x(j)$) converges to the x in the $\mathcal{D}_\alpha(A)$-topology. Hence x belongs to $\mathcal{D}_\alpha(A)$. We then have $E_k(x(j))=x_k$, for $k\in Y_j$, so that $E_k(x)=x_k$ by the continuity of E_k.

COROLLARY 2.9. *Any element* x *of* $\mathcal{D}_\alpha(A)$ *can be expanded as*

$$x=\sum_{k\in Z^n}x_k:\ \textit{convergence in } \mathcal{D}_\alpha(A)\textit{-topology}$$

where the sequence x_k, $k\in Z^n$ *in* A *satisfies the following two conditions*:

(i) $E_k(x_k)=x_k$ *and hence* $E_k(x_l)=0$ *for* $l\neq k$.

(ii) $\{x_k\}_{k\in Z^n}$ *is rapidly decreasing.*

3. The Hilbert C^*-modules.

Let B be the C^*-subalgebra consisting of all fixed elements under the action α, that is,

$$B = \{a \in A \mid \alpha_t(a) = a, \text{ for all } t \in T^n\}.$$

We define a B-valued inner product on A by

$$\langle x, y\rangle_r = E_0(x^*y), \qquad x, y \in A .$$

By completing A under the norm $\|x\|_{r,2} = \|\langle x, x\rangle_r\|^{1/2}$, we see that the completed space naturally has a Hilbert C^*-right B-module structure (cf. [**Bl**]). It is also a left $\mathcal{D}_\alpha(A)$-module in an evident way. We denote it by $L^2_{\alpha r}(A)$. We can similarly construct a Hilbert C^*-left B-module $L^2_{\alpha l}(A)$ by using the following B-valued inner product

$$\langle x, y\rangle_l = E_0(xy^*), \qquad x, y \in A .$$

In this section, we mainly treat the Hilbert C^*-right B-module $L^2_{\alpha r}(A)$. Similar discussion works for $L^2_{\alpha l}(A)$. Throughout this section, we write $\langle\,,\rangle_r$ and $\|\cdot\|_{r,2}$ as $\langle\,,\rangle$ and $\|\cdot\|_2$ respectively.

Let us extend the action α on A to that on $L^2_{\alpha r}(A)$. It is possible because

$$\|\alpha_t(a)\|_2 = \|a\|_2, \qquad a \in A,\ t \in T^n .$$

We also write the extended automorphism as α on $L^2_{\alpha r}(A)$. For any $x \in L^2_{\alpha r}(A)$ and $k \in Z^n$, we define the k-th Fourier component by

$$E_k(x) = \int_{T^n} \alpha_t(x) e_{-k}(t) d\sigma_n(t), \qquad k \in Z^n.$$

The element $E_k(a)$ is defined first as a member of $L^2_{\alpha r}(A)$. But we will see that it belongs to the algebra A and hence to the smooth algebra $\mathcal{D}_\alpha(A)$. We first notice that the inequality below

$$\|E_k(x)\|_2 \leqq \|x\|_2, \qquad x \in L^2_{\alpha r}(A),\ k \in Z^n . \tag{3.1}$$

because of the inequality

$$\left\|\int_{T^n} \alpha_t(x) e_{-k}(t) d\sigma_n(t)\right\|_2 \leqq \int_{T^n} \|\alpha_t(x) e_{-k}(t)\|_2 d\sigma_n(t).$$

LEMMA 3.1. $\|E_k(a)\| \leqq \|a\|_2$, $a \in \mathcal{D}_\alpha(A)$, $k \in Z^n$.

PROOF. For $a \in \mathcal{D}_\alpha(A)$, by using Corollary 2.9, we see

$$E_0(a^*a) \geqq E_k(a)^* E_k(a), \qquad k \in Z^n ,$$

so that $\|E_k(a)\| \leqq \|a\|_2$.

LEMMA 3.2. *For any* $x \in L^2_{\alpha r}(A)$, *and* $k \in Z^n$, $E_k(x)$ *belongs to* A *and hence to* $\mathcal{D}_\alpha(A)$.

PROOF. Take a sequence $x_j \in \mathcal{D}_\alpha(A)$ satisfying $\lim_{j\to\infty} \|x_j - x\|_2 = 0$. By the inequality (3.1), we see that $E_k(x_j)$ converges to $E_k(x)$ in $\|\cdot\|_2$-norm as j tends to infinity. On the other hand, we see, by Lemma 3.1, that the sequence $\{E_k(x_j)\}_j$ is a Cauchy sequense in C^*-norm of A so that there exists a C^*-norm limit a_k in A. The inequality below

$$\|a_k - E_k(x_j)\|_2 \leqq \|a_k - E_k(x_j)\|$$

shows that $a_k = E_k(x)$. Thus we have $E_k(x) \in A$. It is easy to see that

$$D^l E_k(x) = (ik_1)^{l_1} \cdots (ik_n)^{l_n} E_k(x)$$

for $l \in Z^n_+$, because $E_k(x) = x * e_{-k}$ and Lemma 2.1.

LEMMA 3.3.

$$\langle E_k(x),\, a\rangle = \langle E_k(x),\, E_k(a)\rangle\,, \qquad x \in A,\ a \in L^2_{\alpha r}(A),\ k \in Z^n\,.$$

PROOF. Take a sequence $b_j \in \mathcal{D}_\alpha(A)$, $j \in N$ such that $\lim_{j\to\infty} \|b_j - a\|_2 = 0$. By the Schwartz type inequality (cf. [**Bl**; Proposition 13.1.3]), we have

$$\langle E_k(x),\, a\rangle = C^*\text{-}norm \lim_{j\to\infty} \langle E_k(x),\, b_j\rangle\,.$$

As each b_j can be expressed as $b_j = \sum_{k \in Z^n} E_k(b_j)$ in the $\mathcal{D}_\alpha(A)$-topology, it follows that

$$\langle E_k(x),\, b_j\rangle = \langle E_k(x),\, E_k(b_j)\rangle$$

because of the continuity of E_k and orthogonal relations. By the Schwartz type inequality and (3.1), we obtain the assertion.

LEMMA 3.4. *Any element* $a \in L^2_{\alpha r}(A)$ *can be uniquely expanded as*

$$a = \sum_{k \in Z^n} a_k\,, \qquad a_k = E_k(a) \in A$$

where the summation converges in $\|\cdot\|_2$*-topology.*

PROOF. Let Y_j be the finite set of Z^n defined in the proof of Lemma 2.8. Put

$$a(j) = \sum_{k \in Y_j} a_k\,.$$

We notice that

$$\langle E_k(x),\, a - a(j)\rangle = 0\,, \qquad \text{for all } x \in A,\ k \in Y_j \tag{3.2}$$

from the previous lemma. Since the $*$-subalgebra $\mathcal{T}_\alpha(A)$ is dense in $L^2_{\alpha r}(A)$ in $\|\cdot\|_2$-topology, for $\varepsilon > 0$, there exist $c_k \in A$ and $j \in N$ such that

$$\|a-\sum_{k\in Y_j}c_k\|_2<\varepsilon\,,\qquad E_j(c_j)=c_j\,.$$

Put $c=\sum_{k\in Y_j}c_k$. Set

$$L_{\alpha r}^2(A)_j=\{\sum_{k\in Y_j}x_k\,|\,x_k\in A,\ E_k(x_k)=x_k\}\,.$$

As $L_{\alpha r}^2(A)_j=\{\sum_{k\in Y_j}E_k(x)\,|\,x\in A\}$, we have $\langle y,\ a-a(j)\rangle=0$, $y\in L_{\alpha r}^2(A)_j$ by the equality (3.2). By using the Pythagorean theorem, it follows that

$$\begin{aligned}\langle a-a(j),\ a-a(j)\rangle &\leqq \langle a-a(j),\ a-a(j)\rangle+\langle a(j)-c,\ a(j)-c\rangle\\ &=\langle a-c,\ a-c\rangle\,.\end{aligned}$$

This means $\|a-a(j)\|_2\leqq\|a-c\|_2<\varepsilon$. Thus we conclude

$$a=L^2\text{-}norm\lim_{j\to\infty}\sum_{k\in Y_j}a_k\,.$$

The uniqueness for the expression follows from the inequality (3.1).

COROLLARY 3.5.
(i) *For any element* $a\in L_{\alpha r}^2(A)$, *we have*

$$\sum_{k\in Z^n}E_k(a)^*E_k(a)=\langle a,\ a\rangle$$

where the summation above converges in C^*-*norm of* A.

(ii) *Conversely, for a sequence* $a_k\in A$, $k\in Z^n$ *with* $E_k(a_k)=a_k$, *if* $\sum_{k\in Z^n}a_k^*a_k$ *converges in* C^*-*norm of* A, *then* $\sum_{k\in Z^n}a_k$ *converges in* L^2-*norm. In this case, if we put* $a=\sum_{k\in Z^n}a_k$, *then* $E_k(a)=a_k$, $k\in Z^n$.

