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On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term

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§ 0. Introduction.

In this paper we are concerned with the decay property of the solutions to the Cauchy problem for the semilinear wave equation with a dissipative term:

$$\text{(P)}\qquad \begin{cases} u_{tt}-\Delta u+u_t+f(u)=0 & \text{in } \boldsymbol{R}^N\times[0, \infty) \\ u(x, 0)=u_0(x) \quad \text{and} \quad u_t(x, 0)=u_1(x), \end{cases}$$

where $f(u)$ is a nonlinear function like $f(u)=|u|^{\alpha}u$, $\alpha>0$.

As far as the existence and the uniqueness are concerned, the dissipative term u_t causes no difficulty and the known results for the usual nondissipative equations remain valid. That is, the problem (P) admits a weak solution $u(t)\in L^{\infty}([0, \infty); H^1\cap L^{\alpha+2})\cap W^{1,\infty}([0, \infty); L^2)$ for each $(u_0, u_1)\in H^1\cap L^{\alpha+2}\times L^2$ (cf. Strauss [**St70**]), and moreover, such a solution belongs to $C([0, \infty); H^1)\cap C^1([0, \infty); L^2)$ and is unique if $0<\alpha<4/(N-2)$ ($0<\alpha<\infty$ if $N=1, 2$) (cf. Ginibre and Velo [**GV85**], Brenner [**Br89**] etc.). Further, roughly speaking, if $0<\alpha<4/(N-2)$ and (u_0, u_1) are smooth, the solution u is also smooth. (Cf. Jörgens [**Jö61**], Pecher [**Pe76**], Brenner and W. v. Wahl [**BW81**] etc.. See also Grillakis [**Gr90**] for the case $N=3$, $\alpha=4$.)

The purpose of this paper is to derive certain decay rates in several energy norms of the solutions of the problem (P) by use of the effect of the dissipative term u_t. When $0<\alpha\leqq 2/(N-2)$, we can employ a rather standard energy method to show, for example, in the typical case $N=3$ and $\alpha=2$,

$$\sum_{i=0}^{m+1}\|D_t^iD_x^{m+1-i}u(t)\|\leqq c_{m+1}(1+t)^{-(m+1)/2},$$

where c_{m+1} is some positive constant depending on $\|u_0\|_{H^{m+1}}+\|u_1\|_{H^m}$.

For the case $2/(N-2)<\alpha<4/(N-2)$, however, our problem is more delicate and we must utilize precise L^p-L^q-estimates for the linear equation. In this case our purpose will be achieved by combining these L^p-L^q-estimates with the energy method. It should be noted that by such a combination we can derive a sharper decay estimate of the energy even for the case $4/N<\alpha\leqq 2/(N-2)$ under the additional condition $(u_0, u_1)\in L^r\times L^r$, $1\leqq r<2$, which is also an object of this paper.

Although our main interest lies in the case $N=3$ and $2\leqq\alpha<4$ we treat for generality the cases $1\leqq N\leqq 6$ and $0<\alpha<4/(N-2)$.

For the linear equation a closely related result was obtained by A. Matsumura [**Ma76**] and there the result was applied to the proof of global existence and decay for some semilinear wave equations with small initial data, while here we make no smallness condition on the data.

§1. Statement of the results.

We denote by D^k, $k=1, 2, \cdots$, any partial differential operators of order k with respect to t and x_i, $i=1, \cdots, N$. In particular D_x and D_t denote such differential operators in space variable $x=(x_1, \cdots, x_N)$ and the time variable t, respectively. Pairs of conjugate indices are written as p, p', where $1/p+1/p'=1$.

We use only standard function spaces H_p^s ($L^p\equiv H_p^0$, $H^s\equiv H_2^s$) equipped with the norm

$$\|u\|_{H_p^s}\equiv\|\mathcal{F}^{-1}\{\langle\xi\rangle^s\hat{u}(\xi)\}\|_p\,,$$

where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $\|u\|_p$ denotes the usual L^p-norm (we use $\|u\|$ for $\|u\|_2$), and $\mathcal{F}$ denotes the Fourier transform:

$$\mathcal{F}\{u(x)\}(\xi)\equiv\hat{u}(\xi)\equiv(2\pi)^{-N/2}\int e^{-i\langle\xi,x\rangle}u(x)dx\,.$$

We summarize the assumptions on the nonlinear term $f(u)$ in the following.

HYPOTHESIS. (i) *$f(u)$ is a continuous function on $\boldsymbol{R}$ and satisfies the conditions*

$$f(u)u\geqq kF(u)\geqq 0\,,\qquad F(u)\equiv 2\int_0^u f(\eta)d\eta\,, \tag{1.1a}$$

for some constant $k>0$ and

$$|f(u)|\leqq k_0|u|^{\alpha+1} \tag{1.1b}$$

for some constants $k_0>0$ and $\alpha>0$.

(ii) *$f(u)$ belongs to $C^1(\boldsymbol{R})$ and satisfies*

$$|f'(u)|\leqq k_1|u|^{\alpha} \tag{1.2}$$

for some constants $k_1>0$ *and* $\alpha>0$.

(iii) $f(u)$ *belongs to* $C^2(\boldsymbol{R})$ *and satisfies*

$$|f''(u)| \leqq k_2|u|^{[\alpha-1]^+} \tag{1.3}$$

for some constants $k_2>0$ *and* $\alpha>0$, *where we use the notation* $a^+\equiv\max\{0, a\}$.

We shall pick up freely appropriate set of assumptions on $f(u)$ from Hyp. (i)-(iii).

Our first two theorems are concerned with the decay of the usual energy

$$E(t) \equiv E(u(t)) \equiv \|u_t(t)\|^2+\|\nabla u(t)\|^2+\int_{\boldsymbol{R}^N} F(u(t))dx .$$

THEOREM 1. (i) *Let* $(u_0, u_1)\in H^1\times L^2$ *and let* Hyp. (i) *be satisfied with* α *such that*

$$0 < \alpha < 4/(N-2) \qquad (0<\alpha<\infty \text{ if } N=1, 2). \tag{1.4a}$$

Then, the solution $u(t)\in C([0, \infty); H^1)\cap C^1([0, \infty); L^2)$ *of* (P) *has the boundedness and decay property*

$$\|u(t)\| \leqq c_1 \quad \text{and} \quad E(t) \leqq c_1(1+t)^{-1}. \tag{1.4b}$$

(ii) *Moreover, if* $1\leqq N\leqq 3$, $(u_0, u_1)\in H^1\cap L^r\times L^2\cap L^r$, $1\leqq r\leqq 2$, *and*

$$4/N < \alpha \leqq 2/(N-2) \qquad (4/N<\alpha<\infty \text{ if } N=1, 2), \tag{1.5a}$$

then we have

$$\|u(t)\| \leqq c_1(1+t)^{-\eta} \quad \text{and} \quad E(t) \leqq c_1(1+t)^{-1-2\eta} \tag{1.5b}$$

with

$$\eta = \frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \tag{1.5c}$$

where c_1 *denotes positive constants depending on* $\|u_0\|_{H^1}+\|u_1\|$ *or* $\|u_0\|_{H^1}+\|u_1\|+\|u_0\|_r+\|u_1\|_r$.

REMARK 1.1. Strauss **[St70]** established the existence of a weak solution $u(t)\in L^\infty([0, \infty); H^1\cap L^{\alpha+2})\cap W^{1,\infty}([0, \infty); L^2)$ of (P) for any $0<\alpha<\infty$ by use of an approximation method. Our result (1.4b) is valid for the solution in **[St70]** without any restriction on α.

When $4/N<\alpha<4/(N-2)$ and $3\leqq N\leqq 6$, the decay rate of $E(t)$ is improved as follows.

THEOREM 2. *Let* $3\leqq N\leqq 6$ *and let* Hyp. (i)-(ii) *be satisfied with* α *such that*

$$4/N < \alpha < 4/(N-2), \tag{1.6a}$$

and further let the initial data (u_0, u_1) *belong to* $H^2 \cap L^r \times H^1 \cap L^r$, $1 \leqq r \leqq 2$. *Then, we have*

$$\|u(t)\| \leqq c_2(1+t)^{-\eta} \quad \text{and} \quad E(t) \leqq c_2(1+t)^{-1-2\eta} \tag{1.6b}$$

with

$$\eta = \min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\}, \tag{1.6c}$$

where c_2 *denotes positive constants depending on* $\|u_0\|_{H^2}+\|u_1\|_{H^1}+\|u_0\|_r+\|u_1\|_r$.

REMARK 1.2. (i) The solution $u(t)$ in Theorem 2 belongs to $\bigcap_{i=0}^{2} C^i([0, \infty); H^{2-i})$ (see the proof of Theorem 3 and Remark 1.3 below).

(ii) For the decay rate η of L^2-norm of solution $u(t)$, we note that

$$\eta = \min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\} = \frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right) \quad \text{if } 1 \leqq N \leqq 4 \text{ and } \alpha > 4/N.$$

When α satisfies the condition:

$$0 < \alpha < 4/(N-2) \qquad (0<\alpha<\infty \text{ if } N=1, 2),$$

the results of Theorems 1 and 2 are summarized in a convenient form

$$\|u(t)\| \leqq c_2(1+t)^{-\eta} \quad \text{and} \quad \|Du(t)\| \leqq c_2(1+t)^{-\theta_1}, \tag{1.7a}$$

where η and θ_1 are defined by

$$\eta = \begin{cases} \min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\} & \text{if } \alpha > 4/N \\ 0 & \text{if } \alpha \leqq 4/N, \end{cases} \tag{1.7b}$$

and

$$\theta_1 = 1/2+\eta. \tag{1.7c}$$

In what follows, we denote by c_{m+1} various positive constants depending on $\|u_0\|_{H^{m+1}}+\|u_1\|_{H^m}+\|u_0\|_r+\|u_1\|_r$.

Our third result reads as follows.

THEOREM 3. *Let* $1 \leqq N \leqq 5$ *and* $(u_0, u_1) \in H^2 \cap L^r \times H^1 \cap L^r$, $1 \leqq r \leqq 2$, *and let* Hyp. (i)-(ii) *be satisfied with* α *such that*

$$4/N \leqq \alpha < 4/(N-2) \qquad (4/N \leqq \alpha < \infty \text{ if } N=1, 2). \tag{1.8a}$$

Then, it holds that

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| + \|\Delta u(t)\| \leqq c_2(1+t)^{-\theta_2} \tag{1.8b}$$

with

(1.8c) $$\theta_2 = 1/2+\theta_1 = 1+\eta\,,$$

where η *is the number given by* (1.7b).

REMARK 1.3. Even in the case $0<\alpha\leqq 4/N$, we can derive some decay rate. Indeed, if $1\leqq N\leqq 4$ and $0<\alpha\leqq 4/N$, or if $N\geqq 5$ and $0<\alpha\leqq 2/(N-2)$, then (1.8b) holds true with θ_2 replaced by

(1.8d) $$\tilde{\theta}_2 = 1/2+\omega\,,$$

where $\omega=\omega(\alpha)$ is

(1.8e) $$\omega = \begin{cases} \dfrac{\alpha}{8} & \text{if } N=1 \\ \dfrac{\alpha}{4} & \text{if } N=2 \text{ and } \alpha=2 \\ \dfrac{\alpha}{4}-\varepsilon & \text{if } N=2 \text{ and } \alpha<2\ (0<\varepsilon\ll 1) \\ \dfrac{\alpha}{4-(N-2)\alpha} & \text{if } N\geqq 3\,. \end{cases}$$

Note that $\omega(\alpha)$ is a monotone increasing function of $\alpha\in(0, 4/N]$ such that $0<\omega\leqq 1/2$. Further, for the case $N\geqq 5$ and $2/(N-2)\leqq\alpha\leqq 4/N$, we could show a similar result to the above with a certain $\omega>0$ by use of an L^q-estimate (see (6.12)).

When $(u_0, u_1)\in H^{m+1}\times H^m$, $m\geqq 2$, we can show by a standard argument that the solution $u(t)$ belongs to $u(t)\in\bigcap_{i=0}^{m+1} C^i([0, \infty); H^{m+1-i})$. Concerning such a smooth solution, we give two theorems.

THEOREM 4. *Let* $1\leqq N\leqq 5$ *and* $(u_0, u_1)\in H^3\cap L^r\times H^2\cap L^r$, $1\leqq r\leqq 2$, *and let* Hyp. (i)-(iii) *be satisfied with* α *such that* (1.8a) *holds. Then, we have*

(1.9a) $$\sum_{i=0}^{3}\|D_t^i D_x^{3-i}u(t)\| \leqq c_3(1+t)^{-\theta_3}$$

with

(1.9b) $$\theta_3 = 1/2+\theta_2 = 3/2+\eta\,,$$

where η *is given by* (1.7b).

REMARK 1.4. When $0<\alpha\leqq 4/N$, we have (1.9a) with θ_3 replaced by

(1.9c) $$\tilde{\theta}_3 = \tilde{\theta}_2+\omega = 1/2+2\omega \quad \text{(see Remark 1.3)}.$$

THEOREM 5. *Let* $1\leqq N\leqq 5$ *and* $(u_0, u_1)\in H^{m+1}\cap L^r\times H^m\cap L^r$, $1\leqq r\leqq 2$, *with* $m\geqq 3$. *Assume that* $f(u)$ *is an m-times continuously differentiable function and satisfies* Hyp. (i)-(iii) *with* α *such that*

(1.10a) $$0 < \alpha < 4/(N-2) \qquad (0<\alpha<\infty \text{ if } N=1,\ 2).$$

When $N=1$, we assume further that $f(u)$ is at least four times differentiable and satisfies

(1.10b) $$|f^{(i)}(u)| \leqq k_i |u|^{[\alpha+1-i]^+} \qquad \text{near } u=0,\ i=3,\ 4.$$

Then, we have

(1.10c) $$\sum_{i=0}^{l+1} \|D_t^i D_x^{l+1-i} u(t)\| \leqq c_{m+1}(1+t)^{-\theta_{l+1}}, \qquad 1\leqq l\leqq m,$$

where we set

(1.10d) $$\theta_{l+1} = \begin{cases} \dfrac{l+1}{2}+\eta & \text{if } 2\leqq N\leqq 5, \text{ or } N=1 \text{ and } \alpha\geqq 4 \\ \dfrac{\alpha}{8}l+\dfrac{1}{2} & \text{if } N=1 \text{ and } \alpha\leqq 4 \end{cases}$$

with η given by (1.7b).

