J. Math. Soc. Japan
Vol. 46, No. 1, 1994

A Poincaré-Birkhoff-Witt theorem for infinite dimensional Lie algebras

Dedicated to Professor Tsunero Takahashi on his sixtieth birthday

By Hideki OMORI*), Yoshiaki MAEDA**) and Akira YOSHIOKA

(Received June 23, 1992)

§ 0. Introduction.

Let $(1<)\lambda_1 \leqq \cdots \leqq \lambda_n \leqq \cdots$ be a series of positive real numbers such that

$$\sum_{n\geqq 1} \lambda_n^{-s_0} < \infty \qquad \text{for some integer } s_0 .$$

For each $n \in \boldsymbol{N}$, formally consider e_n to be an eigenvector corresponding to the eigenvalue λ_n. Define for any $s \in \boldsymbol{Z}$

$$\mathfrak{g}^s = \{p = \sum_{n\in \boldsymbol{N}} a_n e_n ;\ a_n \in \boldsymbol{C},\ \sum_{n\in \boldsymbol{N}} |a_n|^2 \lambda_n^{2s} < \infty\}.$$

$\mathfrak{g}^s$ is a Hilbert space for every $s \in \boldsymbol{Z}$ with the norm $\|p\|_s^2 = \sum_{n\in \boldsymbol{N}} |a_n|^2 \lambda_n^{2s}$. The inclusion mapping $\iota: \mathfrak{g}^s \to \mathfrak{g}^{s-1}$ is a compact operator for every $s \in \boldsymbol{Z}$. Set $\mathfrak{g} = \bigcap_s \mathfrak{g}^s$. $\{\mathfrak{g}, \mathfrak{g}^s ; s \in \boldsymbol{Z}\}$ will be called a *Sobolev chain*. Set $\mathfrak{g}^* = \bigcup_s \mathfrak{g}^s$. As $\mathfrak{g}^{-s}$ is the dual space of $\mathfrak{g}^s$, $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$.

We denote by $C^\infty(\mathfrak{g}^s)$ the commutative algebra of all C^∞ functions on $\mathfrak{g}^s$. Since $C^\infty(\mathfrak{g}^{s-1}) \subset C^\infty(\mathfrak{g}^s)$, we set $C^\infty(\mathfrak{g}^*) = \bigcap_s C^\infty(\mathfrak{g}^s)$. Any $u \in \mathfrak{g}$, regarded as a linear function on $\mathfrak{g}^*$, is an element of $C^\infty(\mathfrak{g}^*)$. Let $(\hat{\circledast}\mathfrak{g}^s)^m$ be the Banach space of all continuous symmetric m-linear mappings of $\mathfrak{g}^{-s} \times \cdots \times \mathfrak{g}^{-s}$ into $\boldsymbol{C}$ with the natural operator norm, $\|\ \|_{-s}$ and set $(\circledast\mathfrak{g})^m = \bigcap_s (\hat{\circledast}\mathfrak{g}^s)^m$ with the projective limit topology. Hence, any element of $(\circledast\mathfrak{g})^m$ can be naturally viewed as an element of $C^\infty(\mathfrak{g}^*)$ as a homogeneous polynomial of degree m. Thus, we define a *polynomial* of degree m as an element of $\sum_{k=0}^m \oplus (\circledast\mathfrak{g})^k$, where we set $(\circledast\mathfrak{g})^0 = \boldsymbol{C}$. Denote by $\mathcal{P}(\mathfrak{g}^*)$ the space of all polynomials on $\mathfrak{g}^*$.

We define the C^∞-topology on $C^\infty(\mathfrak{g}^*)$, i. e. the C^∞ uniform topology on each compact subset: a basis of neighborhoods of 0 is given by the family $\{N(K, m, s, \varepsilon)\}$ for compact subsets $K \subset \mathfrak{g}^*$, non-negative integers m, integers s and $\varepsilon > 0$, where

$$N(K, m, s, \varepsilon) = \{f \in C^\infty(\mathfrak{g}^*) ;\ \|(d^k f)(p)_-\|_s < \varepsilon, \text{ for } \forall p \in K,\ 0 \leqq \forall k \leqq m\},$$

where $(d^k f)(p)$ is the k-differential of f regarded as an element of $(\circledast\mathfrak{g})^k$.

In the following, we denote $C^\infty(\mathfrak{g}^*)$ with the C^∞ topology by $\mathfrak{a}$ for simplicity.

---

*) This research was partially supported by Grant-in-Aid for Scientific Research (No. 2505-05452012), Ministry of Education, Science and Culture.

**) This research was partially supported by Grant-in-Aid for Scientific Research (No. 011-1081052), Ministry of Education, Science and Culture.

$\mathfrak{a}$ is a topological algebra over $\boldsymbol{C}$.

We are now interested in "deforming" $\mathfrak{a}$ to a noncommutative but associative algebra.

Introducing a formal parameter ν, we consider the direct product

$$\mathfrak{a}[[\nu]] = \prod_{n=0}^{\infty} \nu^n \mathfrak{a}$$

with the direct product topology. We want to define a continuous product $*$ on $\mathfrak{a}[[\nu]]$ with the following properties:

(A.1) $*: \mathfrak{a}[[\nu]] \times \mathfrak{a}[[\nu]] \to \mathfrak{a}[[\nu]]$ is an associative product.

(A.2) ν commutes with any element of $\mathfrak{a}[[\nu]]$ and $1 * \tilde{f} = \tilde{f} * 1 = \tilde{f}$ for any $\tilde{f} \in \mathfrak{a}[[\nu]]$.

For a product $*$ on $\mathfrak{a}[[\nu]]$ with (A.1~2), we set for any $f, g \in \mathfrak{a}$,

$$f * g = \sum_{m=0}^{\infty} \nu^m \pi_m(f, g), \qquad \pi_m(f, g) \in \mathfrak{a}.$$

By (A.1~2), we see for any $f, g, h \in \mathfrak{a}$,

$$(0.1) \quad \begin{cases} (\square_m) \quad \sum_{k+l=m} \pi_k(\pi_l(f, g), h) = \sum_{k+l=m} \pi_k(f, \pi_l(g, h)), \quad \forall m \geqq 0, \\ \qquad \pi_0(f, 1) = \pi_0(1, f) = f, \ \pi_m(f, 1) = \pi_m(1, f) = 0, \quad \forall m > 0. \end{cases}$$

A continuous m-linear mapping $\pi: \mathfrak{a} \times \cdots \times \mathfrak{a} \to \mathfrak{a}$ is called an *m-differential operator* of order k, if at any $p \in \mathfrak{g}^*$, $\pi(f_1, \cdots, f_m)(p) = 0$ holds whenever $(f_1, \cdots, f_m)$ satisfies $(d^{k+1}(f_1 f_2 \cdots f_m))(p) = 0$.

Now suppose $\mathfrak{g}$ is a topological Lie algebra with Lie bracket $[\ ,\ ]'$. For any $f, g \in \mathfrak{a}$, $df(p)$, $dg(p)$ are elements of $\mathfrak{g}^{**} = \mathfrak{g}$ for any $p \in \mathfrak{g}^*$, and $(df)_*: \mathfrak{g}^* \to \mathfrak{g}$ is a C^∞ mapping, i.e. $df: \mathfrak{g}^{-s} \to \mathfrak{g}^t$ is C^∞ for any s, t. Thus, we may define $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ by

$$\{f, g\}(p) = [df(p), dg(p)]'(p).$$

It is obvious that $(\mathfrak{a}, \{\ ,\ \})$ is a Poisson algebra.

DEFINITION 1. $(\mathfrak{a}[[\nu]], *)$ is called a *deformation quantization of* $\mathfrak{a}$ if $*$ satisfies (A.1~2) and the following (A.3~4):

(A.3) $\pi_0(f, g) = fg$ (the usual product) and $\pi_1(f, g) = -(1/2)\{f, g\}$ for any $f, g \in \mathfrak{a}$.

(A.4) π_m is a bidifferential operator of order $2m$ and $\pi_m(f, g) = (-1)^m \pi_m(g, f)$.

Our main theorem of this paper is as follows:

THEOREM A. *There exists a deformation quantization* $(\mathfrak{a}[[\nu]], *)$ *of* $\mathfrak{a}$ *such that* $\pi_m(\mathfrak{g}, \mathfrak{g}) = 0$ *for any* $m \geqq 2$. *Moreover,* $\mathcal{P}(\mathfrak{g}^*)[[\nu]]$ *is a subalgebra of* $(\mathfrak{a}[[\nu]], *)$.

Thus, the quantized algebra $(\mathfrak{a}[[\nu]], *)$ naturally contains the universal enveloping algebra of the Lie algebra $\mathfrak{g}_\nu$ i.e. the Lie algebra generated by $\mathfrak{g}$ and ν with the relations $[X, Y] = \nu[X, Y]'$.

For any $k \in \boldsymbol{N}$, let x_k be the linear function on $\mathfrak{g}^*$ defined by $x_k(p)=\langle e_k, p\rangle_0$. $x_1, \cdots, x_k, \cdots$ are elements of $\boldsymbol{C}^\infty(\mathfrak{g}^*)$.

In the quantized algebra $(\mathfrak{a}[[\nu]], *)$, we have

$$x_i * x_j = x_i x_j + \frac{1}{2}\nu[x_i, x_j]', \quad \text{so} \quad x_i * x_j - x_j * x_i = \nu[x_i, x_j]'.$$

Hence, the above theorem extends the Poincaré-Birkhoff-Witt theorem for finite dimensional Lie algebras.

The method of proof of our main theorem is as follows: suppose we have $\{\pi_0, \pi_1, \cdots, \pi_{m-1}\}$ satisfying $(\square_s)$ in (0.1) for $0 \leqq s \leqq m-1$. Our problem is to construct π_m such that $(\square_s)$ is satisfied for $s=m$.

For multi-indices $\alpha=(\alpha_1, \cdots, \alpha_k, \cdots)$, we set $|\alpha|=\sum \alpha_k$. For α with $|\alpha|<\infty$, we set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \cdots$. We shall first construct $\pi_m(x^\alpha, x^\beta)$ for monomials x^α, x^β, and then applying Taylor's formula. To show key properties of π_m, we use the following polynomial approximation theorem:

THEOREM B. *The space of all polynomials is dense in $C^\infty(\mathfrak{g}^*)$ in the C^∞ topology.*

The condition $\lim_{n\to\infty} \lambda_n = \infty$ is essentially used in this theorem.

Note that the assumption $\sum_{n\geqq 1} \lambda_n^{-s_0} < \infty$, for some integer s_0, is crucial for Theorem A. In fact, for a separable Hilbert space E, let $H=E\oplus E\oplus \boldsymbol{C}$ be an infinite dimensional Heisenberg Lie algebra with the skew-symmetric continuous bilinear mapping $\theta: (E\oplus E)\times(E\oplus E) \to \boldsymbol{C}$ given by $\theta((u, v), (u', v'))=\langle u, v'\rangle - \langle v, u'\rangle$. Then, $f((u, v, c))=\|u\|^2$, $g((u, v, c))=\|v\|^2$ are polynomials of degree 2 on $H^*=H$, but the $*$-product $f*g$ diverges (cf. [**OMY1**] (2.9)). Thus, there is no deformation quantization of $C^\infty(H)$.

If $\mathfrak{g}$ is the Lie algebra of all C^∞ vector fields on a compact manifold, then Theorem A can be applied for $\mathfrak{g}$. Thus, there are several applications including quantizations on coadjoint orbits, which will be given in forthcoming papers.

§ 1. Smooth functions on $\mathfrak{g}^*$.

1.1. Polynomial approximation theorem.

First, we note the following:

LEMMA 1.1. *There exists an increasing series of compact subsets $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ such that $\cup K_n = \mathfrak{g}^*$. For any compact subset $K \subset \mathfrak{g}^*$, there is K_n containing K.*

PROOF. For any positive integer s, let D_{-s} be the open ball in $\mathfrak{g}^{-s}$ of radius s. It is easy to see that $D_{-s} \subset D_{-s-1} \subset \cdots$. Since the inclusion mapping ι is compact, D_{-s} is a relatively compact subset of $\mathfrak{g}^{-s-1}$, and hence of $\mathfrak{g}^*$. Set

$K_s=\overline{D_{-s}}$ in $\mathfrak{g}^*$.

Let $p\in\mathfrak{g}^*$. By the definition of $\mathfrak{g}^*$, there exists s such that $p\in\mathfrak{g}^{-s}$. Suppose $\|p\|_{-s}<m$ for a positive integer m. Setting $n=\max\{s, m\}$, we have $p\in D_{-n}$.

