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Group actions and deformations for harmonic maps

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Introduction.

From the theory of integrable systems it is known that harmonic maps from a Riemann surface to a Lie group may be studied by infinite dimensional methods (cf. [**ZM**], [**ZS**]). This was clarified considerably by the papers [**Uh**], [**Se**], especially in the case of maps from the Riemann sphere S^2 to the unitary group U_n. The basic connection with infinite dimensional methods is the correspondence between harmonic maps $S^2 \to G$ and "extended solutions" $S^2 \to \Omega G$, where G is any compact Lie group and ΩG is its (based) loop group. In [**Uh**] this was used in two ways (in the case $G=U_n$):

(1) to introduce a *group action* of matrix valued rational functions on harmonic maps, and

(2) to prove a *factorization theorem* for harmonic maps, which unifies and extends many of the known results on the classification of harmonic maps from S^2 into various homogeneous spaces.

In [**Se**] it was shown that the factorization theorem can be proved very naturally by using the "Grassmannian model" of ΩG, which is an identification of ΩG with a certain infinite dimensional Grassmannian (see [**PS**]). In this paper we shall show how the group action may be interpreted in terms of the Grassmannian model. The advantages of this point of view are that the geometrical nature of the action is emphasized, and that calculations become easier. We shall illustrate this by giving some applications to deformations of harmonic maps. By using some elementary ideas from Morse theory, we obtain new results on the connectedness of spaces of harmonic maps, a subject which has been studied recently by various *ad hoc* methods (for example, in [**Ve1**], [**Ve2**], [**Ve3**], [**Lo**], [**Kt**]).

The paper is arranged as follows. In § 1 we give the basic definitions, including that of a "generalized Birkhoff pseudo-action". The latter is an action of k-tuples of loops γ on extended solutions Φ, denoted by $(\gamma, \Phi) \mapsto \gamma^{\#}\Phi$. This definition involves a Riemann-Hilbert factorization (a generalization of the Birkhoff factorization for loops), and is an example of a "dressing action" in the theory of integrable systems. Because the factorization cannot always be carried out, the action is defined only for certain γ and Φ, so we call it a pseudo-action.

Nevertheless, it is possible to establish some general properties of the action by using contour integral formulae, and we shall use these to show that the most important case of a generalized Birkhoff pseudo-action is precisely the one introduced by Uhlenbeck. In § 2 we go on to show that the Uhlenbeck action on harmonic maps $S^2 \to U_n$ of fixed energy "collapses" to the pseudo-action of a finite dimensional group. This collapsing phenomenon has been described from a different point of view in [**AJS**], [**AS1**], [**AS2**], [**JK**].

The Grassmannian model and its relevance for harmonic maps are reviewed in § 3. From this point of view there is a natural action of the complex group ΛG^C on extended solutions, where G^C is the complexification of G and ΛG^C is its (free) loop group. This action is denoted by $(\gamma, \Phi) \mapsto \gamma^\natural \Phi$. Elementary properties of this action—which really is an action, not a pseudo-action—are given in § 4. In particular, it is easy to see that this action, like the Uhlenbeck action, collapses to an action of a finite dimensional Lie group.

Our first main result appears in § 5, where we show that the actions $\sharp$ and $\natural$ are essentially the same, despite their very different definitions. The essential point here is that the explicit Riemann-Hilbert factorization needed for $\sharp$ is incorporated into the definition of the Grassmannian needed for $\natural$. This result explains the similarities between the properties of the action $\sharp$ (described in § 1 and § 2) and the properties of the action $\natural$ (described in § 3 and § 4). In particular, it "explains" and extends Theorem 9.4 of [**Uh**].

In § 6, we discuss applications of the action $\natural$ to deformations of harmonic maps. A one-parameter subgroup $\{\gamma_t\}$ of ΛG^C gives rise to a deformation $\Phi_t = \gamma_t{}^\natural \Phi$ of an extended solution Φ. This deformation has a simple geometrical interpretation: it is the result of applying the gradient flow of a suitable Morse-Bott function on ΩG to the extended solution Φ. Hence, we obtain a new extended solution $\Phi_\infty = \lim_{t\to\infty} \Phi_t$ which takes values (almost everywhere) in a critical manifold of this Morse-Bott function. In general, Φ_∞ has a finite number of (removable) singularities. This illustrates the well-known fact (see [**SU**]) that a sequence of harmonic maps (of S^2) has a convergent subsequence over the complement of a finite set, the latter being points at which "bubbling off" occurs. We shall give some examples where the singularities do not occur, so that Φ_∞ is joined to Φ by a continuous path in the space of extended solutions. The main example is the following. Let $\varphi : S^2 \to U_n$ be a harmonic map, with corresponding normalized extended solution $\Phi = \sum_{\alpha=0}^m T_\alpha \lambda^\alpha$ (this notation will be explained later). Then we have (see Theorem 6.2):

(A) Assume that rank $T_0(z) \geqq 2$ for all z. Then φ can be deformed continuously to a harmonic map $\psi : S^2 \to U_{n-1}$.

It is well known that harmonic maps into an inner symmetric space G/K may be studied as a special case of harmonic maps into G (by making use of a totally geodesic embedding of G/K into G). So our method can be used to produce continuous deformations of harmonic maps from S^2 to G/K, for various G/K. We shall give two examples, namely $G/K=\boldsymbol{C}P^n$ and $G/K=S^n$. In the first case we shall show:

(B) The number of connected components of the space of harmonic maps $S^2\to\boldsymbol{C}P^n$ is independent of n, if $n\geqq 2$.

This can be obtained as a consequence of the method for (A), but we shall also give a direct proof (Theorem 6.5). We conjecture that the space of harmonic maps $S^2\to\boldsymbol{C}P^n$ of fixed energy and degree is connected. By (B), it would suffice to verify this conjecture in the case $n=2$.*) In the case $G/K=S^n$, for $n\geqq 4$, we shall use the same method to give a new proof of the following fact (Theorem 6.7; see also [**Lo**], [**Ve3**], [**Kt**]):

(C) The space of harmonic maps $S^2\to S^n$ of fixed energy is connected.

The proof we give is quite elementary and does not depend on §1-§5 of this paper (though it was motivated by the method used for (A)).

Most of our results in §6 generalize to the case of extended solutions $M\to\Omega G$, where M is any compact connected Riemann surface. In particular, the results on the connected components of harmonic maps from S^2 into S^n or $\boldsymbol{C}P^n$ generalize to the case of *isotropic* harmonic maps into S^n or *complex isotropic* harmonic maps into $\boldsymbol{C}P^n$. In fact, since our method primarily involves the target space, one may go even further and obtain similar results on pluriharmonic maps of compact connected complex manifolds (cf. [**OV**]).

Finally, we make some concluding remarks on the two main ingredients of this paper, i.e. group actions and deformations. First, it should be emphasized that the group actions discussed here do not represent a new idea. It is a well-known principle in other contexts to convert from real to complex geometry, in order to reveal a larger (complex) symmetry group. (Here, one converts from harmonic maps into a Riemannian manifold to "horizontal" holomorphic maps into a complex manifold.) Indeed, as mentioned above, the action $\sharp$ had its origins in the theory of integrable systems, while examples of the action $\natural$ have been treated explicitly in [**Gu**] and have been alluded to by other authors. Our contribution to this topic (in §5) is the unification of the two actions. Second, the results of §6 concerning deformations are essentially independent of §1-§5, although we feel that the group action provides some motivation for these deformations. From a practical point of view, the deformations have two main features. One is the connection with Morse theory which allows us to predict easily the end result of the deformations. The other is that the horizontality condition, which is sometimes hard to deal with directly, is never needed explicitly in our calculations.

*) This has been done recently by A. Crawford (McGill University).

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§1. Extended solutions and generalized Birkhoff pseudo-actions.

Let M be a connected Riemann surface or, more generally, a connected complex manifold. Let G be a compact connected Lie group equipped with a bi-invariant Riemannian metric and let $\mathfrak{g}$ denote its Lie algebra. If necessary, we choose a realization for the complexification G^C of G as a subgroup of some general linear group $GL_n(\boldsymbol{C})$, with $G = G^C \cap U(n)$. Let μ denote the Maurer-Cartan form of G^C. For a smooth map $\varphi: M \to G^C$, set $\varphi^*\mu = \alpha = \alpha' + \alpha''$, where α' and α'' are the (1, 0)-component and (0, 1)-component of α, respectively.

DEFINITION. The map $\varphi: M \to G^C$ is said to be *(pluri) harmonic* if and only if $\bar\partial\alpha' = \partial\alpha''$.

If $\varphi(M) \subseteqq G$, then this definition coincides with the usual definition (see (8.5) of [**EL**], §2 of [**OV**]). We shall call such a map φ a *real harmonic map*.

For each $\lambda \in \boldsymbol{C}^* = \boldsymbol{C} \setminus \{0\}$, consider the 1-form on M with values in $\mathfrak{g}^C$ given by

$$\alpha_\lambda = \frac{1}{2}(1-\lambda^{-1})\alpha' + \frac{1}{2}(1-\lambda)\alpha'',$$

and consider the first order linear partial differential equation

$$(*) \qquad \Phi_\lambda{}^*\mu = \alpha_\lambda\,,$$

for a map $\Phi_\lambda: M \to G^C$. Using an embedding $G^C \to GL_n(\boldsymbol{C})$, this equation may be written as

$$(**) \qquad \begin{cases} \partial\Phi_\lambda = \dfrac{1}{2}(1-\lambda^{-1})\Phi_\lambda\alpha' \\[2ex] \bar\partial\Phi_\lambda = \dfrac{1}{2}(1-\lambda)\Phi_\lambda\alpha''. \end{cases}$$

DEFINITION. A family of solutions Φ_λ, $\lambda \in \boldsymbol{C}^*$, to $(*)$ or $(**)$ is called an *extended solution* ([**Uh**]) or an *extended (pluri) harmonic map* ([**OV**]).

The fundamental observation, proved in [**Uh**] for harmonic maps, and extended in [**OV**] to pluriharmonic maps, is:

THEOREM 1.1. *Assume that* $\mathrm{Hom}(\pi_1(M), G) = \{e\}$. *Choose a base point* z_0

of M and a map $\sigma: \boldsymbol{C}^*\to G^C$. Let $\varphi: M\to G^C$ be a (real pluri) harmonic map. Then there exists a unique extended solution $\Phi: M\times \boldsymbol{C}^* \to G^C$ such that $\Phi_\lambda(z_0)=\sigma(\lambda)$. Conversely, if Φ is an extended solution, then $\Phi_{-1}: M\to G^C$ is a (pluri) harmonic map. $\square$

Moreover, the extended solution Φ (obtained from σ and φ) necessarily satisfies $\Phi_{-1}=a\varphi$, where $a=\sigma(-1)\varphi(z_0)^{-1}$.

Let φ be a real harmonic map. If we choose σ satisfying $\sigma(1)=e$ and $\sigma(S^1)\subseteqq G$, then $\Phi_1\equiv e$ and $\Phi_\lambda(M)\subseteqq G$ for any $\lambda\in S^1=\{\lambda\in \boldsymbol{C}^* \mid |\lambda|=1\}$. (For example, we may choose $\sigma\equiv e$.) In this case we call Φ a *real extended solution.*

The smooth loop group of G is defined by:

$$\Omega G=\{\gamma: S^1 \longrightarrow G \mid \gamma \text{ smooth, } \gamma(1)=e\}.$$

Let $\pi: \Omega G\to G$ be the map $\pi(\gamma)=\gamma(-1)$. A real extended solution Φ can be considered as a map into ΩG; conversely, if $\Phi: M\to \Omega G$ satisfies $(*)$ or $(**)$ for $\lambda\in S^1$, then the same argument as for Theorem 1.1 shows that the map $\varphi=\pi\circ\Phi: M\to G$ is (pluri) harmonic. Because of this we shall (with abuse of notation) use the term "real extended solution" for any map $\Phi: M\to \Omega G$ satisfying $(*)$ or $(**)$.