REMARK 1.5. By Theorem 4 we know that the solutions $u(t)$ are uniformly bounded in $L^\infty(\boldsymbol{R}^N)$-norm, and under the assumptions of Theorem 5 we can show

$$|f^{(i)}(u(t))| \leqq c_3(\|u(t)\|_\infty)|u(t)|^{3-i}, \qquad i=0,\ 1,\ 2,\ 3.$$

From this, we note that α in Hyp. (i)-(iii) may be chosen as

$$\alpha > 4/N \quad \text{if } 3\leqq N\leqq 5 \quad \text{and} \quad \alpha \geqq 4/N \quad \text{if } N=2.$$

In particular, we can always take

$$\eta = \frac{N}{2}\Big(\frac{1}{r}-\frac{1}{2}\Big) \qquad \text{if } 3\leqq N\leqq 5.$$

REMARK 1.6. (i) When $N\geqq 6$, the results of Theorems 1-5 hold under a restricted condition $0<\alpha\leqq 2(N-1)/(N-2)(N-3)$ $(<4/(N-2))$ (see (7.4b)), though Theorem 5 must be modified a little when $3\leqq m<[N/2]+1$.

(ii) If we relax the inequalities in Hyp. (i)-(iii) as

(1.1)′ $$|f(u)| \leqq k_0(|u|^{\alpha_1+1}+|u|^{\alpha_2+1})$$

(1.2)′ $$|f'(u)| \leqq k_1(|u|^{\alpha_1}+|u|^{\alpha_2})$$

(1.3)′ $$|f''(u)| \leqq k_2(|u|^{[\alpha_1-1]^+}+|u|^{[\alpha_2-1]^+})$$

with $0<\alpha_i<4/(N-2)$, $i=1,\ 2$, respectively, then all the results are valid if α_i, $i=1,\ 2$, satisfy the conditions on α.

§ 2. Some Lemmas.

We use the following lemmas. The first one is well known.

LEMMA 2.1 (Gagliardo-Nirenberg). *Let* $1 \leqq r < p \leqq \infty$, $1 \leqq q \leqq p$ *and* $m \geqq 0$. *Then, the inequality*

$$\|v\|_p \leqq c\|D_x^m v\|_q^\theta \|v\|_r^{1-\theta} \qquad \text{for } v \in H_q^m \cap L^r \tag{2.1a}$$

holds with some $c>0$ *and*

$$\theta = \Big(\frac{1}{r}-\frac{1}{p}\Big)\Big(\frac{m}{N}+\frac{1}{r}-\frac{1}{q}\Big)^{-1}, \tag{2.1b}$$

provided that $0<\theta \leqq 1$ ($0<\theta<1$ *if* $1<q<\infty$ *and* $m-N/q$ *is a nonnegative integer*).

The second one is useful in deriving decay rate of the energy.

LEMMA 2.2. *Let* $\phi(t)$ *be a nonnegative function on* $\boldsymbol{R}^+=[0, \infty)$, *satisfying*

$$\sup_{t \leqq s \leqq t+1} \phi(s)^{1+\alpha} \leqq k_0(1+t)^\beta \{\phi(t)-\phi(t+1)\} + h(t) \tag{2.2a}$$

for some $k_0>0$, $\alpha>0$, $\beta<1$ *and a function* $h(t)$ *with*

$$0 \leqq h(t) \leqq k_1(1+t)^{-\gamma} \tag{2.2b}$$

for some $k_1>0$ *and* $\gamma>0$. *Then,* $\phi(t)$ *has a decay property*

$$\phi(t) \leqq c_0(1+t)^{-\theta}, \qquad \theta = \min\Big\{\frac{1-\beta}{\alpha}, \frac{\gamma}{1+\alpha}\Big\}, \tag{2.2c}$$

where c_0 *denotes a positive constant depending on* $\phi(0)$ *and other known constants.*

PROOF. The proof is given in Nakao [**Na78**] under a little stronger assumption $h(t)=o(t^{-\gamma})$, $\gamma=(1+\alpha)(1-\beta)/\alpha$, as $t \to \infty$. Here, we give another proof which improves it as above.

We may assume $\gamma/(1+\alpha) \leqq (1-\beta)/\alpha$, i.e., $0<\gamma \leqq (1+\alpha)(1-\beta)/\alpha$. Suppose that (2.2c) was false. Then, for any large $K>0$, there exists $T>1$ such that

$$\phi(t) \leqq K(1+t)^{-\gamma/(1+\alpha)} \qquad \text{for } 0 \leqq t \leqq T-1/2$$

and

$$\phi(T) \geqq K(1+T)^{-\gamma/(1+\alpha)}.$$

(We can easily prove $\phi(t)^{1+\alpha} \leqq \max\{ck_1, c'k_0\phi(0)+c''k_1\} < \infty$.) Taking $t=T-1$ in the inequality (2.2a) we see

$$K^{1+\alpha}(1+T)^{-\gamma} \leqq k_0 T^\beta \{KT^{-\gamma/(1+\alpha)} - K(1+T)^{-\gamma/(1+\alpha)}\} + k_1(1+T)^{-\gamma}$$

and, taking K so large as $K> \min\{2k_1, 1\}$,

$$(2.3)\qquad K^{1+\alpha}(1+T)^{-\gamma} \leqq 2k_0 K T^{\beta}\{T^{-\gamma/(1+\alpha)}-(1+T)^{-\gamma/(1+\alpha)}\}$$
$$\leqq 2k_0 K(1+T)^{\beta-\gamma/(1+\alpha)}\{(1+T^{-1})^{\gamma/(1+\alpha)}-1\}$$
$$\leqq C_* K(1+T)^{\beta-\gamma/(1+\alpha)-1}$$

for some $C_*>0$. Since $\gamma \leqq (1+\alpha)(1-\beta)/\alpha$ we have from (2.3)

$$K^{\alpha} \leqq C_*(1+T)^{\beta-\alpha\gamma/(1+\alpha)-1} \leqq C_*\,,$$

which is a contradiction if we choose $K>C_*^{1/\alpha}$. Q.E.D.

The third one is convenient in deriving L^q-estimate.

LEMMA 2.3. *Let $y(t)$ be a nonnegative function on $[0, T)$, $T>0$ (possibly $T=\infty$), and satisfy the integral inequality*

$$(2.4a)\qquad y(t) \leqq k_0(1+t)^{-\alpha}+k_1\int_0^t (1+t-s)^{-\beta}(1+s)^{-\gamma}y(s)^{\mu}ds$$

for some k_0, $k_1>0$, α, β, $\gamma \geqq 0$ and $0 \leqq \mu < 1$. Then

$$(2.4b)\qquad y(t) \leqq c(1+t)^{-\theta}$$

for some constant $c>0$ and

$$(2.4c)\qquad \theta = \min\left\{\alpha,\ \beta,\ \frac{\gamma}{1-\mu},\ \frac{\beta+\gamma-1}{1-\mu}\right\},$$

with the following exceptional case: If $\alpha \geqq \hat{\theta}$ and $(\beta+\gamma-1)/(1-\mu)=\hat{\theta} \leqq 1$, where

$$(2.4d)\qquad \hat{\theta} = \min\left\{\beta,\ \frac{\gamma}{1-\mu}\right\},$$

then

$$(2.4e)\qquad y(t) \leqq c(1+t)^{-\hat{\theta}}(\log(2+t))^{1/(1-\mu)}\,.$$

REMARK 2.1. Once we know $y(t)$ is bounded function, we can apply Lemma 2.3 also to the case $\mu=1$. In particular, if $\gamma>0$ and $\beta+\gamma-1>0$, we obtain (2.4b) with

$$(2.4f)\qquad \theta = \min\{\alpha,\ \beta\}.$$

We note that even for the exceptional case, (2.4b) is valid if θ is replaced by $\theta-\varepsilon$, $0<\varepsilon \ll 1$.

PROOF. The case $\mu=0$ is well known. Although the case $0<\mu<1$ also seems to be known, we give a proof for completeness. We define $M(t)$ by

$$(2.5)\qquad M(t) \equiv \sup_{0 \leqq s \leqq t}\{(1+s)^{\theta}y(s)\}.$$

Then, we have from (2.4a) and (2.5) that

$$y(t) \leqq k_0(1+t)^{-\alpha}+k_1\int_0^t(1+t-s)^{-\beta}(1+s)^{-\gamma-\mu\theta}dsM(t)^\mu$$
$$\leqq k_0(1+t)^{-\alpha}+c(1+t)^{-\theta^*}M(t)^\mu$$

with a constant $c>0$ and $\theta^*=\{\beta, \gamma+\mu\theta, \beta+\gamma+\mu\theta-1\}$, where we assumed that $\beta\neq 1$ and $\gamma+\mu\theta\neq 1$. Here, it is easy to see $\min\{\alpha, \theta^*\}=\theta$ and hence

$$(1+t)^\theta y(t) \leqq k_0+cM(t)^\mu .$$

Since $0<\mu<1$, this inequality implies $M(t)\leqq C<\infty$, which is equivalent to (2.4b). The exceptional case where $\beta=1$ or $\gamma+\mu\theta=1$ can be proved in a similar way.

Q.E.D.

§ 3. L^p-L^q-estimates for the linear equation.

In this and the next sections we consider the linear wave equation with a dissipative term:

$$\begin{cases} u_{tt}-\Delta u+u_t = f(x, t) & \text{in } \boldsymbol{R}^N\times[0, \infty) \\ u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) . \end{cases} \tag{3.1}$$

The result for the linear equation (3.1) seems to be interesting itself apart from nonlinear problem.

The solution $u(t)$ of the problem (3.1) is given through Fourier transform:

$$\hat{u}(\xi, t) = \hat{u}_L(\xi, t)+\hat{I}_f(\xi, t) . \tag{3.2a}$$

Here, we define

$$\begin{cases} \hat{u}_L(\xi, t) = \dfrac{1}{2}(\phi_1(\xi, t)+\phi_2(\xi, t))\hat{u}_0(\xi)+\phi_2(\xi, t)\hat{u}_1(\xi) \\ \hat{I}_f(\xi, t) = \displaystyle\int_0^t\phi_2(\xi, t-s)\hat{f}(\xi, s)ds , \end{cases} \tag{3.2b}$$

where we set

$$\begin{cases} \phi_1(\xi, t) = e^{-t/2}\{e^{t(1-4|\xi|^2)^{1/2}/2}+e^{-t(1-4|\xi|^2)^{1/2}/2}\} \\ \phi_2(\xi, t) = \dfrac{e^{-t/2}}{(1-4|\xi|^2)^{1/2}}e^{t(1-4|\xi|^2)^{1/2}/2}-e^{-t(1-4|\xi|^2)^{1/2}/2} . \end{cases} \tag{3.2c}$$

Using the formula (3.2) we shall derive L^p-L^q-estimates for $u(t)$. The argument itself is rather standard and related to the nondissipative case (cf. Brenner [**Br75**], Pecher [**Pe76**], Mochizuki [**Mo84**] etc.). Indeed, recently Racke [**Ra90**] has given some L^p-L^q-estimates for the dissipative equation in more general setting including (3.1) with $f(x, t)\equiv 0$ for the case $N=3$. For our purpose, however, we need a little more preciser estimate than those in [**Ra90**]. See also

Matsumura [**Ma76**] for a closely related result.

We take $\chi_i \in C^\infty(\boldsymbol{R}^N)$, $i=1, 2, 3$, such that

$$\chi_1(\xi) = \begin{cases} 1 & \text{if } |\xi| \leqq 1/4 \\ 0 & \text{if } |\xi| \geqq 1/3\,, \end{cases}$$

$$\chi_3(\xi) = \begin{cases} 0 & \text{if } |\xi| \leqq 2/3 \\ 1 & \text{if } |\xi| \geqq 3/4\,, \end{cases}$$

and

$$\chi_2(\xi) = 1 - \chi_1(\xi) - \chi_3(\xi), \qquad \text{i.e.,} \quad \sum_{i=1}^{3} \chi_i(\xi) = 1\,.$$

Our main step is the following.

PROPOSITION 3.1. *Let v belong to $C_0^\infty(\boldsymbol{R}^N)$. Then, we have:*

(i) *For $1 \leqq p \leqq 2$, $2 \leqq q \leqq \infty$, and $\gamma \in \boldsymbol{R}$,*

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\chi_1(\xi)\hat{v}(\xi)\}\|_{H_q^\gamma} \leqq c(1+t)^{-N(1/p-1/q)/2}\|v\|_p\,, \qquad i=1, 2\,. \tag{3.3}$$

(ii) *For $1 \leqq p \leqq 2$ and $\gamma \in \boldsymbol{R}$,*

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\chi_2(\xi)\hat{v}(\xi)\}\|_{H_{p'}^\gamma} = ce^{-\nu t}\|v\|_p\,, \qquad i=1, 2, \tag{3.4}$$

with some $0 < \nu < 1/2$.

(iii) *For $1 \leqq p \leqq 2$, $\gamma \in \boldsymbol{R}$ and an integer $k \geqq 0$,*

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\chi_3(\xi)\hat{v}(\xi)\}\|_{H_{p'}^\gamma} \leqq ct^{-(N-1)(1/p-1/2)}e^{-\nu t}\|D_x^k v\|_p\,, \qquad i=1, 2\,, \tag{3.5a}$$

with some $0 < \nu < 1/2$, provided that

$$\gamma + (N+1)\Big(\frac{1}{p} - \frac{1}{2}\Big) \leqq \begin{cases} k & \textit{if } i=1 \\ k+1 & \textit{if } i=2\,. \end{cases} \tag{3.5b}$$

(iv) *For $\gamma \in \boldsymbol{R}$,*

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\chi_3(\xi)\hat{v}(\xi)\}\|_{H^\gamma} \leqq \begin{cases} ce^{-\nu t}\|v\|_{H^\gamma} & \textit{if } i=1 \\ ce^{-\nu t}\|v\|_{H^{\gamma-1}} & \textit{if } i=2 \end{cases} \tag{3.6}$$

with some $0 < \nu < 1/2$.