Let $K\subset\mathfrak{g}^*$ be a compact subset. Suppose for each positive n, there exists $p_n\in K$ such that $p_n\in\mathfrak{g}^*-K_n$. By taking a subsequence if necessary, there exists $p_0\in\mathfrak{g}^*$ such that $p_0\in\mathfrak{g}^*-D_{-n}$ for any n. This contradicts the above fact. □

PROOF OF THEOREM B. Consider now a C^∞ function f on $\mathfrak{g}^*$. Let K be an arbitrary fixed compact subset of $\mathfrak{g}^*$. By Lemma 1.1, one may assume that $K\subset D_{-n}$ for some n. Since D_{-n} is relatively compact in $\mathfrak{g}^{-l}$ for any $l>n$ and f is C^∞ on $\mathfrak{g}^{-l}$, for any ε and N, there exists $\delta>0$ such that if $\|p-q\|_{-l}<\delta$, then $\|d^jf(p)-d^jf(q)\|_{-l}<\varepsilon$ for any $0\leqq j\leqq N$.

Let $\boldsymbol{R}^m$ be the subspace of $\mathfrak{g}$ spanned by $e_1, \cdots, e_m$ and π_m the projection of $\mathfrak{g}^*$ onto $\boldsymbol{R}^m$. We regard π_m as a linear mapping of $\mathfrak{g}^*$ into itself. For any point $p=\sum a_ie_i$ of D_{-n}, set $p_m=\pi_m(p)$ $(=\sum_{i=1}^m a_ie_i)$. Then

$$\|p-p_m\|_{-1} < n\lambda_m^{-l+n}$$

for any $p\in D_{-n}$. Since $\lim\lambda_m=\infty$, taking m so large that $n\lambda_m^{-l+n}<\delta$, we find that f is approximated on K by π_m^*f.

By the polynomial approximation theorem on $\boldsymbol{R}^m$, we see that on K, π_m^*f is approximated by a series of polynomials on $\mathfrak{g}^*$. Thus, the space of all polynomials is dense in $C^\infty(\mathfrak{g}^*)$ in the C^∞ topology. □

1.2. Tensor products and differential operators.

For a Sobolev chain $\{\mathfrak{g}, \mathfrak{g}^s; s\in\boldsymbol{Z}\}$, we introduced the tensor products $(\hat{\circledast}\mathfrak{g}^s)^m$ as the Banach space of all continuous symmetric m-linear mappings of $\mathfrak{g}^{-s}\times\cdots\times\mathfrak{g}^{-s}$ into $\boldsymbol{C}$ with the natural operator norm, and set $(\circledast\mathfrak{g})^m=\bigcap_s(\hat{\circledast}\mathfrak{g}^s)^m$ with the projective limit topology. For $L\in(\hat{\circledast}\mathfrak{g}^s)^m$, setting $\|L\|_{-s}=\sup_{\|x\|_{-s}=1}|L(x, \cdots, x)|$ defines a Banach norm on $(\hat{\circledast}\mathfrak{g}^s)^m$.

On the other hand, let $(\circledast\mathfrak{g}^s)^m$ be the usual symmetric tensor product of $\mathfrak{g}^s$ as a Hilbert space, that is, any element $a\in(\circledast\mathfrak{g}^s)^m$ can be written as $a=\sum a_{i_1\cdots i_m}e_{i_1}\circledast\cdots\circledast e_{i_m}$ with the Hilbert norm $|a|_s$ defined by

$$|a|_s^2=\sum|a_{i_1\cdots i_m}|^2\lambda_{i_1}^{2s}\cdots\lambda_{i_m}^{2s}. \tag{1.1}$$

Obviously, the dual space of $(\circledast\mathfrak{g}^s)^m$ is $(\circledast\mathfrak{g}^{-s})^m$.

There is a natural continuous inclusion of $(\circledast\mathfrak{g}^s)^m$ into $(\hat{\circledast}\mathfrak{g}^s)^m$. Moreover, by the assumption that $\sum_{n\geq 1}\lambda_n^{-s_0}<\infty$, we see also that there is a continuous inclusion of $(\hat{\circledast}\mathfrak{g}^s)^m$ into $(\circledast\mathfrak{g}^{s-s_0/2})^m$. Hence $(\circledast\mathfrak{g})^m$ coincides with the inverse limit of $(\circledast\mathfrak{g}^s)^m$. Taking its dual, we see that the dual space of $(\circledast\mathfrak{g})^m$ is $\bigcup_s(\circledast\mathfrak{g}^{-s})^m$ with the inductive limit topology, which will be denoted by $(\circledast\mathfrak{g}^*)^m$.

For multi-indices $\alpha=(\alpha_1, \cdots, \alpha_k, \cdots)$, we set $|\alpha|=\sum\alpha_k$. For α such that $|\alpha|<\infty$, we set $\alpha!=\alpha_1!\alpha_2!\cdots\alpha_k!\cdots$, and

$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}\cdots, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_k}^{\alpha_k}\cdots.$$

For any $t\in \boldsymbol{Z}$, $\lambda^{t\alpha}=\lambda_1^{t\alpha_1}\lambda_2^{t\alpha_2}\cdots\lambda_k^{t\alpha_k}\cdots$.

$\sum_{|\alpha|=m}(1/\alpha!)a_\alpha x^\alpha$ is a homogeneous polynomial of degree m on $\mathfrak{g}^*$ if and only if

$$\sum_{|\alpha|=m}|a_\alpha|^2\lambda^{2s\alpha}<\infty$$

for any $s>0$. For any $f\in\mathfrak{a}$, $d^l f(p)$ is a continuous symmetric l-linear mapping of $\mathfrak{g}^{-s}\times\cdots\times\mathfrak{g}^{-s}$ into $\boldsymbol{C}$ for any s, hence $d^l f(p)\in(\circledast\mathfrak{g}^s)^l$ for any s. It follows that $d^l f(p)\in(\circledast\mathfrak{g})^l$. We define the norm $|d^l f(p)|_s$ by

$$|d^l f(p)|_s^2 = \sum_{|\gamma|=l}|\partial^\gamma f|^2(p)\lambda^{2s\gamma}. \tag{1.2}$$

The following is easy to see by the converse of Taylor's theorem:

LEMMA 1.2. *$f\in\mathfrak{a}$, if and only if $|d^l f(p)|_s<\infty$ for any non-negative integer l and any integer s, and $d^l f(p)$ is continuous with respect to $p\in\mathfrak{g}^*$.*

It is easy to see that any l-differential operator π of order d has the expression

$$\pi = \sum_{|\alpha+\cdots+\delta|\le d}\pi_{\alpha,\cdots,\delta}\underbrace{\partial^\alpha\otimes\cdots\otimes\partial^\delta}_{l}.$$

For any linear differential operator $L=\sum_{|\alpha|\le m}a_\alpha\partial^\alpha$ of order m mapping $\mathfrak{a}$ to itself, by evaluation at each $p\in\mathfrak{g}^*$, L defines a continuous linear mapping

$$L_p = \sum_{|\alpha|\le m}a_\alpha(p)\partial^\alpha : \sum_{k=0}^m \oplus(\circledast\mathfrak{g})^k \longrightarrow \boldsymbol{C}.$$

Thus, $L_p\in\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^*)^k$. This implies that

$$L_p\in\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^{-s})^k \qquad \text{for some } s=s(p).$$

Since L is a differential operator of order m, $p\mapsto L_p$ is a C^∞ mapping of $\mathfrak{g}^*$ into $\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^*)^k$. In particular, for any N, $(d^N L_*)_p\in(\circledast\mathfrak{g})^N\otimes\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^*)^k$. This implies that for any t, there exists $s=s(t)$ such that $(d^N L_*)_p\in(\circledast\mathfrak{g}^t)^N\otimes\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^{-s})^k$.

The continuity of $(d^N L_*)_p$ implies that for any $p\in\mathfrak{g}^*$ and for any integers t, $N\geqq 0$, there exist $s=s(t, N, p)$ and a neighborhood V_p of p in $\mathfrak{g}^{-s}$ such that $p\mapsto(d^N L_*)_p$ is a continuous mapping of V_p into $(\circledast\mathfrak{g}^t)^N\otimes\sum_{k=0}^m\oplus(\circledast\mathfrak{g}^{-s})^k$.

Similarly, we have the following criterion:

LEMMA 1.3. *$\pi=\sum_{|\alpha+\beta|\le m}(1/\alpha!\beta!)\pi_{\alpha,\beta}\partial^\alpha\otimes\partial^\beta$, $\pi_{\alpha,\beta}\in\mathfrak{a}$, is a bidifferential operator of order m, if and only if $\pi_{\alpha,\beta}$ satisfies for any non-negative integers*

t, N and for any $p\in\mathfrak{g}^*$,

(i) *there is an integer $s=s(t, N, p)>0$ such that*

$$\sum_{|\gamma|=N}\sum_{\alpha,\beta}|\partial^{\gamma}\pi_{\alpha,\beta}(p)|^2\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)}<\infty,$$

(ii) *for any $\varepsilon>0$, there exist $s=s(N, t, p, \varepsilon)$ and a neighborhood V_p of p in $\mathfrak{g}^{-s}$ such that*

$$\sum_{|\gamma|=N}\sum_{\alpha,\beta}|\partial^{\gamma}\pi_{\alpha,\beta}(p)-\partial^{\gamma}\pi_{\alpha,\beta}(q)|^2\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)}<\varepsilon, \qquad \text{for any } q\in V_p.$$

PROOF. Suppose π is a bidifferential operator of order m. Then, we have

$$\pi_{\alpha,\beta}(p)=\pi((x-x(p))^{\alpha},\ (x-x(p))^{\beta})(p).$$

At every $p\in\mathfrak{g}^*$, by the same argument as above π induces

$$\pi_p=\sum_{|\alpha+\beta|\leq m}\frac{1}{\alpha!\beta!}\pi_{\alpha,\beta}(p)\partial^{\alpha}\otimes\partial^{\beta}\in\Big(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k\Big)\otimes\Big(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k\Big). \tag{1.3}$$

The differentiability of π_p gives the first inequality. The continuity of $p\mapsto(d^N\pi_*)_p$ yields the second one.

Conversely, given $\pi_{\alpha,\beta}\in\mathfrak{a}$, $|\alpha+\beta|\leqq m$, satisfying (i) and (ii), we define π_p by (1.3). Then, by (i), we have

$$\pi_p\in\Big(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k\Big)\otimes\Big(\sum_{k=0}^{m}\oplus(\circledast\mathfrak{g}^*)^k\Big) \tag{1.4}$$

for any $p\in\mathfrak{g}^*$. The second inequality (ii) gives the smoothness of $p\mapsto\pi_p$. Note that $\pi(f, g)(p)=\pi_p(f, g)$ for any f, $g\in\mathfrak{a}$ and $\pi(f, g)(p)$ depends only on $\partial^{\alpha}f(p)$, $\partial^{\beta}g(p)$ for $|\alpha+\beta|\leqq m$. Thus, $\pi(f, g)\in\mathfrak{a}$ by (i) and (ii). It is easy to see that π gives a continuous bilinear mapping of $\mathfrak{a}\times\mathfrak{a}$ into $\mathfrak{a}$. □

For any $f\in\mathfrak{a}$ and $p\in\mathfrak{g}^*$, we see that $f=f(p)+\sum_{1\leq i<\infty}F_i(x, p)(x_i-x_i(p))$, where $F_i(x, p)=\int_0^1(\partial f/\partial x_i)(x(p)+t(x-x(p)))dt$. By Lemma 1.3, we have the following:

LEMMA 1.4. *Let π be a bidifferential operator of order m. Then, the operator L defined by*

$$L(f)(p)=\sum_{i=1}^{\infty}\pi(F_i,\ x_i-x_i(p))(p)$$

is a linear differential operator of order m.

Note that a similar criterion is available for 3-differential operators. If π, π' are bidifferential operators of order m, m' respectively, then $\pi(f, \pi'(g, h))$ is a 3-differential operator of order $m+m'$. If $E(f, g, h)$ is a 3-differential operator of order m, then $E(x_i, f, x_l)$ is a linear differential operator of order

$m-2$ with respect to f.

§ 2. Algebraic preliminaries.

To introduce the obstructions R_m given in § 3, we prepare some algebraic tools, called Hochschild and de Rham-Chevalley coboundary operators. This notion is given in a purely algebraic manner. So, in this section, we do not specify $\mathfrak{a}$ and take it only as an abstract topological vector space.

2.1. Hochschild coboundary operators.

Let $\mathfrak{a}$ be a topological vector space over $\boldsymbol{C}$. Denote by $C^p(\mathfrak{a})$, $p \geqq 1$, the space of all continuous p-linear mappings of $\mathfrak{a} \times \cdots \times \mathfrak{a}$ to $\mathfrak{a}$. We denote by $AC^p(\mathfrak{a})$ and $SC^p(\mathfrak{a})$ ($p \geqq 1$) the set of the alternative and the symmetric p-linear mappings, respectively. If $p=0$, we set $C^0(\mathfrak{a})=AC^0(\mathfrak{a})=SC^0(\mathfrak{a})=\mathfrak{a}$.