PROOF. (i) For $a\in L_{\alpha r}^2(A)$, let $a(j)$, $j\in Z$ be the sequence defined in the proof of the preceding lemma, which converges to a in L^2-norm topology. Since we have

$$\langle a(j),\ a(j)\rangle=\sum_{k\in Y_j}E_k(a)^*E_k(a)\,,$$

it follows that

$$\begin{aligned}\|\sum_{k\in Y_j}E_k(a)^*E_k(a)-\langle a,\ a\rangle\| &=\|\langle a(j),\ a(j)\rangle-\langle a,\ a\rangle\|\\ &\leqq\|a(j)\|_2\|a(j)-a\|_2+\|a(j)-a\|_2\|a\|_2\,.\end{aligned}$$

Thus we obtain

$$C^*\text{-}norm\lim_{j\to\infty}\sum_{k\in Y_j}E_k(a)E_k(a)=\langle a,\ a\rangle\,.$$

(ii) Conversely, suppose $E_k(a_k)=a_k$ and $\lim_{j\to\infty}\sum_{k\in Y_j}a_k^*a_k$ converges in C^*-norm of A. Put $a(j)=\sum_{k\in Y_j}a_k$. Then we have for $i>j$

$$\|a(i)-a(j)\|_2 = \|\sum_{k\in Y_i\setminus Y_j} a_k^* a_k\| .$$

As we have $\lim_{i,j\to\infty}\|\sum_{k\in Y_i\setminus Y_j} a_k^* a_k\|=0$, the sequence $\{a(j)\}_{j\in N}$ is a Cauchy sequence in L^2-norm, whose limit in $L^2_{\alpha r}(A)$ is nothing but $\sum_{k\in Z^n} a_k$.

PROPOSITION 3.6.

(i) *For any element* $a\in L^2_{\alpha r}(A)$, *there uniquely exists a sequence* $a_k\in A$, $k\in Z^n$ *with* $E_k(a_k)=a_k$ *satisfying*

$$a = \sum_{k\in Z^n} a_k : L^2\textit{-norm convergence}$$

and

$$\langle a, a\rangle = \sum_{k\in Z^n} a_k^* a_k : C^*\textit{-norm convergence.}$$

(ii) *For elements* $a, b\in L^2_{\alpha r}(A)$, *we have*

$$\langle a, b\rangle = \sum_{k\in Z^n} a_k^* b_k : C^*\textit{-norm convergence}$$

where

$$a = \sum_{k\in Z^n} a_k , \qquad b = \sum_{k\in Z^n} b_k .$$

(iii) *Conversely, for a sequence* $a_k\in A$, $k\in Z^n$ *with* $E_k(a_k)=a_k$, *such that*

$$\sum_{k\in Z^n} a_k^* a_k : C^*\textit{-norm convergence,}$$

there uniquely exists an element $a\in L^2_{\alpha r}(A)$ *such that*

$$a = \sum_{k\in Z^n} a_k : L^2\textit{-norm convergence.}$$

4. The C^*-distributions.

DEFINITION. A *C^*-left (periodic) distribution* ξ is a continuous left B-module map from $\mathcal{D}_\alpha(A)$ to B under the $\mathcal{D}_\alpha(A)$-topology and C^*-norm topology on B. Namely, it satisfies

$$\xi(bx) = b\xi(x), \qquad b\in B,\ x\in\mathcal{D}_\alpha(A).$$

A C^*-right (periodic) distribution is similarly defined as a continuous right B-module map from $\mathcal{D}_\alpha(A)$ to B.

We denote by $\mathcal{D}'_{\alpha l}(A)$ (resp. $\mathcal{D}'_{\alpha r}(A)$) the set of all C^*-left (resp. right) distributions. We mainly deal with C^*-left distributions and sometimes simply call them C^*-distributions unless we specify.

We first equip $\mathcal{D}'_{\alpha l}(A)$ with left $\mathcal{D}_\alpha(A)$ and right B-module structure as in the following way:

$$(a\xi b)(x)=\xi(xa)b, \qquad \xi\in\mathcal{D}'_{\alpha l}(A),\ a\in\mathcal{D}_\alpha(A),\ b\in B,\ x\in\mathcal{D}_\alpha(A).$$

It is clear that the above one $a\xi b$ becomes a C^*-distribution again.

We next equip $\mathcal{D}'_{\alpha l}(A)$ with a topology which is induced by the pointwise norm convergence. Namely, ξ_j converges to ξ if and only if for each $x\in\mathcal{D}_\alpha(A)$, $\xi_j(x)$ converges to $\xi(x)$ in the C^*-norm on B. We call it $\mathcal{D}'$-topology. We will define an action of the compact abelian group T^n on $\mathcal{D}'_{\alpha l}(A)$ by

$$\alpha_t(\xi)(x)=\xi(\alpha_{-t}(x)), \qquad x\in\mathcal{D}_\alpha(A),\ t\in T^n.$$

As an analogue of the classical case, we will consider an embedding $L^2_{\alpha r}(A)$ into $\mathcal{D}'_{\alpha l}(A)$.

Definition. For any $x\in L^2_{\alpha r}(A)$, we define a C^*-distribution L_x by

$$L_x(y)=\langle y^*, x\rangle_r, \qquad y\in\mathcal{D}_\alpha(A),\ x\in L^2_{\alpha r}(A).$$

It is easy to see that L_x gives a C^*-distribution and the map L preserves the left $\mathcal{D}_\alpha(A)$ and right B-module structure on $L^2_{\alpha r}(A)$. Namely we have

$$aL_xb=L_{axb}, \qquad a\in\mathcal{D}_\alpha(A),\ x\in L^2_{\alpha r}(A),\ b\in B.$$

Moreover, we have

$$\alpha_t(L_x)=L_{\alpha_t(x)}, \qquad x\in L^2_{\alpha r}(A),\ t\in T^n.$$

Thus we sometimes identify the Hilbert module $L^2_{\alpha r}(A)$ with its image in $\mathcal{D}'_{\alpha l}(A)$ through the map L. Since the original C^*-algebra A is embedded into $L^2_{\alpha r}(A)$, all elements of A can be naturally regarded as C^*-distributions.

Definition (Differentiation). For any $\xi\in\mathcal{D}'_{\alpha l}(A)$ and a multi-index $l=(l_1,\cdots,l_n)\in Z^n_+$, we define its l-th differentiation $D^l\xi$ (as a C^*-left distribution) by

$$(D^l\xi)(x)=(-1)^{|l|}\xi(D^l(x)), \qquad \text{for } x\in\mathcal{D}_\alpha(A).$$

It is immediate to see that $D^l\xi$ is a C^*-left distribution again.

Lemma 4.1. $D^lL_x=L_{D^lx}$, $x\in\mathcal{D}_\alpha(A)$, $l\in Z^n_+$.

Proof. We denote by D_i the i-th partial derivative on $\mathcal{D}'_{\alpha l}(A)$. As we have $L_a(y)=E_0(ya)$, for $a,\ y\in\mathcal{D}_\alpha(A)$, it follows that

$$0=\delta_i(E_0(yx))=E_0(\delta_i(y)x)+E_0(y\delta_i(x))=-(D_iL_x)(y)+L_{\delta_i(x)}(y)$$

and hence $(D_iL_x)(y)=L_{\delta_i(x)}(y)$.

This lemma says that the original derivations D^l, $l\in Z^n_+$ on the domain $\mathcal{D}_\alpha(A)$ can be extended on the whole distributions beyond $\mathcal{D}_\alpha(A)$. Therefore any element a of the C^*-algebra A can be differentiable in the C^*-distribution sense. It is a matter of course that the resulting element $D^l a$ does not necessarily belong to A.

LEMMA 4.2. *For all $\xi\in\mathscr{D}'_{\alpha l}(A)$ and $1\leqq i\leqq n$, we have*

$$\mathscr{D}'\text{-}\lim_{h\to 0}\frac{\alpha_{h\varepsilon_i}(\xi)-\xi}{h}=D_i\xi .$$

where $h\in R$ and $\varepsilon_i=(0,\cdots,0,1,0,\cdots,0)\in Z^n_+$.

PROOF. For $x\in\mathscr{D}_\alpha(A)$, we have

$$\left\|\left(\frac{\alpha_{h\varepsilon_i}(\xi)-\xi}{h}\right)(x)-(D_i\xi)(x)\right\|=\left\|\xi\left(\frac{\alpha_{-h\varepsilon_i}(x)-x}{-h}-\delta_i(x)\right)\right\|.$$

As $x\in\mathscr{D}_\alpha(A)$, $(\alpha_{-h\varepsilon_i}(x)-x)/-h$ converges to $\delta_i(x)$ in $\mathscr{D}_\alpha(A)$-topology. Hence we get the assertion.

The C^*-left distribution space $\mathscr{D}'_{\alpha l}(A)$ has left $\mathscr{D}_\alpha(A)$-module structure as we cited before. This module structure behaves well for differentiations. Namely, we can easily show the following Leibnitz formula.

LEMMA 4.3.

$$D_i(a\xi)=\delta_i(a)\xi+aD_i\xi,\qquad \xi\in\mathscr{D}'_{\alpha l}(A),\ a\in\mathscr{D}_\alpha(A),\ 1\leqq i\leqq n .$$

and hence

$$D^l(a\xi)=\sum_{k\leq l}c_{lk}D^{l-k}(a)D^k\xi .$$

where c_{lk} is a constant for l, $k\in Z^n_+$ and $k\leqq l$ means $k_i\leqq l_i$ for $1\leqq i\leqq n$.