PROOF. Noting that

$$|\phi_i(\xi, t)\chi_1(\xi)| \leqq ce^{-t|\xi|^2}\chi_1(\xi)\,, \qquad i=1, 2\,,$$

and using the Hausdorff-Young inequality we have

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\chi_1(\xi)\hat{v}(\xi)\}\|_{H_q^\gamma} = \|\mathcal{F}^{-1}\{\langle\xi\rangle^\gamma\phi_i(\xi, t)\chi_1(\xi)\hat{v}(\xi)\}\|_q \tag{3.7a}$$

$$\leqq c\|\phi_i(\xi,\ t)\chi_1(\xi)\hat{v}(\xi)\|_{q'}\leqq c\left\{\int e^{-t|\xi|^2q'}\chi_1(\xi)^{q'}|\hat{v}(\xi)|^{q'}d\xi\right\}^{1/q'}$$

$$\leqq c\left\{\int e^{-t|\xi|^2p'q'/(p'-q')}d\xi\right\}^{(p'-q')/p'q'}\|\hat{v}(\xi)\|_{p'}$$

$$\leqq ct^{-N(p'-q')/2p'q'}\|\hat{v}(\xi)\|_{p'}\leqq ct^{-N(1/p-1/q)/2}\|v\|_p\,,\qquad i=1,\ 2\,.$$

We also observe for $i=1,\ 2$,

$$\|\mathcal{F}^{-1}\{\phi_i(\xi,\ t)\chi_1(\xi)\hat{v}(\xi)\}\|_{H_q^\gamma} \tag{3.7b}$$

$$\leqq c\left(\int\chi_1(\xi)^{q'}|\hat{v}(\xi)|^{q'}d\xi\right)^{1/q'}\leqq c\|\hat{v}(\xi)\|_{p'}\leqq c\|v\|_p\,.$$

The inequality (3.3) follows from (3.7a-b).

Next, we shall show (3.4). For this, it suffices to show

$$\|\langle\xi\rangle^\gamma\phi_i(\xi,\ t)\chi_2(\xi)\|_\infty\leqq ce^{-\nu t}\,,\qquad i=1,\ 2.$$

The case $i=1$ is easier and we treat the case $i=2$. Now, we have

$$\|\langle\xi\rangle^\gamma\phi_2(\xi,\ t)\chi_2(\xi)\|_\infty \tag{3.8}$$

$$\leqq cte^{-t/2}\sup_{1/4\leqq|\xi|\leqq 3/4}\left|\frac{e^{t(1-4|\xi|^2)^{1/2}/2}-e^{-t(1-4|\xi|^2)^{1/2}/2}}{t(1-4|\xi|^2)^{1/2}/2}\right|\leqq ce^{-\nu t}$$

with some $0<\nu<1/2$, which proves (3.4).

Next, we shall prove (3.5). The cases $i=1,\ 2$ are proved similarly, and we consider the case $i=2$ only. From

$$\phi_2(\xi,\ t)=\frac{e^{-t/2}}{i(4|\xi|^2-1)^{1/2}}\{e^{it|\xi|}\psi(\xi,\ t)-e^{-it|\xi|}\psi(\xi,\ -t)\}$$

with $\psi(\xi,\ t)\equiv e^{-it(2|\xi|^2-1)^{1/2}/2}$, we see that

$$\|\mathcal{F}^{-1}\{\phi_2(\xi,\ t)\chi_3(\xi)\hat{v}(\xi)\}\|_{H_{q'}^\gamma}=\|\mathcal{F}^{-1}\{\langle\xi\rangle^\gamma\phi_2(\xi,\ t)\chi_3(\xi)\hat{v}(\xi)\}\|_{p'} \tag{3.9}$$

$$\leqq e^{-t/2}\left\{\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\frac{\langle\xi\rangle^\gamma}{i(4|\xi|^2-1)^{1/2}}\psi(\xi,\ t)e^{it|\xi|}\hat{v}(\xi)\right\}\right\|_{p'}\right.$$

$$\left.+\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\frac{\langle\xi\rangle^\gamma}{i(4|\xi|^2-1)^{1/2}}\psi(\xi,\ -t)e^{-it|\xi|}\hat{v}(\xi)\right\}\right\|_{p'}\right\}\equiv e^{-t/2}\{I_1^2+I_2^2\}\,.$$

The first term I_1^2 on the right hand side of (3.9) is estimated as follows. Since $\gamma+(N+1)(1/p-1/2)\leqq k+1$, we can choose

$$\frac{\langle\xi\rangle^\gamma|\xi|^{(N+1)(1/p-1/2)}}{(4|\xi|^2-1)^{1/2}|\xi|^k}$$

as a Fourier multiplier on $\operatorname{supp}\chi_3(\xi)\subset\{|\xi|\geqq 2/3\}$ (cf. [**Hö60**], [**Mi65**, p. 232]) to get

$$I_1^2 \leqq c\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)|\xi|^k\phi(\xi, t)\frac{e^{it|\xi|}}{|\xi|^{(N+1)(1/p-1/2)}}\hat{v}(\xi)\right\}\right\|_{p'}.$$

Next, we take $\chi\in C^\infty(\boldsymbol{R})$ such that

$$\chi(s)=\begin{cases}0 & \text{if } |s|\leqq 1/4\\ 1 & \text{if } |s|\geqq 1/2\,.\end{cases}$$

Then, we can choose $|\xi|^k/\sum_{j=1}^N \chi(\xi_j)|\xi_j|^k$ as a Fourier multiplier on supp $\chi_3(\xi)$ to get

$$I_1^2 \leqq c\sum_{j=1}^N\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\chi(\xi_j)|\xi_j|^k\phi(\xi, t)\frac{e^{it|\xi|}}{|\xi|^{(N+1)(1/p-1/2)}}\hat{v}(\xi)\right\}\right\|_{p'}.$$

Moreover, we regard $\chi(\xi_j)|\xi_j|^k/\xi_j^k$ as a Fourier multiplier to get

$$\begin{aligned} I_1^2 &\leqq c\sum_{j=1}^N\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\phi(\xi, t)\frac{e^{it|\xi|}}{|\xi|^{(N+1)(1/p-1/2)}}\xi_j^k\hat{v}(\xi)\right\}\right\|_{p'} \\ &= c\sum_{j=1}^N\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\phi(\xi, t)\frac{e^{it|\xi|}}{|\xi|^{(N+1)(1/p-1/2)}}\widehat{D_{x_j}^k v}(\xi)\right\}\right\|_{p'}. \end{aligned}$$

Finally, we can choose $\phi(\xi, t)$ as a Fourier multiplier on supp $\chi_3(\xi)$ to get

$$I_1^2 \leqq c(1+t)^N\sum_{j=1}^N\left\|\mathcal{F}^{-1}\left\{\chi_3(\xi)\frac{e^{it|\xi|}}{|\xi|^{(N+1)(1/p-1/2)}}\widehat{D_{x_j}^k v}(\xi)\right\}\right\|_{p'}.$$

At this stage we can apply an estimate used for the L^p-$L^{p'}$-estimation of the usual wave equation (cf. [**Br75**], [**Pe76**]) to get

$$\begin{aligned} I_1^2 &\leqq c(1+t)^N t^{-2N(1/p-1/2)+(N+1)(1/p-1/2)}\|D_x^k v\|_p \\ &= c(1+t)^N t^{-(N-1)(1/p-1/2)}\|D_x^k v\|_p\,. \end{aligned}$$

The second term of (3.9) is treated in the same way and we obtain the desired estimate (3.5), $i=2$.

Finally we note that (3.6) follows easily from the Parseval identity. Q.E.D.

Summing up the above estimates (3.3)-(3.6) in Proposition 3.1, we have the following.

COROLLARY 3.1. (L^p-$L^{p'}$-estimate). *For* $1\leqq p\leqq 2$, $\gamma\in\boldsymbol{R}$ *and* $v\in C_0^\infty(\boldsymbol{R}^N)$, *it holds*

$$\|\mathcal{F}^{-1}\{\phi_i(\xi, t)\hat{v}(\xi)\}\|_{H^\gamma_{p'}}\leqq c\{(1+t)^{-N(1/p-1/2)}+t^{-(N-1)(1/p-1/2)}e^{-\nu t}\}\|v\|_p, \quad i=1, 2,$$

with some $0<\nu<1/2$, *provided that* $\gamma+(N+1)(1/p-1/2)\leqq i-1$, $i=1, 2$.

Applying Proposition 3.1 to the equation (3.1) or using similar argument as in the proof of Proposition 3.1, we can obtain the desired L^q-estimate of the solution $u(t)$ of the linear equation (3.1).

PROPOSITION 3.2. (i) *Let* $(u_0, u_1)\in H^1\cap L^r\times L^2\cap L^r$, $1\leqq r\leqq 2$, *and let* $f\in L^\infty_{loc}([0, \infty); L^p\cap L^2)$, $1\leqq p\leqq 2$. *Then, the solution* $u(t)$ *of* (3.1) *satisfies*

$$\begin{aligned}\|u(t)\|_2 \leqq\ & c(1+t)^{-N(1/r-1/2)/2}(\|u_0\|_r+\|u_1\|_r)+ce^{-\nu t}(\|u_0\|_2+\|u_1\|_2)\\ &+c\int_0^t(1+t-s)^{-N(1/p-1/2)/2}\|f(s)\|_p ds+c\int_0^t e^{-\nu(t-s)}\|f(s)\|_2 ds\end{aligned}\tag{3.10}$$

with some $0<\nu<1/2$.

(ii) *Let* $(u_0, u_1)\in H^{m+1}\cap L^r\times H^m\cap L^r$, $1\leqq r\leqq 2$, $m\geqq 0$ *and let* $f\in L^\infty_{loc}([0, \infty); L^{p_1})$, $D_x^k f\in L^\infty_{loc}([0, \infty); L^p)$, $1\leqq p_1$, $p\leqq 2$, $k\geqq 0$ *being an integer. Then, the solution* $u(t)$ *satisfies*

$$\begin{aligned}\|u(t)\|_q \leqq\ & c(1+t)^{-N(1/r-1/q)/2}(\|u_0\|_r+\|u_1\|_r)+ce^{-\nu t}(\|u_0\|_{H^{m+1}}+\|u_1\|_{H^m})\\ &+c\int_0^t(1+t-s)^{-N(1/p_1-1/q)/2}\|f(s)\|_{p_1} ds\\ &+c\int_0^t(t-s)^{-(N-1)(1/p-1/2)}e^{-\nu(t-s)}\|D_x^k f(s)\|_p ds\,,\end{aligned}\tag{3.11a}$$

with some $0<\nu<1/2$, *provided that*

$$\frac{m+1}{N}\geqq\frac{1}{2}-\frac{1}{q},\quad 2\leqq q\leqq\infty\,,\quad\left(\frac{m+1}{N}>\frac{1}{2}\ \ \textit{if}\ q=\infty\right),\tag{3.11b}$$

and there exists $\gamma>0$ *such that*

$$\begin{cases}1-\dfrac{1}{p}-\dfrac{\gamma}{N}\leqq\dfrac{1}{q}\leqq 1-\dfrac{1}{p},\quad\left(1-\dfrac{1}{p}-\dfrac{\gamma}{N}<0\ \ \textit{if}\ q=\infty\right),\\ \gamma+(N+1)\left(\dfrac{1}{p}-\dfrac{1}{2}\right)\leqq k+1\,.\end{cases}\tag{3.11c}$$

PROOF. The proof of (i) is easier and we shall give the proof of (ii) only. Recall that $u(t)=u_L(t)+I_f(t)$ (see (3.2)). We assume $2\leqq q<\infty$. (The case $q=\infty$ is treated quite similarly by a trivial modification.) First, we use the Sobolev embedding theorem that

$$H^{m+1}\subset L^q\qquad\text{if }\frac{m+1}{2}\geqq\frac{2}{N}-\frac{1}{q}\,.$$

Then, we see by Proposition 3.1

$$\begin{aligned}\|u_L(t)\|_q\leqq\ & \|\mathcal{F}^{-1}\{\phi_1(\xi, t)\chi_1(\xi)\hat{u}_0(\xi)/2\}\|_q\\ &+\|\mathcal{F}^{-1}\{\phi_2(\xi, t)\chi_1(\xi)(\hat{u}_0(\xi)/2+\hat{u}_1(\xi))\}\|_q\\ &+c\sum_{i=2}^{3}\{\|\mathcal{F}^{-1}\{\phi_1(\xi, t)\chi_i(\xi)\hat{u}_0(\xi)/2\}\|_{H^{m+1}}\\ &+\|\mathcal{F}^{-1}\{\phi_2(\xi, t)\chi_i(\xi)(\hat{u}_0(\xi)/2+\hat{u}_1(\xi))\}\|_{H^{m+1}}\}\end{aligned}$$

$$\leqq c(1+t)^{-N(1/r-1/q)/2}(\|u_0\|_r+\|u_1\|_r)+ce^{-\nu t}(\|u_0\|_2+\|u_1\|_2)$$
$$+ce^{-\nu t}(\|u_0\|_{H^{m+1}}+\|u_1\|_{H^m}).$$

Next, we use

$$H^{\gamma}_{p'}\subset L^q \quad \text{if } 1-\frac{1}{p}-\frac{\gamma}{N}\leqq\frac{1}{q}\leqq 1-\frac{1}{p}, \qquad \gamma>0. \tag{3.12}$$

Then, we see by Proposition 3.1

$$\|I_f(t)\|_q\leqq c\int_0^t\{\|\mathcal{F}^{-1}\{\phi_2(\xi,s)\chi_1(\xi)\hat{f}(\xi,s)\}\|_q \tag{3.13}$$
$$+\|\mathcal{F}^{-1}\{\phi_2(\xi,s)\chi_2(\xi)\hat{f}(\xi,s)\}\|_{H^{\gamma_2}_{p_2'}}$$
$$+\|\mathcal{F}^{-1}\{\phi_2(\xi,s)\chi_3(\xi)\hat{f}(\xi,s)\}\|_{H^{\gamma}_{p'}}\}ds$$
$$\leqq c\int_0^t\{(1+t-s)^{-N(1/p_1-1/q)/2}\|f(s)\|_{p_1}+e^{-\nu(t-s)}\|f(s)\|_{p_2}$$
$$+(t-s)^{-(N-1)(1/p-1/2)}e^{-\nu(t-s)}\|D_x^kf(s)\|_p\}ds,$$

where k is a nonnegative integer, p_1 is any number with $1\leqq p_1\leqq 2$, (γ_2, p_2) should satisfy the condition (3.12), and (γ, p) should satisfy the condition (3.12) and $\gamma+(N+1)(1/p-1/2)\leqq k+1$. When $1\leqq p\leqq 2$ and $2\leqq q<\infty$, there always exists γ satisfying (3.12). Hence, we can take $p_2=p_1$ $(1\leqq p_1\leqq 2)$ in (3.13) and this gives (3.11). Q.E.D.