For any $\pi \in C^2(\mathfrak{a})$, we define the *Hochschild coboundary operator* δ_π: $C^p(\mathfrak{a}) \to C^{p+1}(\mathfrak{a})$, $p \geqq 1$, by

$$(2.1)\qquad \begin{aligned}(\delta_\pi F)(v_1, \cdots, v_{p+1}) &= \pi(v_1, F(v_2, \cdots, v_{p+1})) \\ &+ \sum_{i=1}^{p} (-1)^i F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_{p+1}) \\ &+ (-1)^{p+1} \pi(F(v_1, \cdots, v_p), v_{p+1})\end{aligned}$$

for $F \in C^p(\mathfrak{a})$, and for $p=0$, we set $(\delta_\pi v)(v_1)=\pi(v_1, v)$ for any $v \in \mathfrak{a}$.

By a direct computation using the linearization, we have the following:

LEMMA 2.1. *For any* $\pi, \pi', \pi'' \in C^2(\mathfrak{a})$, *we have*

$$\delta_\pi \pi' = \delta_{\pi'} \pi, \qquad \delta_\pi I = \pi, \qquad (I = \textit{identity}) \text{ and } \delta_\pi \delta_\pi \pi = 0,$$
$$\mathfrak{S}_{(\pi, \pi', \pi'')} \delta_\pi \delta_{\pi'} \pi'' = 0,$$

where $\mathfrak{S}_{(\pi,\pi',\pi'')}$ *means the cyclic summation with respect to* π, π', π''.

$\delta_\pi \pi = 0$, if and only if $(\mathfrak{a}, \pi)$ is an associative algebra. If $(\mathfrak{a}, \pi)$ is an associative algebra, then $\delta_\pi^2 F=0$, for any $F \in C^p(\mathfrak{a})$ (cf. [**Mc**]). In particular, $\delta_\pi^2 I = \delta_\pi \pi = 0$. Therefore, $\delta_\pi^2 = 0$ is equivalent to $\delta_\pi \pi = 0$.

Let $(\mathfrak{a}, \pi_0)$ be any associative algebra. Suppose $\pi_0, \pi_1, \cdots, \pi_{k-1} \in C^2(\mathfrak{a})$ satisfy $(\square_l)$ in (0.1) for any integer l such that $0 \leqq l \leqq k-1$. We denote $\delta_i = \delta_{\pi_i}$ for simplicity. We consider the equation $(\square_k)$, which is equivalent to

$$(2.2)\qquad \delta_0 \pi_k = -Q_k, \qquad \text{where } Q_k = \frac{1}{2} \sum_{i+j=k,\, i,\, j \geqq 1} \delta_i \pi_j .$$

Since $\delta_0^2=0$ by the associativity of π_0, if (2.2) can be solved, then the right hand side must satisfy $\delta_0 Q_k = 0$. At the first glance, this looks like a necessary condition for $(\mathfrak{a}, \pi_0)$ to be deformed associatively, but in fact this is fulfilled automatically. Namely, we have

PROPOSITION 2.2. *Let* $(\mathfrak{a}, \pi_0)$ *be any associative algebra. If* $\pi_0, \pi_1, \cdots, \pi_{k-1} \in C^2(\mathfrak{a})$ *satisfy* $(\square_l)$ *for any integer* l *such that* $0 \leqq l \leqq k-1$, *then* $\pi_0, \cdots, \pi_{k-1}$ *satisfy also* $\delta_0 Q_k = 0$.

Proof is seen in [**OMY2**], Proposition 1.3.

2.2. *p*-derivations.

For $\pi \in C^2(\mathfrak{a})$, we define $\partial_i^\pi : C^p(\mathfrak{a}) \to C^{p+1}(\mathfrak{a})$ $(1 \leqq i \leqq p)$, $p \geqq 1$, by

$$\begin{aligned}(\partial_i^\pi F)(v_1, \cdots, v_{p+1}) = \pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) \\ -F(v_1, \cdots, \pi(v_i, v_{i+1}), \cdots, v_{p+1}) \\ +\pi(F(v_1, \cdots, \hat{v}_{i+1}, \cdots, v_{p+1}), v_{i+1})\end{aligned} \tag{2.3}$$

for any $F \in C^p(\mathfrak{a})$.

We call $F \in C^p(\mathfrak{a})$ a *p-derivation with respect to* π, if $\partial_j^\pi F = 0$ for any j, $(1 \leqq j \leqq p)$. By $Der^p(\mathfrak{a}, \pi)$, we denote the space of all p-derivations with respect to π. Set also

$$\mathcal{A}^p(\mathfrak{a}, \pi) = AC^p(\mathfrak{a}) \cap Der^p(\mathfrak{a}, \pi).$$

We define mappings σ_p, $\mathfrak{c}_p$: $C^p(\mathfrak{a}) \to C^p(\mathfrak{a})$ by

$$(\sigma_p F)(v_1, v_2, \cdots, v_{p-1}, v_p) = F(v_p, v_{p-1}, \cdots, v_2, v_1), \tag{2.4}$$

$$(\mathfrak{c}_p F)(v_1, v_2, \cdots, v_{p-1}, v_p) = F(v_p, v_1, v_2, \cdots, v_{p-1}). \tag{2.5}$$

Since $\mathfrak{c}_3^3 = 1$, we have

$$(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\mathfrak{c}_3) = 0, \tag{2.6}$$

$$(1-\mathfrak{c}_3+\mathfrak{c}_3^2)(1+\mathfrak{c}_3) = 2. \tag{2.7}$$

The following formulas are useful for later computations:

LEMMA 2.3. (i) *For any* $\pi \in C^2(\mathfrak{a})$ *and* $F \in C^p(\mathfrak{a})$, *we have*

$$\delta_\pi \sigma_p F = (-1)^{p+1} \sigma_{p+1} \delta_{\sigma_2 \pi} F,$$

$$\partial_j^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1} \partial_{j+1}^\pi F \ (1 \leqq j \leqq p-1), \qquad \partial_p^\pi \mathfrak{c}_p F = \mathfrak{c}_{p+1}^2 \partial_1^\pi F.$$

(ii) *In particular, if* $\pi \in SC^2(\mathfrak{a})$, *we have*

$$\delta_\pi F = \sum_{1 \leq i \leq p} (-1)^{i-1} \partial_i^\pi F, \qquad \partial_j^\pi \sigma_p F = \sigma_{p+1} \partial_{p+1-j}^\pi F \ (1 \leqq j \leqq p).$$

(iii) *If* $\pi \in SC^2(\mathfrak{a})$ *and* $\delta_\pi \pi = 0$, *we have*

$$(\partial_j^\pi - \partial_{j+1}^\pi)\partial_j^\pi = 0 \qquad \textit{for} \quad 1 \leqq j \leqq p.$$

2.3. de Rham-Chevalley coboundary operators.

For any $\pi \in AC^2(\mathfrak{a})$, we define the *Chevalley coboundary operator* $d_\pi: AC^p(\mathfrak{a}) \to AC^{p+1}(\mathfrak{a})$ by

$$(2.8) \qquad (d_\pi F)(v_1, \cdots, v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \pi(v_i, F(v_1, \cdots, \hat{v}_i, \cdots, v_{p+1})) + \sum_{i<j} (-1)^{i+j} F(\pi(v_i, v_j), v_1, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{p+1}).$$

By a direct computation using the linearization, we have

LEMMA 2.4. *For any $\pi, \pi', \pi'' \in AC^2(\mathfrak{a})$,*

$$d_\pi \pi' = d_{\pi'} \pi, \quad d_\pi I = \pi, \quad (I = \textit{identity}), \textit{ and } d_\pi d_\pi \pi = 0,$$

$$\mathop{\mathfrak{S}}_{(\pi, \pi', \pi'')} d_\pi d_{\pi'} \pi'' = 0, \quad (d_\pi \pi)(u, v, w) = 2 \mathop{\mathfrak{S}}_{(u, v, w)} \pi(u, \pi(v, w)).$$

By the last identity in Lemma 2.4, $d_\pi \pi = 0$ if and only if $(\mathfrak{a}, \pi)$ is a Lie algebra. If $(\mathfrak{a}, \pi)$ is a Lie algebra, then $d_\pi^2 F = 0$ for any $F \in AC^p(\mathfrak{a})$ (cf. [**Ma**]). Therefore, $d_\pi^2 = 0$ is equivalent to $d_\pi \pi = 0$.

In the following, we use the notations

$$(2.9) \qquad \pi^\pm(u, v) = \frac{1}{2} \{\pi(u, v) \pm \pi(v, u)\},$$

for $\pi \in C^2(\mathfrak{a})$.

DEFINITION 2.5. For $\pi_0, \cdots, \pi_{m-1} \in C^2(\mathfrak{a})$, we set

$$(2.10) \qquad \begin{cases} Q_m = \dfrac{1}{2} \displaystyle\sum_{i+j=m,\, i, j \geqq 1} \delta_i \pi_j, \text{ (cf. (2.2))} \\ R_m = \dfrac{1}{2} \displaystyle\sum_{i+j=m,\, i, j \geqq 1} d_i^- \pi_j^-, \end{cases}$$

where $d_i^- = d_{\pi_i^-}$.

By Proposition 2.2, we have $\delta_0 Q_k = 0$, if $\pi_0, \pi_1, \cdots, \pi_{k-1}$ satisfy $(\square_l)$ $0 \leqq l \leqq k-1$.

Assume that $(\mathfrak{a}, \pi_0, \pi_1)$ is a Poisson algebra, i. e. $\pi_0 \in SC^2(\mathfrak{a})$, $\pi_1 \in AC^2(\mathfrak{a})$ such that $\delta_0 \pi_0 = 0$, $\delta_0 \pi_1 = 0$, $d_1 \pi_1 = 0$.

We easily have

$$d_{\pi_1} \mathcal{A}^p(\mathfrak{a}, \pi_0) \subset d_{\pi_1} \mathcal{A}^{p+1}(\mathfrak{a}, \pi_0), \qquad d_{\pi_1}^2 = 0.$$

Thus, we can give the following p-th cohomology group $H^p(\mathfrak{a}, \pi_0, \pi_1)$ of the cochain complex

$$\cdots \longrightarrow \mathcal{A}^p(\mathfrak{a}, \pi_0) \xrightarrow{d_{\pi_1}} \mathcal{A}^{p+1}(\mathfrak{a}, \pi_0) \longrightarrow \cdots,$$

which is called the *de Rham-Chevalley cohomology group* of the Poisson algebra. By a similar manner as in Proposition 2.2, we have the following:

PROPOSITION 2.6. *Suppose* $(\mathfrak{a}, \pi_0, \pi_1)$ *is a Poisson algebra. If* $\pi_0, \cdots, \pi_{k-1} \in C^2(\mathfrak{a})$ *satisfy* $(\square_l)$ *for* $0 \leqq l \leqq k-1$, *then* $R_l=0$ *for* $2 \leqq l \leqq k-1$ *and* $d_1^- R_k=0$.

Proof is seen in [**OMY2**], Propositions 3.2-3.3.

§ 3. Jacobi identities.

3.1. The obstruction R_m.

Let $\mathfrak{a}=C^\infty(\mathfrak{g}^*)$ and assume the following:

(H.1) Set $\pi_0(f, g)=fg$, $\pi_1(f, g)=-(1/2)\{f, g\}$. Furthermore, $\pi_2, \cdots, \pi_{m-1} \in C^2(\mathfrak{a})$ are given so that $(\square_l)$: $\sum_{i+j=l}\delta_i\pi_j=0$ for any l, $0 \leqq l \leqq m-1$.

(H.2) $\pi^+_{\text{odd}}=\pi^-_{\text{even}}=0$ and $\pi_s(x_i, x_j)=0$ for $2 \leqq s \leqq m-1$.

(H.3) π_s is a bidifferential operator of order $2s$ for any $0 \leqq s \leqq m-1$.

Remark that if m is odd, then $R_m=0$. $R_m(f, g, h)$ is a 3-differential operator of order $2m$.