It is known that ΩG has the structure of an infinite dimensional homogeneous Kähler manifold (see [**PS**]). There is a left-invariant complex structure J such that the $(+i)$-eigenspace of J is the subspace spanned by the elements $(\lambda^{-k}-1)\mathfrak{g}^C (k=1, 2, \cdots)$, under the identification $T_e^C\Omega G\cong \Omega\mathfrak{g}^C$. The condition $(*)$ or $(**)$ may be written

$$\Phi^*\mu(T_{1,0}M)=\Phi^{-1}d\Phi(T_{1,0}M)\subseteqq(\lambda^{-1}-1)\mathfrak{g}^C.$$

In particular, we see that any extended solution $\Phi: M\to\Omega G$ is holomorphic relative to J.

Following [**Uh**], we say that a harmonic map φ has finite uniton number if there is an extended solution Φ such that $\pi\circ\Phi=a\varphi$ for some $a\in G^C$ and $\Phi(\lambda)=\sum_{\alpha=0}^m T_\alpha\lambda^\alpha$ (for some m). The least such integer m is called the *minimal uniton number* of φ (or of Φ). The next fundamental result is that any harmonic map which admits a corresponding real extended solution has finite uniton number:

THEOREM 1.2 ([**Uh**]). *Assume that M is compact. Let $\Phi: M\to \Omega U_n$ be an extended solution. Then there exists a loop $\gamma\in\Omega U_n$ and a non-negative integer $m\leqq n-1$ such that* (i) $\gamma\Phi(\lambda)=\sum_{\alpha=0}^m T_\alpha\lambda^\alpha$, (ii) $\mathrm{Span}\{\mathrm{Im}\,T_0(z)\mid z\in M\}=\boldsymbol{C}^n$. *Here m is equal to the minimal uniton number of Φ_{-1}.* $\square$

We shall refer to property (ii) as the *Uhlenbeck normalization.*

Now we discuss the group action studied by Uhlenbeck, and its generalizations. The idea of a "dressing action" (see, for example, [**ZM**], [**ZS**], [**Uh**], [**BG**]) is as follows. Let $\mathcal{G}$ be a group and $\mathcal{G}_1$, $\mathcal{G}_2$ two subgroups of $\mathcal{G}$ with $\mathcal{G}=\mathcal{G}_1\mathcal{G}_2$ and $\mathcal{G}_1\cap\mathcal{G}_2=\{e\}$, where e is the identity element of $\mathcal{G}$. For any $g\in\mathcal{G}$, we have a unique decomposition $g=g_1g_2$, $g_1\in\mathcal{G}_1$, $g_2\in\mathcal{G}_2$. For g, $h\in\mathcal{G}$, define $g^{\#}h$ by $g^{\#}h=gh(h^{-1}gh)_2^{-1}=h(h^{-1}gh)_1$. If g, g', $h\in\mathcal{G}$, then we have $g^{\#}(g'^{\#}h)=(gg')^{\#}h$, so this defines an action of $\mathcal{G}$ on itself.

Let T^C be the complexification of a maximal torus T of G. Let $U_+=\{\lambda\in S^2\,|\,|\lambda|<1\}$ and $U_-=\{\lambda\in S^2\,|\,|\lambda|>1\}$ in the Riemann sphere $S^2=\boldsymbol{C}\cup\{\infty\}$. Set

$\Lambda G^C=\{\gamma:S^1\to G^C\,|\,\gamma\text{ smooth}\}$,

$\Lambda_+G^C=\{\gamma\in\Lambda G^C\,|\,\gamma$ extends continuously to a holomorphic map $U_+\to G^C\}$,

$\Lambda_-G^C=\{\gamma\in\Lambda G^C\,|\,\gamma$ extends continuously to a holomorphic map $U_-\to G^C\}$,

$\Lambda_-^*G^C=\{\gamma\in\Lambda_-G^C\,|\,\gamma(1)=e\}$,

$\Delta G^C=\{\delta\in\Lambda G^C\,|\,\delta:S^1\to T^C\subseteqq G^C$ is a homomorphism$\}$.

The following fact is known as the Birkhoff decomposition ([**PS**]): the map

$$\Lambda_-G^C\times\Delta G^C\times\Lambda_+G^C\longrightarrow\Lambda G^C,\qquad(\gamma_-,\delta,\gamma_+)\longrightarrow\gamma_-\delta\gamma_+$$

is surjective. Moreover, $\Lambda_-^*G^C\times\Lambda_+G^C$ maps diffeomorphically to $\Lambda_-G^C\Lambda_+G^C$, which is an open dense subset of the identity component of ΛG^C. We shall now take $\mathcal{G}=\Lambda G^C$, $\mathcal{G}_1=\Lambda_-^*G^C$, $\mathcal{G}_2=\Lambda_+G^C$ in the definition of dressing action. Since $\mathcal{G}_1\mathcal{G}_2$ is not quite equal to $\mathcal{G}$ here, we use the term "pseudo-action":

DEFINITION. The *Birkhoff pseudo-action* of ΛG^C in itself is defined by $\gamma^{\#}\delta=\gamma\delta(\delta^{-1}\gamma\delta)_+^{-1}=\delta(\delta^{-1}\gamma\delta)_-\in\Lambda G^C$, for γ, $\delta\in\Lambda G^C$ with $\delta^{-1}\gamma\delta\in\Lambda_-^*G^C\Lambda_+G^C$.

We can also consider "generalized Birkhoff pseudo-actions" ([**BG**]). Let $C_1,\cdots,C_k$ be oriented circles of radius r on the Riemann sphere $S^2=\boldsymbol{C}\cup\{\infty\}$. Let I_i and E_i denote the interior and exterior of C_i for each $i=1,\cdots,k$. Set $C=C_1\cup\cdots\cup C_k$, $I=I_1\cup\cdots\cup I_k$ and $E=E_1\cap\cdots\cap E_k$. We assume in addition that $\bar{I}_i\cap\bar{I}_j=\emptyset$ for $i\neq j$ and $1\in E$. Let

$$\Lambda^{1,\cdots,k}G^C=\{\gamma:C\longrightarrow G^C\,|\,\gamma\text{ smooth}\},$$

which is isomorphic to a direct product of k copies of ΛG^C. Set

$\Lambda_EG^C=\{\gamma\in\Lambda^{1,\cdots,k}G^C\,|\,\gamma$ extends continuously to a holomorphic map $E\to G^C\}$,

$\Lambda_IG^C=\{\gamma\in\Lambda^{1,\cdots,k}G^C\,|\,\gamma$ extends continuously to a holomorphic map $I\to G^C\}$,

$\Lambda_E^*G^C=\{\gamma\in\Lambda_EG^C\,|\,\gamma(1)=e\}$,

$\Delta^{1,\cdots,k}G^C=\{\delta\in\Lambda^{1,\cdots,k}G^C\,|\,\delta:C\longrightarrow T^C\subseteqq G^C$ is a homomorphism$\}$.

(To say that the map δ is a homomorphism means that it can be written in the form $\delta(\lambda)=(\{(\lambda-c_i)/r\}^{b_1},\cdots,\{(\lambda-c_i)/r\}^{b_n})$ for $\lambda\in C_i=\{\lambda\in S^2\,|\,|\lambda-c_i|=r\}$.) There is an analogue of the Birkhoff decomposition in this situation, namely (see [**BG**]): $\Lambda^{1,\cdots,k}G^C=\Lambda_EG^C\Delta G^C\Lambda_IG^C$. Moreover, under the multiplication map, $\Lambda_E^*G^C\times\Lambda_IG^C$ is diffeomorphic to $\Lambda_EG^C\Lambda_IG^C$, which is an open dense subset of the identity component of $\Lambda^{1,\cdots,k}G^C$. If we take $\mathcal{G}=\Lambda^{1,\cdots,k}G^C$, $\mathcal{G}_1=\Lambda_E^*G^C$, $\mathcal{G}_2=\Lambda_I G^C$ in the definition of a dressing action, we obtain:

DEFINITION. The *generalized Birkhoff pseudo-action* of $\Lambda^{1,\cdots,k}G^C$ on itself is defined by $\gamma^{\#}\delta=\gamma\delta(\delta^{-1}\gamma\delta)_I^{-1}=\delta(\delta^{-1}\gamma\delta)_E\in\Lambda^{1,\cdots,k}G^C$, for γ, $\delta\in\Lambda^{1,\cdots,k}G^C$ with $\delta^{-1}\gamma\delta\in\Lambda_E^*G^C\Lambda_I G^C$.

The main reason for studying such pseudo-actions is:

PROPOSITION 1.3 ([**ZM**], [**ZS**], [**Uh**], [**BG**]). *Let $g\in\Lambda^{1,\cdots,k}G^C$ and let Φ be an extended solution. If $\Phi^{-1}(z)g\Phi(z)\in\Lambda_E^*G^C\Lambda_I G^C$ for each $z\in M$, then the map $g^{\#}\Phi$ is also an extended solution.* □

(We assume that Φ_λ is defined for all λ in some region which includes C. For example, this is the case if C does not contain the points 0, ∞ and if we choose $\sigma\equiv e$ in Theorem 1.1.) The pseudo-action of $\Lambda^{1,\cdots,k}G^C$ on extended solutions gives rise to a pseudo-action on harmonic maps, by means of the formula $g^{\#}(\pi\circ\Phi)=\pi\circ g^{\#}\Phi$. This is not quite well-defined, as the extended solution Φ corresponding to a harmonic map $M\to G$ is determined only up to left translation in ΩG. However, the non-uniqueness will be of no consequence in this article.

Let us impose now the following "reality conditions": (1) the equator S^1 is contained in E, (2) 0, $\infty\in I$, and (3) $C=C_1\cup\cdots\cup C_k$ is preserved by the transformation $\lambda\to\bar{\lambda}^{-1}$. We call an element $g\in\Lambda^{1,\cdots,k}G^C$ *real* if $g(\bar{\lambda}^{-1})^*=g(\lambda)^{-1}$ for each $\lambda\in C$. It is easy to check that $g^{\#}\Phi$ is a real extended solution if g and Φ are real. We denote by $\Lambda_R^{1,\cdots,k}G^C$ the subgroup of real elements of $\Lambda^{1,\cdots,k}G^C$, and by $\Lambda_{E,R}G^C$, $\Lambda_{E,R}^*G^C$, $\Lambda_{I,R}G^C$, $\Delta_R G^C$ the subgroups of real elements of $\Lambda_E G^C$, $\Lambda_E^*G^C$, $\Lambda_I G^C$, ΔG^C.

We shall now give a contour integral expression for the generalized Birkhoff pseudo-action of $\Lambda^{1,\cdots,k}G^C$ on $\Lambda_E G^C$. Note that for $\delta\in\Lambda_E G^C$ the formula for $\gamma^{\#}\delta$ simplifies to $\gamma^{\#}\delta=\gamma\delta(\gamma\delta)_I^{-1}=(\gamma\delta)_E$.

LEMMA 1.4. *Let $g\in\Lambda^{1,\cdots,k}G^C$ and $h\in\Lambda_E G^C$. Assume that $h^{-1}gh\in\Lambda_E^*G^C\Lambda_I G^C$, so that $g^{\#}h\in\Lambda_E^*G^C$ is well-defined. Then*

$$(g^{\#}h)(\lambda)-h(\lambda)=\frac{\lambda-1}{2\pi i}\int_C\frac{h(\lambda)h^{-1}(\mu)(g^{-1}(\mu)-e)(g^{\#}h)(\mu)}{(\mu-1)(\mu-\lambda)}d\mu$$

for each $\lambda\in E$.