We next study convolution products on the C^*-distributions. In general, convolutions between C^*-distributions can not be defined. However, it is possible to define convolutions between C^*-distributions and the classical distributions as follows. Let $\mathscr{D}'(T^n)$ be the set of all classical distributions on T^n. It is well known that a distribution φ in $\mathscr{D}'(T^n)$ can be regarded as a slowly increasing sequence $\{\varphi(e_k)\}_{k\in Z^n}$ of its Fourier coefficients.

DEFINITION (Convolutions). For $\xi\in\mathscr{D}'_{\alpha l}(A)$, $a\in\mathscr{D}_\alpha(A)$ and $\varphi\in\mathscr{D}'(T^n)$, we define the convolutions $a*\varphi$ and $\xi*\varphi$ by

$$a*\varphi=\sum_{k\in Z^n}E_k(a)\varphi(e_k),\qquad (\xi*\varphi)(x)=\xi(x*\tilde\varphi),\quad x\in\mathscr{D}_\alpha(A)$$

where $\tilde\varphi\in\mathscr{D}'(T^n)$ is a distribution whose Fourier coefficients are given by $\{\varphi(e_{-k})\}_{k\in Z^n}$.

The following lemma is routine.

LEMMA 4.4. *For $\xi\in\mathscr{D}'_{\alpha l}(A)$, $a\in\mathscr{D}_\alpha(A)$ and $\varphi\in\mathscr{D}'(T^n)$, we have*

(i) $a*\varphi\in\mathcal{D}_{\alpha}(A)$, $\xi*\varphi\in\mathcal{D}_{\alpha l}'(A)$.

(ii) $D^{l}(\xi*\varphi)=(D^{l}\xi)*\varphi=\xi*(D^{l}\varphi)$, $l\in Z_{+}^{n}$.

5. Fourier series for C^*-distributions.

We define Fourier series for C^*-distributions. As in our following definition, each component of them are defined as C^*-distributions.

DEFINITION (Fourier components for C^*-distributions). For $\xi\in\mathcal{D}_{\alpha l}'(A)$, $k\in Z^{n}$, we define k-th Fourier component $\mathcal{E}_{k}(\xi)$ of ξ as a C^*-left distribution by

$$\mathcal{E}_{k}(\xi)(x)=\xi(E_{-k}(x)), \qquad x\in\mathcal{D}_{\alpha}(A).$$

It is easy to see that $\mathcal{E}_{k}(\xi)\in\mathcal{D}_{\alpha l}'(A)$. The following lemma is immediate from the definition.

LEMMA 5.1.

$$\mathcal{E}_{k}(\xi)(E_{l}(x))\equiv\begin{cases}\mathcal{E}_{k}(\xi)(x), & l=k,\\ 0, & l\neq k,\end{cases} \qquad \mathcal{E}_{k}(\mathcal{E}_{l}(\xi))\equiv\begin{cases}\mathcal{E}_{k}(\xi), & l=k,\\ 0, & l\neq k,\end{cases}$$

where $\xi\in\mathcal{D}_{\alpha l}'(A)$, l, $k\in Z^{n}$, $x\in\mathcal{D}_{\alpha}(A)$.

LEMMA 5.2. $\mathcal{E}_{k}(L_{x})=E_{k}(x)$, $x\in L_{\alpha r}^{2}(A)$, $k\in Z^{n}$.

PROOF. We have for $y\in\mathcal{D}_{\alpha}(A)$,

$$\mathcal{E}_{k}(L_{x})(y)=L_{x}(E_{-k}(y))=\langle y^{*}, E_{k}(x)\rangle_{r}=L_{E_{k}(x)}(y)$$

because of the orthogonal expansion $x=\sum_{k\in Z^{n}}E_{k}(x)$ in $L_{\alpha r}^{2}(A)$.

This lemma shows that the Fourier components of C^*-distributions are extended notations for those of the elements of the C^*-algebra A and of the Hilbert C^*-module $L_{\alpha r}^{2}(A)$.

We henceforth assume that A is nondegenerately represented on a Hilbert space H on which α is spatial. This means that α is written as $\alpha_{t}=\mathrm{Ad}\,u_{t}$, $t\in T^{n}$ for some strongly continuous unitary representation u on H of T^{n}. This assumption is always achieved by embedding A into the reduced crossed product $A\times_{\alpha}T^{n}$. Hence α is uniquely extended on the weak operator closure A'' of A as a σ-weakly continuous action of T^{n}, which is also denoted by α.

We will show that each Fourier component of a C^*-distribution can be regarded as an element of A'' and hence a C^*-distribution can be characterized in terms of a sequence of A''.

For $k\in Z^{n}$ let, $A^{\alpha}(k)$ be the k-th spectral subspace for the C^*-dynamical system (A, α, Z^{n}), which means

$$A^{\alpha}(k)=\{a\in A\,|\,\alpha_{t}(a)=e^{it\cdot k}a,\ t\in Z^{n}\}=E_{k}(A).$$

We put for each $k \in Z^n$

$$\mathfrak{m}_\alpha(k) = \left\{ \sum_{l=1}^{m} x(l)y(l)^* \mid x(l),\ y(l) \in A^\alpha(k),\ 1 \leqq l \leqq m,\ m \in N \right\}.$$

Namely $\mathfrak{m}_\alpha(k)$ is the set of all finite linear combinations of elements of the form xy^*, x, $y \in A^\alpha(k)$. It is easy to see that $\mathfrak{m}_\alpha(k)$ becomes a 2-sided ideal of B. We fix $k \in Z^n$ henceforth.

LEMMA 5.3. *There exists a net* $\{u_\lambda\}_{\lambda \in \Lambda}$ *of elements of* $\mathfrak{m}_\alpha(k)$ *such that*

$$\lim_\lambda \|xu_\lambda - x\| = 0, \qquad x \in A^\alpha(-k),$$

$$0 \leqq u_\lambda \leqq u_\mu, \quad \lambda \leqq \mu \quad \textit{and} \quad \|u_\lambda\| \leqq 1.$$

PROOF. By [**Ta**; Theorem 7.4], we can take a net $\{u_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathfrak{m}_\alpha(k)$ which forms a (right) approximate identity for the C^*-norm closure $\overline{\mathfrak{m}_\alpha(k)}$ of $\mathfrak{m}_\alpha(k)$. It follows that for $x \in A^\alpha(-k)$

$$\|xu_\lambda - x\|^2 = \|(u_\lambda - 1)^* x^* x(u_\lambda - 1)\| \leqq 2\|x^* x(u_\lambda - 1)\|.$$

Thus we have $\lim_\lambda \|xu_\lambda - x\| = 0$.

LEMMA 5.4. *A continuous left B-module map ζ from $A^\alpha(-k)$ to B satisfies the following inequality*

$$\left\| \sum_{i=1}^{m} z(i)\zeta(w(i)^*) \right\| \leqq \|\zeta\| \left\| \sum_{n=1}^{m} z(i)w(i)^* \right\|, \quad z(i),\ w(i) \in A^\alpha(k),\ 1 \leqq i \leqq m,\ m \in N.$$

PROOF. The identity below

$$\left(\sum_{i=1}^{m} z(i)\zeta(w(i)^*) \right)^* \left(\sum_{j=1}^{m} z(j)\zeta(w(j)^*) \right) = \zeta\left(\sum_{i=1}^{m} \zeta(w(i)^*)^* z(i)^* \sum_{j=1}^{m} z(j)w(j)^* \right)$$

implies that

$$\begin{aligned} \left\| \sum_{i=1}^{m} z(i)\zeta(w(i)^*) \right\|^2 &\leqq \|\zeta\| \left\| \sum_{i=1}^{m} \zeta(w(i)^*)^* z(i)^* \cdot \sum_{j=1}^{m} z(j)w(j)^* \right\| \\ &\leqq \|\zeta\| \left\| \sum_{i=1}^{m} z(i)\zeta(w(i)^*) \right\| \left\| \sum_{j=1}^{m} z(j)w(j)^* \right\|. \end{aligned}$$

Hence we have the desired inequality.

Since (A'', α, T^n) is a W^*-dynamical system, the k-th spectral subspace $(A'')^\alpha(k)$ associated with this action is similarly defined as the previous ones. The expectation E_k is uniquely extended on A'' as a σ-weakly continuous expectation from A'' to $(A'')^\alpha(k)$, which is also denoted by E_k.

LEMMA 5.5. *The spectral subspace $A^\alpha(k)$ is dense in $(A'')^\alpha(k)$ in the weak operator topology.*

PROOF. Since the extended expectation E_k from A'' to $(A'')^{\alpha}(k)$ is weakly continuous, we easily have the desired assertion from the fact that A is dense in A'' in the topology.

We then have the following proposition.