Let Q_m be given in (2.2). Under the assumptions (H.1)~(H.3), we want to solve the equation $\delta_0\pi_m=-Q_m$ (cf. (2.2)). By remarking $\sigma_2=\mathfrak{c}_2$, and using Lemma 2.3, the above equation is rewritten as

$$(3.1)\qquad \begin{cases} (1-\mathfrak{c}_3)\partial_2^0\pi_m^+ = -\delta_0\pi_m^+ = -\dfrac{1}{2}(1-\sigma_3)\delta_0\pi_m = \dfrac{1}{2}(1-\sigma_3)Q_m\,, \\[2ex] (1+\mathfrak{c}_3)\partial_2^0\pi_m^- = -\delta_0\pi_m^- = -\dfrac{1}{2}(1+\sigma_3)\delta_0\pi_m = \dfrac{1}{2}(1+\sigma_3)Q_m\,, \end{cases}$$

where $\partial_i^{\pi_0}=\partial_i^0$. By (2.7), the equation (3.1) splits into two equations:

$$(3.2)\qquad \partial_2^0\pi_m^- = \frac{1}{4}(1-\mathfrak{c}_3+\mathfrak{c}_3^2)(1+\sigma_3)Q_m\,,$$

$$(3.3)\qquad (1-\mathfrak{c}_3)\partial_2^0\pi_m^+ = \frac{1}{2}(1-\sigma_3)Q_m\,.$$

Assume (3.1) has a solution π_m. By applying Lemma 2.3, and (2.6), (2.7), in addition to $\delta_0 Q_m=0$, Q_m must satisfy the following consistency conditions for (3.2-3):

$$(3.4)\qquad (\partial_2^0-\partial_3^0)(1-\mathfrak{c}_3+\mathfrak{c}_3^2)(1+\sigma_3)Q_m=0\,,$$

$$(3.5)\qquad (1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_m=0\,.$$

However, (3.4) is not a new condition. Namely, we have the following;

LEMMA 3.1. *If* $\delta_0 Q=0$ *for* $Q \in C^3(\mathfrak{a})$, *then* (3.4) *is satisfied.*

Proof is seen in Appendix 6.1.

Next, we consider (3.5), the consistency condition for (3.3).

LEMMA 3.2. $(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_m=4R_m$. *Thus, the consistency condition of* (3.3) *is* $R_m=0$.

PROOF. Since $\delta_i=\delta_i^+ +\delta_i^-$, where $\delta_i^\pm=\delta_{\pi_i^\pm}$, we see by the definition of Q_m, that

$$Q_m=\frac{1}{2}\sum_{i+j=m,\, i,\, j\geq 1}(\delta_i^+\pi_j^+ +\delta_i^-\pi_j^-)+\sum_{i+j=m,\, i,\, j\geq 1}\delta_i^+\pi_j^- . \tag{3.6}$$

Note $\sigma_3\delta_i^+\pi_j^-=\delta_i^+\pi_j^-$, $\sigma_3\delta_i^+\pi_j^+=-\delta_i^+\pi_j^+$, $\sigma_3\delta_i^-\pi_j^-=-\delta_i^-\pi_j^-$ by Lemma 2.3. Then, we have

$$\begin{cases} Q_m-\sigma_3 Q_m = \sum_{i+j=m,\, i,\, j\geq 1}(\delta_i^+\pi_j^+ +\delta_i^-\pi_j^-), \\ Q_m+\sigma_3 Q_m = 2\sum_{i+j=m,\, i,\, j\geq 1}\delta_i^+\pi_j^- . \end{cases} \tag{3.7}$$

By (2.2), (3.7) and Lemma 2.4, we have

$$\begin{aligned}(1+\mathfrak{c}_3+\mathfrak{c}_3^2)(1-\sigma_3)Q_m(f,\, g,\, h) &= 4\sum_{i+j=m,\, i,\, j\geq 1}\mathop{\mathfrak{S}}_{(f,\, g,\, h)}\pi_i^-(f,\, \pi_j^-(g,\, h)) \\ &= 4R_m(f,\, g,\, h). \quad \square \end{aligned} \tag{3.8}$$

3.2. Cohomological property for R_m.

By Lemma 3.2, $R_m=0$ must hold for π_m to exist. First, recall the following fact whose proof is seen in [**OMY2**], Theorem 3.4.

THEOREM 3.3. *Suppose* $\pi_2, \cdots, \pi_{m-1}\in C^2(\mathfrak{a})$ *satisfy* (H.1)$\sim$(H.3). *Then,*

$$\partial_j^0 R_m=0, \qquad \text{for} \quad j=1,\, 2,\, 3 \quad i.e. \quad R_m\in \mathcal{A}_3(\mathfrak{a},\, \pi_0).$$

Hence, by Proposition 2.6 R_m *is a de Rham-Chevalley* 3*-cocycle.*

Using Theorem 3.3, we have

COROLLARY 3.4. *Assume that* (H.1)$\sim$(H.3) *hold for* $\mathfrak{a}=C^\infty(\mathfrak{g}^*)$. *Then,* $R_m=0$.

PROOF. $\pi_l(x_i,\, x_j)=0$ for $l\geqq 2$. By the 3-derivation property and by the polynomial approximation theorem, we have only to check the quantities

$$R_m(x_i,\, x_j,\, x_k)=\mathop{\mathfrak{S}}_{(i,\, j,\, k)}\pi_{m-1}^-(x_i,\, \pi_1^-(x_j,\, x_k)).$$

R_2 always vanishes because $d_{\pi_1}\pi_1=0$. Hence, if $\pi_1(x_i,\, x_j)=c_{ij}+\sum_k c_{ij}^k x_k$, then $R_m=0$. $\square$

REMARK. We shall call $R_m=0$ *the Jacobi identities.*

For the convenience sake, in what follows, we use the notation:

$$
(3.9)\quad \begin{cases} f\cdot g=\pi_0(f, g), \qquad \langle f, g\rangle^{\pm}_m=\pi^{\pm}_m(f, g), \qquad (m\geqq 1), \\ \langle f, g\cdot\langle h, t\rangle^{\pm}\rangle^{\pm}_m=\sum_{i+j=m, i, j\geqq 1}\pi^{\pm}_i(f, g\cdot\pi^{\pm}_j(h, t)) \qquad (m\geqq 2), \\ \langle\langle f, \langle g, h\rangle^{\pm}\rangle^{\pm}, t\rangle^{\pm}_m=\sum_{a+b+c=m, a, b, c\geqq 1}\pi^{\pm}_a(\pi^{\pm}_b(f, \pi^{\pm}_c(g, h)), t) \qquad (m\geqq 3), \\ \langle\langle f, g\rangle^{\pm}, \langle h, t\rangle^{\pm}\rangle^{\pm}_m=\sum_{a+b+c=m, a, b, c\geqq 1}\pi^{\pm}_a(\pi^{\pm}_b(f, g), \pi^{\pm}_c(h, t)) \qquad (m\geqq 3). \end{cases}
$$

Now, we shall discuss the cases m=even and m=odd separately.

(E) Case $m=2k$: The equations (3.2-3) for $\pi_{2k}=\pi^+_{2k}+\pi^-_{2k}$ are rewritten as follows:

$$
(3.10)\quad \begin{cases} (a)\quad (1-\mathfrak{c}_3)\partial^0_2\pi^+_{2k}=\dfrac{1}{2}\sum_{i+j=2k, i, j\geqq 1}(\delta^+_i\pi^+_j+\delta^-_i\pi^-_j) \\ (b)\quad \partial^0_2\pi^-_{2k}=0, \end{cases}
$$

where we used (3.7). One may set $\pi^-_{2k}=0$, for this is the trivial solution of (3.10, (b)). By a little careful computation together with the definition of $\delta^+_i\pi^+_j$, $\delta^-_i\pi^-_j$, we see that (3.10, (a)) is equivalent to the following:

$$
(3.11)\qquad \pi^+_{2k}(f, gh)-\pi^+_{2k}(h, gf)=E_{2k}(f, g, h),
$$

where

$$
(3.12)\qquad \begin{aligned} E_{2k}(f, g, h)=&\ \pi^+_{2k}(f, g)h-\pi^+_{2k}(h, g)f \\ &+\langle\langle f, g\rangle^+, h\rangle^+_{2k}-\langle\langle h, g\rangle^+, f\rangle^+_{2k} \\ &-\langle\langle h, f\rangle^-, g\rangle^-_{2k}. \end{aligned}
$$

$E_{2k}(f, g, h)$ is a 3-differential operator of order $4k$.

(O) Case $m=2l+1$: The equations (3.2-3) are changed into

$$
(3.13)\quad \begin{cases} (a)\quad \partial^0_2\pi^-_{2l+1}=\dfrac{1}{4}(1-\mathfrak{c}_3+\mathfrak{c}^2_3)(1+\sigma_3)Q_{2l+1} \\ (b)\quad (1-\mathfrak{c}_3)\partial^0_2\pi^+_{2l+1}=\dfrac{1}{2}\sum_{i+j=2l+1, i, j\geqq 1}(\delta^+_i\pi^+_j+\delta^-_i\pi^-_j). \end{cases}
$$

By (H.2), the right hand side of (3.13, (b)) vanishes. In what follows we set $\pi^+_{2l+1}=0$.

§ 4. Construction of π_{odd}.

In this section, we prove the following:

THEOREM 4.1. *Let* $l\geqq 1$. *Under the assumptions* (H.1-3), *there exists* $\pi_{2l+1}\in AC^2(\mathfrak{a})$ *such that* $\sum_{i+j=2l+1, i, j\geqq 0}\delta_i\pi_j=0$ *and* π_{2l+1} *is a bidifferential operator of order* $2(2l+1)$ *satisfying* $\pi_{2l+1}(\pi_i, x_j)=0$.

Let x_k be the linear functional on $\mathfrak{g}^*$ defined by $x_k(p)=\langle e_k, p\rangle_0$ and set

$$\pi_{2l+1}^{-}(x_i,\ x_j) = 0. \tag{4.1}$$

4.1. Construction of π_{odd}^{-}.

First, we show how to construct π_{2l+1}. By (3.7), we see that (3.13, (a)) is equivalent to

$$\begin{aligned}\pi_{2l+1}^{-}(f,\ gh) &= g\pi_{2l+1}^{-}(f,\ h)+\pi_{2l+1}^{-}(f,\ g)h \\ &+\langle\langle f,\ g\rangle^{-},\ h\rangle_{2l+1}^{+}+\langle\langle f,\ h\rangle^{-},\ g\rangle_{2l+1}^{+}-\langle f,\ \langle g,\ h\rangle^{+}\rangle_{2l+1}^{-}.\end{aligned} \tag{4.2}$$

Setting $\zeta_j=x_j-x_j(p)$, we have

$$g(x) = g(p)+\sum_{j\geq 1} G_j(x,\ p)\zeta_j,$$

where $G_j(x,\ p)=\int_0^1(\partial g/\partial x_j)(p+t(x-p))dt$. Putting $f=x_i$ in (4.2), we get

$$\begin{aligned}\pi_{2l+1}^{-}(x_i,\ g)(p) = \sum_{j\geq 1}\{&\langle\langle x_i,\ G_j\rangle^{-},\ x_j\rangle_{2l+1}^{+}(p) \\ &+\langle\langle x_i,\ x_j\rangle^{-},\ G_j\rangle_{2l+1}^{+}(p)-\langle x_i,\ \langle G_j,\ x_j\rangle^{+}\rangle_{2l+1}^{-}(p)\}.\end{aligned} \tag{4.3}$$

Remark that $\partial_x^\alpha G_j(x,\ p)|_{x=p}=(1/(|\alpha|+1))(\partial_x^\alpha\partial_{x_j}g)(p)$. By the assumptions (H.1-3) and Lemma 1.4, the right hand side (4.3) is a linear differential operator of order $4l+1$ with respect to g.

Define $\pi_{2l+1}^{-}(h,\ x_i)$ by

$$\pi_{2l+1}^{-}(h,\ x_i) = -\pi_{2l+1}^{-}(x_i,\ h). \tag{4.4}$$

By (4.2), we have

$$\begin{aligned}\pi_{2l+1}^{-}(f,\ g)(p) = \sum_{j\geq 1}\Big\{&\frac{\partial g}{\partial x_j}(p)\pi_{2l+1}^{-}(f,\ x_j)(p) \\ &+\langle\langle f,\ G_j\rangle^{-},\ x_j\rangle_{2l+1}^{+}(p)+\langle\langle f,\ x_j\rangle^{-},\ G_j\rangle_{2l+1}^{+}(p)-\langle f,\ \langle G_j,\ x_j\rangle^{+}\rangle_{2l+1}^{-}(p)\Big\}.\end{aligned} \tag{4.5}$$

By a similar proof as in Lemma 1.4, the right hand side of (4.5) is a bidifferential operator of order $2(2l+1)$ with respect to $f,\ g$.

Thus, we obtain $\pi_{2l+1}^{-}(f,\ g)$ for any $f,\ g\in\mathfrak{a}$. However, we only see that $\pi_{2l+1}^{-}(x_i,\ x_j)=0$ for $l\geq 1$ and $\pi_{2l+1}^{-}(x_i,\ h)=-\pi_{2l+1}^{-}(h,\ x_i)$.