Remark 3.1. (i) When we use (3.11a), the additional condition

$$(N-1)\left(\frac{1}{p}-\frac{1}{2}\right)<1, \quad \text{i.e., } p>\frac{2(N-1)}{N+1} \tag{3.14}$$

is made for the convergence of the last integral.

(ii) Furthermore, when $k=1$ and $N\geqq 3$, for this p in (3.14), we can take any q such that

$$\frac{1}{2}-\frac{2}{N}+\frac{1}{N}\left(\frac{1}{p}-\frac{1}{2}\right)\leqq\frac{1}{q}\leqq 1-\frac{1}{p} \tag{3.15}$$

($q=\infty$ is possible only if $1/2-2/N+(1/N)(1/p-1/2)<0$). In particular, we can take $q=\infty$ if $N=3$.

§4. Energy decay for the linear equation.

In this section we shall derive some difference inequalities concerning the energy $E(t)\equiv\|u_t(t)\|^2+\|\nabla u(t)\|^2$ for the linear equation (3.1), which is useful in deriving the decay rate of the solutions for the nonlinear equation.

Proposition 4.1. *Let $f(t)\in L^2_{loc}([0,\infty); L^2)$ and let $u(t)$ be the solution of* *the problem* (3.1) *which belongs to* $C([0, \infty); H^1)\cap C^1([0, \infty); L^2)$. *Then, the following inequality holds*:

$$\sup_{t\leq s\leq t+1} E(s) \leqq c\{D(t)^2+(D(t)+\delta(t)) \sup_{t\leq s\leq t+1} \|u(s)\|+\delta(t)^2\}, \tag{4.1a}$$

where $c>0$ *is a constant and*

$$D(t)^2 \equiv E(t)-E(t+1)+\delta(t)^2 \quad \text{and} \quad \delta(t)^2 \equiv \int_t^{t+1} \|f(s)\|^2 ds. \tag{4.1b}$$

(*Nonnegativity of* $D(t)^2$ *easily follows from* (4.2) *below.*)

PROOF. The proof is essentially included in Nakao [**Na83**] and we sketch it briefly. Multiplying the equation by u_t and integrating over $\boldsymbol{R}^N$, we have

$$\frac{1}{2}\frac{d}{dt}E(t)+\|u_t(t)\|^2 = (f(t), u_t(t)), \qquad (\cdot, \cdot)=(\cdot, \cdot)_{L^2}, \tag{4.2}$$

which implies

$$\int_t^{t+1} \|u_t(s)\|^2 ds \leqq cD(t)^2. \tag{4.3}$$

Thus, there exist $t_1\in[t, t+1/4]$, $t_2\in[t+3/4, t+1]$ such that

$$\|u_t(t_i)\| \leqq 2cD(t), \qquad i=1, 2.$$

Next, multiplying the equation by u and integrating over $\boldsymbol{R}^N\times[t_1, t_2]$, we have

$$\begin{aligned} \int_{t_1}^{t_2} \|\nabla u(s)\|^2 ds &= \int_{t_1}^{t_2} \|u_t(s)\|^2 ds+(u_t(t_1), u(t_1))-(u_t(t_2), u(t_2)) \\ &\quad -\int_{t_1}^{t_2} (u_t(s), u(s))ds+\int_{t_1}^{t_2} (f(s), u(s))ds \\ &\leqq D(t)^2+c(D(t)+\delta(t)) \sup_{t\leq s\leq t+1} \|u(s)\| \equiv A(t)^2. \end{aligned} \tag{4.4}$$

It follows from (4.3) and (4.4) that there exists $t^*\in[t_1, t_2]$ such that

$$E(t^*) \leqq 2\int_{t_1}^{t_2} E(s)ds \leqq cA(t)^2.$$

Returning to (4.2), we obtain easily

$$\begin{aligned} \sup_{t\leq s\leq t+1} E(s) &\leqq E(t^*)+\int_t^{t+1} \|u_t(s)\|^2 ds+\int_t^{t+1} \|f(s)\|\|u_t(s)\| ds \\ &\leqq c\{D(t)^2+(D(t)+\delta(t)) \sup_{t\leq s\leq t+1} \|u(s)\|+\delta(t)^2\}. \end{aligned}$$

Q.E.D.

In application of Proposition 4.1 to the nonlinear equation we will take $-f'(u)u_t$, $-D^k f(u)$, etc. for $f(t)$, and the following is convenient.

PROPOSITION 4.2. *In the inequality* (4.1a), *we assume further*

$$\|u(t)\|^2 \leqq k_0(1+t)^{-a} \tag{4.5a}$$

and

$$\|f(t)\|^2 \leqq k_1\{(1+t)^{-b}+(1+t)^{-c}E(t)^{\mu}+(1+t)^{-d}E(t)\} \tag{4.5b}$$

with some k_0, $k_1>0$, $a\geqq 0$, b, c, $d>0$ *and* $0<\mu<1$. *Then,* $E(t)$ *has the decay property*

$$E(t) \leqq c_1(1+t)^{-\theta}\,, \tag{4.5c}$$

where c_1 *is a constant depending on* $E(0)$ *and other known constants, and* $\theta>0$ *is given by*

$$\theta = \min\left\{1+a,\ b,\ \frac{a+b}{2},\ \frac{c}{1-\mu},\ \frac{a+c}{2-\mu},\ a+d\right\}. \tag{4.5d}$$

PROOF. It is clear from (4.2) and (4.5b) that $E(t)$ is locally bounded, i.e.,

$$E(t) \leqq c(E(0),\,T) \equiv c_0 < \infty\,, \qquad 0\leqq t\leqq T\,, \tag{4.6}$$

for any $T>0$. Now, we see from (4.1) that

$$\begin{aligned}\sup_{t\leq s\leq t+1} E(s)^2 &\leqq c\{D(t)^2+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}D(t)^2\\ &\quad +c\delta(t)^2\{\delta(t)^2+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}\\ &\leqq c\{E(t)+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}\{E(t)-E(t+1)\}\\ &\quad +c\delta(t)^2E(t)+c\delta(t)^2\{\delta(t)^2+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}\end{aligned}$$

and

$$\begin{aligned}\sup_{t\leq s\leq t+1} E(s)^2 &\leqq c\{E(t)+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}\{E(t)-E(t+1)\}\\ &\quad +c\delta(t)^2\{\delta(t)^2+\sup_{t\leq s\leq t+1}\|u(s)\|^2\}\\ &\leqq c\{E(t)+(1+t)^{-a}\}\{E(t)-E(t+1)\}\\ &\quad +c\{(1+t)^{-2b}+(1+t)^{-a-b}+(1+t)^{-2c}E(t)^{2\mu}\\ &\quad +(1+t)^{-a-c}E(t)^{\mu}+(1+t)^{-a-d}E(t)+(1+t)^{-2d}E(t)^2\}.\end{aligned}$$

Hence, there exists $T_0>0$ such that if $t>T_0$,

$$\sup_{t\leq s\leq t+1} E(s)^2 \leqq c\{E(t)+(1+t)^{-a}\}\{E(t)-E(t+1)\}+c(1+t)^{-\gamma} \tag{4.7a}$$

with

(4.7b) $$\gamma = \min\left\{2b,\ a+b,\ \frac{2c}{1-\mu},\ \frac{2(a+c)}{2-\mu},\ 2(a+d)\right\}.$$

Here, we observe that $E(t)$ is bounded on $[0, \infty)$. Indeed, if $E(t) \leqq E(t+1)$ for some $t \geqq T_0$, then (4.7) implies

$$\sup_{t \leq s \leq t+1} E(s)^2 \leqq c(1+t)^{-\gamma} \leqq C_*^2 < \infty$$

and we conclude

$$E(t) \leqq \max\left\{\sup_{T_0 \leq s \leq T_0+1} E(s),\ C_*\right\},$$

that is, by use of (4.6),

(4.8) $$E(t) \leqq c(E(0)) < \infty \qquad \text{on } [0, \infty).$$

In what follows, we denote by c_1 various positive constants depending on $E(0)$. By (4.8) and (4.7) we see first

$$\sup_{t \leq s \leq t+1} E(s)^2 \leqq c_1 \{E(t) - E(t+1)\} + c(1+t)^{-\gamma}$$

and hence, by Lemma 2.2,

(4.9) $$E(t) \leqq c_1(1+t)^{-\theta_1}, \qquad \theta_1 = \min\{1, \gamma/2\}.$$

Returning to (4.7) and using the estimate (4.9) just obtained we see

$$\sup_{t \leq s \leq t+1} E(s)^2 \leqq c_1(1+t)^{-\min\{a, \theta_1\}} \{E(t) - E(t+1)\} + c(1+t)^{-\gamma}$$

and hence, Lemma 2.2 yields

$$E(t) \leqq c_1(1+t)^{-\theta_2}, \qquad \theta_2 = \min\{1 + \min\{a, \theta_1\}, \gamma/2\}.$$

Repeating this procedure indefinitely, we have

$$E(t) \leqq c_1(1+t)^{-\theta_m}, \qquad \theta_m = \min\{1 + \min\{a, \theta_{m-1}\}, \gamma/2\}$$

for $m=2, 3, 4, \cdots$. Since $\theta_m = \min\{1+a, \gamma/2\}$ for large m and we arrive at the estimate (4.5c-d). Q.E.D.

Now, let us return to the linear equation (3.1) and the inequality (4.1a).

PROPOSITION 4.3. (i) *Suppose that* $f \equiv 0$ *and* $(u_0, u_1) \in H^1 \cap L^r \times L^2 \cap L^r$, $1 \leqq r \leqq 2$. *Then, the solution* $u(t)$ *of* (3.1) *meets the decay property*:

(4.10a) $$\|u(t)\| \leqq c_1(1+t)^{-N(1/r-1/2)/2} \quad \textit{and} \quad E(t) \leqq c_1(1+t)^{-1-N(1/r-1/2)}.$$

(ii) *Moreover, if* $(u_0, u_1) \in H^{m+1} \cap L^r \times H^m \cap L^r$, $1 \leqq r \leqq 2$, $m \geqq 1$, *we have*

(4.10b) $$\sum_{i=0}^{m+1} \|D_t^i D_x^{m+1-i} u(t)\| \leqq c_{m+1}(1+t)^{-(m+1)/2-N(1/r-1/2)/2}.$$

PROOF. (4.10a) follows immediately from Propositions 3.2 and 4.2. Next, differentiating the equation, we get

$$U_{tt}-\Delta U+U_t=0$$

with $U(x, t)=Du(x, t)$. Since

$$\|U(t)\|^2=\|Du(t)\|^2 \leqq c_1(1+t)^{-2\theta_1}, \qquad 2\theta_1=1+N\Big(\frac{1}{r}-\frac{1}{2}\Big),$$

Proposition 4.2 implies

$$\|U_t(t)\|^2+\|\nabla U(t)\|^2 \leqq c_2(1+t)^{-1-2\theta_1}.$$

Repeating this argument we obtain (4.10b). Q.E.D.

§ 5. Proof of Theorem 1.

Now, we are in a position to treat the nonlinear equation:

(5.1a)
$$\begin{cases} u_{tt}-\Delta u+u_t+f(u)=0 & \text{in } \boldsymbol{R}^N\times[0, \infty) \\ u(x, 0)=u_0(x) \quad \text{and} \quad u_t(x, 0)=u_1(x). \end{cases}$$

Setting

(5.1b)
$$E(t)\equiv\|u_t(t)\|^2+\|\nabla u(t)\|^2+\int_{\boldsymbol{R}^N}F(u(t))dx, \qquad F(u)\equiv 2\int_0^u f(\eta)d\eta,$$

the same argument yielding (4.1a) can be applied to (5.1) without any essential changes and (4.1a) remains valid for $E(t)$ defined above and for $\delta(t)\equiv 0$ (i.e., $f(t)\equiv 0$). To show the boundedness of $\|u(t)\|$, we first note that

(5.2)
$$E(t)+\int_0^t\|u_t(s)\|^2ds \leqq E(0)<\infty,$$

which follows by multiplying (5.1) by u_t and integrating over $\boldsymbol{R}^N\times[0, \infty)$. Further, multiplying (5.1) by u we have

$$\frac{d}{dt}\Big\{(u_t(t), u(t))+\frac{1}{2}\|u(t)\|^2\Big\}-\|u_t(t)\|^2+\|\nabla u(t)\|^2+\int_{\boldsymbol{R}^N}f(u(t))u(t)dx=0$$

and by Hyp. (i),

$$\frac{1}{2}\|u(t)\|^2 \leqq \frac{1}{2}\|u_0\|^2+(u_0, u_1)+\|u_t(t)\|\|u(t)\|+\int_0^t\|u_t(s)\|^2ds$$

which together with (5.2) implies

(5.3)
$$\|u(t)\| \leqq c_1(E(0))<\infty.$$

Thus, applying Proposition 4.2 with $f(t)\equiv 0$ and $a=0$ to (5.1), we obtain

(5.4)
$$E(t) \leqq c_1(1+t)^{-1}, \quad \text{and} \quad \|Du(t)\| \leqq c_1(1+t)^{-1/2},$$

where D denotes D_t and D_x. The former part of Theorem 1 is now proved.