PROOF. By using Cauchy's Integral Theorem, we obtain

$$(h^{-1}gh)_E(\lambda)-e=\frac{\lambda-1}{2\pi i}\int_C\frac{((h^{-1}gh)^{-1}(\mu)-e)(h^{-1}gh)_E(\mu)}{(\mu-1)(\mu-\lambda)}d\mu.$$

Multiplying by $h(\lambda)$ on the left, we obtain the required formula. □

Using this lemma, we derive a formula for the infinitesimal action of $\Lambda^{1,\cdots,k}G^C$ on $\Lambda_E G^C$. Let $\{g_t\}_{|t|<\varepsilon}$ be a curve in $\Lambda^{1,\cdots,k}G^C$ with $g_0=e$ and set $V=(d/dt)g_t|_{t=0}\in\Lambda^{1,\cdots,k}\mathfrak{g}^C$. Let $h\in\Lambda_E G^C$. Note that for each t sufficiently close to 0, $h^{-1}g_t h\in\Lambda_E^* G^C\Lambda_I G^C$ and hence $g_t^\# h\in\Lambda_E^* G^C$ is defined. Set

$$V_h^\# = \frac{d}{dt}g_t^\# h|_{t=0} \in T_h\Lambda_E G^C.$$

PROPOSITION 1.5. *For each $\lambda\in E$, we have*

$$dL_h^{-1}(V_h^\#)(\lambda) = -\frac{\lambda-1}{2\pi i}\int_C \frac{h^{-1}(\mu)V(\mu)h(\mu)}{(\mu-1)(\mu-\lambda)}d\mu.$$

Here L_h denotes left translation by h in the group $\Lambda_E G^C$.

PROOF. Replace g by g_t in the formula of Lemma 1.4. By differentiating at $t=0$, we obtain the required formula. $\square$

COROLLARY 1.6. *Assume that $0\in I_1$, $\infty\in I_2$. If $g\in\Lambda_I G^C$ satisfies $g|_{I_i}=e$ for $i=1, 2$ and $h\in\Lambda_E G^C$ extends to a holomorphic map $\boldsymbol{C}^*=S^2\setminus\{0, \infty\}\to G^C$, then $g^\# h$ exists and $g^\# h=h$.* $\square$

Thus, if Φ is a real extended solution, which without loss of generality we may assume is defined for all $\lambda\in\boldsymbol{C}^*$, then it is only necessary to consider generalized Birkhoff pseudo-actions with $C=C_1\cup C_2$, where C_1, C_2 are circles around 0, ∞ respectively.

§2. Properties of the Uhlenbeck pseudo-action.

In this section we shall study the pseudo-action introduced by Uhlenbeck in [**Uh**]. It can be regarded as the generalized Birkhoff pseudo-action given by the choice of circles

$$C_0^\varepsilon = \{\lambda\in S^2 \mid |\lambda|=\varepsilon\}, \qquad C_\infty^\varepsilon = \left\{\lambda\in S^2 \mid |\lambda| = \frac{1}{\varepsilon}\right\},$$

where $0<\varepsilon<1$. We shall call it the *Uhlenbeck pseudo-action*. This is the simplest choice which is compatible with the reality conditions, and by Corollary 1.6 it contains the essential features of all the other choices.

We shall write $\Lambda^\varepsilon G^C$ for $\Lambda^{1,2}G^C$, where $C_1=C_0^\varepsilon$, $C_2=C_\infty^\varepsilon$. Using the notation of the previous section, we have $C=C_1\cup C_2$, $I=I_1\cup I_2$ and $E=S^2\setminus C\cup I_1\cup I_2$, where

$$I_1 = \{\lambda\in S^2 \mid |\lambda|<\varepsilon\}, \qquad I_2 = \left\{\lambda\in S^2 \mid |\lambda| > \frac{1}{\varepsilon}\right\}.$$

We have subgroups $\Lambda_E G^C$, $\Lambda_E^* G^C$, $\Lambda_I G^C$ of $\Lambda^\varepsilon G^C$ as in the previous section.

We denote by $\Lambda_{R}^{\varepsilon}G^{C}$ the subgroup of all real elements γ of $\Lambda^{\varepsilon}G^{C}$, namely elements satisfying the reality condition $\gamma(\bar{\lambda}^{-1})^{*}=\gamma(\lambda)^{-1}$ on C.

Let

$$\mathcal{G}=\{g: U\to G^{C}\,|\,g \text{ holomorphic in some neighbourhood } U \text{ of } \{0, \infty\}\},$$
$$\mathcal{G}_{R}=\{g\in\mathcal{G}\,|\,g(\bar{\lambda}^{-1})^{*}=g(\lambda)^{-1} \text{ for all } \lambda\}.$$

Note that $\mathcal{G}$ and $\mathcal{G}_{R}$ are connected. Let

$$\mathcal{A}=\{g\in\mathcal{G}\,|\,g \text{ extends to a } G^{C}\text{-valued rational function on } S^{2}\},$$
$$\mathcal{A}_{R}=\{g\in\mathcal{A}\,|\,g(\bar{\lambda}^{-1})^{*}=g(\lambda)^{-1} \text{ for all } \lambda\}.$$

For each ε with $0<\varepsilon<1$, we consider $\Lambda_{I}G^{C}$ and $\Lambda_{I,R}G^{C}$ as subgroups of $\mathcal{G}$ and $\mathcal{G}_{R}$, respectively. We then have

$$\bigcup_{0<\varepsilon<1}\Lambda_{I}G^{C}=\mathcal{G},\qquad \bigcup_{0<\varepsilon<1}\Lambda_{I,R}G^{C}=\mathcal{G}_{R}.$$

Denote by $\mathrm{Lie}(\mathcal{G})$ and $\mathrm{Lie}(\mathcal{G}_{R})$ the Lie algebras of $\mathcal{G}$ and $\mathcal{G}_{R}$, respectively. For each integer $k\geqq 0$ or $k=\infty$, let

$$\begin{aligned}\mathrm{Lie}(\mathcal{G})_{k}=\{V\in\mathrm{Lie}(\mathcal{G})\,|\,V(\lambda)&=\sum_{\alpha\geqq k}V_{\alpha}^{(0)}\lambda^{\alpha} \text{ around } 0,\\ V(\lambda)&=\sum_{\alpha\geqq k}V_{-\alpha}^{(\infty)}\lambda^{-\alpha} \text{ around } \infty\}.\end{aligned}$$

Then $\mathrm{Lie}(\mathcal{G})_{k}$ is an ideal of $\mathrm{Lie}(\mathcal{G})$ and $\mathrm{Lie}(\mathcal{G})_{k}\subseteqq\mathrm{Lie}(\mathcal{G})_{k-1}$, $\mathrm{Lie}(\mathcal{G})_{0}=\mathrm{Lie}(\mathcal{G})$. Let $\mathcal{G}_{k}$ be the analytic subgroup of $\mathcal{G}$ generated by the Lie algebra $\mathrm{Lie}(\mathcal{G})_{k}$, which is a connected closed normal subgroup of $\mathcal{G}$. (Thus, $\mathrm{Lie}(\mathcal{G}_{k})=\mathrm{Lie}(\mathcal{G})_{k}$.) The quotient complex Lie algebra $\mathrm{Lie}(\mathcal{G})/\mathrm{Lie}(\mathcal{G})_{k}$ has complex dimension $2k\dim_{C}\mathfrak{g}^{C}$. We have a sequence of surjective Lie group homomorphisms: $\mathcal{G}/\mathcal{G}_{k}\to\mathcal{G}/\mathcal{G}_{k-1}$ $(k=1, 2, \cdots)$. Set $\mathrm{Lie}(\mathcal{G})_{k,R}=\mathrm{Lie}(\mathcal{G})_{R}\cap\mathrm{Lie}(\mathcal{G})_{k}$, which is a real Lie algebra. The Lie algebra $\mathrm{Lie}(\mathcal{G})_{k,R}$ generates an analytic subgroup $\mathcal{G}_{k,R}$ of $\mathcal{G}_{R}$, which is a real Lie algebra. The Lie algebra $\mathrm{Lie}(\mathcal{G})_{k,R}$ generates an analytic subgroup $\mathcal{G}_{k,R}$ of $\mathcal{G}_{R}$, which is a connected closed normal subgroup of $\mathcal{G}_{R}$. The quotient real Lie algebra $\mathrm{Lie}(\mathcal{G}_{R})/\mathrm{Lie}(\mathcal{G})_{k,R}$ has real dimension $2k\dim\mathfrak{g}$.

For each integer $k\geqq 0$ or $k=\infty$, we set $\mathcal{A}_{k}=\mathcal{A}\cap\mathcal{G}_{k}$, $\mathcal{A}_{k,R}=\mathcal{A}_{R}\cap\mathcal{G}_{k}$. Note that $\mathcal{A}_{k}$ is a closed normal subgroup of $\mathcal{A}$.

PROPOSITION 2.1. (i) *For each* k *with* $0\leqq k<\infty$, *the natural injective homomorphism of* $\mathcal{A}$ *into* $\mathcal{G}$ *induces a Lie group isomorphism of* $\mathcal{A}/\mathcal{A}_{k}$ *onto* $\mathcal{G}/\mathcal{G}_{k}$. (ii) *For each* k *with* $0\leqq k<\infty$, *the natural injective homomorphism of* $\mathcal{A}_{R}$ *into* $\mathcal{G}_{R}$ *induces a Lie group isomorphism of* $\mathcal{A}_{R}/\mathcal{A}_{k,R}$ *onto* $\mathcal{G}_{R}/\mathcal{G}_{k,R}$.

PROOF. Denote by σ and $d\sigma$ the Lie group homomorphism $\mathcal{A}/\mathcal{A}_{k}\to\mathcal{G}/\mathcal{G}_{k}$ and its derivative, respectively. We have only to show that σ is surjective. Let V be any element of $\mathrm{Lie}(\mathcal{G})$. We take the Taylor expansions of V around

0 and ∞: $V(\lambda)=\sum_{\alpha\geq 0}V_{\alpha}^{(0)}\lambda^{\alpha}$ around 0, and $V(\lambda)=\sum_{\alpha\geq 0}V_{-\alpha}^{(\infty)}\lambda^{-\alpha}$ around ∞. By the method of indeterminate coefficients, we can find $U\in\mathrm{Lie}(\mathcal{A})$ such that $U(\lambda)=\sum_{\alpha=0}^{k-1}V_{\alpha}^{(0)}\lambda^{\alpha}+\sum_{\alpha=k}^{\infty}U_{\alpha}^{(0)}\lambda^{\alpha}$ around 0 and $U(\lambda)=\sum_{\alpha=0}^{k-1}V_{-\alpha}^{(\infty)}\lambda^{-\alpha}+\sum_{\alpha=k}^{\infty}U_{-\alpha}^{(\infty)}\lambda^{-\alpha}$ around ∞. Hence $U-V\in\mathrm{Lie}(\mathcal{A}_k)$, namely $U\equiv V \bmod \mathrm{Lie}(\mathcal{A}_k)$. Thus $d\sigma$ is surjective. Since $\mathcal{G}/\mathcal{G}_k$ is connected, σ is also surjective. This proves (i). The proof of (ii) is similar. $\square$

For each integer $k\geq 0$ or $k=\infty$, let

$$\mathcal{X}_k=\{\gamma: \boldsymbol{C}^*\longrightarrow G^C \mid \gamma \text{ holomorphic, } \gamma(1)=e, \text{ and } \gamma(\lambda)=\sum_{|\alpha|\leq k}A_\alpha\lambda^\alpha,\ \gamma^{-1}(\lambda)=\sum_{|\alpha|\leq k}B_\alpha\lambda^\alpha\}$$

$$\mathcal{X}_{k,R}=\{\gamma\in\mathcal{X}_k \mid \gamma(\bar{\lambda}^{-1})^*=\gamma(\lambda)^{-1} \text{ for all } \lambda\}.$$

Similary, let

$$\mathcal{X}_k^+=\left\{\gamma: \boldsymbol{C}^*\longrightarrow G^C \mid \gamma \text{ holomorphic, } \gamma(1)=e, \text{ and } \gamma(\lambda)=\sum_{\alpha=0}^{k}A_\alpha\lambda^\alpha,\ \gamma^{-1}(\lambda)=\sum_{\alpha=0}^{k}B_{-\alpha}\lambda^{-\alpha}\right\}$$

$$\mathcal{X}_{k,R}^+=\{\gamma\in\mathcal{X}_k^+ \mid \gamma(\bar{\lambda}^{-1})^*=\gamma(\lambda)^{-1} \text{ for all } \lambda\}.$$

We can consider $\mathcal{X}_{k,R}$ and $\mathcal{X}_{k,R}^+$ as subspaces of ΩG. Set $\mathcal{X}=\mathcal{X}_\infty$ and $\mathcal{X}_R=\mathcal{X}_{\infty,R}$. The point of these definitions is that a harmonic map of finite uniton number gives rise to an extended solution with values in $\mathcal{X}_{k,R}^+$, for some k.