4.2. Skewness of π_{2l+1}.

To prove Theorem 4.1, we only show the following:

PROPOSITION 4.2. *$\pi_{2l+1}^{-}(f,\ h)$ given by (4.5) is skew-symmetric.*

PROOF. By the polynomial approximation theorem, we have only to show the skewness for polynomials. Thus in what follows, we assume the following:

$(\mathrm{S})_s$ $\quad \pi_{2l+1}^{-}(x^\alpha,\ x^\beta) = -\pi_{2l+1}^{-}(x^\beta,\ x^\alpha)$ for any $\alpha,\ \beta$ such that $|\alpha+\beta|\leqq s$.

Consider $\pi_{2l+1}^{-}(x^\alpha,\ x^\beta)$ such that $|\alpha+\beta|=s+1$. If either of $|\alpha|,\ |\beta|$ is 1, then

(4.4) shows the skew-symmetricity. We now show $(\mathrm{S})_{s+1}$ for $|\alpha|, |\beta| \geqq 2$. Since π^-_{2l+1} is a continuous bilinear mapping, it is enough to show that

$$\pi^-_{2l+1}(x^\alpha x^{\alpha'}, x^\beta x^{\beta'}) = -\pi^-_{2l+1}(x^\beta x^{\beta'}, x^\alpha x^{\alpha'}) \quad \text{for} \quad |\alpha|, |\alpha'|, |\beta|, |\beta'| \geqq 1.$$

For simplicity, set $f=x^\alpha$, $g=x^{\alpha'}$, $h=x^\beta$, $t=x^{\beta'}$. By the assumption $(\mathrm{S})_s$, one obtains

$$\pi^-_{2l+1}(fg, h) = -\pi^-_{2l+1}(h, fg), \quad \pi^-_{2l+1}(f, gh) = -\pi^-_{2l+1}(gh, f), \quad \text{etc.}. \tag{4.6}$$

By (4.2), we have

$$\begin{aligned}\pi^-_{2l+1}(fg, ht) = {} & \pi^-_{2l+1}(fg, h)t + \pi^-_{2l+1}(fg, t)h + \langle\langle fg, h\rangle^-, t\rangle^+_{2l+1} \\ & + \langle\langle fg, t\rangle^-, h\rangle^+_{2l+1} - \langle fg, \langle h, t\rangle^+\rangle^-_{2l+1}.\end{aligned}$$

Using (4.2), and the assumption $(\mathrm{S})_s$, we have

$$\begin{aligned}\pi^-_{2l+1}(fg, ht) = {} & \pi^-_{2l+1}(f, h)gt + \pi^-_{2l+1}(g, h)ft + \pi^-_{2l+1}(f, t)gh + \pi^-_{2l+1}(g, t)fh \\ & - t\langle\langle h, f\rangle^-, g\rangle^+_{2l+1} - t\langle\langle h, g\rangle^-, f\rangle^+_{2l+1} + t\langle h, \langle f, g\rangle^+\rangle^-_{2l+1} \\ & - h\langle\langle t, f\rangle^-, g\rangle^+_{2l+1} - h\langle\langle t, g\rangle^-, f\rangle^+_{2l+1} + h\langle t, \langle f, g\rangle^+\rangle^-_{2l+1} \\ & + \langle\langle fg, h\rangle^-, t\rangle^+_{2l+1} + \langle\langle fg, t\rangle^-, h\rangle^+_{2l+1} - \langle fg, \langle h, t\rangle^+\rangle^-_{2l+1}.\end{aligned} \tag{4.7}$$

The first line of the right hand side of (4.7) is skew-symmetric under the permutation of $(f, g, h, t) \to (h, t, f, g)$, which we shall denote by σ. Let $\mathfrak{S}$ denote $1+\sigma$. Then, using (4.2) and applying the assumption to the last line of (4.7), we have the following:

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathfrak{S}\pi^-_{2l+1}(fg, ht) =} \\
-\mathfrak{S}t\langle\langle h, f\rangle^-, g\rangle^+_{2l+1} & -\mathfrak{S}t\langle\langle h, g\rangle^-, f\rangle^+_{2l+1} & +\mathfrak{S}t\langle h, \langle f, g\rangle^+\rangle^-_{2l+1} \\
 & & \qquad\blacktriangle \\
-\mathfrak{S}h\langle\langle t, f\rangle^-, g\rangle^+_{2l+1} & -\mathfrak{S}h\langle\langle t, g\rangle^-, f\rangle^+_{2l+1} & +\mathfrak{S}h\langle t, \langle f, g\rangle^+\rangle^-_{2l+1} \\
 & & \qquad\blacktriangle \\
 & -\mathfrak{S}f\langle g, \langle h, t\rangle^+\rangle^-_{2l+1} & -\mathfrak{S}g\langle f, \langle h, t\rangle^+\rangle^-_{2l+1} \\
 & \qquad\blacktriangle & \qquad\blacktriangle \\
+\mathfrak{S}\langle\langle f, g\rangle^+, \langle h, t\rangle^+\rangle^-_{2l+1} & & \\
\qquad\blacklozenge & & \\
 & +\mathfrak{S}\langle\langle\langle h, t\rangle^+, f\rangle^-, g\rangle^+_{2l+1} & +\mathfrak{S}\langle\langle\langle h, t\rangle^+, g\rangle^-, f\rangle^+_{2l+1} \\
 & \qquad\blacktriangledown & \qquad\blacktriangledown \\
-\mathfrak{S}\langle\langle\langle h, f\rangle^-, g\rangle^+, t\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle h, g\rangle^-, f\rangle^+, t\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle f, g\rangle^+, h\rangle^-, t\rangle^+_{2l+1} \\
 & & \qquad\blacktriangledown \\
-\mathfrak{S}\langle\langle\langle t, f\rangle^-, g\rangle^+, h\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle t, g\rangle^-, f\rangle^+, h\rangle^+_{2l+1} & -\mathfrak{S}\langle\langle\langle f, g\rangle^+, t\rangle^-, h\rangle^+_{2l+1} \\
 & & \qquad\blacktriangledown \\
+\mathfrak{S}\langle f\langle g, h\rangle^-, t\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f, h\rangle^-, t\rangle^+_{2l+1} & \\
+\mathfrak{S}\langle f\langle g, t\rangle^-, h\rangle^+_{2l+1} & +\mathfrak{S}\langle g\langle f, t\rangle^-, h\rangle^+_{2l+1} &
\end{array}$$

The terms marked by ▲, ▼, ◆ are cancelled out. Denoting by σ_{12}, σ_{34} the permutations $(f, g, h, t) \to (g, f, h, t)$, $(f, g, h, t) \to (f, g, t, h)$ respectively, we have:

$$\mathfrak{S}\pi^-_{2l+1}(fg, ht) = -\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\{t\langle\langle h, f\rangle^-, g\rangle^+_{2l+1} + \langle\langle\langle h, f\rangle^-, g\rangle^+, t\rangle^+_{2l+1} - \langle f\langle g, h\rangle^-, t\rangle^+_{2l+1}\}. \tag{4.8}$$

Substitute the equality (ε_{2l}) given in Appendix 6.2 to the last term of (4.8), where we remark that (ε_{2l}) is valid for any π^+_m such that $m \leqq 2l$. Note that

$$\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})S_a(f, \pi^-_b(g, h), t) = 0. \tag{4.9}$$

By a little complicated calculation, we have

$$\begin{aligned} \mathfrak{S}\pi^-_{2l+1}(fg, ht) &= -\frac{1}{3}\mathfrak{S}(1+\sigma_{34})(1+\sigma_{12})\langle f, \langle t, \langle g, h\rangle^-\rangle^-\rangle^-_{2l+1} \\ &= \frac{1}{3}\langle t, \langle f, \langle g, h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f, \langle t, \langle g, h\rangle^-\rangle^-\rangle^-_{2l+1} \\ &+ \frac{1}{3}\langle t, \langle g, \langle f, h\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g, \langle t, \langle f, h\rangle^-\rangle^-\rangle^-_{2l+1} \\ &+ \frac{1}{3}\langle h, \langle f, \langle g, t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle f, \langle h, \langle g, t\rangle^-\rangle^-\rangle^-_{2l+1} \\ &+ \frac{1}{3}\langle h, \langle g, \langle f, t\rangle^-\rangle^-\rangle^-_{2l+1} - \frac{1}{3}\langle g, \langle h, \langle f, t\rangle^-\rangle^-\rangle^-_{2l+1}. \end{aligned} \tag{4.10}$$

We see by (3.8) that

$$\begin{aligned} &\langle t, \langle f, \langle g, h\rangle^-\rangle^-\rangle^-_{2l+1} - \langle f, \langle t, \langle g, h\rangle^-\rangle^-\rangle^-_{2l+1} \\ &= -\langle\langle g, h\rangle^-, \langle t, f\rangle^-\rangle^-_{2l+1} + R_{2l}(t, f, \pi^-_1(g, h)). \end{aligned}$$

Substituting these to (4.10), we have

$$\begin{aligned} \mathfrak{S}\pi^-_{2l+1}(fg, ht) &= \frac{1}{3}R_{2l}(t, f, \pi^-_1(g, h)) + \frac{1}{3}R_{2l}(\pi^-_1(t, f), g, h) \\ &+ \frac{1}{3}R_{2l}(t, g, \pi^-_1(f, h)) + \frac{1}{3}R_{2l}(\pi^-_1(t, g), f, h) \\ &= 0, \end{aligned} \tag{4.11}$$

because $R_m = 0$ by Corollary 3.4. Proposition 4.2 is thereby proved. □

§5. The construction of π_{even}.

The goal of this section is as follows

THEOREM 5.1. *Assume* (H.1)~(H.3) *for* $m=2k$. *There exists* $\pi_{2k} \in SC^2(\mathfrak{a})$

such that $\sum_{i+j=2k}\delta_i\pi_j=0$, *and* π_{2k} *is a bidifferential operator of order* $4k$.

Notice at first that several existence theorems which will be given in what follows for monomials x^α, x^β etc. are evenly valid for monomials $(x-x(p))^\alpha$, $(x-x(p))^\beta$ etc. for any $p\in\mathfrak{g}^*$ by usual parallel displacements.

5.1. Induction for constructing π_{ev}.

To construct π_{2k}^+, we work at first on monomials of $x_1, \cdots, x_n, \cdots$. We set

$$\pi_{2k}^+(x_i, x_j)=0, \quad (k\geqq 1). \tag{5.1}$$

For multi-indices α, β, we construct $\pi_{2k}^+(x^\alpha, x^\beta)$ inductively.

Assume the following:

$(\mathrm{B})_s$ $\pi_{2k}^+(x^\alpha, x^\beta)$ are obtained for any x^α, x^β such that $|\alpha+\beta|\leqq s$, and these satisfy (3.10), and $\pi_{2k}^+(x^\alpha, x^\beta)=\pi_{2k}^+(x^\beta, x^\alpha)$.

In what follows, we put unknown quantities $\pi_{2k}^+(x^\alpha, x^\beta)$ by $\varpi_{2k}^+(x^\alpha, x^\beta)$ for $|\alpha+\beta|=s+1$. Under $(\mathrm{B})_s$, we want at first to obtain $\varpi_{2k}^+(x_i, x^\gamma)$ for $|\gamma|+1=s+1$.

Use the following notation:

$$(x^\alpha)\in x^\mu, \qquad (x^\alpha, x^\beta, x^\gamma)\in x^\mu, \quad \text{etc.},$$

if there exist x^δ, $x^{\delta'}$ such that $x^\alpha x^\delta=x^\mu$, $x^\alpha x^\beta x^\gamma x^{\delta'}=x^\mu$, etc..

Now, for any (x_i, x^β, x_j) such that $x_i x_j x^\beta=x^\mu$, (3.10, (a)) is read as follows:

$$\varpi_{2k}^+(x_i, x^\beta x_j)-\varpi_{2k}^+(x_j, x^\beta x_i)=E_{2k}(x_i, x^\beta, x_j), \tag{5.2}$$

where E_{2k} is defined by (3.12). Set the right hand side of (5.2) by $A_{ij}(=-A_{ji})$. Under the assumption $(\mathrm{B})_s$, A_{ij}'s known quantities.

5.2. Left extremals.

We now assume that x^μ is fixed as $|\mu|=s+1$. $\varpi_{2k}^+(x_i, x^\beta x_j)$ depends only on i such that $(x_i)\in x^\mu$. Set

$$T_i=\varpi_{2k}^+(x_i, x^\beta x_j). \tag{5.3}$$

Then, (5.2) is nothing but an over determined linear system

$$T_i-T_j=A_{ij} \qquad \text{for} \quad (x_i, x_j)\in x^\mu.$$

This can be solved if and only if A_{ij} satisfy

$$A_{ij}+A_{jh}+A_{hi}=0 \qquad \text{for any} \quad (x_i, x_j, x_h)\in x^\mu. \tag{5.4}$$

First of all, we remark the following:

PROPOSITION 5.2. *For any fixed* x^μ *such that* $|\mu|=s+1$, *the solubility condition* (5.4) *is satisfied.*

Proof is seen in Appendix 6.2.