PROPOSITION 5.6. *For a continuous left B-module map ζ from $A^{\alpha}(-k)$ to B, there exists a unique element $e_k(\zeta)$ in $(A'')^{\alpha}(k)$ such that*

$$\zeta(x) = xe_k(\zeta), \quad x \in A^{\alpha}(-k) \quad \text{and} \quad \|e_k(\zeta)\| = \|\zeta\| .$$

PROOF. Take a net $\{u_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathfrak{m}_{\alpha}(k)$ as in Lemma 5.3. They are of the form

$$u_\lambda = \sum_{i=1}^{m(\lambda)} z_\lambda(i) w_\lambda(i)^* , \qquad \lambda \in \Lambda$$

for some $z_\lambda(i)$, $w_\lambda(i) \in A^{\alpha}(k)$, and $m(\lambda) \in N$. Put

$$v_\lambda = \sum_{i=1}^{m(\lambda)} z_\lambda(i) \zeta(w_\lambda(i)^*), \qquad \lambda \in \Lambda .$$

Then we have

$$\|v_\lambda\| \leqq \|\zeta\| \|u_\lambda\| \leqq \|\zeta\|$$

by Lemma 5.4 so that $\{v_\lambda\}_{\lambda \in \Lambda}$ is a bounded net in A. Thus we can take a weak limit point $e_k(\zeta)$ in A'' which satisfies $\|e_k(\zeta)\| \leqq \|\zeta\|$. As v_λ belongs to $A^{\alpha}(k)$, $e_k(\zeta)$ belongs to $(A'')^{\alpha}(k)$ because $(A'')^{\alpha}(k)$ is weakly closed in A''. Since we see for $x \in A^{\alpha}(-k)$

$$xv_\lambda = \sum_{i=1}^{m(\lambda)} \zeta(xz_\lambda(i) w_\lambda(i)^*) = \zeta(xu_\lambda),$$

it follows that

$$xe_k(\zeta) = \zeta(x), \qquad x \in A^{\alpha}(-k)$$

because of the continuity of ζ and Lemma 5.3. The inequality $\|\zeta\| \leqq \|e_k(\zeta)\|$ is direct from the above equality so that we have $\|\zeta\| = \|e_k(\zeta)\|$.

For an element $e \in (A'')^{\alpha}(k)$ with $xe=0$, $x \in A^{\alpha}(k)$, we obtain $ye=0$, $y \in (A'')^{\alpha}(k)$, by Lemma 5.5 so that $e^*e=0$. Therefore $e_k(\zeta)$ is unique.

As each Fourier component $\mathcal{E}_k(\xi)$, $k \in Z^n$ of a C^*-distribution $\xi \in \mathcal{D}'_{\alpha l}(A)$ is a continuous left B-module map from $A^{\alpha}(-k)$ to B, we thus obtain

COROLLARY 5.7. *For each C^*-distribution $\xi \in \mathcal{D}'_{\alpha l}(A)$, there exists a unique sequence $E_k(\xi) \in (A'')^{\alpha}(k)$, $k \in Z^n$, satisfying*

$$\mathcal{E}_k(\xi)(x) = xE_k(\xi), \qquad x \in A^{\alpha}(-k).$$

Namely, each C^*-distribution of Fourier components of a C^*-distribution can be regarded as an element of the weak operator closure A'' of the C^*-algebra A.

DEFINITION. A C^*-distribution $\xi \in \mathcal{D}'_{\alpha l}(A)$ is said to be of *order* $m \in N$, if there exists a constant $c>0$ such that

$$\|\xi(x)\| \leqq c \sum_{l \in Z^n_+, |l| \leqq m} \|D^l x\|, \qquad x \in \mathcal{D}_\alpha(A).$$

EXAMPLE 5.8. For $a \in L^2_{\alpha r}(A)$ and $l \in Z^n_+$, the C^*-distribution $D^l L_a$ is of order $|l|$. In fact, we have

$$\|(D^l L_a)(x)\| \leqq \|D^l x\| \|a\|, \qquad x \in \mathcal{D}_\alpha(A).$$

LEMMA 5.9. *For any $\xi \in \mathcal{D}'_{\alpha l}(A)$, there exists $m \in N$ such that ξ is of order m.*

PROOF. Suppose that for any $m \in N$, ξ is not of order m. Hence there exists $x_m \in \mathcal{D}_\alpha(A)$ such that

$$\|\xi(x_m)\| > (m+1) \sum_{l \in Z^n_+, |l| \leqq m} \|D^l x_m\|.$$

Put

$$y_m = \frac{1}{m+1} \cdot \frac{1}{\sum_{l \in Z^n_+, |l| \leqq m} \|D^l x_m\|} \cdot x_m$$

so that

$$\|\xi(y_m)\| = \frac{1}{m+1} \cdot \frac{\|\xi(x_m)\|}{\sum_{l \in Z^n_+, |l| \leqq m} \|D^l x_m\|} > \ .$$

On the other hand,

$$\sum_{l \in Z^n_+, |l| \leqq m} \|D^l y_m\| = \frac{1}{m+1}.$$

Hence

$$\|D^l y_m\| \leqq \frac{1}{m+1} \qquad \text{for } l \in Z^n_+ \text{ satisfying } |l| \leqq m.$$

Thus $\{y_m\}_{m \in N}$ converges to zero in $\mathcal{D}_\alpha(A)$-topology although $\|\xi(y_m)\| \geqq 1$, which is a contradiction for the continuity of ξ.

DEFINITION. A sequence $a_k \in (A'')^\alpha(k)$, $k \in Z^n$ is said to be *slowly increasing* if there exist $m \in N$ and a constant $c>0$ such that

$$\|a_k\| \leqq c \sum_{l \in Z^n_+, |l| \leqq m} |k_1|^{l_1} \cdots |k_n|^{l_n} \qquad \text{for all } k=(k_1, \cdots, k_n) \in Z^n.$$

The next lemma is clear.

LEMMA 5.10. *For a sequence $a_k \in (A'')^\alpha(k)$, $k \in Z^n$, the following two condi-*

tions are equivalent:

(i) $\{a_k\}_{k\in Z^n}$ *is slowly increasing.*

(ii) *There exist* $m'\in N$ *and a constant* $c'>0$ *such that*

$$\|a_k\| \leqq c'(|k_{i_1}|\cdots|k_{i_p}|)^{m'}, \qquad \text{for all } k\in Z^n$$

where for $k=(k_1, \cdots, k_n)$ *only* $k_{i_1}, \cdots, k_{i_p}$ *are nonzero components.*

LEMMA 5.11. *A sequence of Fourier components of a C*-distribution is slowly increasing.*

PROOF. For $\xi\in\mathcal{D}'_{\alpha l}(A)$, there exists $E_k(\xi)\in(A'')^\alpha(k)$, $k\in Z^n$ such that

$$\mathcal{E}_k(\xi)(x) = xE_k(\xi), \qquad x\in A^\alpha(-k).$$

We further assume that ξ is of order $m(\in N)$ and hence

$$\|\xi(x)\| \leqq c \sum_{l\in Z_+^n, |l|\leq m} \|D^l x\|, \qquad x\in\mathcal{D}_\alpha(A).$$

Thus we have for $k\in Z^n$

$$\|E_{-k}(x)E_k(\xi)\| \leqq c \sum_{l\in Z_+^n, |l|\leq m} |k_1|^{l_1}\cdots|k_n|^{l_n}\|E_{-k}(x)\|.$$

By the continuity of the extended expectation E_k on A'' and Kaplansky's density theorem, we can find a net $x_\gamma\in A$, $\gamma\in\Gamma$ such that

$$\text{strong-}\lim_\gamma E_k(x_\gamma) = E_k(\xi) \quad\text{and}\quad \|E_k(x_\gamma)\| \leqq \|E_k(\xi)\|, \quad \gamma\in\Gamma.$$

Thus we have

$$\|E_{-k}(x_\gamma)E_k(\xi)\| \leqq c \sum_{l\in Z_+^n, |l|\leq m} |k_1|^{l_1}\cdots|k_n|^{l_n}\|E_k(\xi)\|$$

and hence

$$\|E_k(\xi)^*E_k(\xi)\| \leqq c \sum_{l\in Z_+^n, |l|\leq m} |k_1|^{l_1}\cdots|k_n|^{l_n}\|E_k(\xi)\|.$$

Therefore we see

$$\|E_k(\xi)\| \leqq c \sum_{l\in Z_+^n, |l|\leq m} |k_1|^{l_1}\cdots|k_n|^{l_n}.$$

For $k=(k_1, \cdots, k_n)\in Z^n$, a finite subset $\operatorname{supp} k\subset\{1, 2, \cdots, n\}$ is defined by

$$\operatorname{supp} k\ni i \qquad \text{if and only if } k_i\neq 0.$$

LEMMA 5.12. *For any slowly increasing sequence* $a_k\in(A'')^\alpha(k)$ *with* $A^\alpha(-k)a_k\in B$, $k\in Z^n$, *there exists a unique C*-left distribution* $\xi\in\mathcal{D}'_{\alpha l}(A)$ *satisfying*

$$E_k(\xi) = a_k, \qquad k\in Z^n.$$

PROOF. By Lemma 5.10, there exist $m\in N$ and $c>0$ such that

$$\|a_k\| \leqq c|k_{i_1}| \cdots |k_{i_p}|^{m-2},$$

$$\text{for all } k=(k_1, \cdots, k_n)\in Z^n, \quad \operatorname{supp} k=\{i_1, \cdots, i_p\}.$$

Put

$$b_k = (-ik_{i_1})^{-m} \cdots (-ik_{i_p})^{-m} a_k, \qquad \text{for } k\neq \mathbf{0}, \text{ and } b_0=0.$$