Uhlenbeck obtained the following theorem by showing that any element of $\mathcal{A}_R$ decomposes into a product of elements of "simplest type", then by showing that the action is defined for any element of simplest type. See also [**Be**].

THEOREM 2.2 ([**Uh**]). *For each $g\in\mathcal{A}_R$ and each $\gamma\in\mathcal{X}_R$, $g^\#\gamma\in\mathcal{X}_R$ is well-defined.* $\square$

We call the action of $\mathcal{A}_R$ on $\mathcal{X}_R$ *the Uhlenbeck action.*

THEOREM 2.3. (i) *If $V\in\mathrm{Lie}(\mathcal{G})_{2k}$ and $\gamma\in\mathcal{X}_k$, then $V_\gamma^\#=0$.* (ii) *If $g\in\mathcal{G}_{2k}$ and $\gamma\in\mathcal{X}_k$, then $g^\#\gamma\in\mathcal{X}_k$ is defined and $g^\#\gamma=\gamma$.*

THEOREM 2.4. (i) *If $V\in\mathrm{Lie}(\mathcal{G})_k$ and $\gamma\in\mathcal{X}_k^+$, then $V_\gamma^\#=0$.* (ii) *If $g\in\mathcal{G}_k$ and $\gamma\in\mathcal{X}_k^+$, then $g^\#\gamma\in\mathcal{X}_k$ is defined and $g^\#\gamma=\gamma$.*

PROOF OF THEOREM 2.3. (i) Let $V\in\mathrm{Lie}(\mathcal{G})_{2k}$ and $\gamma\in\mathcal{X}_k$. Then we have $\gamma(\lambda)=\sum_{|\alpha|\leq k}A_\alpha\lambda^\alpha$ and $\gamma^{-1}(\lambda)=\sum_{|\alpha|\leq k}B_\alpha\lambda^\alpha$ for $\lambda\in\boldsymbol{C}^*$. By Proposition 1.5 we have, for $\lambda\in S^1$,

$$dL_\gamma^{-1}(V_\gamma^\#)(\lambda)=-\frac{\lambda-1}{2\pi i}\left\{\int_{C_0}\frac{\gamma^{-1}(\mu)V(\mu)\gamma(\mu)}{(\mu-1)(\mu-\lambda)}d\mu+\int_{C_\infty}\frac{\gamma^{-1}(\mu)V(\mu)\gamma(\mu)}{(\mu-1)(\mu-\lambda)}d\mu\right\}.$$

Denote by (A) and (B) the first term and the second term on the right-hand side of this formula. By assumption we have

$$V(\lambda)=\sum_{\alpha\geqq 2k}V_{\alpha}^{(0)}\lambda^{\alpha}\quad\text{on } \bar{I}_1,$$

$$V(\lambda)=\sum_{\alpha\geqq 2k}V_{-\alpha}^{(\infty)}\lambda^{-\alpha}\quad\text{on } \bar{I}_2.$$

On the circle C_0, we have

$$\gamma^{-1}(\mu)V(\mu)\gamma(\mu)=\sum_{|\alpha|\leq k,\,|\alpha'|\leq k,\,\beta\geq 2k}B_{\alpha'}V_{\beta}^{(0)}A_{\alpha}\mu^{\alpha'+\beta+\alpha}.$$

Write

$$\frac{1}{(\mu-1)(\mu-\lambda)}=\sum_{\alpha''\geqq 0}a_{\alpha''}^{(0)}\mu^{\alpha''}$$

around 0. Then the first integrand is

$$\sum_{\alpha''\geqq 0,\,|\alpha|\leq k,\,|\alpha'|\leq k,\,\beta\geqq 2k}a_{\alpha''}^{(0)}B_{\alpha'}V_{\beta}^{(0)}A_{\alpha}\mu^{\alpha''+\alpha'+\beta+\alpha}.$$

Since $\alpha''+\alpha'+\beta+\alpha\geqq 0$, in particular $\alpha''+\alpha'+\beta+\alpha\neq -1$, we obtain (A)$=0$. On the circle C_∞, we have

$$\gamma^{-1}(\mu)V(\mu)\gamma(\mu)=\sum_{|\alpha|\leq k,\,|\alpha'|\leq k,\,\beta\geq 2k}B_{\alpha'}V_{-\beta}^{(\infty)}A_{\alpha}\mu^{\alpha'-\beta+\alpha}.$$

Write

$$\frac{1}{(\mu-1)(\mu-\lambda)}=\sum_{\alpha''\geqq 2}a_{-\alpha''}^{(\infty)}\mu^{-\alpha''}$$

around ∞. Then the second integrand is

$$\sum_{\alpha''\geqq 2,\,|\alpha|\leq k,\,|\alpha'|\leq k,\,\beta\geqq 2k}a_{-\alpha''}^{(\infty)}B_{\alpha'}V_{-\beta}^{(\infty)}A_{\alpha}\mu^{-\alpha''+\alpha'-\beta+\alpha}.$$

Since $-\alpha''+\alpha'-\beta+\alpha\leqq -2+k-2k+k=-2$, in particular $-\alpha''+\alpha'-\beta+\alpha\neq -1$, we obtain (B)$=0$.

(ii) By (i), there is a neighbourhood $\mathcal{U}$ of e in $\mathcal{G}_{2k}$ such that $g^{\#}\gamma$ exists and $g^{\#}\gamma=\gamma$ for each $\gamma\in\mathcal{U}$. Since the group $\mathcal{G}_{2k}$ is connected, $\mathcal{G}_{2k}$ is generated by elements of $\mathcal{U}$. Hence we obtain (ii). □

PROOF OF THEOREM 2.4. Let $V\in\mathrm{Lie}(\mathcal{G})_k$ and $\gamma\in\mathcal{X}_k^{+}$. Then we have $\gamma(\lambda)=\sum_{\alpha=0}^{k}A_{\alpha}\lambda^{\alpha}$ and $\gamma^{-1}(\lambda)=\sum_{\alpha=0}^{k}B_{-\alpha}\lambda^{-\alpha}$ for $\lambda\in\boldsymbol{C}^*$. By assumption, we have

$$V(\lambda)=\sum_{\alpha\geqq k}V_{\alpha}^{(0)}\lambda^{\alpha}\quad\text{on } \bar{I}_1,$$

$$V(\lambda)=\sum_{\alpha\geqq k}V_{-\alpha}^{(\infty)}\lambda^{-\alpha}\quad\text{on } \bar{I}_2.$$

As in the proof of Theorem 2.3, the first integrand in the expression for $dL_{\gamma}^{-1}(V_{\gamma}^{\#})$ is

$$\sum_{\alpha''\geqq 0,\, 0\leqq\alpha\leqq k,\, 0\leqq\alpha'\leqq k,\, \beta\geqq k} a_{\alpha''}^{(0)} B_{-\alpha'} V_{\beta}^{(0)} A_{\alpha} \mu^{\alpha''-\alpha'+\beta+\alpha}.$$

Since $\alpha''-\alpha'+\beta+\alpha\geqq 0-k+k+0=0$, in particular $\alpha''-\alpha'+\beta+\alpha\neq -1$, we obtain (A)=0. The second integrand is

$$\sum_{\alpha''\geqq 2,\, 0\leqq\alpha\leqq k,\, 0\leqq\alpha'\leqq k,\, \beta\geqq k} a_{-\alpha''}^{(\infty)} B_{-\alpha'} V_{-\beta}^{(\infty)} A_{\alpha} \mu^{-\alpha''-\alpha'-\beta+\alpha}.$$

Since $-\alpha''-\alpha'-\beta+\alpha\leqq -2+0-k+k=-2$, in particular $-\alpha''-\alpha'-\beta+\alpha\neq -1$, we obtain (B)=0. This proves (i). By the same argument as in the proof of Theorem 1.3, (ii) follows from (i). $\square$

Theorem 2.4 implies that, for each k with $0\leqq k<\infty$, the pseudo-actions of the infinite dimensional Lie groups $\mathcal{A}_{R}$ and $\mathcal{G}_{R}$ on $\mathcal{X}_{k,R}^{+}$ collapse to the pseudo-actions of the finite dimensional Lie groups $\mathcal{A}_{R}/\mathcal{A}_{k,R}$ and $\mathcal{G}_{R}/\mathcal{G}_{k,R}$, respectively. Moreover, by Theorem 2.2 and Proposition 2.1, we see that these pseudo-actions are in fact actions. In §5 we shall prove by a different argument that the pseudo-action of $\mathcal{G}_{R}/\mathcal{G}_{k,R}$ on $\mathcal{X}_{k,R}^{+}$ is an action, i.e. without using Theorem 2.2.

§3. The natural action.

In this section we study a different group action on the space of extended solutions $M\to\Omega G$. This approach depends on recognising explicitly the role of the loop group ΩG. It is well known that ΩG enjoys many of the properties of a *finite dimensional* generalized flag manifold (or Kähler C-space); one reason for this is that ΩG arises as an orbit of the "adjoint action" for the Lie group $S^1\ltimes\Lambda G$. The semi-direct product here is defined with respect to the action of S^1 on the free loop group $\Lambda G=\mathrm{Map}(S^1, G)$ by rotation of the loop parameter. (That is, $(e^{2\pi i\varphi}, \gamma(e^{2\pi it}))\cdot(e^{2\pi i\psi}, \delta(e^{2\pi it}))=(e^{2\pi i(\varphi+\psi)}, \gamma(e^{2\pi it})\delta(e^{2\pi i(t-\varphi)}))$.) Indeed, the isotropy subgroup of the point $(i, 0)\in i\boldsymbol{R}\ltimes\Lambda\mathfrak{g}$ is the group $S^1\times G$, so

$$\Omega G=\frac{\Lambda G}{G}\cong\frac{S^1\ltimes\Lambda G}{S^1\times G}.$$

The analogy can be strengthened by introducing the "Grassmannian model" of ΩG (see [**PS**], Chapters 7, 8). This is a submanifold of an infinite dimensional Grassmannian on which $S^1\ltimes\Lambda G$ acts transitively, with isotropy subgroup $S^1\times G$, and it provides a geometrical basis for the above identification. We shall review briefly this construction.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $\boldsymbol{C}^n$. Let $H^{(n)}$ be the Hilbert space $L^2(S^1, \boldsymbol{C}^n)=\mathrm{Span}\{\lambda^i e_j \mid i\in\boldsymbol{Z},\ j=1, \cdots, n\}$, and let H_+ be the subspace Span $\{\lambda^i e_j \mid i\geqq 0,\ j=1, \cdots, n\}$. The group ΩU_n acts naturally on $H^{(n)}$ by multiplication, and we have a map from ΩU_n to the Grassmannian of all closed linear subspaces of $H^{(n)}$, given by $\gamma\mapsto\gamma H_+=\{\gamma f \mid f\in H_+\}$. It is easy to see that this map is injective. Regarding the image, one has:

THEOREM 3.1 ([**PS**]). *The image $Gr_{\infty}^{(n)}$ of the map $\gamma\mapsto\gamma H_+$ consists of all closed linear subspaces W of $H^{(n)}$ which satisfy*

(1) $\lambda W\subseteqq W$,

(2) *the orthogonal projections $W\to H_+$ and $W\to(H_+)^\perp$ are respectively Fredholm and Hilbert-Schmidt, and*

(3) *the images of the orthogonal projections $W^\perp\to H_+$ and $W\to(H_+)^\perp$ consist of smooth functions.*

Moreover, if $\gamma\in\Omega U_n$ and $W=\gamma H_+$, then $\deg(\det\gamma)$ *is minus the index of the orthogonal projection operator $W\to H_+$.* $\square$

This is the Grassmannian model of ΩU_n.