By Proposition 5.2, T_i is given by

$$(5.5)\qquad T_i = \frac{1}{n(\mu)}\sum_l A_{il} + K_{2k}(x^\mu),$$

where $n(\mu)$ is the number of (l) such that $(x_l) \in x^\mu$, and

$$K_{2k}(x^\mu) = \text{arbitrary element of } C^\infty(\mathfrak{g}^*) \text{ depending only on } x^\mu.$$

We choose simply $K_{2k}=0$ in what follows.

For a fixed μ such that $|\mu|=s+1$, we define a set of pairs of multi-indices by

$$S_\mu = \{(\alpha, \beta);\ \alpha+\beta = \mu,\ |\alpha| \geqq 1,\ |\beta| \geqq 1\}.$$

For any i, $i \geqq 1$, we denote $\langle i \rangle = (0, \cdots, 0, 1, 0, \cdots)$. An element $(\langle i \rangle, \mu - \langle i \rangle)$ (resp. $(\mu - \langle i \rangle, \langle i \rangle)$) will be called a *left extremal point* (resp. a *right extremal point*) of S_μ.

For a fixed x^μ, set $\mu(i) = \mu - \langle i \rangle$, $\mu(i, j) = \mu - \langle i \rangle - \langle j \rangle$ for any (x_i), $(x_i, x_j) \in x^\mu$. Then, we have by (5.5)

$$(5.6)\qquad \begin{aligned} &\varpi^+_{2k}(x_i - x_i(p), (x - x(p))^{\mu(i)}) \\ &= \frac{1}{n(\mu)}\sum_j E_{2k}(x_i - x_i(p), (x - x(p))^{\mu(i,j)}, x_j - x_j(p)) \qquad \forall p \in \mathfrak{g}^*. \end{aligned}$$

LEMMA 5.3. *Let* $L_i(f)(p) = \sum_\alpha \varpi^+_{2k}(x_i - x(p), (x - x(p))^\alpha)(p)\partial^\alpha f(p)$ *by using* $\varpi^+_{2k}(x_i - x_i(p), (x - x(p))^\alpha)$ *obtained by* (5.6) *for any* $(x_i - x(p), (x - x(p))^\alpha)$. *Then,* L_i *is a linear differential operator of order* $4k-1$ *for any* i.

PROOF. Replace $\varpi^+_{2k}(x_i - x(p), (x - x(p))^\alpha)(p)$ in $L_i(f)(p)$ by the right hand side of (5.6) and remark that $E_{2k}(x_i - x_i(p), (x - x(p))^{\alpha - \langle j \rangle}, x_j - x_j(p))(p)$ involves only the terms $\langle\langle\ ,\ \rangle^\pm, \rangle^\pm_{2k}$. Since $\langle\langle\ ,\ \rangle^\pm, \rangle^\pm_{2k}$ is a 3-differential operator of order $4k$ by the assumptions (H.1)-(H.3), L_i satisfies that at every $p \in \mathfrak{g}^*$ that

$$\varpi^+_{2k}(x_i, (x - x(p))^\alpha)(p) = 0 \qquad \text{for} \quad |\alpha| > 4k-1.$$

By using the similar criterion of Lemma 1.3 for 3-differential operators $\langle\langle\ ,\ \rangle^\pm, \rangle^\pm_{2k}$, we have that there is an integer s such that

$$\sum_{|\mu| < 4k} |\varpi^+_{2k}(x_i, (x - x(p))^\mu)(p)|^2 \lambda^{-2s\mu} < \infty.$$

Similarly, for any $\varepsilon > 0$, and for any $p \in \mathfrak{g}^*$, there is a neighborhood V_p of p and an integer $s > 0$ such that for any $q \in V_p$,

$$\sum_\mu |\varpi^+_{2k}(x_i, (x - x(q))^\mu)(q) - \varpi^+_{2k}(x_i, (x - x(p))^\mu)(p)|^2 \lambda^{-2s\mu} < \varepsilon.$$

Now, assume that

(1) For a fixed integer $l-1$ and an arbitrary t, there is $s=s(l-1, t)$ such that

$$\sum_{|\gamma|=l-1}\sum_{\mu}|\partial^{\gamma}\varpi_{2k}^{+}(x_i-x(p), (x-x(p))^{\mu})(p)|^2\lambda^{2t\gamma}\lambda^{-2s\mu}<\infty.$$

(2) For any $\varepsilon>0$, t, and for any $p\in\mathfrak{g}^*$, there is a neighborhood V_p of p and an integer $s=s(l-1, t, V_p)$ such that for any $q\in V_p$,

$$\sum_{|\gamma|=l-1}\sum_{\mu}|\partial^{\gamma}\varpi_{2k}^{+}(x_i, (x-x(q))^{\mu})(q)-\partial^{\gamma}\varpi_{2k}^{+}(x_i, (x-x(p))^{\mu})(p)|^2\lambda^{2t\gamma}\lambda^{-2s\mu}<\varepsilon.$$

We shall show that same inequalities as (1), (2) hold for l. Recall (3.11), and we see that $(\partial^{\gamma}E_{2k}(x_i-x_i(p), (x-x(p))^{\alpha}, x_j-x_j(p))(p)$ involves the partial derivatives $\partial^{\beta}\varpi_{2k}^{+}$ up to only $|\beta|\leq l-1$. Hence, the assumptions (1), (2) can be applied. Other terms are written as $\langle\langle\ ,\ \rangle^{\pm},\ \rangle_{2k}^{\pm}$. By using the similar criterion as in Lemma 1.3 for 3-differential operators $\langle\langle\ ,\ \rangle^{\pm},\ \rangle_{2k}^{\pm}$, we obtain the lemma.

□

5.3. Bridges.

Using the left extremal points, we shall construct $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$ for the pair of multi-indices (α, β) with $\alpha+\beta=\mu$,

DEFINITION 5.4. For pairs of multi-indices (α, β) and (α', β') such that there is γ with $\alpha'=\alpha+\gamma$, $\beta'=\beta-\gamma$, and $\alpha+\beta=\alpha'+\beta'=\mu$. The *bridge relation* $(Br)_{\gamma}$ from (α, β) to (α', β') is the following:

$$(Br)_{\gamma}\qquad \varpi_{2k}^{+}(x^{\alpha'}, x^{\beta'})-\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})=-E_{2k}(x^{\alpha}, x^{\gamma}, x^{\beta'}),$$

where

$$\begin{aligned}E_{2k}(x^{\alpha}, x^{\gamma}, x^{\beta'})&=\pi_{2k}^{+}(x^{\alpha}, x^{\gamma})x^{\beta'}-x^{\alpha}\pi_{2k}^{+}(x^{\gamma}, x^{\beta'})\\&\quad+\langle\langle x^{\alpha}, x^{\gamma}\rangle^{+}, x^{\beta'}\rangle_{2k}^{+}-\langle x^{\alpha}, \langle x^{\gamma}, x^{\beta'}\rangle^{+}\rangle_{2k}^{+}\\&\quad-\langle x^{\gamma}, \langle x^{\alpha}, x^{\beta'}\rangle^{-}\rangle_{2k}^{-}\qquad \text{(cf. (3.12))}.\end{aligned}$$

If $(\alpha, \beta), (\alpha', \beta')\in S_{\mu}$ have the bridge relation $(Br)_{\gamma}$, we denote by $(\alpha, \beta)\overset{\gamma}{\rightsquigarrow}(\alpha', \beta')$ (or $(x^{\alpha}, x^{\beta})\overset{\gamma}{\rightsquigarrow}(x^{\alpha'}, x^{\beta'})$).

Note that if $(\alpha, \beta)\overset{\gamma}{\rightsquigarrow}(\alpha', \beta')$, then $(\beta', \alpha')\overset{\gamma}{\rightsquigarrow}(\beta, \alpha)$, which is called the *dual bridge relation* to $(\alpha, \beta)\overset{\gamma}{\rightsquigarrow}(\alpha', \beta')$. The following lemma shows that any chain of bridges from a point of S_{μ} to another can be replaced by a direct bridge:

LEMMA 5.5. *For $(\alpha, \beta+\gamma+\gamma')$, $(\alpha+\gamma, \beta+\gamma')$, $(\alpha+\gamma+\gamma', \beta)\in S_{\mu}$, the relations $(\alpha, \beta+\gamma+\gamma')\overset{\gamma}{\rightsquigarrow}(\alpha+\gamma, \beta+\gamma')$ and $(\alpha+\gamma, \beta+\gamma')\overset{\gamma'}{\rightsquigarrow}(\alpha+\gamma+\gamma', \beta)$ generate the relation $(\alpha, \beta+\gamma+\gamma')\overset{\gamma+\gamma'}{\rightsquigarrow}(\alpha+\gamma+\gamma', \beta)$.*

PROOF. Let $f=x^{\alpha}$, $g=x^{\gamma}$, $h=x^{\gamma'}$, $k=x^{\beta}$ for the simplicity. By Proposition 2.2, we see that $\delta_0 Q_{2k}=0$. Using (3.6) and Corollary 3.4, we have

(5.7) $$Q_{2k}(a, b, c) = \langle a, \langle b, c\rangle^{+}\rangle_{2k}^{+} - \langle\langle a, b\rangle^{+}, c\rangle_{2k}^{+} + \langle b, \langle a, c\rangle^{-}\rangle_{2k}^{-}.$$

The bridge relations $(Br)_{\gamma}$, $(Br)_{\gamma'}$, $(Br)_{\gamma+\gamma'}$ are written as follows:

$$\begin{cases} -f\pi_{2k}^{+}(g, ht) + \varpi_{2k}^{+}(fg, ht) - \varpi_{2k}^{+}(f, ght) + \pi_{2k}^{+}(f, g)ht = Q_{2k}(f, g, ht), \\ -fg\pi_{2k}^{+}(h, t) + \varpi_{2k}^{+}(fgh, t) - \varpi_{2k}^{+}(fg, ht) + \pi_{2k}^{+}(fg, h)t = Q_{2k}(fg, h, t), \\ -f\pi_{2k}^{+}(gh, t) + \varpi_{2k}^{+}(fgh, t) - \varpi_{2k}^{+}(f, ght) + \pi_{2k}^{+}(f, gh)t = Q_{2k}(f, gh, t). \end{cases}$$

Computing $-(Br)_{\gamma} - (Br)_{\gamma'} + (Br)_{\gamma+\gamma'}$, we get

(5.8) $$\begin{aligned} &f(\delta_0\pi_{2k}^{+})(g, h, t) + (\delta_0\pi_{2k}^{+})(f, g, h)t \\ &\qquad = -Q_{2k}(f, g, ht) - Q_{2k}(fg, h, t) + Q_{2k}(f, gh, t). \end{aligned}$$

By the assumption $(\mathrm{B})_s$, we have

$$(\delta_0\pi_{2k}^{+})(g, h, t) = -Q_{2k}(g, h, t), \qquad (\delta_0\pi_{2k}^{+})(f, g, h) = -Q_{2k}(f, g, h).$$

Hence, (5.8) is

$$-fQ_{2k}(g, h, t) - Q_{2k}(f, g, h)t = -Q_{2k}(fg, h, t) + Q_{2k}(f, gh, t) - Q_{2k}(f, g, ht).$$

This holds because of $\delta_0 Q_{2k} = 0$. □

Note that by (5.7), we see easily that

(5.9) $$\mathop{\mathfrak{S}}_{(f, g, h)} Q_{2k}(f, g, h) = 0.$$

By a similar manner, we have

LEMMA 5.6. *If there are relations*

$$(\langle i\rangle, \mu - \langle i\rangle) \overset{\gamma}{\rightsquigarrow} (\alpha, \beta), \qquad (\langle j\rangle, \mu - \langle j\rangle) \overset{\gamma'}{\rightsquigarrow} (\alpha, \beta),$$

then the computation of $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$ *does not depend on* $(Br)_{\gamma}$ *and* $(Br)_{\gamma'}$, *where the initial conditions for the bridges are given by* (5.3), (5.5).