Since we see for $k\neq \mathbf{0}$,

$$\|b_k\| \leqq c|k_{i_1}|^{-2} \cdots |k_{i_p}|^{-2},$$

it follows that

$$\begin{aligned}\sum_{k\in Z^n} \|b_k\| &\leqq c \sum_{k\in Z^n} |k_{i_1}|^{-2} \cdots |k_{i_p}|^{-2} \\ &= c\Big(\sum_{k_1\in Z^n,\, k_1\neq 0} |k_1|^{-2}\Big) \cdots \Big(\sum_{k_n\in Z^n,\, k_n\neq 0} |k_n|^{-2}\Big) \\ &= c\Big(\frac{\pi^2}{3}\Big)^n.\end{aligned}$$

Thus $\sum_{k\in Z^n} b_k$ converges in C^*-norm of A. For any finite set $\Delta=\{j_1, \cdots, j_q\} \subset \{1, 2, \cdots, n\}$, we put

$$b_\Delta = \sum_{l\in Z^n,\, \operatorname{supp} l=\Delta} b_l,$$

(which absolutely converges in C^*-norm as in the above discussion)

$$D^\Delta = D_{j_1} \cdots D_{j_q}, \qquad (D^\Delta)^m = D_{j_1}^m \cdots D_{j_q}^m$$

where D_i means the differentiation on $\mathcal{D}'_{\alpha l}(A)$ corresponding to the i-th component of Z_+^n. We define a C^*-distribution ξ by the finite sum:

$$\xi = \sum_{\Delta\subset\{1,2,\cdots,n\},\, \Delta\neq\varnothing} (D^\Delta)^m L_{b_\Delta} + L_{a_0}.$$

For $x\in\mathcal{D}_\alpha(A)$, and $\Delta=\{j_1, \cdots, j_q\}\subset\{1, 2, \cdots, n\}$, we have

$$(D^\Delta)^m L_{b_\Delta}(E_{-k}(x)) = (-ik_{j_1})^m \cdots (-ik_{j_q})^m L_{b_\Delta}(E_{-k}(x))$$

which is zero unless $\{j_1, \cdots, j_q\}\subset \operatorname{supp} k$. Since we see

$$L_{b_\Delta}(E_{-k}(x)) = E_0(E_{-k}(x)b_\Delta) = \begin{cases} E_{-k}(x)b_k, & \operatorname{supp} k=\Delta, \\ 0, & \operatorname{supp} k\neq\Delta, \end{cases}$$

we have for $k\in Z^n$ with $\operatorname{supp} k=\Delta$

$$(D^\Delta)^m L_{b_\Delta}(E_{-k}(x)) = E_{-k}(x)a_k = L_{a_k}(x).$$

As we have $L_{a_0}(E_{-k}(x))=0$, for $k\neq\mathbf{0}$, we obtain

$$\mathcal{E}_k(\xi)(x) = \sum_{\Delta\subset\{1,2,\cdots\},\, \Delta\neq\varnothing} (D^\Delta)^m L_{b_\Delta}(E_{-k}(x)) + L_{a_0}(E_{-k}(x)) = L_{a_k}(x).$$

Hence we have $E_k(\xi)=a_k$ for $k\in Z^n$, $k\neq\mathbf{0}$. In the case of $k=\mathbf{0}$, it is easy to see that

$$(D^{\Delta})^m L_{b_\Delta}(E_{-k}(x)) = 0, \qquad \text{for } \Delta \neq \emptyset,\ L_{a_0}(E_{-k}(x)) = E_0(x a_0)$$

so that $E_0(\xi)=a_0$. Thus we conclude $E_k(\xi)=a_k$, for all $k\in Z^n$.

We summarize the above discussions as in the following way.

THEOREM 5.13.

(i) *For a C^*-left distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$, the sequence a_k, $k\in Z^n$ of the Fourier components of ξ is taken as $a_k\in(A'')^{\alpha}(k)$, $k\in Z^n$ and satisfies the following conditions:*

(5.1) $$A^{\alpha}(-k)a_k \subset B\,, \qquad k\in Z^n$$

and

(5.2) $$\{\|a_k\|\}_{k\in Z^n} \textit{ is slowly increasing.}$$

(ii) *Conversely, for a sequence $a_k\in(A'')^{\alpha}(k)$, $k\in Z^n$ satisfying the above two conditions* (5.1) *and* (5.2), *there exists a unique C^*-left distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$ whose Fourier components are given by the sequence $\{a_k\}_{k\in Z^n}$.*

In the proof of Lemma 5.12, we reach the next theorem for C^*-distributions.

THEOREM 5.14. *For any C^*-distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$, there exists a family of elements $b_\Delta\in A(\Delta)(=\sum_{k\in Z^n,\,\mathrm{supp}\,k=\Delta}(A'')^{\alpha}(k))$ for $\Delta\subset\{1, 2, \cdots, n\}$ such that*

$$\xi = \sum_{\Delta\subset\{1,2,\cdots,n\},\,\Delta\neq\emptyset} (D^{\Delta})^m b_\Delta + b_0\,,$$

where differentiations are taken in C^*-distribution sense.

By the continuity of ξ and Corollary 2.9, we have

PROPOSITION 5.15. *For any C^*-distribution ξ and a smooth element $x\in\mathcal{D}_\alpha(A)$, we have*

$$\xi(x) = \sum_{k\in Z^n} \boldsymbol{E}_{-k}(x)E_k(\xi)$$

where the summation converges in C^*-norm of B.

COROLLARY 5.16. *For any C^*-distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$, we put*

$$\xi_j = \sum_{k\in Y_j} E_k(\xi) \qquad \textit{where } Y_j=\{k=(k_1, \cdots, k_n)\in Z^n \mid |k_1|+\cdots+|k_n|\leq j\}.$$

Then the sequence of C^*-distributions $\{\xi_j\}_{j=0}^{\infty}\subset\mathcal{D}'_{\alpha l}(A)$ converges to ξ in $\mathcal{D}'$-topology.

As seen in the classical case, convolutions between C^*-distributions and ordinary distributions, cited in the previous section, can be written in terms of the Fourier components as follows:

PROPOSITION 5.17. *For* $\xi\in\mathcal{D}'_{\alpha l}(A)$ *and* $\varphi\in\mathcal{D}'(T^n)$, *we have*

$$E_k(\xi*\varphi)=E_k(\xi)\varphi(e_k), \qquad k\in Z^n.$$

For a C^*-distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$, a Fourier component $E_k(\xi)$ does not necessarily come from the C^*-algebra A. We however have many cases in which all Fourier components of all C^*-distributions come from A. We specify such cases as in the following way.

DEFINITION (locally self-dual spectrum and local self-duality). The k-th spectral subspace $A^\alpha(k)$ is called a *left self-dual module* if for any continuous (C^*-norm on A and C^*-norm on B) left B-module map ζ from $A^\alpha(k)$ to B, there exists an element $a_{-k}\in A^\alpha(-k)$ satisfying

$$\zeta(x)=xa_{-k}, \qquad x\in A^\alpha(k).$$

We define the *left locally self-dual spectrum* $\Omega_l(\alpha)$ for an action α by

$$\Omega_l(\alpha)=\{k\in Z^n \mid A^\alpha(k) \text{ is not left self-dual}\}.$$

This invariant of actions on C^*-algebras can be defined for actions of general compact abelian groups. The action α is said to be *left locally self-dual* if all spectral subspaces $A^\alpha(k)$ are left self-dual, that is, $\Omega_l(\alpha)=\emptyset$. The right locally self-duality for the action are similarly defined, we mainly deal with left ones. We indeed have $\Omega_l(\alpha)=-\Omega_r(\alpha)$. In what follows, local self-duality always means left local self-duality unless we specify.

There are, of course, many examples of C^*-dynamical systems with local self-duality. But, in Section 8, we will also see an example of an action of a unital C^*-algebra which is not locally self-dual.

By the previous discussions, we have the following proposition.

PROPOSITION 5.18. *Suppose that the action* α *is locally self-dual. Then we have*

(i) *For any C^*-left distribution* ξ, *there uniquely exists a slowly increasing sequence* $a_k\in A^\alpha(k)$, $k\in Z^n$ *satisfying*

$$\xi(x)=xa_k, \qquad x\in A^\alpha(-k),\ k\in Z^n.$$

(ii) *Any C^*-left distribution is a finite sum of finite order derivatives of elements of A in C^*-distribution sense.*

We notice that, by [**Ik**], there exists a W^*-dynamical system $(M, \bar{\alpha}, T^n)$ associated with a C^*-dynamical system (A, α, T^n) such that A is σ-weakly dense in M, the restriction of $\bar{\alpha}$ to A is α and it is universal in a sense. The k-th spectral subspace $A^{\alpha}(k)$ is σ-weakly dense in the k-th spectral subspace $M^{\bar{\alpha}}(k)$ by [**Ik**; Proposition 2], our previous discussions still go well in replacing $(A'')^{\alpha}(k)$ by $M^{\bar{\alpha}}(k)$. Namely, we can take Fourier components of a C^*-distribution as elements of $M^{\bar{\alpha}}(k)$. Hence we can express Theorem 5.13 as in the following way.

THEOREM 5.19.