Now suppose G is a compact connected Lie group with trivial centre. Via the adjoint representation, we may consider G as a subgroup of U_n (where $n=\dim G$) and ΩG as a subgroup of ΩU_n. The Hilbert space $H^{(n)}$ inherits the structure of a Lie algebra from $\mathfrak{g}^C$, and its Hermitian inner product arises from the Killing form of $\mathfrak{g}$.

COROLLARY 3.2 ([**PS**]). *The image of ΩG under the map $\gamma\mapsto\gamma H_+$ consists of all closed linear subspaces W of $H^{(n)}$ which satisfy*

(1) $\lambda W\subseteqq W$,

(2) *the orthogonal projections $W\to H_+$ and $W\to(H_+)^\perp$ are respectively Fredholm and Hilbert-Schmidt, and*

(3) *W^{sm} is a subalgebra of the Lie algebra $H^{(n)}$, where W^{sm} is the space of smooth functions in W, and*

(4) $\overline{W}^\perp=\lambda W$. $\square$

This is the Grassmannian model of ΩG. If G' is any locally isomorphic group, we can obtain a Grassmannian model for $\Omega G'$, because it suffices to give a model for the identity component, and the identity components of ΩG and $\Omega G'$ may be identified. In particular, this shows that one has a Grassmannian model for any compact semisimple Lie group.

The complexified group ΛG^C also acts transitively on the Grassmannian model, with isotropy subgroup Λ_+G^C at H_+. Hence one obtains the identification

$$\Omega G\cong\frac{\Lambda G^C}{\Lambda_+G^C}.$$

It follows that $\Lambda G^C=\Omega G\cdot\Lambda_+G^C$. Since $\Omega G\cap\Lambda_+G^C=\{e\}$, we have a factorization theorem: any $\gamma\in\Lambda G^C$ can be written as $\gamma=\gamma_u\gamma_+$, where γ_u, γ_+ are uniquely defined elements of ΩG, Λ_+G^C respectively. If $\gamma\in\Lambda G^C$, we shall write $[\gamma]$ for the coset $\gamma(\Lambda_+G^C)\in\Lambda G^C/\Lambda_+G^C\cong\Omega G$. Thus, the natural action of ΛG^C on ΩG, denoted by the symbol $\natural$, may be written

$$\gamma^{\natural}\delta = [\gamma\delta] = (\gamma\delta)_u .$$

DEFINITION. Let $\Phi: M \to \Omega G$ be an extended solution. Let $\gamma \in \Lambda G^C$. We define the *natural action* of γ on Φ by $\gamma^{\natural}\Phi = [\gamma\Phi] = (\gamma\Phi)_u$.

Let $\Phi: M \to \Omega G$ be a smooth map. By the Grassmannian model, this may be identified with a map $W: M \to Gr_{\infty}^{G}$, where $W(z) = \Phi(z)H_+$. The extended solution equations for Φ are equivalent to the conditions

$$\text{(1)} \qquad \frac{\partial}{\partial \bar{z}} C^{\infty}W \subseteq C^{\infty}W$$

$$\text{(2)} \qquad \frac{\partial}{\partial z} C^{\infty}W \subseteq C^{\infty}\lambda^{-1}W$$

where $C^{\infty}W$ denotes the space of (locally defined) smooth maps $f: M \to H^{(n)}$ with $f(z) \in W(z)$ for all z. The first condition is simply the condition that Φ be holomorphic. The second condition is a horizontality condition on the derivative of Φ (this terminology will be explained in the next section).

PROPOSITION 3.3. *Let $\Phi: M \to \Omega G$ be an extended solution. Let $\gamma \in \Lambda G^C$. Then $\gamma^{\natural}\Phi$ is also an extended solution.*

PROOF. Let $W: M \to Gr_{\infty}^{G}$ be the map corresponding to Φ; thus γW corresponds to $\gamma^{\natural}\Phi$. If W satisfies equations (1) and (2), then so does γW, as multiplication by γ commutes with differentiation with respect to z or $\bar{z}$ and with multiplication by λ^{-1}. □

To understand this action, it is helpful to consider the following concrete examples. We shall show later that these examples represent special cases of the action.

EXAMPLE 3.4. Let $\varphi: M \to U_n$ be a harmonic map with minimal uniton number 1. Then $\varphi = \pi \circ \Phi$, where $\Phi: M \to Gr_k(\boldsymbol{C}^n)$ is a holomorphic map (for some k), and where $\pi: Gr_k(\boldsymbol{C}^n) \to U_n$ is a totally geodesic embedding. More explicitly, there exists some $a \in U_n$ such that $\varphi(z) = a(\pi_{\Phi(z)} - \pi_{\Phi(z)}^{\perp})$, where $\pi_{\Phi(z)}$ denotes the orthogonal projection $\boldsymbol{C}^n \to \Phi(z)$ with respect to the Hermitian inner product of $\boldsymbol{C}^n$. The embedding $\pi: Gr_k(\boldsymbol{C}^n) \to U_n$ is then given by $V \mapsto a(\pi_V - \pi_V^{\perp})$. Conversely, any map φ of this form (with Φ non-constant) is a harmonic map with minimal uniton number 1. Since the standard action of the complex group $GL_n(\boldsymbol{C}) = \mathrm{Aut}(\boldsymbol{C}^n)$ on $Gr_k(\boldsymbol{C}^n)$ is holomorphic, we obtain an action of $GL_n(\boldsymbol{C})$ on holomorphic maps $M \to Gr_k(\boldsymbol{C}^n)$. Thus, an element A of $GL_n(\boldsymbol{C})$ gives rise to a new holomorphic map $A^{\natural}\varphi = \pi(A\Phi)$.

EXAMPLE 3.5. It is well known (see [**EL**]) that all harmonic maps $\varphi: S^2 \to \boldsymbol{C}P^n$ are of the form $\varphi = \pi \circ \Phi$, where $\Phi: S^2 \to F_{r,r+1}(\boldsymbol{C}^{n+1})$ is (a) holomorphic with respect to the natural complex structure of $F_{r,r+1}(\boldsymbol{C}^{n+1})$, and (b) horizontal with respect to the projection $\pi: F_{r,r+1}(\boldsymbol{C}^{n+1}) \to \boldsymbol{C}P^n$. Here, $F_{r,r+1}(\boldsymbol{C}^{n+1})$ is the space of flags of the form $\{0\} \subseteqq E_r \subseteqq E_{r+1} \subseteqq \boldsymbol{C}^{n+1}$. Conversely, given a holomorphic horizontal map Φ, the map $\varphi = \pi \circ \Phi$ is harmonic. If the flag corresponding to $\Phi(z)$ is denoted by $\{0\} \subseteqq W_r(z) \subseteqq W_{r+1}(z) \subseteqq \boldsymbol{C}^{n+1}$, then the holomorphicity and horizontality conditions are

$$(1) \qquad \frac{\partial}{\partial \bar{z}} C^\infty W_i \subseteqq C^\infty W_i, \qquad i = r, r+1$$

$$(2) \qquad \frac{\partial}{\partial z} C^\infty W_r \subseteqq C^\infty W_{r+1}.$$

The standard action of $GL_{n+1}(\boldsymbol{C})$ on $F_{r,r+1}(\boldsymbol{C}^{n+1})$ preserves both these conditions because of the linearity of the derivative. Hence for any $A \in GL_{n+1}(\boldsymbol{C})$, we obtain a new harmonic map $A^\natural \varphi = \pi(A\Phi)$. This action of $GL_{n+1}(\boldsymbol{C})$ on harmonic maps $S^2 \to \boldsymbol{C}P^n$ was studied in [**Gu**].

More generally, if M is a Riemann surface, complex isotropic harmonic maps $\varphi: M \to \boldsymbol{C}P^n$ correspond to holomorphic horizontal maps $\Phi: M \to F_{r,r+1}(\boldsymbol{C}^{n+1})$. Thus, we obtain an action of $GL_{n+1}(\boldsymbol{C})$ on complex isotropic harmonic maps.

EXAMPLE 3.6. There is a similar description of harmonic maps from S^2 to S^n or $\boldsymbol{R}P^n$. It suffices to consider harmonic maps $\varphi: S^2 \to \boldsymbol{R}P^{2n}$, as the other cases can be deduced from this one. Let Z_n be the space of (complex) n-dimensional subspaces V of $\boldsymbol{C}^{2n+1}$ such that V and $\bar{V}$ are orthogonal with respect to the standard Hermitian inner product of $\boldsymbol{C}^{2n+1}$, i.e. such that V is "isotropic". There is a projection map $\pi: Z_n \to \boldsymbol{R}P^{2n}$, which associates to V the $(+1)$-eigenspace of the operator $x \mapsto \bar{x}$ on $(V \oplus \bar{V})^\perp$. It is known (see [**Ca1**], [**Ca2**], [**Ba**]) that such harmonic maps are of the form $\varphi = \pi \circ \Phi$ where $\Phi: S^2 \to Z_n$ is a holomorphic map which is horizontal with respect to π. The holomorphicity and horizontality conditions are

$$(1) \qquad \frac{\partial}{\partial \bar{z}} C^\infty \Phi \subseteqq C^\infty \Phi$$

$$(2) \qquad \frac{\partial}{\partial z} C^\infty \Phi \perp C^\infty \bar{\Phi}.$$

The standard action of $SO_{2n+1}^{\boldsymbol{C}}$ on Z_n preserves both these conditions, hence we obtain an action of $SO_{2n+1}^{\boldsymbol{C}}$ on harmonic maps.

More generally, if M is a Riemann surface, isotropic harmonic maps from M into S^{2n} or $\boldsymbol{R}P^{2n}$ correspond to holomorphic horizontal maps $\Phi: M \to Z_n$, and we obtain an action of $SO_{2n+1}^{\boldsymbol{C}}$ on such maps.

The harmonic maps arising in these three examples fit into a more general framework, described in [**Br**], [**BR**], which we shall recall briefly. Let G/H be a generalized flag manifold, i. e. the orbit of a point P of $\mathfrak{g}$ under the adjoint representation. It is well-known that the complex group G^C acts transitively on G/H. If G_P is the isotropy subgroup at P, then we have an identification $G/H \cong G^C/G_P$. This endows G/H with a complex structure, and the holomorphic tangent bundle of G/H may be identified with the homogeneous bundle $G^C \times_{G_P} (\mathfrak{g}^C/\mathfrak{g}_P)$.

Without essential loss of generality (see [**BR**]) we may assume that the linear endomorphism $\operatorname{ad} P$ on $\mathfrak{g}^C$ has eigenvalues in $i\boldsymbol{Z}$. If the (il)-eigenspace is denoted by $\mathfrak{g}_l$, then one has $\mathfrak{g}_0 = \mathfrak{h}^C$, $\mathfrak{g}_P = \bigoplus_{i \leq 0} \mathfrak{g}_i$, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. Let $\mathfrak{k}^C = \bigoplus_{i \text{ even}} \mathfrak{g}_i$. Then $(\mathfrak{g}^C, \mathfrak{k}^C)$ is a symmetric pair, and (up to local isomorphism) one obtains a symmetric space G/K, where $K = \{g \in G \mid g(\exp \pi P) = (\exp \pi P) g\}$.