PROOF. One may assume that $i \neq j$. Since there are bridges, (x^{α}, x^{β}) must be given in the shape $(x_i x_j h, x^{\beta})$. We set $t = x^{\beta}$ for simplicity. Then, $(Br)_{\gamma}$, $(Br)_{\gamma'}$ are written as follows:

(5.10) $$\varpi_{2k}^{+}(x_i x_j h, t) = \varpi_{2k}^{+}(x_i, x_j ht) + x_i\pi_{2k}^{+}(x_j h, t) - \pi_{2k}^{+}(x_i, x_j h)t + Q_{2k}(x_i, x_j h, t),$$

(5.11) $$\varpi_{2k}^{+}(x_j x_i h, t) = \varpi_{2k}^{+}(x_j, x_i ht) + x_j\pi_{2k}^{+}(x_i h, t) - \pi_{2k}^{+}(x_j, x_i h)t + Q_{2k}(x_j, x_i h, t).$$

We have only to show the right hand side of (5.10)-(5.11) vanishes. Note that $\varpi_{2k}^{+}(x_i, x^{\alpha})$ satisfies (5.2). By (5.2), we have

(5.12) $$\begin{aligned} &\varpi_{2k}^{+}(x_i, htx_j) - \varpi_{2k}^{+}(x_j, htx_i) \\ &\qquad = -x_i\pi_{2k}^{+}(ht, x_j) + \pi_{2k}^{+}(x_i, ht)x_j - Q_{2k}(x_i, ht, x_j). \end{aligned}$$

Using (5.12), we compute the right hand side of (5.11). So, the right hand side of (5.10)-(5.11) is

$$
\begin{aligned}
(5.13)\qquad & x_i(\pi^+_{2k}(x_j h,\ t)-\pi^+_{2k}(ht,\ x_j)) \\
& +x_j(\pi^+_{2k}(x_i,\ ht)-\pi^+_{2k}(x_i h,\ t)) \\
& +t(\pi^+_{2k}(x_j,\ x_i h)-\pi^+_{2k}(x_i,\ x_j h)) \\
& +Q_{2k}(x_i,\ x_j h,\ t)-Q_{2k}(x_j,\ x_i h,\ t)-Q_{2k}(x_i,\ ht,\ x_j).
\end{aligned}
$$

By the assumption $(B)_s$, (5.13) is

$$
\begin{aligned}
& x_i Q_{2k}(x_j,\ h,\ t)-x_j Q_{2k}(x_i,\ h,\ t)-tQ_{2k}(x_j,\ h,\ x_i) \\
& \quad +Q_{2k}(x_i,\ x_j h,\ t)+Q_{2k}(t,\ x_i h,\ x_j)+Q_{2k}(x_j,\ ht,\ x_i).
\end{aligned}
$$

Recalling the definition of $\delta_0 Q_{2k}$ and using (5.9), we see that the above quantity is

$$
(5.14)\qquad (\delta_0 Q_{2k})(x_i,\ x_j,\ h,\ t)-(\delta_0 Q_{2k})(x_j,\ x_i,\ h,\ t)=0. \quad \square
$$

5.4. Right extremals.

As we have shown in 5.2, we have obtained $\varpi^+_{2k}(x_i,\ x^\alpha)$ for $\alpha+\langle i\rangle=\mu,\ |\mu|=s+1$. Next, we shall determine $\varpi^+_{2k}(x^\alpha,\ x_i)$ for $\alpha+\langle i\rangle=\mu,\ |\mu|=s+1$. Given $(x^\alpha,\ x_i)$, there are a pair $(x_j,\ x^\beta)$ and a multi-index γ such that $(x_j,\ x^\beta)\overset{\gamma}{\rightsquigarrow}(x^\alpha,\ x_i)$. Thus, we can get $\varpi^+_{2k}(x^\alpha,\ x_i)$ by $(Br)_\gamma$. By Lemma 5.6, $\varpi^+_{2k}(x^\alpha,\ x_i)$ is independent of the choice of γ and $(x_j,\ x^\beta)$. We now show that $\varpi^+_{2k}(x_i,\ x^\alpha)=\varpi^+_{2k}(x^\alpha,\ x_i)$.

First of all, we easily have

LEMMA 5.7. *For any i, j and a multi-index α, we have*

$$
(5.15)\qquad \varpi^+_{2k}(x^\alpha x_i,\ x_j)=\varpi^+_{2k}(x_j,\ x^\alpha x_i).
$$

PROOF. Consider a bridge relation $(\langle i\rangle,\ \alpha+\langle j\rangle)\overset{\alpha}{\rightsquigarrow}(\alpha+\langle i\rangle,\ \langle j\rangle)$ and we have

$$
(5.16)\qquad \varpi^+_{2k}(x^\alpha x_i,\ x_j)=\varpi^+_{2k}(x_i,\ x^\alpha x_j)-E_{2k}(x_i,\ x^\alpha,\ x_j)
$$

by $(Br)_\alpha$. On the other hand, we write down (5.2) for $(x_j,\ x^\alpha x_i)$:

$$
(5.17)\qquad \varpi^+_{2k}(x_j,\ x^\alpha x_i)=\varpi^+_{2k}(x_i,\ x^\alpha x_j)+A_{ji}.
$$

Combining (5.16) with (5.17), we have (5.15). $\square$

Using Lemma 5.7, we have:

LEMMA 5.8. *$\varpi^+_{2k}(x_i,\ x^\alpha)=\varpi^+_{2k}(x^\alpha,\ x_i)$ for any i and α.*

5.5. Determination for $\varpi^+_{2k}(x^\alpha,\ x^\beta)$.

To determine $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$, we choose a left extremal point (x_i, x^{δ}) such that $(x_i, x^{\delta})\overset{\gamma}{\rightsquigarrow}(x^{\alpha}, x^{\beta})$. Thus, we put $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$ by $(Br)_{\gamma}$, which also does not depend on the choice of γ and (x_i, x^{δ}).

We now prove

PROPOSITION 5.9. *Under the assumptions* (HE.1-3), $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$ *can be constructed so that they satisfy* $(Br)_{\gamma}$, $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})=\varpi_{2k}^{+}(x^{\beta}, x^{\alpha})$, *and* ϖ_{2k}^{+} *is a bidifferential operator of order* $4k$.

PROOF. Using the bridge relation

$$\tag{5.18} \begin{cases} \varpi_{2k}^{+}(x^{\gamma+\langle i\rangle}, x^{\beta})-\varpi_{2k}^{+}(x_i, x^{\gamma+\beta}) = -E_{2k}(x_i, x^{\gamma}, x^{\beta}), \\ \varpi_{2k}^{+}(x^{\gamma+\beta}, x_i)-\varpi_{2k}^{+}(x^{\beta}, x^{\gamma+\langle i\rangle}) = -E_{2k}(x^{\beta}, x^{\gamma}, x_i). \end{cases}$$

Hence, we have $\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})=\varpi_{2k}^{+}(x^{\beta}, x^{\alpha})$ for $|\alpha+\beta|=s+1$. This implies that for any α, β, γ with $\alpha+\beta+\gamma=\mu$, the equation $(Br)_{\gamma}$ is equal to that of (3.11) substituted by $f=x^{\alpha}$, $g=x^{\gamma}$, $h=x^{\beta}$. Then, we get the first and the second part of Proposition 5.9. This construction can be applied for monomials $(x-x(p))^{\alpha}$, $(x-x(q))^{\beta}$, etc..

To prove the last part, remark that

$$\begin{aligned} &\varpi_{2k}^{+}((x-x(p))^{\alpha}, (x-x(p))^{\beta}) \\ &\quad = \varpi_{2k}^{+}(x_i, (x-x(p))^{\alpha+\beta-\langle i\rangle})-E_{2k}(x_i, (x-x(p))^{\alpha-\langle i\rangle}, (x-x(p))^{\beta}), \end{aligned}$$

for an $(x_i)\in x^{\alpha}$. By a similar proof as in Lemma 5.3, we have the desired result. Namely, we obtain by induction that ϖ_{2k}^{+} satisfies that for any l, t, there is an integer $s=s(l, t)$ such that

$$\sum_{|\gamma|=l}\sum_{\alpha,\beta}|\partial^{\gamma}\varpi_{2k}^{+}((x-x(p))^{\alpha}, (x-x(p))^{\beta})(p)|^{2}\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)} < \infty,$$

and that for any $\varepsilon>0$ and l, t, there is a neighborhood V_p of p in $\mathfrak{g}^*$ and s such that for any $q\in V_p$,

$$\begin{aligned} \sum_{|\gamma|=k}\sum_{\alpha,\beta}|\partial^{\gamma}\varpi_{2k}^{+}((x-x(p))^{\alpha}, (x-x(p))^{\beta})(p)-\partial^{\gamma}\varpi_{2k}^{+}((x-x(q))^{\alpha}, (x-x(q))^{\beta})(q)|^{2} \\ \times\lambda^{2t\gamma}\lambda^{-2s(\alpha+\beta)} < \varepsilon. \quad \square \end{aligned}$$

We now put $\pi_{2k}^{+}(x^{\alpha}, x^{\beta})=\varpi_{2k}^{+}(x^{\alpha}, x^{\beta})$. The symmetricity of π_{2k}^{+} is obtained by the polynomial approximation theorem and Proposition 5.9. Theorem 5.1 is thereby proved, and we obtain Theorem A.

§ 6. Appendix.

6.1. Proof of Lemma 3.1.

If $\delta_0 Q=0$, then $\delta_0(1+\sigma_3)Q=0$ by Lemma 2.3. Set $Q^{+}=(1/2)(1+\sigma_3)Q$. Note that $\delta_0=\partial_1^0-\partial_2^0+\partial_3^0$ by Lemma 2.3, (ii). Thus, we have $(\partial_2^0-\partial_3^0)Q^+=\partial_1^0Q^+$. Using Lemma 2.3, we have $(\partial_2^0-\partial_3^0)c_3^2=c_4^3(\partial_1^0-\partial_2^0)$. So, we get

$$(\partial_2^0-\partial_3^0)c_3^2Q^+ = -c_4^3\partial_3^0Q^+ .$$

Hence,

$$(\partial_2^0-\partial_3^0)(1-c_3+c_3^2)Q^+ = \partial_1^0Q^+-(\partial_2^0-\partial_3^0)c_3Q^+-c_4^3\partial_3^0Q^+ . \tag{6.1}$$

Evaluating the right hand side of (6.1) at (f, g, h, t), we have

$$\begin{aligned} &\underset{\blacktriangle}{f}\cdot Q^+(g, h, t)-Q^+(f\cdot g, h, t)+\underline{Q^+(f, h, t)\cdot g} \\ &-g\cdot Q^+(t, f, h)+Q^+(t, f, g\cdot h)-\underline{Q^+(h\cdot t, f, g)+Q^+(h, f, g)\cdot t} \\ &-t\cdot Q^+(f, h, g)+Q^+(g, h, t\cdot f)-Q^+(g, \underset{\blacktriangle}{h}, t)\cdot f , \end{aligned} \tag{6.2}$$

where $f\cdot g=\pi_0(f, g)$. The terms marked by $\blacktriangle$ are trivially cancelled. Use $\sigma_3Q^+=Q^+$, $\delta_0Q=0$, to the underlined terms of (6.2). Then, these terms are changed into $Q^+(g\cdot f, h, t)-Q^+(g, f\cdot h, t)$. Hence (6.2) is

$$-Q^+(g, f\cdot h, t)-g\cdot Q^+(t, f, h)+Q^+(t, f, g\cdot h)-t\cdot Q^+(f, h, g)+Q^+(g, h, t\cdot f).$$

Using $\sigma_3Q^+=Q^+$ to $Q^+(g, h, t\cdot f)$, we see that (6.2) is $-(\delta_0Q^+)(t, f, h, g)=0$. □

6.2. Proof of Proposition 5.2.

We shall show that (5.4) is satisfied under the assumptions (H. 1-2). For that purpose, we shall investigate (3.11) more precisely. For any fixed (f, g, h), (3.11) can be regarded as a linear system with unknowns $\pi_{2k}^+(f, gh)$, $\pi_{2k}^+(g, hf)$, $\pi_{2k}^+(h, fg)$:

$\pi_{2k}^+(f, gh)$	$\pi_{2k}^+(g, hf)$	$\pi_{2k}^+(h, fg)$	
1	0	-1	: $E_{2k}(f, g, h)$
-1	1	0	: $E_{2k}(g, h, f)$
0	-1	1	: $E_{2k}(h, f, g)$

The solubility condition of the above linear system is satisfied by virtue of $R_{2k}=0$. Set

$$S_{2k}(f, g, h) = \sum_{(f, g, h)} \pi_{2k}^+(f, gh). \tag{6.3}$$

Then, $S_{2k}\in SC^3(\mathfrak{a})$. By using (3.12), the solution of the linear system is written as follows:

$$\pi_{2k}^+(f, gh) = \frac{1}{3}S_{2k}(f, g, h)+\frac{1}{3}\pi_{2k}^+(f, g)h+\frac{1}{3}\pi_{2k}^+(f, h)g-\frac{2}{3}f\pi_{2k}^+(g, h) \tag{ε_{2k}}$$

$$+\frac{1}{3}\langle\langle f, g\rangle^{+}, h\rangle_{2k}^{+}+\frac{1}{3}\langle\langle f, h\rangle^{+}, g\rangle_{2k}^{+}-\frac{2}{3}\langle\langle g, h\rangle^{+}, f\rangle_{2k}^{+}$$

$$+\frac{1}{3}\langle\langle f, g\rangle^{-}, h\rangle_{2k}^{-}+\frac{1}{3}\langle\langle f, h\rangle^{-}, g\rangle_{2k}^{-}.$$

All others are obtained by the cyclic permutation of (f, g, h). Note also that the above formula can be applied for π_m^+ such that $m\leqq 2k-1$.