To show the latter part we must improve the estimate (5.3). For this, we utilize (3.10) with $p=1$ in Proposition 3.2 and Hyp. (i) to get

$$\|u(t)\| \leqq c_1(1+t)^{-N(1/r-1/2)/2}+c\int_0^t(1+t-s)^{-N(1-1/2)/2}\|u(s)\|_{\alpha+1}^{\alpha+1}ds \\ +c\int_0^t e^{-\nu(t-s)}\|u(s)\|_{2(\alpha+1)}^{\alpha+1}ds\,. \tag{5.5}$$

Here, to estimate the second and the third terms on the right hand side of (5.5), we need the following well known interpolation estimates.

CLAIM 1. (i) *Let α satisfy the condition*:

$$1 \leqq \alpha \leqq (N+2)/(N-2) \qquad (1\leqq\alpha<\infty \text{ if } N=1, 2). \tag{5.6a}$$

Then, it holds

$$\|u(t)\|_{\alpha+1}^{\alpha+1} \leqq c\|\nabla u(t)\|^{N(\alpha-1)/2}\|u(t)\|^{\mu_0}, \qquad \mu_0\equiv\alpha+1-N(\alpha-1)/2, \tag{5.6b}$$

where we note that $\mu_0>1$ if $N=1$, 2, and $\mu_0\geqslant1$ for $\alpha\leqslant N/(N-2)$ if $N\geqq3$.

(ii) *Let α satisfy the condition*:

$$0 \leqq \alpha \leqq 2/(N-2) \qquad (0\leqq\alpha<\infty \text{ if } N=1, 2). \tag{5.7a}$$

Then, it holds

$$\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c\|\nabla u(t)\|^{N\alpha/2}\|u(t)\|^{\alpha+1-N\alpha/2}\,. \tag{5.7b}$$

First, using the estimates (5.3), (5.4) and (5.6), we see that

$$\|u(t)\|_{\alpha+1}^{\alpha+1} \leqq c\|\nabla u(t)\|^{N(\alpha-1)/2}\|u(t)\|^{\mu_0} \leqq c_1(1+t)^{-N(\alpha-1)/4}\|u(t)\|\,. \tag{5.8}$$

(Note that from the assumption of Theorem 1 (ii) we see $\mu_0>1$, and therefore $\|u(t)\|^{\mu_0}\leqq c_1\|u(t)\|$ by (5.3).) Next, using the estimates (5.3), (5.4) and (5.7), we see that

$$\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c\|\nabla u(t)\|^{N\alpha/2}\|u(t)\|^{\alpha+1-N\alpha/2} \leqq c_1(1+t)^{-N\alpha/4}. \tag{5.9}$$

Thus, it follows from (5.5), (5.8) and (5.9) that

$$\|u(t)\| \leqq c_1(1+t)^{-N(1/r-1/2)/2}+c_1\int_0^t(1+t-s)^{-N/4}(1+s)^{-N(\alpha-1)/4}\|u(s)\|ds \\ +c_1\int_0^t e^{-\nu(t-s)}(1+s)^{-N\alpha/4}ds\,. \tag{5.10}$$

Here, from $1\leqq r\leqq2$ and $\alpha>4/N>1$ (if $1\leqq N\leqq3$), we see

$$N(\alpha-1)/4>0, \quad N/4+N(\alpha-1)/4-1>0 \quad \text{and} \quad N\alpha/4>N/4\geqq\frac{N}{2}\Big(\frac{1}{r}-\frac{1}{2}\Big).$$

Thus, applying Lemma 2.3 (see Remark 2.1) to (5.10), we have

$$\|u(t)\| \leqq c_1(1+t)^{-\eta}, \qquad \eta = \frac{N}{2}\Big(\frac{1}{r}-\frac{1}{2}\Big).$$

Moreover, applying Proposition 4.2 with $f(t)\equiv 0$ and $a=2\eta$ to (5.1), we can obtain

$$E(t) \leqq c_1(1+t)^{-1-2\eta}, \quad \text{and} \quad \|Du(t)\| \leqq c_1(1+t)^{-1/2-\eta}.$$

The proof of Theorem 1 is now complete.

§ 6. Proof of Theorem 2.

We consider the nonlinear equation (5.1). Here, we make the hypotheses Hyp. (i)-(ii) with $N/4 \leqq \alpha < 4/(N-2)$, $3 \leqq N \leqq 6$, and $(u_0, u_1) \in H^2 \cap L^r \times H^1 \cap L^r$, $1 \leqq r \leqq 2$. For the proof of Theorem 2 we first derive L^{q*}-estimate of the solution $u(t)$. Let us define q_* by

$$q_* = \begin{cases} \infty & \text{for } N=3 \\ \dfrac{2N(N-1)}{(N-2)(N-3)}+\varepsilon & \text{for } 4 \leqq N \leqq 6, \end{cases} \tag{6.1}$$

where $\varepsilon > 0$ is sufficiently small. We note that $H^2 \subset L^{q*}$.

PROPOSITION 6.1. *Under the above conditions (i.e., the condition of Theorem 2), the solution $u(t) \in \bigcap_{i=0}^2 C^i([0, \infty); H^{2-i})$ satisfies*

$$\|u(t)\|_{q*} \leqq c_2(1+t)^{-N(1/2-1/q*)/2}, \tag{6.2}$$

where q^* is defined by (6.1).

REMARK 6.1. After Theorem 2 is proved, this estimate (6.2) is in fact improved a little (see Proposition 6.2).

PROOF. Let us take p_* such that

$$\frac{1}{p_*} = \frac{N+1}{2(N-1)} - \varepsilon = \frac{1}{2} + \frac{1}{N-1} - \varepsilon, \quad \text{i.e.,} \quad p_* = \frac{2(N-1)}{N+1} + \varepsilon, \tag{6.3}$$

where $\varepsilon > 0$ should be chosen very small. Thus, we see $1 < p_* < 2$ and (3.14) in Remark 3.1 is satisfied. For this choice of p_*, q_* in (6.1) is given by

$$\begin{cases} q_* = \infty & \text{if } N=3 \\ \dfrac{1}{q_*} = \dfrac{1}{2} - \dfrac{2}{N} + \dfrac{1}{N}\Big(\dfrac{1}{p_*} - \dfrac{1}{2}\Big) & \text{if } 4 \leqq N \leqq 6 \text{ (see (3.15))}. \end{cases} \tag{6.4}$$

We utilize (3.11) in Proposition 3.2 with $q=q_*$, $p=p_*$ and $k=1$ to get

$$\|u(t)\|_{q*} \leqq c_2(1+t)^{-N(1/r-1/q*)/2}+c\int_0^t (1+t-s)^{-N(1/p_1-1/q*)/2}\|u(s)\|_{p_1(\alpha+1)}^{\alpha+1} ds \tag{6.5}$$

$$+c\int_0^t (t-s)^{-\delta}e^{-\nu(t-s)}\|\,|u(s)|^{\alpha}D_x u(s)\|_{p*}ds$$

with $\delta\equiv(N-1)(1/p_*-1/2)$, $0<\delta<1$, and $1\leqq p_1\leqq 2$. Here, to estimate the second term on the right hand side of (6.5), we need the following interpolation estimate in addition to Claim 1.

CLAIM 2. *Let $N\geqq 3$ and let α satisfy the condition*:

$$\frac{2}{N-2} \leqq \alpha \leqq \left(\frac{4}{N-2} <\right) \frac{2(2N-3)}{(N-2)(N-3)}. \tag{6.6a}$$

Then, it holds that

$$\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c\|\nabla u(t)\|^{2\beta_1}\|u(t)\|_{q*}^{\mu_1} \tag{6.6b}$$

with β_1 and μ_1 satisfying

$$2\beta_1 = \left(\frac{1}{2}-\frac{\alpha+1}{q_*}\right)\left(\frac{N-2}{2N}-\frac{1}{q_*}\right)^{-1}, \qquad \mu_1 = \alpha+1-2\beta_1\,. \tag{6.6c}$$

Moreover, we have

$$\beta_1+\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q_*}\right)\mu_1 = \frac{N\alpha}{4}, \tag{6.6d}$$

and

$$\beta_1+\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q_*}\right) > 1 \qquad \textit{if } \alpha\leqq 4/(N-2). \tag{6.6e}$$

PROOF OF CLAIM 2. If $2N/(N-2)\leqq q\leqq q_*$, then we have by the Gagliardo-Nirenberg inequality

$$\|u(t)\|_q \leqq c\|\nabla u(t)\|^{\theta}\|u(t)\|_{q*}^{1-\theta}, \qquad \theta=\left(\frac{1}{q}-\frac{1}{q_*}\right)\left(\frac{N-2}{2N}-\frac{1}{q_*}\right)^{-1}.$$

Here, taking $q=2(\alpha+1)$, we see

$$q-\frac{2N}{N-2} = 2\left(\alpha-\frac{2}{N-2}\right) \geqq 0 \qquad \text{if } \alpha\geqq\frac{2}{N-2}$$

and

$$q_*-q = 2\left(\frac{2(2N-4)}{(N-2)(N-3)}-\alpha\right)+\varepsilon > 0 \qquad \text{if } \alpha\leqq\frac{2(2N-3)}{(N-2)(N-3)}$$

which implies (6.6a-c). The relations (6.6d) and (6.6e) are checked directly.

To estimate the third term on the right hand side of (6.5), we need the following interpolation estimates for $3\leqq N\leqq 6$.

CLAIM 3. (i) *Let α satisfy the condition*

$$\frac{2}{N-1}\ \left(<\frac{2}{N-2}\right)\leqq\alpha<\frac{2N}{(N-1)(N-2)}. \tag{6.7a}$$

Then, it holds that

$$\||u(t)|^{\alpha}D_x u(t)\|_{p*}\leqq c\|\nabla u(t)\|^{1+2\beta*}\|u(t)\|^{\alpha-2\beta*} \tag{6.7b}$$

with

$$2\beta_*=N\Big(\frac{\alpha+1}{2}-\frac{1}{p_*}\Big). \tag{6.7c}$$

Moreover, we have

$$\beta_*+\frac{1}{2}>\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)\qquad \textit{if}\ \ \alpha\geqq 4/N. \tag{6.7d}$$

(ii) *Let α satisfy the condition*

$$\frac{2N}{(N-1)(N-2)}\leqq\alpha<\Big(\frac{4}{N-2}\leqq\Big)\ \frac{2N}{(N-2)(N-3)}. \tag{6.8a}$$

Then, it holds that

$$\||u(t)|^{\alpha}D_x u(t)\|_{p*}\leqq c\|\nabla u(t)\|^{1+2\beta_2}\|u(t)\|_{q*}^{\mu_2} \tag{6.8b}$$

with

$$2\beta_2=\Big(\Big(\frac{1}{p_*}-\frac{1}{2}\Big)-\frac{\alpha}{q_*}\Big)\Big(\frac{N-2}{2N}-\frac{1}{q_*}\Big)^{-1}\quad \text{and}\quad \mu_2=\alpha-2\beta_2. \tag{6.8c}$$

Moreover, we have

$$\beta_2+\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)\mu_2=\beta_* \tag{6.8d}$$

with β_* given by (6.7c), *and also*

$$0\leqq\mu_2<1\qquad \textit{if}\ \ \alpha<4/(N-2). \tag{6.8e}$$

PROOF OF CLAIM 3. Noting that $1<p_*<2$, we can use the Hölder inequality to get

$$\||u(t)|^{\alpha}D_x u(t)\|_{p*}\leqq c\|\nabla u(t)\|\|u(t)\|_q^{\alpha},\qquad \frac{\alpha}{q}=\frac{1}{p_*}-\frac{1}{2}. \tag{6.9}$$

Here, if $2\leqq q\leqq 2N/(N-2)$ we see further

$$\|u(t)\|_q^{\alpha}\leqq c\|\nabla u(t)\|^{\alpha\theta}\|u(t)\|^{\alpha(1-\theta)} \tag{6.10}$$

with

$$\alpha\theta=\alpha N\Big(\frac{1}{2}-\frac{1}{q}\Big)=N\Big(\frac{\alpha+1}{2}-\frac{1}{p_*}\Big)\equiv 2\beta_*.$$

Since $q=(N-1)\alpha+\varepsilon$, the condition $2\leqq q\leqq 2N/(N-2)$ is equivalent to

$$\frac{2}{N-1}-\varepsilon\leqq\alpha\leqq\frac{2N}{(N-1)(N-2)}-\varepsilon,$$

and this is certainly valid under our condition on α in (i) if we choose a sufficiently small $\varepsilon>0$, which implies (6.7a-c). Moreover, we see

$$\left(\beta_*+\frac{1}{2}\right)-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{q_*}\right)=\frac{N\alpha}{4}-1+\varepsilon>0 \qquad \text{if } \alpha\geqq 4/N$$

(see (6.7c), (6.4) and (6.3)), which implies (6.7d). If $2N/(N-2)<q\leqq q_*$ we have, instead of (6.10),

$$\|u(t)\|_q^\alpha\leqq c\|\nabla u(t)\|^{\alpha\theta}\|u(t)\|_{q*}^{\alpha(1-\theta)} \tag{6.11}$$

with

$$\alpha\theta=\alpha\left(\frac{1}{q}-\frac{1}{q_*}\right)\left(\frac{N-2}{2N}-\frac{1}{q_*}\right)^{-1}\equiv 2\beta_2\,.$$

The condition $2/(N-2)\leqq q\leqq q_*$ is equivalent to

$$\frac{2N}{(N-1)(N-2)}-\varepsilon\leqq\alpha\leqq\frac{2N}{(N-2)(N-3)}-\varepsilon$$

and satisfied under the condition on α in (ii), which implies (6.8a-d). Finally, we note that

$$1-\mu_2=\frac{N-1}{2}\left(\frac{4}{N-2}-\alpha\right)-\varepsilon>0 \qquad \text{if } \alpha<4/(N-2),$$

which implies (6.8e).