(i) *For a C^*-left distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$, the sequence a_k, $k\in Z^n$ of the Fourier components of ξ is taken as $a_k\in M^{\bar{\alpha}}(k)$, $k\in Z^n$ and satisfies the conditions* (5.1) *and* (5.2).

(ii) *Conversely, for a sequence $a_k\in M^{\alpha}(k)$, $k\in Z^n$ satisfying the conditions* (5.1) *and* (5.2), *there exists a unique C^*-left distribution $\xi\in\mathcal{D}'_{\alpha l}(A)$ whose Fourier components are given by the sequence $\{a_k\}_{k\in Z^n}$.*

6. The dual $L^2_{\alpha l}{}'(A)$.

In this section, we will study the dual $L^2_{\alpha l}{}'(A)$ of the Hilbert C^*-left B-module $L^2_{\alpha l}(A)$ cited in Section 3. We will show that $L^2_{\alpha l}{}'(A)$ can be regarded as a subclass of $\mathcal{D}'_{\alpha l}(A)$ and their members are characterized in terms of their Fourier components.

We denote by $L^2_{\alpha l}{}'(A)$ the set of all left B-module homomorphisms from $L^2_{\alpha l}(A)$ to B which is continuous in $\|\cdot\|_{l,2}$-norm on $L^2_{\alpha l}(A)$ and C^*-norm on B.

For $\eta\in L^2_{\alpha l}{}'(A)$, we put

$$\|\eta\| = \sup_{y\in L^2_{\alpha l}(A),\, \|y\|_{l,2}=1} \|\eta(y)\| .$$

We equip $L^2_{\alpha l}{}'(A)$ with left A (and hence $\mathcal{D}_{\alpha}(A)$) and right B-module structure as in the following way:

$$(a\eta b)(y) = \eta(ya)b , \qquad \eta\in L^2_{\alpha l}{}'(A),\ y\in L^2_{\alpha l}(A),\ a\in A,\ b\in B .$$

We notice that, in the above expression, $L^2_{\alpha l}(A)$ naturally has right A-module structure because of the inequality

$$\|ya\|_{l,2} \leqq \|a\|\|y\|_{l,2}, \qquad a\in A,\ y\in L^2_{\alpha l}(A) .$$

The $*$-involution in A can be extended as an anti-module isometry between $L^2_{\alpha l}(A)$ and $L^2_{\alpha r}(A)$. We also write the extended $*$-involution between them by $*$. Set

$$\mathcal{L}_x(y) = \langle y^*, x\rangle_r , \qquad x\in L^2_{\alpha r}(A),\ y\in L^2_{\alpha l}(A) .$$

LEMMA 6.1. *For any $x\in L^2_{\alpha r}(A)$, we have*

(i) $$\mathcal{L}_x \in L^2_{\alpha l}{}'(A)$$

(ii) $$\|\mathcal{L}_x\| = \|x\|_{r,2}$$

(iii) $$a\mathcal{L}_x b = \mathcal{L}_{axb}, \qquad a\in A,\ b\in B .$$

COROLLARY 6.2.

$$\textit{The map } \mathcal{L}: x\in L^2_{\alpha r}(A) \longrightarrow \mathcal{L}_x \in L^2_{\alpha l}{}'(A)$$

is an isometry which preserves left A and right B-module structure.

LEMMA 6.3. *Any element of* $L^2_{\alpha l}{}'(A)$ *can be regarded as C^*-left distribution of order* 0.

PROOF. Since we have $\|y\|_{l,2}\leqq\|y\|$, for $y\in\mathcal{D}_\alpha(A)$, the assertion is clear from the boundedness of an element of $L^2_{\alpha l}{}'(A)$.

By the continuity of an element of $L^2_{\alpha l}{}'(A)$ and Proposition 5.15, the next lemma is easily proved.

LEMMA 6.4. *For* $y\in L^2_{\alpha l}(A)$ *and* $\eta\in L^2_{\alpha l}{}'(A)$, *we have*

$$\eta(y) = \sum_{k\in Z^n} E_{-k}(y)E_k(\eta): C^*\text{-norm convergence in } B$$

where $E_k(\eta)$ *is the k-th Fourier component of* η *as C^*-left distribution.*

PROPOSITION 6.5. *Suppose that the action* α *is locally self-dual. For any C^*-left distribution* $\eta\in\mathcal{D}'_{\alpha l}(A)$, *the following two conditions are equivalent*:

(i) $\eta\in L^2_{\alpha l}{}'(A)$ *(Namely,* η *can be extended on* $L^2_{\alpha l}(A)$ *as an element of* $L^2_{\alpha l}{}'(A)$).

(ii) *There exists a constant* $K>0$ *such that*

$$\Big\|\sum_{k\in Z^n|k|\leq m} E_k(\eta)^*E_k(\eta)\Big\| < K, \qquad \text{for all } m\in N.$$

where $|k|=|k_1|+\cdots+|k_n|$ *for* $k=(k_1,\cdots,k_n)\in Z^n$.

PROOF. (ii) $\Rightarrow$ (i) For any $y\in L^2_{\alpha l}(A)$, put $y_m=\sum_{k\in Y_m} E_k(y)$, where Y_m is a subset of Z^n defined in the proof of Lemma 2.8. Thus we have for $i<j$,

$$\begin{aligned}
\|\eta(y_j)-\eta(y_i)\| &= \Big\|\sum_{i<|k|\leq j} E_{-k}(y)E_k(\eta)\Big\| \\
&= \Big\|\Big\langle \sum_{i<|k|\leq j} E_{-k}(y),\ \sum_{i<|k|\leq j} E_k(\eta)^*\Big\rangle_l\Big\| \\
&\leqq \Big\|\sum_{i<|k|\leq j} E_k(y)\Big\|_{l,2}\Big\|\sum_{i<|k|\leq j} E_k(\eta)^*\Big\|_{l,2} \\
&= \Big\|\sum_{i<|k|\leq j} E_k(y)E_k(y)^*\Big\|^{1/2}\Big\|\sum_{i<|k|\leq j} E_k(\eta)^*E_k(\eta)\Big\|^{1/2} \\
&\leqq \|\langle y_j, y_j\rangle_l-\langle y_i, y_i\rangle_l\|^{1/2}K^{1/2} \\
&= \|y_j-y_i\|_{l,2}K^{1/2} .
\end{aligned}$$

Hence we can define $\eta(y)$ by $\eta(y)=\lim_{j\to\infty}\eta(y_j)$ where the limit is taken in C^*-norm of B. Since $\|\eta(y_j)\|\leqq K^{1/2}\|y_j\|_{l,2}$, we have $\|\eta(y)\|\leqq K^{1/2}\|y\|_{l,2}$, so that η can be extended on $L^2_{\alpha l}(A)$ as an element of $L^2_{\alpha l}{}'(A)$.

(i) $\Rightarrow$ (ii) Assume that $\eta\in L^2_{\alpha l}{}'(A)$, so that there exists $K>0$ such that $\|\eta(y)\|\leqq K\|y\|_{l,2}$, $y\in L^2_{\alpha l}(A)$. For any $m\in N$, we put $z_m=\sum_{k\in Y_m}E_k(\eta)^*$. Note that $z_m\in\mathcal{D}_\alpha(A)$ and $\eta(z_m)=\sum_{k\in Y_m}E_k(\eta)^*E_k(\eta)$. Now we have

$$\|z_m\|_{l,2}=\|\sum_{k\in Y_m}E_k(\eta)^*E_k(\eta)\|^{1/2}.$$

Hence we obtain

$$\|\sum_{k\in Y_m}E_k(\eta)^*E_k(\eta)\|\leqq K\|\sum_{k\in Y_m}E_k(\eta)^*E_k(\eta)\|^{1/2}$$

so that we conclude

$$\|\sum_{k\in Y_m}E_k(\eta)^*E_k(\eta)\|\leqq K^2.$$

REMARK. In general, for a sequence $a_k\in A$, $k\in Z^n$, satisfying $E_k(a_k)=a_k$, the following two conditions

(1) $\sum_{k\in Z^n}a_k^*a_k$ converges in C^*-norm,

(2) there exists $K>0$ such that

$$\|\sum_{k\in Z^n,\,|k|\leq m}a_k^*a_k\|<K,\qquad\text{for all } m\in N,$$

are clearly different. The implication from (1) to (2) is immediate but the other one does not necessarily hold. Hence the inclusion relation through $\mathcal{L}$

$$L^2_{\alpha r}(A)\subset L^2_{\alpha l}{}'(A)$$

is proper in many cases. We will see such an example later.

DEFINITION. An action α of T^n is said to be *globally self-dual* if the Hilbert C^*-right B-module $L^2_{\alpha r}(A)$ is self-dual: $L^2_{\alpha r}(A)=L^2_{\alpha l}{}'(A)$, namely the above implication from (2) to (1) always holds.

It is clear that global self-duality automatically implies local self-duality whereas the converse does not hold (cf. Corollary 7.2).

The following proposition is immediate from Proposition 6.5 and [**Li**; Theorem 4.1 (ii)].

PROPOSITION 6.6. *Suppose that the fixed point algebra B is primitive. Then the following conditions are equivalent:*

(i) *The action α is globally self-dual.*

(ii) *There exists $m\in N$ such that the linear subspace*

$$\sum_{k \in Z^n, |k|>m} A^\alpha(k)^* A^\alpha(k) = Span\{a^*b \mid a, b \in A^\alpha(k), |k|>m\}$$

is finite dimensional.