The natural map $\pi : G/H \to G/K$ is a "twistor fibration"; it gives rise to a relation between harmonic maps $M \to G/K$ and holomorphic maps $M \to G/H$. The simplest aspect of this relation may be expressed in terms of the *super-horizontal distribution*, which is by definition the holomorphic subbundle $G^C \times_{G_P} (\mathfrak{g}_P \oplus \mathfrak{g}_1/\mathfrak{g}_P)$ of $G^C \times_{G_P} (\mathfrak{g}^C/\mathfrak{g}_P) \cong T_{1,0} G/H$. A holomorphic map $\Phi : M \to G/H$ is said to be super-horizontal if it is tangential to the super-horizontal distribution. It is shown in [**Br**], [**BR**] that:

(†) If Φ is holomorphic and super-horizontal, then $\varphi = \pi \circ \Phi$ is harmonic.

Clearly the action of G^C preserves holomorphicity and super-horizontality. Hence we obtain an action of G^C on the set of those harmonic maps $M \to G/K$ which are of the above form. This is precisely the action described in Examples 3.5 and 3.6, since in those cases it turns out that $\mathfrak{g}_i = 0$ for $|i| > 2$, hence (for holomorphic maps) the concepts of horizontality and super-horizontality coincide. (This is also, trivially, the action of Example 3.4, where $K = H$.)

Before leaving these examples, we make some brief comments on further generalizations. It is possible to weaken the hypothesis of super-horizontality in (†). Indeed, in [**BR**], it is shown that holomorphicity and horizontality, or the even weaker condition of "J_2-holomorphicity", implies that φ is harmonic. In the case $M = S^2$, one then has a converse to (†), namely that any harmonic map $\varphi : S^2 \to G/K$ is of the form $\varphi = \pi \circ \Phi$ for some J_2-holomorphic map $\Phi : S^2 \to G/H$, for a suitable twistor fibration $\pi : G/H \to G/K$. These generalizations are not so useful from the point of view of the action of G^C, because neither holomorphicity and horizontality nor J_2-holomorphicity are preserved by this action in general. On the other hand, there is a natural filtration of $T_{1,0} G/H$ by the holomorphic subbundles $T^{(l)} = G^C \times_{G_P} (\bigoplus_{i \leq l} \mathfrak{g}_i)/\mathfrak{g}_P$. Let us say that a holomorphic map $\Phi : M \to G/H$ is *l-holomorphic* if it is tangential to $T^{(l)}$. Thus, a 1-holomorphic map is a holomorphic super-horizontal map; an ∞-holomorphic map is simply a holomorphic map. Clearly the action of G^C preserves l-holomorphicity. However, the relevance of this remark depends on the answer to the question: what is the geometrical significance of the maps $\varphi=\pi\circ\Phi$, where Φ is l-holomorphic?

Finally, we shall explain why the actions in the above examples are special cases of the natural action of ΛG^C on extended solutions. Because of the previous discussion, it suffices to do this for the action of G^C on 1-holomorphic maps $\Phi: M\to G/H$, where $G/H=\mathrm{Ad}(G)P$. First, let us define a loop $\gamma_P\in\Omega G$ by $\gamma_P(\lambda)=\exp 2\pi tP$, where $\lambda=e^{2\pi it}$. Then G/H may be realized as a submanifold of ΩG, namely as the orbit of γ_P under conjugation by G. The associated symmetric space G/K may be realized as a submanifold of G, namely as the conjugacy class of $\exp\pi P$. Thus, the twistor fibration $\pi: G/H\to G/K$ is just a restriction of the map $\pi: \Omega G\to G$ (evaluation at -1):

$$\begin{array}{ccc} G/H & \longrightarrow & \Omega G \\ \downarrow & & \downarrow \\ G/K & \longrightarrow & G \end{array}$$

Recall that we have the identifications $T_P^c G/H=\bigoplus_{l\neq 0}\mathfrak{g}_l$, and $T_{\gamma_P}^c\Omega G\cong T_e^c\Omega G\cong \bigoplus_{l\neq 0}(\lambda^l-1)\mathfrak{g}^C$.

LEMMA 3.7. *The derivative at P of the embedding $G/H\to\Omega G$ identifies $\mathfrak{g}_l$ with $(\lambda^{-l}-1)\mathfrak{g}_l$.*

PROOF. Let $U\in\mathfrak{g}_l$. This corresponds to the initial tangent vector to the curve $\mathrm{Ad}(\exp sU)P$ through P in $G/H=\mathrm{Ad}(G)P$, i.e. to the curve $(\exp sU)\gamma_P(\exp sU)^{-1}$ through γ_P in ΩG. By left translation we obtain the curve $\gamma_P^{-1}(\exp sU)\gamma_P(\exp sU)^{-1}$ through e in ΩG. Now,

$$\begin{aligned} \gamma_P^{-1}(\exp sU)\gamma_P(\exp sU)^{-1} &= \exp \mathrm{Ad}[\exp(-2\pi tP)]sU(\exp sU)^{-1} \\ &= \exp(e^{-2\pi t\,\mathrm{ad}P}sU)\exp(-sU) \\ &= \exp(e^{-2\pi til}sU)\exp(-sU) \\ &= \exp(s(\lambda^{-l}-1)U)\,. \end{aligned}$$

The initial tangent vector of this curve is $(\lambda^{-l}-1)U$. □

In particular, the super-horizontal distribution of $T_{1,0}G/H$ maps into the subbundle of $T_{1,0}\Omega G$ defined by $(\lambda^{-1}-1)\mathfrak{g}^C$, so we obtain:

PROPOSITION 3.8. *Via the embedding $G/H\to\Omega G$, a holomorphic super-horizontal map into G/H goes to an extended solution into ΩG. Moreover, the action of G^C on G/H corresponds to the action of the subgroup G^C of ΛG^C on ΩG.* □

More generally, the concept of l-holomorphicity for a map $\Phi: M\to G/H$ may be interpreted in terms of the corresponding map $M\to\Omega G$. Let us say that a holomorphic map $M\to\Omega G$ is *l-holomorphic* if it is tangential to the (holomorphic) subbundle $\mathcal{H}^{(l)}$ of $T_{1,0}\Omega G$ defined by $\bigoplus_{1\le i\le l}(\lambda^{-i}-1)\mathfrak{g}^C$. Thus, l-holomorphic maps interpolate between extended solutions ($l=1$) and general holomorphic maps ($l=\infty$). By Lemma 3.7, l-holomorphic maps into G/H go (via the embedding $G/H\to\Omega G$) to l-holomorphic maps into ΩG. If Φ is l-holomorphic, and $\gamma\in\Lambda G^C$, then $\gamma^{\sharp}\Phi$ is clearly also l-holomorphic. As in the finite dimensional case, however, the geometrical significance of maps $\varphi=\pi\circ\Phi$: $M\to G$, where Φ is l-holomorphic, is not clear.

In contrast to the actions described in § 1 and § 2, the natural action is very easy to work with. In particular, it has the advantage that it is always well defined (so we have an action, rather than a pseudo-action). In the next section we shall give some elementary properties of this action.

§ 4. Properties of the natural action.

In this section we shall always take M to be a compact Riemann surface and $G=U_n$.

The version of the extended solution equations used in the last section is due to Segal (see [**Se**]), who used it to give a new proof of the factorization theorem of [**Uh**] for harmonic maps $S^2\to U_n$, and of the classification theorem (see [**EL**]) for harmonic maps $S^2\to \boldsymbol{C}P^n$. We shall review Segal's approach here, before discussing further properties of the natural action. The main technical result is the following version of Theorem 1.2:

THEOREM 4.1 ([**Se**]). *Let $\Phi: M\to\Omega U_n$ be an extended solution. Then there exists a loop $\gamma\in\Omega U_n$ and a non-negative integer m such that the map $W=\gamma\Phi H_+$ satisfies* (i) $\lambda^m H_+\subseteq W(z)\subseteq H_+$, *for all* $z\in M$, *and* (ii) $\mathrm{Span}\{W(z)\mid z\in M\}=H_+$. *Moreover, $m\leqq n-1$.* □

The extended solution $\gamma\Phi$ is said to be *normalized*. For example, let f: $M\to Gr_k(\boldsymbol{C}^n)$ be a holomorphic map, so that $\varphi=\pi_f-\pi_f^{\perp}$: $M\to U_n$ is a harmonic map (as in Example 3.4). Then the corresponding extended solution $\Phi=\pi_f+\lambda\pi_f^{\perp}$ is normalized if and only if f is "full", i. e. $\mathrm{Span}\{f(z)\mid z\in M\}=\boldsymbol{C}^n$. It can be shown that condition (ii) of Theorem 4.1 is equivalent to the Uhlenbeck normalization (condition (ii) of Theorem 1.2).

Let us assume that W corresponds to a normalized extended solution, as in the theorem. Then there is a *canonical flag* associated to W, namely

$$\lambda^m H_+ \subseteq W = W_{(m)} \subseteq W_{(m-1)} \subseteq \cdots \subseteq W_{(0)} = H_+$$

where $W_{(i)}: M\to Gr_\infty^{(n)}$ is the holomorphic map defined by $W_{(i)}=\lambda^{-(m-i)}W\cap H_+$. Strictly speaking, this formula defines a holomorphic map with a finite number of removable singularities, but we shall use the notation $W_{(i)}$ to mean the map obtained by removing these. The canonical flag satisfies the following conditions:

$$\lambda W_{(i)} \subseteqq W_{(i+1)} \tag{0}$$

$$\frac{\partial}{\partial \bar{z}} C^\infty W_{(i)} \subseteqq C^\infty W_{(i)} \tag{1}$$

$$\frac{\partial}{\partial z} C^\infty W_{(i)} \subseteqq C^\infty W_{(i-1)}. \tag{2}$$

In fact, these equations are *equivalent* to the extended solution equations for Φ, as the holomorphicity condition for W is given by (1), and the horizontality condition for W follows from (2) and the definition of W_{m-1}. It is an immediate consequence that each map $W_{(i)}$ satisfies the extended solution equations. Hence, by the Grassmannian model, we have $W_{(i)}=\Phi_{(i)}H_+$ for some extended solution $\Phi_{(i)}$.

From Example 3.5, we see that condition (2) can be interpreted as saying that the (holomorphic) map $(W_{(i)}, W_{(i-1)})$ is horizontal with respect to the map $(E_{(i)}, E_{(i-1)})\mapsto E_{(i)}^\perp\cap E_{(i-1)}$. This is why the equation $(\partial/\partial z)C^\infty W\subseteqq C^\infty\lambda^{-1}W$ is called the horizontality condition. Each map $W_{(i)}^\perp\cap W_{(i-1)}$ is a harmonic map into a Grassmannian.

From condition (0) we have $\lambda W_{(i-1)}\subseteqq W_{(i)}\subseteqq W_{(i-1)}$, so we can derive some further information. The map $W_{(i-1)}/\lambda W_{(i-1)}$ defines a holomorphic vector bundle on M, and multiplication by $\Phi_{(i-1)}^{-1}$ defines a smooth isomorphism of this bundle with the trivial bundle $M\times H_+/\lambda H_+\cong M\times \boldsymbol{C}^n$. Through this isomorphism, the map $W_{(i)}/\lambda W_{(i-1)}$ corresponds to a map Ψ_i, and we have $\Phi_{(i)}=\Phi_{(i-1)}\Psi_i$. Each map Ψ_i is necessarily of the form $\pi_{f_i}+\lambda\pi_{f_i}^\perp$, where f_i is a map from M to a Grassmannian. By construction, f_i is holomorphic with respect to a complex structure which is obtained by "twisting" the standard complex structure by $\Phi_{(i-1)}$. Hence we have the factorization theorem: Φ can be written as $\Phi=\Psi_1\cdots\Psi_m$, where $\Psi_i=\pi_{f_i}+\lambda\pi_{f_i}^\perp$, and each sub-product $\Psi_1\cdots\Psi_i$ is an extended solution.