Now, we are in a position to complete the proof of the L^{q*}-decay estimate (6.2).

As the first step, we shall show that the solution $u(t)$ is uniformly bounded in L^{q*}, i.e.,

$$\|u(t)\|_{q*}\leqq c_2<\infty \qquad \text{if } 2/(N-1)\leqq\alpha<4/(N-2)\,. \tag{6.12}$$

We set

$$p_1=\begin{cases}1 & \text{if } \alpha>1\\ \dfrac{2}{\alpha+1} & \text{if } \alpha\leqq 1\,.\end{cases} \tag{6.13}$$

Then, we see, noting (6.1),

$$\frac{N}{2}\left(\frac{1}{p_1}-\frac{1}{q_*}\right)>1 \quad\text{and}\quad 2\leqq p_1(\alpha+1)\leqq\frac{2N}{N-2}\,. \tag{6.14}$$

It follows from (6.14), (5.3) and (5.4) that

(6.15a) $$\|u(t)\|_{p_1(\alpha+1)} \leqq c\|u(t)\|_{H^1} \leqq c_1 < \infty .$$

Also, in the case (i) of Claim 3, we have

(6.15b) $$\| |u(t)|^\alpha D_x u(t)\|_{p*} \leqq c\|u(t)\|_{H^1}^{\alpha+1} \leqq c_1 < \infty ,$$

while in the case (ii) of Claim 3

(6.16) $$\| |u(t)|^\alpha D_x u(t)\|_{p*} \leqq c\|\nabla u(t)\|^{1+2\beta_2}\|u(t)\|_{q*}^{\mu_2} \leqq c_1\|u(t)\|_{q*}^{\mu_2}, \quad \mu_2<1 ,$$

which follows from (6.8) and (5.4). Thus, from (6.5), (6.14), (6.15) and (6.16), we have, in both cases (i), (ii) of Claim 3,

$$\|u(t)\|_{q*} \leqq c_2(1+t)^{-N(1/r-1/q*)/2}+c_1\int_0^t(1+t-s)^{-N(1/p_1-1/q*)/2}ds$$

$$+c_1\int_0^t(t-s)^{-\delta}e^{-\nu(t-s)}(1+\|u(s)\|_{q*}^{\mu_2})ds \quad (0\leqq\mu_2<1),$$

which implies (6.12) by Lemma 2.3.

As the second step, we shall show that if the solution $u(t)$ satisfies

(6.17) $$\|u(t)\| \leqq c_2(1+t)^{-\eta}$$

(6.18) $$\|Du(t)\| \leqq c_2(1+t)^{-1/2-\eta}$$

with

$$\eta=0 \quad \text{or} \quad \eta=\min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\},$$

then it holds that

(6.19) $$\|u(t)\|_{q*} \leqq c_2(1+t)^{-N(1/2-1/q*)/2-\eta} .$$

If we can prove the above assertion, the estimate (6.2) follows by taking $\eta=0$ (see (5.3), (5.4) or Theorem 1). To show (6.19), we take $p_1=2$ on the interval $[t/2, t]$ in the second integral of (6.5) to get

(6.20) $$\|u(t)\|_{q*} \leqq c_2(1+t)^{-N(1/r-1/q*)/2}+c\int_0^{t/2}(1+t-s)^{-N(1/p_1-1/q*)/2}\|u(s)\|_{p_1(\alpha+1)}^{\alpha+1}ds$$

$$+c\int_{t/2}^t(1+t-s)^{-N(1/2-1/q*)/2}\|u(s)\|_{2(\alpha+1)}^{\alpha+1}ds$$

$$+c\int_0^t(t-s)^{-\delta}e^{-\nu(t-s)}\| |u(s)|^\alpha D_x u(s)\|_{p*}ds$$

$$\equiv I_1+I_2+I_3+I_4$$

with $\delta=(N-1)(1/p_*-1/2)$ and $1\leqq p_1\leqq 2$. (Such a division of the integration interval is justified by the procedure to derive (6.5).)

We claim

$$(6.21) \qquad I_2 \leqq c_2(1+t)^{-\zeta}, \qquad \zeta = \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta\,.$$

First, we assume $\alpha>1$. Then, taking $p_1=1$ (see (6.13)) and applying (5.6) in the case (i) of Claim 1, we have from (6.17), (6.18) that

$$\|u(t)\|_{p_1(\alpha+1)}^{\alpha+1} \leqq c_2(1+t)^{-\lambda}, \qquad \lambda=N(\alpha-1)/4+(\alpha+1)\eta\,.$$

Hence,

$$I_2 \leqq c_2(1+t)^{-N(1-1/q_*)/2}\int_0^{t/2}(1+s)^{-\lambda}ds\,.$$

When $\lambda=1$ we see $r\neq 1$, i.e., $\eta<N/4$ and

$$I_2 \leqq c_2(1+t)^{-N(1/2-1/q_*)/2-\eta}(1+t)^{-(N/4-\eta)}\log(2+t) \leqq c_2(1+t)^{-\zeta}\,.$$

In the case $\lambda>1$ or $\lambda<1$, we can show (6.21) by a direct calculation. Next, we shall consider the case $\alpha\leqq 1$. Taking $p_1=2/(\alpha+1)$ (see (6.13)), we have from (6.17) that

$$\|u(t)\|_{p_1(\alpha+1)}^{\alpha+1} = \|u(t)\|^{\alpha+1} \leqq c_2(1+t)^{-(\alpha+1)\eta}\,.$$

Then, we see that

$$I_2 \leqq c_2(1+t)^{-N((\alpha+1)/2-1/q_*)/2}\int_0^{t/2}(1+s)^{-(\alpha+1)\eta}ds\,.$$

From this we can derive (6.21) by a direct calculation, where we note that $\eta<N\alpha/4$ if $(\alpha+1)\eta=1$.

Concerning the term I_3 in (6.20), we claim

$$(6.22) \qquad I_3 \leqq c_2(1+t)^{-\zeta}(1+M(t)^{\mu}), \qquad \zeta = \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta\,,$$

for some $0\leqq\mu<1$, where $M(t)$ is defined by

$$(6.23) \qquad M(t) \equiv \sup_{0\leqq s\leqq t}\{(1+s)^{\zeta}\|u(s)\|_{q_*}\}\,.$$

First, we assume $\alpha\leqq 2/(N-2)$. Then, it follows from (5.7), (6.17) and (6.18) that

$$\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c_2(1+t)^{-N\alpha/4-(\alpha+1)\eta}\,.$$

Noting $N(1/2-1/q_*)/2<1$, we can show

$$I_3 \leqq c_2(1+t)^{-N(1/2-1/q_*)/2-\eta}(1+t)^{-(N\alpha/4-1)-\alpha\eta} \leqq c_2(1+t)^{-\zeta}\,,$$

which implies the estimate (6.22) with $\mu=0$. When $2/(N-2)<\alpha<4/(N-2)$, it follows from (6.6) and (6.18) that

$$(6.24) \qquad \|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c_2(1+t)^{-\beta_1-2\beta_1\eta}\|u(t)\|_{q_*}^{\mu_1}$$

with μ_1 defined by (6.6b). If $\mu_1<1$, then

$$I_3 \leqq c_2(1+t)^{-\rho_1}\int_{t/2}^{t}(1+t-s)^{-N(1/2-1/q_*)/2}ds M(t)^{\mu_1}$$

with

$$\rho_1 = \beta_1+2\beta_1\eta+\Big(\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta\Big)\mu_1$$

$$= \beta_1+\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)\mu_1+(2\beta_1+\mu_1)\eta = \frac{N\alpha}{4}+(\alpha+1)\eta \quad \text{(see (6.6c-d))}.$$

Since $N(1/2-1/q_*)/2<1$, we easily have

$$I_3 \leqq c_2(1+t)^{-N(1/2-1/q_*)/2-\eta}(1+t)^{-(N\alpha/4-1)-\alpha\eta}M(t)^{\mu_1} \leqq c_2(1+t)^{-\zeta}M(t)^{\mu_1},$$

which implies the estimate (6.22) with $\mu=\mu_1<1$. If $\mu_1\geqq 1$, we have from the uniform estimate (6.12) and (6.24) that

$$I_3 \leqq c_2(1+t)^{-\rho}\int_{t/2}^{t}(1+t-s)^{-N(1/2-1/q_*)/2}ds M(t)^{\mu_\varepsilon}$$

with $\mu_\varepsilon=1-\varepsilon$, $0<\varepsilon\ll 1$, and $\rho=\beta_1+2\beta_1\eta+(N(1/2-1/q_*)/2+\eta)\mu_\varepsilon$. Here, we note that

$$\rho+\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)-1 = \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta+\Big\{\beta_1+\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)-1\Big\}+2\beta_1\eta-O(\varepsilon)$$

$$> \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta = \zeta \qquad \text{(see (6.6e))}.$$

Thus, it holds again

$$I_3 \leqq c_2(1+t)^{-\zeta}M(t)^{\mu_\varepsilon}, \qquad \mu_\varepsilon = 1-\varepsilon(<1),$$

which implies the estimate (6.22) with $\mu=\mu_\varepsilon$.

Concerning the last term I_4 in (6.20), we claim

$$\text{(6.25)} \qquad I_4 \leqq c_2(1+t)^{-\zeta}(1+M(t)^{\mu}), \qquad \zeta = \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta$$

with some $0\leqq\mu<1$, where $M(t)$ is defined by (6.23). First, assume that $\alpha< 2N/(N-1)(N-2)$. Then, it follows from (6.7), (6.17) and (6.18) that

$$\|\,|u(t)|^{\alpha}D_x u(t)\|_{p_*} \leqq c_2(1+t)^{-(\beta_*+1/2)-(\alpha+1)\eta} \leqq c_2(1+t)^{-\zeta} \qquad \text{(see (6.7d))}.$$

Thus, we have

$$I_4 \leqq c_2\int_0^t (t-s)^{-\delta}e^{-\nu(t-s)}(1+s)^{-\zeta}ds \leqq c_2(1+t)^{-\zeta}$$

which implies the estimate (6.25) with $\mu=0$. When $2N/(N-1)(N-2)\leqq\alpha< 4/(N-2)$, it follows from (6.8) and (6.18) that

$$\|\,|u(t)|^{\alpha}D_x u(t)\|_{p_*} \leqq c_2(1+t)^{-\beta_2-1/2-(1+2\beta_2)\eta}\|u(t)\|_{q_*}^{\mu_2} \leqq c_2(1+t)^{-\rho_2}M(t)^{\mu_2}$$

with $0\leqq\mu_2<1$ in (6.8e) and ρ_2 defined by

$$\rho_2=\beta_2+1/2+(1+2\beta_2)\eta+\Big(\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta\Big)\mu_2$$

$$=\beta_*+1/2+\eta+\alpha\eta \quad \text{(see (6.8c-d))} > \frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta \quad \text{(see (6.7d))}.$$

Thus, it holds that

$$I_4\leqq c_2(1+t)^{-\zeta}M(t)^{\mu_2}, \qquad 0\leqq\mu_2<1,$$

which implies the estimate (6.25) with $\mu=\mu_2$.

Consequently, summing up the above estimates (6.21), (6.22) and (6.25), we obtain from (6.20) that

$$\|u(t)\|_{q_*}\leqq c_2(1+t)^{-\zeta}(1+M(t)^{\mu})$$

with some $0\leqq\mu<1$, and hence

$$M(t)\leqq c_2(1+M(t)^{\mu}).$$

Thus, we conclude that

$$M(t)\equiv\sup_{0\leq s\leq t}\{(1+s)^{\zeta}\|u(s)\|_{q_*}\}\leqq c_2<\infty, \qquad \zeta=\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{q_*}\Big)+\eta,$$

which implies of course (6.19). This completes the proof of Proposition 6.1.

Q.E.D.

On the basis of L^{q_*}-estimate (6.2), we can derive the decay rate for $\|u(t)\|$ and consequently the decay estimate for $E(t)$, which will complete the proof of Theorem 2.