PROOF. (i) $\Rightarrow$ (ii) If the Arveson spectrum $Sp(\alpha)$ is infinite and for any $m \in N$

$$\sum_{k \in Z^n, |k|>m} A^\alpha(k)^* A^\alpha(k)$$

is not finite dimensional, $L^2_{\alpha r}(A)$ becomes an infinitely generated Hilbert C^*-module over the infinite dimensional C^*-algebra B. It contradicts to [**Li**; Theorem 4.1 (ii)].

(ii) $\Rightarrow$ (i) This implication is easy from Proposition 3.6 and Proposition 6.5.

By summing the previous discussions, we reach the following inclusion relations as left $\mathcal{D}_\alpha(A)$ and right B-modules

$$\mathcal{D}_\alpha(A) \subset L^2_{\alpha r}(A) \subset L^2_{\alpha l}{}'(A) \subset \mathcal{D}'_{\alpha l}(A).$$

This sequence of inclusion relations can be regarded as a non-commutative version of Gelfand triplet.

7. Model of C^*-distribution.

For an action β of Z on a unital C^*-algebra B, its dual action $\hat{\beta}$ of T on $B \times_\beta Z$ is locally self-dual and gives a clear example for C^*-distributions. In fact, put $A = B \times_\beta Z$ and $\alpha = \hat{\beta}$. In considering A as the reduced crossed product, we denote by u the generating unitary of the left regular representation of Z corresponding to the positive generator 1 of Z. As the n-th spectral subspace $A^\alpha(n)$ is of the form $A^\alpha(n) = Bu^n$ for $n \in Z$, any continuous left B-module map ζ from $A^\alpha(n)$ to B can be written as

$$\zeta(x) = x a_{-n}, \qquad x \in A^\alpha(n)$$

by putting $a_{-n} = u^{-n}\zeta(u^n)$ so that a_{-n} belongs to $A^\alpha(-n)$. This shows that α is locally self-dual.

In this section, we will present a good model for C^*-distributions which is not given by a dual action.

Let $\mathcal{O}_\infty$ be the Cuntz algebra with infinite generators of isometries $\{S_i\}_{i \in Z}$ (cf. [**Cu**]). Namely, $\mathcal{O}_\infty$ be the C^*-algebra generated by isometries S_i indexed by the integer group Z, which satisfies $\sum_{i \in Z} S_i S_i^* = 1$, where the convergence in the summation is taken in the strong operator topology on a Hilbert space.

We define an action α of the 1-dimensional torus T on $\mathcal{O}_\infty$ by

$$\alpha_\lambda(S_n) = \lambda^n S_n, \qquad n \in Z, \ \lambda \in C, \ |\lambda| = 1$$

so that we have a C^*-dynamical system $(\mathcal{O}_\infty, \alpha, T)$. One will know that this triplet can be regarded as a good example of a non-commutative version of the ordinary torus T in C^*-distribution theory.

Let δ be the infinitesimal generator of the action α and E the expectation from $\mathcal{O}_\infty$ to the fixed point algebra $\mathcal{O}_\infty^\alpha$ defined by

$$E(X) = \int_T \alpha_\lambda(X) d\lambda, \qquad X \in \mathcal{O}_\infty .$$

As we will see that the generators S_n (or S_n^*), $n \in Z$ behave as a "basis" in Fourier expansion of elements of $\mathcal{O}_\infty$, we can take Fourier coefficients of elements of $\mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)$ as elements of $\mathcal{O}_\infty^\alpha$.

DEFINITION. For $X \in \mathcal{D}_\alpha(\mathcal{O}_\infty)$, $n \in Z$, we define the n-th left (resp. right) Fourier coefficient by

$$F_n^l(X) = E(XS_n^*) \in \mathcal{O}_\infty^\alpha, \qquad (\text{resp. } F_n^r(X) = E(S_n X) \in \mathcal{O}_\infty^\alpha).$$

Note that the following identities hold

$$F_n^l(X) S_n S_n^* = F_n^l(X), \qquad S_n S_n^* F_n^r(X) = F_n^r(X)$$

and

$$E_n(X) = F_n^l(X) S_n = S_{-n}^* F_{-n}^r(X)$$

where $E_n(X)$ is the n-th Fourier component in the sense of Section 2.

It is clear that the action α on $\mathcal{O}_\infty$ is locally self-dual. Not only the Fourier components of a C^*-distribution associated to this dynamical system can be defined as elements of $\mathcal{O}_\infty$ but also the Fourier coefficients of it can be given by elements of $\mathcal{O}_\infty^\alpha$. We can indeed define the n-th right Fourier coefficient of $\xi \in \mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)$ by

$$F_n^r(\xi) = \xi(S_n) \in \mathcal{O}_\infty^\alpha, \qquad n \in Z$$

so that we have $S_n S_n^* F_n^r(\xi) = F_n^r(\xi)$ and $E_{-n}(\xi) = S_n^* F_n^r(\xi)$.

As seen in the previous discussions, each classes of the following quadruplet

$$\mathcal{D}_\alpha(\mathcal{O}_\infty) \subset L^2_{\alpha r}(\mathcal{O}_\infty) \subset L^2_{\alpha l}{}'(\mathcal{O}_\infty) \subset \mathcal{D}'_{\alpha l}(\mathcal{O}_\infty) \tag{7.1}$$

can be characterized in terms of decay of their Fourier components. Hence, in this case, we know the following proposition by the previous results.

PROPOSITION 7.1. *For a sequence $a_n \in \mathcal{O}_\infty^\alpha$, $n \in Z$ with $a_n = S_n S_n^* a_n$, we put $\xi = \sum_{n \in Z} S_n^* a_n$ (formal sum). Then we have*

(i) *$\xi \in \mathcal{D}_\alpha(\mathcal{O}_\infty)$ if and only if $\{a_n\}_{n \in Z}$ is rapidly decreasing.*

(ii) *$\xi \in L^2_{\alpha r}(\mathcal{O}_\infty)$ if and only if $\sum_{n \in Z} a_n^* a_n$ converges in C^*-norm.*

(iii) *$\xi \in L^2_{\alpha l}{}'(\mathcal{O}_\infty)$ if and only if there exists a constant $K > 0$ such that for any $l \in N$, $\|\sum_{|n| < l} a_n^* a_n\| < K$.*

(iv) $\xi\in\mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)$ *if and only if* $\{a_n\}_{n\in Z}$ *is slowly increasing.*

COROLLARY 7.2. *The above all inclusion relations* (7.1) *are proper.*

PROOF. (i) $L^2_{\alpha r}(\mathcal{O}_\infty)\subset L^2_{\alpha l}{}'(\mathcal{O}_\infty)$: Let δ_1 be the ordinary delta function at 1 on T. The induced C^*-left distribution $\xi^l_{\delta_1}$ has of the form

$$\xi^l_{\delta_1}(X)=\sum_{n\in Z}F^l_n(X), \qquad X\in\mathcal{D}_\alpha(\mathcal{O}_\infty).$$

Since $F^r_n(\xi^l_{\delta_1})=S_nS^*_n$, the summation $\sum_{n\in Z}F^r_n(\xi^l_{\delta_1})^*F^r_n(\xi^l_{\delta_1})$ does not converge in C^*-norm whereas

$$\|\sum_{|n|\leq l}F^r_n(\xi^l_{\delta_1})^*F^r_n(\xi^l_{\delta_1})\|\leqq 1, \qquad \text{for all } l\in N.$$

Hence we see

$$\xi^l_{\delta_1}\in L^2_{\alpha l}{}'(\mathcal{O}_\infty)\setminus L^2_{\alpha r}(\mathcal{O}_\infty).$$

(ii) $L^2_{\alpha l}{}'(\mathcal{O}_\infty)\subset\mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)$: Put $a_n=nS_nS^*_n$, $n\in Z$. As a_n, $n\in Z$ is slowly increasing, there exists a C^*-left distribution ξ_a whose Fourier coefficients are $\{a_n\}_{n\in Z}$. It is clear that $\|\sum_{|n|\leq l}F^r_n(\xi_a)^*F^r_n(\xi_a)\|$ is not bounded on $l\in N$ so that we have

$$\xi_a\in\mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)\setminus L^2_{\alpha l}{}'(\mathcal{O}_\infty).$$

We remark that the local self-duality for the action α does not necessarily imply the global self-duality (the self-duality for the Hilbert C^*-module $L^2_{\alpha r}(\mathcal{O}_\infty)$) from the above corollary.

One of the most excellent point of this model is the fact that the classical theory of distributions on T is absorbed in this model. In fact, let $\mathcal{D}'(T)$ be the ordinary distributions on the torus T (cf. [**Be**]). For a classical distribution $\varphi\in\mathcal{D}'(T)$, we can define an associated C^*-left distribution $\xi^l_\varphi\in\mathcal{D}'_{\alpha l}(\mathcal{O}_\infty)$ as

$$\xi^l_\varphi(X)=\sum_{n\in Z}F^l_n(X)\varphi(e_n)\in\mathcal{O}^\alpha_\infty, \qquad X\in\mathcal{D}_\alpha(\mathcal{O}_\infty)$$

where the summation above converges in the C^*-norm of $\mathcal{O}^\alpha_\infty$.