Suppose $(x_i, x_j, x_h)\in x^{\mu}$, i.e. there is a monomial g such that $x_i x_j x_h g=x^{\mu}$. By (3.12), we have

$$A_{ij}+A_{jh}+A_{hi}=\sum_{(i,j,h)}[\pi_{2k}^{+}(x_i, gx_h)x_j-\pi_{2k}^{+}(x_j, gx_h)x_i \tag{6.4}$$
$$+\langle\langle x_i, gx_h\rangle^{+}, x_j\rangle_{2k}^{+}-\langle\langle x_j, gx_h\rangle^{+}, x_i\rangle_{2k}^{+}$$
$$+\langle\langle x_i, x_j\rangle^{-}, gx_h\rangle_{2k}^{-}]$$
$$=(1)+(2)+(3),$$

where

$$(1)=\sum_{(i,j,h)} x_i\{\pi_{2k}^{+}(x_h, gx_j)-\pi_{2k}^{+}(x_j, gx_h)\}=\sum_{(i,j,h)} x_i E_{2k}(x_h, g, x_j)$$
$$(2)=\sum_{(i,j,h)}\langle x_i, \langle x_h, gx_j\rangle^{+}-\langle x_j, gx_h\rangle^{+}\rangle_{2k}^{+}$$
$$(3)=\sum_{(i,j,h)}\langle\langle x_i, x_j\rangle^{-}, gx_h\rangle_{2k}^{-}.$$

Recalling (3.8) and using (4.2) for the term (3), we have

$$(3)=\sum x_h\langle\langle x_i, x_j\rangle^{-}, g\rangle_{2k}^{-} \tag{6.5}$$
$$+\sum\langle\langle\langle x_i, x_j\rangle^{-}, g\rangle^{-}, x_h\rangle_{2k}^{+}-\sum\langle\langle x_i, x_j\rangle^{-}, \langle g, x_h\rangle^{+}\rangle_{2k}^{-},$$

where we used

$$\sum\langle\langle\langle x_i, x_j\rangle^{-}, x_h\rangle^{-}, g\rangle_{2k}^{+}=\sum_{a+b=2k,\, a,b\geq 1}\pi_a^{+}(R_b(x_i, x_j, x_h), g)=0.$$

From (3.12), we have

$$(1)=\sum x_i\{\langle\langle x_h, g\rangle^{+}, x_j\rangle_{2k}^{+}-\langle\langle x_j, g\rangle^{+}, x_h\rangle_{2k}^{+}\} \tag{6.6}$$
$$+\sum x_i\langle\langle x_h, x_j\rangle^{-}, g\rangle_{2k}^{-}.$$

Note that in (1)+(3) the last term of (6.6) and the first term of (6.5) are cancelled out. Use (3.11-12) to (2), and remark that $R_m=0$. Then, we see

(6.7)

$$A_{ij}+A_{jh}+A_{hi}$$
$$=\sum\langle\langle g, x_h\rangle^{+}, \langle x_i, x_j\rangle^{-}\rangle_{2k}^{-}+\sum\langle\langle\langle x_i, x_j\rangle^{-}, g\rangle^{-}, x_h\rangle_{2k}^{+}$$
$$+\sum x_i\{\langle\langle x_h, g\rangle^{+}, x_j\rangle_{2k}^{+}-\langle\langle x_j, g\rangle^{+}, x_h\rangle_{2k}^{+}\}+\sum\langle x_i, \langle x_h, g\rangle^{+}x_j-\langle x_j, g\rangle^{+}x_h\rangle_{2k}^{+}$$
$$+\sum\langle x_i, \langle\langle x_h, g\rangle^{+}, x_j\rangle^{+}-\langle\langle x_j, g\rangle^{+}, x_h\rangle^{+}\rangle_{2k}^{+}+\sum\langle x_i, \langle\langle x_h, x_j\rangle^{-}, g\rangle^{-}\rangle_{2k}^{+}.$$

Note that the second term and the last term of the right hand side of (6.7) are cancelled out. We now use (ε_{2k}) to the second term of the second line in (6.7). After a little complicated rearrangement of the terms, we have

(6.8)

$$
\begin{aligned}
&A_{ij}+A_{jh}+A_{hi}\\
&=\mathfrak{S}x_i\cdot\langle\langle x_h, g\rangle^+, x_j\rangle^+_{2k}-\mathfrak{S}x_i\cdot\langle\langle x_j, g\rangle^+, x_h\rangle^+_{2k}+\mathfrak{S}\langle\langle g, x_h\rangle^+, \langle x_i, x_j\rangle^-\rangle^-_{2k}\\
&\quad+\mathfrak{S}\langle x_i, \langle x_j, \langle x_h, g\rangle^+\rangle^+\rangle^+_{2k}-\mathfrak{S}\langle x_i, \langle x_h, \langle x_j, g\rangle^+\rangle^+\rangle^+_{2k}\\
&\quad+\frac{1}{3}\underset{\bigstar}{\mathfrak{S}\sum_{a+b=2k}S_a(x_i, \pi_b^+(x_h, g), x_j)}-\frac{1}{3}\underset{\bigstar}{\mathfrak{S}\sum_{a+b=2k}S_a(x_i, \pi_b^+(x_j, g), x_h)}\\
&\quad+\frac{1}{3}\underset{\blacktriangle}{\mathfrak{S}\langle x_i, x_j\rangle^+\cdot\langle x_h, g\rangle^+}+\frac{1}{3}\mathfrak{S}x_j\cdot\langle x_i, \langle x_h, g\rangle^+\rangle^+_{2k}-\frac{2}{3}\mathfrak{S}x_i\cdot\langle x_j, \langle x_h, g\rangle^+\rangle^+_{2k}\\
&\quad-\frac{1}{3}\underset{\blacktriangle}{\mathfrak{S}\langle x_i, x_h\rangle^+\cdot\langle x_j, g\rangle^+}-\frac{1}{3}\mathfrak{S}x_h\cdot\langle x_i, \langle x_j, g\rangle^+\rangle^+_{2k}+\frac{2}{3}\mathfrak{S}x_i\cdot\langle x_h, \langle x_j, g\rangle^+\rangle^+_{2k}\\
&\quad+\frac{1}{3}\underset{\blacklozenge}{\mathfrak{S}\langle\langle x_i, x_j\rangle^+, \langle x_h, g\rangle^+\rangle^+_{2k}}+\frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_h, g\rangle^+\rangle^+, x_j\rangle^+_{2k}\\
&\qquad\qquad\qquad-\frac{2}{3}\mathfrak{S}\langle x_i, \langle x_j, \langle x_h, g\rangle^+\rangle^+\rangle^+_{2k}\\
&\quad-\frac{1}{3}\underset{\blacklozenge}{\mathfrak{S}\langle\langle x_i, x_h\rangle^+, \langle x_j, g\rangle^+\rangle^+_{2k}}-\frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_j, g\rangle^+\rangle^+, x_h\rangle^+_{2k}\\
&\qquad\qquad\qquad+\frac{2}{3}\mathfrak{S}\langle x_i, \langle x_h, \langle x_j, g\rangle^+\rangle^+\rangle^+_{2k}\\
&\quad+\frac{1}{3}\mathfrak{S}\langle\langle x_i, x_j\rangle^-, \langle x_h, g\rangle^+\rangle^-_{2k}+\frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_h, g\rangle^+\rangle^-, x_j\rangle^-_{2k}\\
&\quad-\frac{1}{3}\mathfrak{S}\langle\langle x_i, x_h\rangle^-, \langle x_j, g\rangle^+\rangle^-_{2k}-\frac{1}{3}\mathfrak{S}\langle\langle x_i, \langle x_j, g\rangle^+\rangle^-, x_h\rangle^-_{2k},
\end{aligned}
$$

where $A^+\cdot B^+$ means $\sum_{a+b=2k, a, b\geq 1}A_a^+B_b^+$. The terms marked by $\blacktriangle$, $\bigstar$, $\blacklozenge$ are cancelled out respectively. Since

$$\mathfrak{S}x_i\cdot\langle\langle x_h, g\rangle^+, x_j\rangle^+_{2k}=\mathfrak{S}x_i\cdot\langle x_j, \langle x_h, g\rangle^+\rangle^+_{2k}=\mathfrak{S}x_h\cdot\langle x_i, \langle x_j, g\rangle^+\rangle^+_{2k},$$

the six terms involving $\cdot$ of (6.8) are cancelled out. Note also that

(6.9)
$$
\begin{aligned}
&\mathfrak{S}\langle\langle x_i, \langle x_j, g\rangle^+\rangle^+, x_h\rangle^+_{2k}=\mathfrak{S}\langle x_i, \langle x_j, \langle x_h, g\rangle^+\rangle^+\rangle^+_{2k}\\
&\mathfrak{S}\langle\langle x_i, x_h\rangle^-, \langle x_j, g\rangle^+\rangle^-_{2k}=-\mathfrak{S}\langle\langle x_i, x_j\rangle^-, \langle x_h, g\rangle^+\rangle^-_{2k}.
\end{aligned}
$$

Finally, (6.8) is reduced to the following:

(6.10)

$$-\frac{1}{3}\textstyle\sum\langle\langle x_i,\ x_j\rangle^-,\ \langle x_h,\ g\rangle^+\rangle_{2k}^- + \frac{1}{3}\textstyle\sum\langle\langle x_i,\ \langle x_h,\ g\rangle^+\rangle^-,\ x_j\rangle_{2k}^-$$

$$-\frac{1}{3}\textstyle\sum\langle\langle x_i,\ \langle x_j,\ g\rangle^+\rangle^-,\ x_h\rangle_{2k}^-$$

$$= -\frac{1}{3}\sum_{(i,j,h)}\{\langle\langle x_i,\ x_j\rangle^-,\ \langle x_h,\ g\rangle^+\rangle_{2k}^- + \langle\langle\langle x_h,\ g\rangle^+,\ x_i\rangle^-,\ x_j\rangle_{2k}^-$$

$$+\langle\langle x_i,\ \langle x_j,\ g\rangle^+\rangle^-,\ x_h\rangle_{2k}^-\}$$

$$= -\frac{1}{3}\sum_{a+b=2k,\, a,b\geq 1}\ \sum_{(i,j,h)} R_a(x_i,\ x_j,\ \pi_b^+(x_h,\ g)) = 0.$$

So, $\varpi_{2k}^+(x_i,\ x^\alpha)$ is obtained by (5.5) for any $(x_i,\ x^\alpha)$ such that $x_i x^\alpha = x^\mu$. Thus, Proposition 5.2 is proved. □

References

[Be] F. A. Berezin, Quantization, Math. USSR-Izv., **8** (1974), 1109-1165.

[BK] R. H. Bruck and E. Kleinfeld, The structure of alternative division rings, Proc. Amer. Math. Soc., **2** (1951), 878-890.

[CG] M. Cahen and S. Gutt, Regular * representations of Lie algebras, Lett. Math. Phys., **6** (1982), 395-404.

[G2] S. Gutt, Some aspects of deformation theory and quantization, Quantum Theories and Geometry, (eds., M. Cahen and M. Flato), Kluwer Academic Publishers, 1988, pp. 77-102.

[Ma] Y. Matsushima, Theory of Lie Algebras, (in Japanese) Gendai Suugaku-kouza 15, Kyouritsu Press.

[Mc] S. MacLane, Homology, Springer, 1963.

[S] R. Schafer, An Introduction to Non Associative Algebras, Academic Press, 1976.

[OMY1] H. Omori, Y. Maeda and A. Yoshioka, Weyl manifolds and deformation quantization, Adv. in Math., **85** (1991), 224-255.

[OMY2] H. Omori, Y. Maeda and A. Yoshioka, On a construction of deformation quantization of Poisson algebras, Proceedings of the Workshop on Geometry and its applications in honor of Morio Obata, (eds., T. Nagano, H. Omori, Y. Maeda and M. Kanai), World Scientific, 1993, pp. 201-218.

[Ve] J. Vey, Déformation du crochet de Poisson sur variété symplectique, Comment. Math. Helv., **50** (1975), 421-454.

[VK] Yu. M. Vorob'ev and V. Karasev, Poisson manifolds and the Schouten bracket, Functional Anal. Appl., **22** (1988), 1-9.

[W] A. Weinstein, The local structure of Poisson manifolds, J. Differential Geom., **18** (1983), 523-557.

Hideki OMORI
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba 278
Japan

Yoshiaki MAEDA
Department of Mathematics
Faculty of Science and Technology
Keio University
Hiyoshi, Yokohama 223
Japan

Akira YOSHIOKA
Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba 278
Japan