COMPLETION OF THE PROOF OF THEOREM 2. It is enough to prove Theorem 2 for the cases

$$\begin{cases}\Big(\frac{4}{N}<\Big)\ \frac{2}{N-2}<\alpha<\frac{4}{N-2} & \text{if } N=3\\ \Big(\frac{2}{N-2}\leqq\Big)\ \frac{4}{N}<\alpha<\frac{4}{N-2} & \text{if } 4\leqq N\leqq 6,\end{cases}$$

and $r\neq2$, i.e., $1\leqq r<2$. First, we shall derive a sharper estimate for $\|u(t)\|$. We utilize (3.10) in Proposition 3.2 to get

$$\text{(6.26)}\qquad \|u(t)\|\leqq c_1(1+t)^{-N(1/r-1/2)/2}+c\int_0^t(1+t-s)^{-N(1/p-1/2)/2}\|u(s)\|_{p(\alpha+1)}^{\alpha+1}ds$$

$$+c\int_0^t e^{-\nu(t-s)}\|u(s)\|_{2(\alpha+1)}^{\alpha+1}ds\equiv I_1+I_2+I_3,$$

where $1\leqq p\leqq2$. The term I_3 is treated as follows. Since $2/(N-2)<\alpha<4/(N-2)$, it follows from (6.6), (5.3) and (6.2) that

$$\|u(t)\|_{2(\alpha+1)}^{\alpha+1} \leqq c\|\nabla u(t)\|^{2\beta_1}\|u(t)\|_{q*}^{\mu_1}$$
$$\leqq c_2(1+t)^{-\beta_1-N(1/2-1/q*)\mu_1/2} = c_2(1+t)^{-N\alpha/4} \quad \text{(see (6.6d))}.$$

Thus, we have

$$(6.27) \qquad I_3 \leqq c_2(1+t)^{-N\alpha/4} \leqq c_2(1+t)^{-\eta}, \qquad \eta = \min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\}.$$

Next, we shall estimate the term I_2. When $\alpha>1$, we take $p=1$ in (6.26). Then, it follows from (5.6) and (5.4) that

$$\|u(t)\|_{\alpha+1}^{\alpha+1} \leqq c\|\nabla u(t)\|^{N(\alpha-1)/2}\|u(t)\|^{\mu_0} \leqq c_1(1+t)^{-N(\alpha-1)/4}\|u(t)\|^{\mu_0}$$

with $\mu_0=\alpha+1-N(\alpha-1)/2$. Thus, we have

$$(6.28) \qquad I_2 \leqq c_1\int_0^t(1+t-s)^{-N/4}(1+s)^{-N(\alpha-1)/4}\|u(s)\|^{\mu_0}ds\,.$$

Hence, we have from (6.26), (6.27) and (6.28) that

$$(6.29) \qquad \|u(t)\| \leqq c_2(1+t)^{-\eta}+c_1\int_0^t(1+t-s)^{-N/4}(1+s)^{-N(\alpha-1)/4}\|u(s)\|^{\mu_0}ds.$$

Here, we note that

$$N(\alpha-1)/4 > 0, \quad N/4+N(\alpha-1)/4 = N\alpha/4 > 1 \quad \text{and} \quad N/4 \geqq \frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right) \geqq \eta.$$

Also, we note that if $\mu_0\geqq 1$, we may replace $\mu_0=1$ (see (5.3)), while in the case $\mu_0<1$, we have

$$\frac{N(\alpha-1)/4}{1-\mu_0} > \frac{N\alpha/4-1}{1-\mu_0} > \eta\,,$$

because $\mu_0<1$ corresponds to $\alpha>N/(N-2)$ and this applies only to the case $N=3$ and $3<\alpha<4$. Thus, applying Lemma 2.3 (see Remark 2.1) to (6.29) we obtain

$$(6.30) \qquad \|u(t)\| \leqq c_2(1+t)^{-\eta}\,.$$

When $\alpha\leqq 1$, we take $1/p=(\alpha+1)/2-2\varepsilon_0/N$ in (6.26), $0<\varepsilon_0\ll 1$. Then, we see

$$\frac{N}{2}\left(\frac{1}{p}-\frac{1}{2}\right) = \frac{N\alpha}{4}-\varepsilon_0 \quad \text{and} \quad 2 < p(\alpha+1) = 2+O(\varepsilon_0) < \frac{2N}{N-2}\,.$$

Here, we have $H^1\subset L^{p(\alpha+1)}$, and

$$\|u(t)\|_{p(\alpha+1)}^{\alpha+1} \leqq c\|\nabla u(t)\|^{(\alpha+1)\theta}\|u(t)\|^{(\alpha+1)(1-\theta)}, \qquad \theta = N\left(\frac{1}{2}-\frac{1}{p(\alpha+1)}\right),$$

where we note that

$$(\alpha+1)\theta = N\left(\frac{\alpha+1}{2}-\frac{1}{p}\right) = 2\varepsilon_0 \quad \text{and} \quad (\alpha+1)(1-\theta) = \alpha+1-2\varepsilon_0 \geqq 1\,.$$

It follows from (5.3) and (5.4) that

$$\|u(t)\|_{p(\alpha+1)}^{\alpha+1} \leqq c_1(1+t)^{-\varepsilon_0}\|u(t)\|^{\alpha+1-2\varepsilon_0} \leqq c_1(1+t)^{-\varepsilon_0}\|u(t)\| .$$

Thus, we have from (6.26) and (6.27) that

$$\|u(t)\| \leqq c_2(1+t)^{-\eta}+c_1\int_0^t(1+t-s)^{-(N\alpha/4-\varepsilon_0)}(1+s)^{-\varepsilon_0}\|u(s)\|ds .$$

We apply Lemma 2.3 (see Remark 2.1) to obtain

(6.31) $\|u(t)\| \leqq c_2(1+t)^{-\eta_1}, \qquad \eta_1 = \min\{\eta, N\alpha/4-\varepsilon_0\}(>0$ by $r\neq 2)$.

Further, we shall improve L^2-decay estimate (6.31). For this, we take $p=2/(\alpha+1)$ in I_2. Then, we see by (6.31)

(6.32) $$\|u(t)\|_{p(\alpha+1)}^{\alpha+1} \leqq c_2(1+t)^{-\alpha\eta_1}\|u(t)\| .$$

Thus, we have from (6.26), (6.27) and (6.32) that

$$\|u(t)\| \leqq c_2(1+t)^{-\eta}+c_2\int_0^t(1+t-s)^{-N\alpha/4}(1+s)^{-\alpha\eta_1}\|u(s)\|ds ,$$

with $\alpha\eta_1>0$ and $N\alpha/4+\alpha\eta_1-1>0$. Applying Lemma 2.3 (see Remark 2.1), we obtain

(6.33) $$\|u(t)\| \leqq c_2(1+t)^{-\eta_2}, \qquad \eta_2 = \min\left\{\eta, \frac{N\alpha}{4}\right\} = \eta .$$

From (6.30) and (6.33) we conclude

$$\|u(t)\| \leqq c_2(1+t)^{-\eta}, \qquad \eta = \min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right), \frac{N\alpha}{4}\right\} .$$

Moreover, applying Proposition 4.2 with $f(t)\equiv 0$ and $a=2\eta$ to (5.1), we obtain

$$E(t) \leqq c_2(1+t)^{-1-2\eta}, \quad \text{and} \quad \|Du(t)\| \leqq c_2(1+t)^{-1/2-\eta} .$$

The proof of Theorem 2 is now finished. Q.E.D.

Finally in this section, we shall state an improved result of Proposition 6.1 for the case $\alpha\geqq 4/N$, which is as follows.

PROPOSITION 6.2. *Let* $1\leqq N\leqq 6$ *and* α *satisfy the following*

(6.34a) $$4/N \leqq \alpha < 4/(N-2) \qquad (4/N\leqq\alpha<\infty \text{ if } N=1, 2).$$

Then, it holds that

(6.34b) $$\|u(t)\|_q \leqq c_2(1+t)^{-N(1/2-1/q)/2-\eta}, \qquad 2\leqq q\leqq q_* ,$$

where q_* *is defined by*

$$q_* = \begin{cases} \infty & \text{for } 1\leqq N\leqq 3 \\ \dfrac{2N(N-1)}{(N-2)(N-3)}+\varepsilon & \text{for } 4\leqq N\leqq 6 , \end{cases}$$

($0<\varepsilon\ll 1$, *see* (6.1)), *and* η *is defined by* (1.7b), *i.e.*,

$$\eta=\min\left\{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right),\ \frac{N\alpha}{4}\right\}\qquad (\eta=0\ \textit{if}\ \alpha=N/4). \tag{6.34c}$$

REMARK 6.2. When $0<\alpha\leqq 4/N$ we can show

$$\|u(t)\|_q\leqq c_1(1+t)^{-N(1/2-1/q)/2}, \tag{6.34d}$$

provided that

$$\begin{cases} 2\leqq q\leqq\infty & \text{if } N=1\\ 2\leqq q<\infty & \text{if } N=2\\ 2\leqq q\leqq 2N/(N-2) & \text{if } N\geqq 3\,. \end{cases} \tag{6.34e}$$

PROOF OF PROPOSITION 6.2. When $3\leqq N\leqq 6$, we have proved Theorem 2, so that (6.17) and (6.18) hold true with η given by (6.34c). Consequently, we have (6.19) with the same η. A simple interpolation, using (6.17) and (6.19), then gives (6.34b).

When $N=1$, (6.34b) follows easily from (1.5) by using the Gagliardo-Nirenberg inequality. The same argument also yields (6.34b) for $N=2$ and $2\leqq q<\infty$. On the other hand, in order to prove (6.34b) for $N=2$ and $q=\infty$, we must carry out a similar computation employed in the proof of Proposition 6.1. In fact, we apply (3.11) with $q=\infty$, $p=1$ and $k=1$ in Proposition 3.2. Since $(N+1)\cdot(1/p-1/2)=3/2$ and $k+1=2$ in this case, we can take $\gamma=1/2$ in (3.11) to get

$$\|u(t)\|_\infty\leqq c_2(1+t)^{-1/r}+c\int_0^t(1+t-s)^{-1/p_1}\|u(s)\|_{p_1(\alpha+1)}^{\alpha+1}ds$$
$$+c\int_0^t(t-s)^{-1/2}e^{-\nu(t-s)}\|\,|u(s)|^\alpha D_x u(s)\|_1 ds\,.$$

Then, repeating a similar argument as in the proof of Proposition 6.1, we get the desired conclusion, the details being omitted.

Finally we note that (3.34d) in Remark 6.2 follows easily from (1.4) by using the Gagliardo-Nirenberg inequality.

§7. Proof of Theorem 3.

On the basis of the estimates (6.34) in Proposition 6.2 (see also Remark 6.2), i.e.,

$$\|u(t)\|_q\leqq c_2(1+t)^{-N(1/2-1/q)/2-\eta}\,,\qquad 2\leqq q\leqq q_*\,,$$

where q_* is given by (6.1), we shall give the estimate for $\|D^2u(t)\|^2$. Setting $U(x,t)=Du(x,t)$, $U(x,t)$ satisfies

$$U_{tt}-\Delta U+U_t=-Df(u)\,.$$

Applying Proposition 4.1, we have

$$D(t)^2\equiv E_2(t)-E_2(t+1)+\delta(t)^2\geqq 0$$

and

(7.1a) $$\sup_{t\leqq s\leqq t+1} E_2(s)\leqq c_2\{D(t)^2+(D(t)+\delta(t))\sup_{t\leqq s\leqq t+1}\|U(s)\|+\delta(t)^2\}\,,$$

where we set

(7.1b) $$E_2(t)\equiv\|U_t(t)\|^2+\|\nabla U(t)\|^2 \quad\text{and}\quad \delta(t)^2\equiv\int_t^{t+1}\|Df(u(s))\|^2ds\,.$$

Now, by Hyp. (ii) we see

(7.2) $$\|Df(u(t))\|^2\leqq c\|\,|u(t)|^{\alpha}Du(t)\|^2\equiv I^2\,.$$

In order to apply Proposition 4.2 with $a=2\theta_1=1+2\eta$ to (7.1), we shall estimate the term I^2 under the condition $4/N\leqq\alpha<4/(N-2)$. When $1\leqq N\leqq 3$, we see from (6.34) with $q=q_*=\infty$ that

(7.3a) $$I^2\leqq c\|u(t)\|_\infty^{2\alpha}\|Du(t)\|^2\leqq c_2(1+t)^{-b}$$

with $b=2\alpha(N/4+\eta)+2\theta_1=2+2\theta_1+2\kappa$, where κ is defined by

(7.3b) $$\kappa\equiv N\alpha/4-1+\alpha\eta \qquad (\geqq 0 \text{ if } \alpha\geqq 4/N)\,.$$

Here, we note that for $a=2\theta_1$

(7.3c) $$\min\left\{1+a,\ b,\ \frac{a+b}{2}\right\}=1+2\theta_1\,.$$

When $N=4$, 5, we see

(7.4a) $$I^2\leqq c\|u(t)\|_{N\alpha}^{2\alpha}\|Du(t)\|_{2N/(N-2)}^2\leqq c\|u(t)\|_{N\alpha}^{2\alpha}\|\nabla Du(t)\|^2\,.$$

Since $4/N\leqq\alpha<4/(N-2)$, we see

(7.4b) $$2<N\alpha<q_*\equiv\frac{2N(N-1)}{(N-2)(N-3)}+\varepsilon \qquad \text{if } N=4,\,5\,,$$

and we can use (6.34) with $q=N\alpha$ to get from the above

(7.4c) $$I^2\leqq c_2(1+t)^{-d}E_2(t), \qquad d=2\alpha\Big(\frac{N}{2}\Big(\frac{1}{2}-\frac{1}{N\alpha}\Big)+\eta\Big)=1+2\kappa\,,$$

where $\kappa\geqq 0$ is defined by (7.3b). Here, we note that for $a=2\theta_1$

(7.4d) $$\min\{1+a,\ a+d\}=1+2\theta_1\,.$$

Any way, we can apply Proposition 4.2 to (7.1) to obtain

$$E_2(t)\equiv\|U_t(t)\|^2+\|\nabla U(t)\|^2\leqq c_2(1+t)^{-2\theta_2}\,,$$

with $2\theta_2=1+2\theta_1=2+2\eta$ (see (7.3), (7.4)). The proof of Theorem 3 is now com plete.

Finally we briefly discuss the case treated in Remark 1.3. Let $1\leqq N\leqq 4$ and $0<\alpha\leqq 4/N$, or let $N\geqq 5$ and $0<\alpha\leqq 2/(N-2)$. Then we have the following estimates for I^2: When $N=1$,

$$I^2\leqq c_1(1+t)^{-b}$$

with $b=1+\alpha/2$, and when $N\geqq 2$,

$$I^2\leqq c_1(1+t)^{-c}E_2(t)^{\mu},$$

where $c=\alpha+1-4\alpha/q$ and $\mu=2\alpha/q$ (q being sufficiently large) if $N=2$, while $c=\alpha+1-\mu$ and $\mu=(N-2)\alpha/2$ if $N\geqq 3$. The assertion of Remark 1.3 can be proved by using these estimates. The details are omitted. Q.E.D.

§8. Proof of Theorem 4.