REMARK 7.3. Keep the above notations. We see for φ, $\phi\in\mathcal{D}'(T)$,

(i) $F^r_n(\xi^l_\varphi)=S_nS^*_n\varphi(e_n)$

(ii) $D\xi^l_\varphi=\xi^l_{D\varphi}$

(iii) $\xi^l_\varphi*\phi=\xi^l_{\varphi*\phi}$

where in (ii) the left side means the differentiation of the C^*-distribution ξ^l_φ whereas the right one denotes the C^*-distribution induced by the differentiation of the classical distribution φ, and in (iii) the left side means the convolution between ξ^l_φ and ϕ whereas the right one denotes the C^*-distribution induced by the convolution $\varphi*\phi$ of the classical distributions φ and ϕ.

8. Other examples.

As we stated in the previous sections, a C^*-dynamical system, which is locally self-dual in our sense, well behaves in our machinery. However, all (unital) C^*-dynamical system are of course not necessarily locally self-dual. We will, in this section, give an example of a C^*-dynamical system which is not locally self-dual.

The C^*-algebra of the C^*-dynamical system is the same as the preceding one, that is the Cuntz algebra $\mathcal{O}_\infty$ with infinite generators. But the action is different, which is called the gauge action (cf. [**Cu**]). The action γ is defined by

$$\gamma_\lambda(S_k) = \lambda S_k\,, \qquad k\in N,\ \lambda\in C,\ |\lambda|=1$$

where $\mathcal{O}_\infty$ is generated by isometries S_k, $k\in N$ with $\sum_{k\in N} S_k S_k^*=1$ whose summation converges in the strong operator topology on a Hilbert space.

For $n\in Z$, we denote by $\mathcal{O}_\infty^\gamma(n)$ the n-th spectral subspace of the action γ. As in [**Cu**], we see

$$\mathcal{O}_\infty^\gamma(n) = \begin{cases} \mathcal{O}_\infty^\gamma S_1^n\,, & n\geqq 1 \\ \mathcal{O}_\infty^\gamma\,, & n=0 \\ S_1^{*(-n)}\mathcal{O}_\infty^\gamma\,, & n\leqq -1 \end{cases}$$

where $\mathcal{O}_\infty^\gamma$ is the fixed point algebra of $\mathcal{O}_\infty$ under the action γ.

The following lemma is straightforward.

LEMMA 8.1. *For a C^*-left distribution ξ with respect to the action γ and $n\geqq 0$, the $-n$-th Fourier component $E_{-n}(\xi)$ is given by*

$$E_{-n}(\xi) = S_1^{*n}\xi(S_1^n) \qquad \in\mathcal{O}_\infty^\gamma(-n)\,.$$

Therefore the spectral subspaces $\mathcal{O}_\infty^\gamma(n)$ for $n\geqq 0$ are left self-dual.

On the other hand, we henceforth show that the other spectral subspaces $\mathcal{O}_\infty^\gamma(n)$ for $n<0$ are not left self-dual.

Let $\langle\,,\rangle_l$ denotes the inner product of the Hilbert C^*-left $\mathcal{O}_\infty^\gamma$-module on $\mathcal{O}_\infty$ as in Section 3. Then we can analogously prove the following lemma.

LEMMA 8.2. *For $n\geqq 1$, any element $X\in\mathcal{O}_\infty^\gamma(-n)$ can be expressed as in the following way*

$$X = \sum_{i_1,\dots,i_n=1}^{\infty} F_{-n}^l(X)_{i_1,\dots,i_n} S_{i_n}^* \cdots S_{i_1}^*$$

where $F_{-n}^l(X)_{i_1,\dots,i_n}=XS_{i_1}\cdots S_{i_n}$ and the above summation converges in L^2-norm on $\mathcal{O}_\infty^\gamma(-n)$.

We notice that L^2-norm on $\mathcal{O}_\infty^\gamma(-n)$ coincides with C^*-norm on $\mathcal{O}_\infty^\gamma(-n)$.

We shall mainly treat $\mathcal{O}^l_\infty(-1)$ for simplicity.

LEMMA 8.3. *Put* $\xi(S_i^*)=S_iS_i^*$, $1\leqq i\leqq\infty$. *For* $X=\sum_{i=1}^\infty F_{-1}^l(X)_iS_i^*$ *in* $\mathcal{O}^l_\infty(-1)$, *we define*

$$\xi(X) = \sum_{i=1}^\infty F_{-1}^l(X)_i\xi(S_i^*).$$

Then ξ *becomes a continuous left B-module map from* $\mathcal{O}^l_\infty(-1)$ *to* B *such that*

$$\|\xi(X)\| \leqq \|X\|.$$

PROOF. Put $X_n=\sum_{i=1}^n F_{-1}^l(X)_iS_i^*$ in $\mathcal{O}^l_\infty(-1)$. As we see $\xi(X_n)=\sum_{i=1}^n F_{-1}^l(X)_iS_i^*\cdot S_iS_iS_i^*$, it follows that by the Schwartz type inequality

$$\begin{aligned}\|\xi(X_n)\| &= \left\|\left\langle \sum_{i=1}^n F_{-1}^l(X)_iS_i^*,\ \sum_{j=1}^n S_jS_j^*S_j^*\right\rangle_l\right\| \\ &\leqq \left\|\sum_{i=1}^n F_{-1}^l(X)_iS_i^*\right\|_{l,2}\left\|\sum_{j=1}^n S_jS_j^*S_j^*\right\|_{l,2} \\ &\leqq \left\|\sum_{i=1}^n F_{-1}^l(X)_iS_i^*\right\|_{l,2}\left\|\sum_{j=1}^n S_jS_j^*\right\|.\end{aligned}$$

Hence we have $\|\xi(X_n)\|\leqq\|X_n\|$ and so that $\|\xi(X)\|\leqq\|X\|$.

We now assume that $\mathcal{O}_\infty$ is represented on a Hilbert space H on which the canonical action α, defined in the previous section, is spatial.

LEMMA 8.4. *Let* ξ *be as in Lemma* 8.3. *If there exists an element* $e_\xi\in\mathcal{O}_\infty$ *such that*

$$\xi(X) = Xe_\xi, \qquad X\in\mathcal{O}^l_\infty(-1),$$

then

$$e_\xi = w\text{-}\lim_{n\to\infty}\sum_{i=1}^n S_i\xi(S_i^*) \quad \left(= w\text{-}\lim_{n\to\infty}\sum_{i=1}^n S_iS_iS_i^*\right)$$

where w-lim *means the limit with respect to the weak operator topology on the Hilbert space H.*

LEMMA 8.5. *The limit* w-$\lim_{n\to\infty}\sum_{i=1}^n S_iS_iS_i^*$ *in the weak operator closure* $\mathcal{O}''_\infty$ *of* $\mathcal{O}_\infty$ *can not belong to the* C^**-algebra* $\mathcal{O}_\infty$.

PROOF. We extend the canonical action α on $\mathcal{O}_\infty$ to the σ-weakly continuous action on the von Neumann algebra $\mathcal{O}''_\infty$. Suppose that the element w-$\lim_{n\to\infty}\sum_{i=1}^n S_iS_iS_i^*$ belongs to $\mathcal{O}_\infty$ which we denoted by f_ξ. Then the k-th Fourier coefficient of f_ξ with respect to the action α is $F_k^l(f_\xi)=S_kS_kS_k^*S_k^*$, $k\in Z$. When we put

$$E_k(f_\xi) = F_k^l(f_\xi)S_k = S_kS_kS_k^*,$$

it is clear that the summation

$$\sum_{k\in Z} E_k(f_\xi)E_k(f_\xi)^* = \sum_{k\in Z} S_k S_k S_k^* S_k^*$$

does not converge in C^*-norm (but converges in weak operator topology). Thus f_ξ can not become an element of $L^2_{al}(\mathcal{O}_\infty)$ and hence of $\mathcal{O}_\infty$, which is a contradiction.

By combining the previous two lemmas, we reach

COROLLARY 8.6. *For the C^*-left distribution ξ defined in Lemma* 8.3, *there does not exist an element Y in $\mathcal{O}_\infty$ such that*

$$\xi(X) = XY\,, \qquad \textit{for all } X \in \mathcal{O}_\infty^\gamma(-1)\,.$$

Since similar discussions work for other spectral subspaces $\mathcal{O}_\infty^\gamma(n)$, $n<0$, we can summarize the above discussions as follows:

PROPOSITION 8.7. *For the gauge action γ of the torus T on the Cuntz algebra $\mathcal{O}_\infty$, the spectral subspaces $\mathcal{O}_\infty^\gamma(n)$, $n<0$, are not left self-dual whereas $\mathcal{O}_\infty^\gamma(n)$, $n\geqq 0$, are left self-dual. Namely, we have*

$$\Omega_l(\gamma) = \{n\in Z \mid n<0\} = -\Omega_r(\gamma)\,.$$

Thus the gauge action on $\mathcal{O}_\infty$ is not locally self-dual.

REMARK 8.8. It is easy to see that the gauge actions on other Cuntz algebras $\mathcal{O}_n$, $n<\infty$ are all locally self-dual.

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