This completes our review of [**Se**], to which the reader is referred for further details. As for the generalization to pluriharmonic maps, we can show that Theorem 4.1 holds also for a compact complex manifold M. Moreover, the above argument for the canonical flag and the factorization also works for the higher dimensional case, if we consider meromorpic maps and coherent sheaves instead of holomorphic maps and holomorphic vector bundles (cf. [**OV**]).

The finiteness properties of extended solutions described above may be expressed in terms of a filtration of the "algebraic loop group" by certain finite dimensional varieties. The algebraic loop group is defined as follows:

DEFINITION. $\Omega_{\mathrm{alg}}U_n = \{\gamma \in \Omega U_n \mid \gamma(\lambda) \text{ is polynomial in } \lambda, \lambda^{-1}\}$.

A Grassmannian model for $\Omega_{\mathrm{alg}}U_n$ may be deduced from that of $Gr_\infty^{(n)}$:

PROPOSITION 4.2 ([**PS**]). *The image of $\Omega_{\mathrm{alg}}U_n$, under the map $\Omega U_n \to Gr_\infty^{(n)}$, is the subspace $Gr_{\mathrm{alg}}^{(n)}$ of $Gr_\infty^{(n)}$ consisting of linear subspaces W which satisfy*

$$\lambda^k H_+ \subseteqq W \subseteqq \lambda^{-k} H_+ \qquad \textit{for some } k.$$

Moreover, if $\gamma \in \Omega_{\mathrm{alg}}U_n$ and $W = \gamma H_+$, then for such minimal k we have $\deg(\det\gamma) = (\dim \lambda^{-k}H_+/W - \dim W/\lambda^k H_+)/2$. □

If we define

$$\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C}) = \{\gamma \in \Lambda GL_n(\boldsymbol{C}) \mid \gamma(\lambda), \gamma(\lambda)^{-1} \text{ are polynomials in } \lambda, \lambda^{-1}\},$$

then we obtain the identification

$$\Omega_{\mathrm{alg}}U_n \cong \frac{\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})}{\Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C})},$$

where $\Lambda_{\mathrm{alg}}U_n$, $\Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C})$ are defined in the obvious way. This is analogous to the identification $\Omega U_n \cong \Lambda GL_n(\boldsymbol{C})/\Lambda_+ GL_n(\boldsymbol{C})$ described in the last section. However, in the case of algebraic loops, one can replace $\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$ by an even larger group, the semi-direct product $\boldsymbol{C}^* \ltimes \Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$, where the action of $\boldsymbol{C}^*$ on $\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$ is given by "re-scaling", i. e. $(v\cdot\gamma)(\lambda) = \gamma(v^{-1}\lambda)$ for all $v \in \boldsymbol{C}^*$, $\gamma \in \Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$. The group $\boldsymbol{C}^*$ also acts on $\Omega_{\mathrm{alg}}U_n$, by

$$v^\natural\gamma = [v\cdot\gamma]$$

where square brackets (as usual) denote cosets in $\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})/\Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C}) \cong \Omega_{\mathrm{alg}}U_n$. We use the "natural" notation for this action, because the formula

$$(v, \gamma)^\natural\delta = \gamma^\natural v^\natural\delta$$

defines an action of $\boldsymbol{C}^* \ltimes \Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$ on $\Omega_{\mathrm{alg}}U_n$, which extends the natural action of $\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$ on $\Omega_{\mathrm{alg}}U_n$. Thus we obtain finally the identifications

$$\Omega_{\mathrm{alg}}U_n \cong \frac{\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})}{\Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C})} \cong \frac{\boldsymbol{C}^* \ltimes \Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})}{\boldsymbol{C}^* \ltimes \Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C})}.$$

Mitchell ([**Mi**]) introduced the following subspaces of $Gr_{\mathrm{alg}}^{(n)}$ (see also § 1 of [**Se**]):

DEFINITION. $F_{n,k} = \{W \subseteqq H^{(n)} \mid \lambda^k H_+ \subseteqq W \subseteqq H_+,\ \lambda W \subseteqq W.\ \dim H_+/W = k\}$.

It can be shown that $F_{n,k}$ is a connected complex algebraic subvariety of the Grassmannian $Gr_{kn-k}(\boldsymbol{C}^{kn})$. Explicitly, if we make the identification $\boldsymbol{C}^{kn}\cong H_+/\lambda^k H_+=\mathrm{Span}\{[\lambda^i e_j]\mid 0\leqq i\leqq k-1,\ 1\leqq j\leqq n\}$, then

$$F_{n,k}\cong\{E\in Gr_{kn-k}(\boldsymbol{C}^{kn})\mid NE\subseteqq E\},$$

where N is the nilpotent operator on $\boldsymbol{C}^{kn}$ given by the multiplication by λ. The space $F_{n,k}$ is preserved by the action of $\Lambda_+GL_n(\boldsymbol{C})$, since (by definition) this group fixes H_+. The action of $\Lambda_+GL_n(\boldsymbol{C})$ on $F_{n,k}$ collapses to the action of the finite dimensional group

$$G_{n,k}=\{X\in GL_{kn}(\boldsymbol{C})\mid XN=NX\}.$$

Indeed, the action of $\Lambda_+GL_n(\boldsymbol{C})$ factors through the homomorphism $\Lambda_+GL_n(\boldsymbol{C})\to GL_n(\boldsymbol{C}[\lambda]/(\lambda^k))$ defined by $\sum_{i\geqq 0}\lambda^i A_i\mapsto\sum_{i=0}^{k-1}\lambda^i A_i$, and we have $GL_n(\boldsymbol{C}[\lambda]/(\lambda^k))\cong G_{n,k}$. This is a complex Lie group of dimension kn^2. The action of $\boldsymbol{C}^*$ also preserves $F_{n,k}$, for the action of an element $u\in\boldsymbol{C}^*$ induces the linear transformation $T_u[\lambda^i e_j]=[u^i\lambda^i e_j]$, and so $T_uN=uNT_u$.

In these terms, we see that a normalized extended solution is a "horizontal" holomorphic map $\Phi: M\to F_{n,k}$, where the integer $k=\deg\det\Phi(z)$ represents the connected component of ΩU_n which contains the image of Φ. The minimal uniton number m satisfies the conditions $m\leqq n-1$, $m\leqq k$. It is known (see [**Mi**]) that $H_2(F_{n,k};\boldsymbol{Z})\cong\boldsymbol{Z}$, so Φ has a topological "degree" d, which (with appropriate choice of orientations) is a non-negative integer. The geometrical significance of d is that it represents the energy of the corresponding harmonic map $\varphi: M\to U_n$ (see [**EL**], [**Va**], [**OV**]). From the discussion above we have:

PROPOSITION 4.3. *The natural action of $\Lambda_+GL_n(\boldsymbol{C})$ on normalized extended solutions Φ preserves*

(1) *the connected component k of ΩU_n containing the image of Φ,*

(2) *the minimal uniton number m of Φ, and*

(3) *the degree d of Φ (i.e. the energy of the corresponding harmonic map). Moreover, for a fixed choice of k, the action of $\Lambda_+GL_n(\boldsymbol{C})$ on normalized extended solutions collapses to the action of the finite dimensional (complex) Lie group $G_{n,k}$.* □

In this proposition, $\Lambda_+GL_n(\boldsymbol{C})$ could be replaced by the group $\boldsymbol{C}^*\ltimes\Lambda_{\mathrm{alg}}^+GL_n(\boldsymbol{C})$, and $G_{n,k}$ by $\boldsymbol{C}^*\ltimes G_{n,k}$; we leave the verification of this to the reader. The existence of an action of $\boldsymbol{C}^*$ on extended solutions was first noticed by Terng (see §7 of [**Uh**]).

§ 5. Relation between the Uhlenbeck pseudo-action and the natural action.

In this section we shall show that the Uhlenbeck pseudo-action discussed in § 2 and the natural action defined in § 3 coincide on harmonic maps of finite uniton number. We begin by considering a special case.

For any ε with $0<\varepsilon<1$, we have an injective homomorphism as real Lie groups

$$\Lambda_+ GL_n(\boldsymbol{C}) \longrightarrow \Lambda_{I,R} GL_n(\boldsymbol{C}) \subseteqq \mathcal{G}_R, \qquad \gamma \longmapsto \hat{\gamma}$$

defined by

$$\hat{\gamma}(\lambda)=\begin{cases} \gamma(\lambda) & \text{for } |\lambda| \leqq \varepsilon, \\ \gamma(\bar{\lambda}^{-1})^{-1*} & \text{for } |\lambda| \geqq 1/\varepsilon \end{cases}$$

for $\gamma \in \Lambda_+ GL_n(\boldsymbol{C})$.

THEOREM 5.1. *If $\gamma \in \Lambda_+ GL_n(\boldsymbol{C})$ and $\delta \in \mathcal{X}_{k,R} \subseteqq \Omega U_n$ for $0 \leqq k \leqq \infty$, then $\hat{\gamma}^{\#}\delta \in \mathcal{X}_{k,R}$ is well-defined and*

$$\gamma^{\natural}\delta = \hat{\gamma}^{\#}\delta .$$

PROOF. By the decomposition $\Lambda GL_n(\boldsymbol{C}) \cong \Omega U_n \cdot \Lambda_+ GL_n(\boldsymbol{C})$, we have $\gamma\delta=(\gamma\delta)_u(\gamma\delta)_+$, where $(\gamma\delta)_u \in \Omega U_n$, $(\gamma\delta)_+ \in \Lambda_+ GL_n(\boldsymbol{C})$. Note that $(\gamma\delta)_u(\lambda)=\gamma(\lambda)\delta(\lambda)(\gamma\delta)_+^{-1}(\lambda)$ extends holomorphically to $\{\lambda \in \boldsymbol{C} \mid 0<|\lambda|<1\}$. Define

$$(\hat{\gamma}\delta)_I(\lambda)=\begin{cases} (\gamma\delta)_+(\lambda) & \text{for } |\lambda| \leqq \varepsilon, \\ \{(\gamma\delta)_+(\bar{\lambda}^{-1})\}^{-1*} & \text{for } |\lambda| \geqq 1/\varepsilon , \end{cases}$$

namely, $(\hat{\gamma}\delta)_I=(\gamma\delta)_+^{\wedge} \in \Lambda_{I,R} GL_n(\boldsymbol{C})$. Define

$$(\hat{\gamma}\delta)_E(\lambda)=\begin{cases} (\gamma\delta)_u(\lambda) & \text{for } 0<|\lambda| \leqq 1, \\ \{(\gamma\delta)_u(\bar{\lambda}^{-1})\}^{-1*} & \text{for } 1 \leqq |\lambda| < \infty . \end{cases}$$

By Painlevé's Theorem we have $(\hat{\gamma}\delta)_E \in \Lambda_{E,R} GL_n(\boldsymbol{C})$, and moreover $(\hat{\gamma}\delta)_E \in \mathcal{X}_{k,R}$, because $(\gamma\delta)_u=\gamma\delta(\gamma\delta)_+^{-1}$, $\delta \in \mathcal{X}_{k,R}$.