We make the hypotheses $(u_0, u_1)\in H^3\cap L^r\times H^2\cap L^r$, $1\leqq r\leqq 2$, and Hyp. (i)-(iii). Setting $D^2u(x, t)=U(x, t)$, $U(x, t)$ satisfies

$$U_{tt}-\Delta U+U_t=-D^2f(u).$$

$U(x, t)=D^2u(x, t)$ can be regarded as $(N+1)\times(N+1)$ matrix valued function and it is clear that Propositions 4.1 and 4.2 are applicable to this $U(x, t)$ with

$$E_3(t)\equiv\|U_t(t)\|^2+\|\nabla U(t)\|^2. \tag{8.1}$$

Now, by Hyp. (ii)-(iii) we see

$$\|D^2f(u)\|^2\leqq c\{\||u|^{[\alpha-1]^+}|Du|^2\|^2+\||u|^{\alpha}D^2u\|^2\}\equiv I_1^2+I_2^2. \tag{8.2}$$

In order to apply Proposition 4.2 with $a=2\theta_2=2+2\eta$ to (8.1), we shall estimate the right hand side of (8.2). Since the term I_2^2 is treated by the same way as in the proof of Theorem 3 (see (7.2)), we have only to estimate the first term I_1^2. When $1\leqq N\leqq 3$, we know $H^2\subset L^\infty$ and hence Theorem 3 (see also Remark 1.3) implies $\|u(t)\|_\infty\leqq c_2<\infty$. Thus, under our hypothesis Hyp. (i)-(iii), we know

$$|f^{(i)}(u(t))|\leqq c_2(\|u(t)\|_\infty)|u(t)|^{2-i},\qquad i=0,\,1,\,2.$$

Noting this fact and taking $q=q_*=\infty$ in (6.34b) we have, if $1\leqq N\leqq 3$,

$$\begin{aligned} I_1^2 &\leqq c\|u(t)\|_\infty^{2[\alpha-1]^+}\|Du(t)\|_4^4 \\ &\leqq c\|u(t)\|_\infty^{2[\alpha-1]^+}\|Du(t)\|^{4(1-N/4)}\|\nabla Du(t)\|^N\leqq c_2(1+t)^{-b} \end{aligned} \tag{8.3a}$$

with $b=2(\alpha-1)(N/4+\eta)+4(1-N/4)\theta_1+N\theta_2=2+2\theta_2+2\kappa$ ($\kappa\geqq 0$ by (7.3b)). Here, we note that for $a=2\theta_2$

$$\min\left\{1+a,\ b,\ \frac{a+b}{2}\right\}=1+2\theta_2\,. \tag{8.3b}$$

When $N=4$, 5, we have

$$I_1^2=c\|\,|u|^{[\alpha-1]^+}|Du|^2\|^2\leqq c\|u\|_{2N/(N-2)}^{[\alpha-1]^+}\|Du\|_p^4,\quad \frac{2}{p}=\frac{1}{2}-\frac{(N-2)[\alpha-1]^+}{2N}, \tag{8.4a}$$
$$\leqq c\|\nabla u\|^{[\alpha-1]^+}\|Du\|_{2N/(N-2)}^{4(1-\theta)}\|\Delta Du\|^{4\theta},\quad \theta=\left(\frac{N-2}{2N}-\frac{1}{p}\right)\left(\frac{2}{N}+\frac{N-2}{2N}-\frac{1}{2}\right)^{-1},$$
$$\leqq c_2(1+t)^{-c}E_3(t)^{\mu}$$

with $c=[\alpha-1]^+\theta_1+4(1-\theta)\theta_2$ and $\mu=2\theta=(N-4)/2+(N-2)[\alpha-1]^+/4N$. Here, we note that $0\leqq\mu=2\theta<1$ if $N=4$, 5 and $4/N\leqq\alpha\leqq 4/(N-2)$, and for $a=2\theta_2$

$$\frac{c}{1-\mu}\geqq\frac{a+c}{2-\mu}=\frac{[\alpha-1]^++2+2\eta}{2-2\theta}+2\theta_2\geqq 1+2\theta_2\,,$$

and hence

$$\min\left\{1+a,\ \frac{c}{1-\mu},\ \frac{a+c}{2-\mu}\right\}=1+2\theta_2 \tag{8.4b}$$

(if $N=4$ and $\alpha=1$ then $\mu=0$). Applying Proposition 4.2 to (8.1), we obtain for $\alpha\geqq 4/N$ that

$$E_3(t)\equiv\|U_t(t)\|^2+\|\nabla U(t)\|^2\leqq c_3(1+t)^{-2\theta_3}$$

with $2\theta_3=1+2\theta_2=3+2\theta_1$ (see (7.3), (7.4), (8.3) and (8.4)).

Finally, we note that the proof of Remark 1.4 is given easily by the proofs of Theorems 3 and 4 combined with Remark 1.3. The proof of Theorem 4 is now complete.

§ 9. Proof of Theorem 5.

Let us proceed to the proof of Theorem 5. By Theorem 4 and Remark 1.4, we know already $\|u(t)\|_\infty\leqq c_3<\infty$ for $0<\alpha<4/(N-2)$ ($0<\alpha<\infty$ if $N=1$, 2). Since $f(u)$ is assumed further to be m-times continuously differentiable with $m\geqq 3$, it holds from Hyp. (i)-(ii) that

$$|f^{(i)}(u(t))|\leqq c_3(\|u(t)\|_\infty)|u(t)|^{3-i},\qquad i=0,1,2,3 \text{ (cf. Remark 1.5)}. \tag{9.1}$$

(Cf. Consider the Taylor expansion of $f(u)$ at $u=0$.) Hence, in this situation we can choose $\alpha>\max\{4/N, 1\}$ if $3\leqq N\leqq 5$ and $\alpha\geqq 4/N=2$ if $N=2$ in all the previous arguments. In particular, we can always take

$$\eta=\frac{N}{2}\left(\frac{1}{r}-\frac{1}{2}\right)\qquad \text{if } 3\leqq N\leqq 5\,.$$

Setting $U(x,t)=D^l u(x,t)$, $U(x,t)$ satisfies

$$U_{tt}-\Delta U+U_t = -D^l f(u), \qquad l=1, 2, \cdots, m.$$

It is easy to see that Propositions 4.1 and 4.2 are applicable to this $U(t)$ with

$$E_{l+1}(t) \equiv \|U_t(t)\|^2+\|\nabla U(t)\|^2. \tag{9.2}$$

Note that

$$D^l f(u) = \sum_{j=1}^{l} f^{(j)}(u) \sum_{\sigma\in S_l} c_{\sigma,j}(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k},$$

where $c_{j,k}$ is a certain constant and

$$S_l = \left\{\sigma=(\sigma_1, \cdots, \sigma_k)\in N^k \,\middle|\, \sum_{i=1}^{k}\sigma_i r_i=l,\ \sum_{i=1}^{k} r_i=j\right\}.$$

We shall prove Theorem 5 by induction. Our estimate is valid for $l=1, 2, 3$ by Theorems 1-4, and we assume that it is valid for less than l $(3\leqq l\leqq m)$.

$$\|D^j u(t)\| \leqq c_m(1+t)^{-\theta_j}, \qquad \theta_j=\frac{j}{2}+\eta, \quad 1\leqq j\leqq l. \tag{9.3}$$

We must show that (9.3) is valid for $j=l+1$. It will be sufficient to consider the case $l=m$, the other cases being treated similarly. First, we note from Theorem 4 that

$$\|u(t)\|_\infty \leqq c\|u(t)\|^{1-N/6}\|D_x^3 u(t)\|^{N/6} \leqq c_3(1+t)^{-A}, \qquad A=N/4+\eta. \tag{9.4}$$

In order to apply Proposition 4.2 with $a=2\theta_m=m+2\eta$ to (9.2), we shall estimate the L^2-norm of $D^m f(u)$. For $3\leqq j\leqq m$, we see

$$\begin{aligned}\|f^{(j)}(u)\sum_\sigma(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2 &\leqq c_3\sum_\sigma\prod_{i=1}^{k}\|D^{\sigma_i}u\|_{2p_ir_i}^{2r_i}\\ &\leqq c_3\sum_\sigma\prod_{i=1}^{k}\|D^{\sigma_i}u\|^{2r_i(1-\xi_i)}\|D^m u\|^{2r_i\xi_i} \leqq c_m(1+t)^{-b},\end{aligned} \tag{9.5a}$$

where $0\leqq\xi_i\leqq 1$ and $1\leqq p_i<\infty$ should be chosen as

$$\xi_i = \frac{1}{2}\Big(1-\frac{1}{p_ir_i}\Big)\frac{N}{m-\sigma_i} \leqq 1 \quad\text{and}\quad \sum_{i=1}^{k}\frac{1}{p_i}=1. \tag{9.5b}$$

Such a choice of a set of $\{\xi_i\}$ and $\{p_i\}$ is possible since, setting

$$\frac{1}{2}\Big(1-\frac{1}{p_i^* r_i}\Big)\frac{N}{m-\sigma_i}=1, \quad \text{i.e.,} \quad p_i^* = \frac{1}{r_i}\frac{N}{[N-2(m-\sigma_i)]^+},$$

we see

$$\sum_{i=1}^{k}\frac{1}{p_i^*} = \sum_{i=1}^{k} r_i\Big[1-\frac{2(m-\sigma_i)}{N}\Big]^+ < 1,$$

details being omitted. Here, b in (9.5a) is given by

$$b = \min_{3\le j\le m} \sum_{i=1}^{k} \{2r_i(1-\xi_i)\theta_{\sigma_i} + 2r_i\xi_i\theta_m\}$$

$$= \min_{3\le j\le m} \sum_{i=1}^{k} \{r_i\xi_i(m-\sigma_i) + \sigma_i r_i + 2r_i\eta\} \qquad \text{(see (9.3))}$$

$$= \min_{3\le j\le m} \{N(j-1)/2 + m + 2j\eta\} \qquad \text{(see (9.5b))}$$

$$= 2 + 2\theta_m + (N-2+4\eta)\,.$$

Quite similarly, we see

$$\|f''(u)D^{\sigma_1}uD^{\sigma_2}u\|^2 \le c_3\|u(t)\|_\infty^2\|D^{\sigma_1}uD^{\sigma_2}u\|^2 \le c_m(1+t)^{-b}$$

and

$$\|f'(u)D^m u(t)\|^2 \le c_3\|u(t)\|_\infty^4\|D^m u(t)\|^2 \le c_m(1+t)^{-b}\,.$$

We observe that for $a=2\theta_m$

$$\min\left\{1+a,\ b,\ \frac{a+b}{2}\right\} = 1+2\theta_m\,. \tag{9.6}$$

Thus, applying Proposition 4.2 to (9.2), we have from (9.6) that

$$E_{m+1}(t) \equiv \|U_t(t)\|^2 + \|\nabla U(t)\|^2 \le c_{m+1}(1+t)^{-2\theta_{m+1}}$$

with $2\theta_{m+1}=1+2\theta_m=1+m+2\eta$, which completes the proof of Theorem 5 for $2\le N\le 5$.

When $N=1$, we make the following induction assumption

$$\|D^j u(t)\| \le c_m(1+t)^{-\theta_j} \tag{9.7a}$$

with

$$\theta_j = \frac{1}{2} + (j-1)\omega + \eta, \qquad 1\le j\le m, \quad \omega = \min\left\{\frac{1}{2}, \frac{\alpha}{8}\right\}, \tag{9.7b}$$

where η is given by (1.7b). First, we note from Proposition 6.2 (see Remark 6.2) that

$$\|u(t)\|_\infty \le c_1(1+t)^{-A}\,, \qquad A=4/N+\eta\,, \tag{9.8}$$

and from the assumption Hyp. (i)-(iii) and (1.10b) that

$$f^{(i)}(u) \le k_i|u|^{[\alpha+1-i]^+}\,, \qquad i=0,\cdots,4 \tag{9.9}$$

with $\alpha\ge 2$ (see (9.1)). For $5\le j\le m$, we see

$$\|f^{(j)}(u)\sum_\sigma (D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2$$

$$\le c_3\|\sum_\sigma (D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2 \le c_m(1+t)^{-b_1}$$

with $b_1=4\omega+2\theta_m+2\kappa_1$, $\kappa_1=2+4\eta-4\omega$, where ω is given in (9.7b). Further, we see

$$\|f^{(4)}(u)\sum_{\sigma}(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2$$

$$\leqq c_3\|u\|_\infty^{2[\alpha-3]^+}\|\sum_{\sigma}(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2\leqq c_m(1+t)^{-b_2}$$

with $b_2=4\omega+2\theta_m+2\kappa_2$, $\kappa_2=3/2+[\alpha-3]^+(1/4+\eta)+3\eta-7\omega/2$ (see (9.1), (9.8) and (9.9)), and

$$\|f^{(3)}(u)\sum_{\sigma}(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2$$

$$\leqq c_3\|u\|_\infty^{2[\alpha-2]^+}\|\sum_{\sigma}(D^{\sigma_1}u)^{r_1}\cdots(D^{\sigma_k}u)^{r_k}\|^2\leqq c_m(1+t)^{-b_3}$$

with $b_3=4\omega+2\theta_m+2\kappa_3$, $\kappa_3=(2+\alpha)/4+\alpha\eta-3\omega$, and

$$\|f''(u)D^{\sigma_1}uD^{\sigma_2}u\|^2\leqq c_3\|u\|_\infty^{2[\alpha-1]^+}\|D^{\sigma_1}uD^{\sigma_2}u\|^2\leqq c_m(1+t)^{-b_4}$$

with $b_4=4\omega+2\theta_m+2\kappa_4$, $\kappa_4=(1+\alpha)/4+\alpha\eta-5\omega/2$. Finally, we see also

$$\|f'(u)D^mu\|^2\leqq c_3\|u\|_\infty^{2\alpha}\|D^mu\|^2\leqq c_m(1+t)^{-b_5}$$

with $b_5=4\omega+2\theta_m+2\kappa_5$, $\kappa_5=\alpha/4+\alpha\eta-2\omega$. By the definition of ω in (9.7b) we see $\kappa_i\geqq 0$, $i=1,\cdots,5$, and observe for $a=2\theta_m$

$$\min\left\{1+a,\ b_i,\ \frac{a+b_i}{2}\right\}=\min\{1+2\theta_m,\ 2\omega+2\theta_m+\kappa_i\}\geqq 2\omega+2\theta_m. \tag{9.10}$$

Thus, applying Proposition 4.2 to (9.2), we obtain from (9.10) that

$$E_{m+1}(t)\equiv\|U_t(t)\|^2+\|\nabla U(t)\|^2\leqq c_{m+1}(1+t)^{-2\theta_{m+1}}$$

with $2\theta_{m+1}=2\omega+2\theta_m=1+2m\omega+2\eta$. The proof of Theorem 5 is now finished.

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