For $0<|\lambda| \leqq \varepsilon$, we have

$$\begin{aligned}(\hat{\gamma}\delta(\hat{\gamma}\delta)_I^{-1})(\lambda) &= \gamma(\lambda)\delta(\lambda)(\gamma\delta)_+^{-1}(\lambda) \\ &= (\gamma\delta)_u(\lambda)=(\hat{\gamma}\delta)_E(\lambda) .\end{aligned}$$

For $1/\varepsilon \leqq |\lambda| < \infty$, we have

$$\begin{aligned}(\hat{\gamma}\delta(\hat{\gamma}\delta)_I^{-1})(\lambda) &= \gamma(\bar{\lambda}^{-1})^{-1*}\delta(\bar{\lambda}^{-1})^{-1*}\{(\gamma\delta)_+(\bar{\lambda}^{-1})\}^{*} \\ &= \{\gamma(\bar{\lambda}^{-1})\delta(\bar{\lambda}^{-1})(\gamma\delta)_+(\bar{\lambda}^{-1})^{-1}\}^{-1*} \\ &= \{(\gamma\delta)_u(\bar{\lambda}^{-1})\}^{-1*}=(\hat{\gamma}\delta)_E(\lambda) .\end{aligned}$$

Hence $\hat{\gamma}\delta(\hat{\gamma}\delta)_I^{-1}=(\hat{\gamma}\delta)_E=\hat{\gamma}^{\#}\delta$. Thus we obtain $\hat{\gamma}^{\#}\delta=(\gamma\delta)_u=\gamma^{\natural}\delta$. $\square$

COROLLARY 5.2. *If $\gamma\in\Lambda_+GL_n(\boldsymbol{C})$ and $\Phi: M\to\Omega U_n$ is an extended solution such that Φ_λ is holomorphic in $\lambda\in\boldsymbol{C}^*$, then we have*

$$\gamma^{\natural}\Phi=\hat{\gamma}^{\#}\Phi .$$

PROOF. By assumption the image of Φ is contained in $\mathcal{X}_{\boldsymbol{R}}=\mathcal{X}_{\infty,\boldsymbol{R}}$. Hence the corollary follows from Theorem 5.1. $\square$

In § 2, we saw that the Uhlenbeck pseudo-action of $\mathcal{G}_{\boldsymbol{R}}$ on $\mathcal{X}_{k,\boldsymbol{R}}$ collapses to the pseudo-action of the finite dimensional Lie group $\mathcal{G}_{\boldsymbol{R}}/\mathcal{G}_{k,\boldsymbol{R}}\cong\mathcal{A}_{\boldsymbol{R}}/\mathcal{A}_{k,\boldsymbol{R}}$, and in § 4 that the natural action of $\Lambda_+GL_n(\boldsymbol{C})$ on $F_{n,k}$ collapses to the action of the Lie group $G_{n,k}$. Evidently, we have $\mathcal{G}_{\boldsymbol{R}}/\mathcal{G}_{k,\boldsymbol{R}}\cong G_{n,k}$ as real Lie groups. From Theorem 5.1, and by using the same argument as was used at the end of § 2, we see that the pseudo-action of $\mathcal{G}_{\boldsymbol{R}}$ (or $\mathcal{G}_{\boldsymbol{R}}/\mathcal{G}_{k,\boldsymbol{R}}$) on extended solutions with finite uniton number is an action, and coincides with the action of $\Lambda_+GL_n(\boldsymbol{C})$ (or $G_{n,k}$). Hence:

COROLLARY 5.3. *The Uhlenbeck pseudo-action of $\mathcal{G}_{\boldsymbol{R}}$ on extended solutions (or harmonic maps) with finite uniton number coincides with the natural action of $\Lambda_+GL_n(\boldsymbol{C})$.* $\square$

§ 6. Deformations of harmonic maps.

Let $\{g_t\}$ be a curve in ΛG^C, i. e. a continuous map $t\mapsto g_t$ from an open interval of $\boldsymbol{R}$ to ΛG^C, with $g_0=e$. Let $\Phi: M\to\Omega G$ be an extended solution. Then the formula

$$\Phi_t=g_t^{\natural}\Phi$$

defines a continuous family of extended solutions passing through Φ (a "deformation" of Φ). For example, we can take $\{g_t\}$ to be a one-parameter subgroup $\{\exp t\beta\}$, for $\beta\in\Lambda\mathfrak{g}^C$. The same observation applies to a curve in $\boldsymbol{C}^*\ltimes\Lambda_{\mathrm{alg}}GL_n(\boldsymbol{C})$, providing that the extended solution Φ takes values in $\Omega_{\mathrm{alg}}G$.

Now, it may happen that $\lim_{t\to\infty}\Phi_t$ exists, even if $\lim_{t\to\infty}g_t$ does not exist, and in this case we obtain an extended solution $\Phi_\infty=\lim_{t\to\infty}\Phi_t$ which is *not*, a priori, of the form $g^{\natural}\Phi$. Some examples of this "completion" process were studied in [**BG**] for the case of the Uhlenbeck action $\#$. By using the action $\natural$, however, we can obtain more detailed information. The reason for this is that, for certain β, the curve $\gamma\mapsto(\exp t\beta)^{\natural}\gamma$ has a simple geometrical interpretation: it is a flow line of the gradient vector field of a natural Morse-Bott function on ΩG.

The basic example of a Morse-Bott function on ΩG is the "perturbed energy functional" $E+cK^Q$, where E is the energy functional

$$E(\gamma)=\frac{1}{2}\int_{S^1}\|\gamma^{-1}\gamma'\|^2,$$

and K^Q is the momentum functional

$$K^Q(\gamma)=\int_{S^1}\langle\!\langle\gamma^{-1}\gamma', Q\rangle\!\rangle,$$

for some fixed $Q\in\mathfrak{g}$, and where c is a non-zero constant. The critical points of $E+cK^Q$ are simply the homomorphisms $S^1\to C(T_Q)$, where $C(T_Q)$ is the centralizer in G of the torus T_Q generated by Q. It is classical that this is a Morse-Bott function. The flow of $-\nabla E$ with respect to the Kähler metric is given by the re-scaling action of the one-parameter semi-group $\{e^{-t}|t\geqq 0\}$, and the flow of $-\nabla K^Q$ is given by the (natural) action of $\{\exp itQ\}$. Hence the flow of $E+cK^Q$ is given by the action of $\{\exp it(i, cQ)|t\geqq 0\}$ (which is contained in $\boldsymbol{C}^*\times G^C$, and hence in $\boldsymbol{C}^*\ltimes\Lambda_{\mathrm{alg}}G^C$).

As a first application, let us consider the case where Q is a regular point of $\mathfrak{g}$. Since Q generates (by definition) a maximal torus T, which is equal to its own centralizer, the critical points are the homomorphisms $S^1\to T$; in particular, they are isolated. The stable manifold of a critical point is a cell in ΩG of finite codimension, the so-called Birkhoff cell (see [**PS**]). If $\Phi: M\to\Omega G$ is a holomorphic map, then $\Phi(z)$ must lie in a single Birkhoff manifold for all but a finite number of points $z\in M$, so we obtain:

PROPOSITION 6.1. *Let $\Phi: M\to\Omega_{\mathrm{alg}}G$ be an extended solution. Then there exists a curve $\{g_t\}$ in $\boldsymbol{C}^*\ltimes\Lambda_{\mathrm{alg}}G^C$ such that*

$$\Phi_\infty(z)=\lim_{t\to\infty} g_t{}^\natural\Phi(z)$$

defines a constant (extended solution) $\Phi_\infty: M\setminus S\to\Omega G$, where the set S consists of a finite number of removable singularities of Φ_∞. □

This is an example of the "bubbling off" phenomenon for harmonic maps ([**SU**]). Proposition 6.1 answers positively the question posed at the end of §7 of [**BG**], namely whether any extended solution can be reduced to a constant map by applying the "modified completion" procedure. However, it is perhaps of more interest to find deformations where the singularities do not occur, and this we shall do next.

For our second application, we shall consider the function K^Q. The set of critical points is $\Omega C(T_Q)$, which is infinite dimensional. However, for the application to extended solutions, we are primarily interested in the restriction of K^Q to the finite dimensional subvariety $F_{n,k}$ (with $G=U_n$). Let us now consider the flow of $-\nabla K^Q$, which is given by the natural action of $\{\exp itQ\}$

on ΩU_n. We may consider $\{\exp itQ\}$ to be a one-parameter subgroup of $\Lambda_+ GL_n(\boldsymbol{C})$, so it preserves $F_{n,k}$. (Indeed, $\{\exp itQ\}$ is a one-parameter subgroup of $GL_n(\boldsymbol{C})=G_{n,1}\subseteqq G_{n,k}$, in the notation of §4.) We shall use this flow, with a suitable choice of Q, to prove:

THEOREM 6.2. *Let $\Phi=\sum_{\alpha=0}^{m} T_\alpha \lambda^\alpha : M\to F_{n,k}$ be a normalized extended solution. If* $\operatorname{rank} T_0(z)\geqq 2$ *for all $z\in M$, then Φ can be deformed continuously through extended solutions to an extended solution $\Psi: M\to\Omega U_{n-1}$.*

PROOF. Let $Q=i\pi_L$ where $\pi_L: \boldsymbol{C}^n\to\boldsymbol{C}^n$ denotes orthogonal projection onto a complex line L in $\boldsymbol{C}^n$. The homomorphism $GL_n(\boldsymbol{C})\to G_{n,1}\to G_{n,k}\subseteqq GL_{kn}(\boldsymbol{C})$ will be denoted by $X\mapsto X'$. Thus, if $\boldsymbol{C}^{kn}$ is identified with $H_+/\lambda^k H_+$ as usual, we have $X'(\lambda^i v)=\lambda^i Xv$ for any $v\in\boldsymbol{C}^n$. Observe that $(\pi_L)'=\pi_{L'}$, where L' is the k-plane $L\oplus\lambda L\oplus\cdots\oplus\lambda^{k-1}L$.

Consider the flow on the Grassmannian $Gr_{kn-k}(\boldsymbol{C}^{kn})$ which is given by the action of the one-parameter subgroup $\{(\exp itQ)'\}$ $(=\{\exp itQ'\})$ of $GL_{kn}(\boldsymbol{C})$. It is well-known that this is the downwards gradient flow of a Morse-Bott function on $Gr_{kn-k}(\boldsymbol{C}^{kn})$, such that

(1) the set of absolute minima is $G^L=\{W\mid L'\subseteqq W\}$, and

(2) the stable manifold of G^L (i.e. the union of the flow lines which terminate on G^L) is $S^L=\{W\mid W^\perp\cap L'=\{0\}\}$.

(These assertions represent a mild generalization of the standard Schubert cell decomposition of a Grassmannian. They are explained in more detail in the Appendix.)

Observe that $G^L\cap F_{n,k}=F_{n-1,k}$ if we take $L=\operatorname{Span}\{e_n\}$. Thus, if the image of the extended solution Φ is contained entirely in $S^L\cap F_{n,k}$, the formula $\Phi_t=(\exp itQ')^{\#}\Phi$ gives a continuous deformation of Φ into $F_{n-1,k}$. To prove the theorem, therefore, it suffices to show that any extended solution satisfying the hypotheses lies in $S^L\cap F_{n,k}$, for *some* line L.

Let Φ be an extended solution satisfying the hypotheses. Let

$$Y^\Phi=\{L\mid \Phi(z)\notin S^L \text{ for some } z\in M\}.$$

Thus, Y^Φ is the set of "bad" lines in $\boldsymbol{C}^n$. We shall show that $\dim_C Y^\Phi<n-1$, which implies immediately that not all lines are "bad".

To do this, note that

$$\begin{aligned}\Phi(z)\notin S^L &\Longleftrightarrow \Phi(z)^\perp\cap L'\neq\{0\}\\ &\Longleftrightarrow \Phi(z)^\perp\cap L\neq\{0\}\\ &\Longleftrightarrow \Phi(z)\subseteq L^\perp\end{aligned}$$

(the middle step follows from the fact that both $\Phi(z)^\perp$ and L' are preserved by the adjoint of multiplication by λ, i.e. by the linear transformation λ^* of $H_+/\lambda^k H_+$ given by $\lambda^*(\lambda^i e_j)=\lambda^{i-1}e_j$, $1\leqq i\leqq k-1$, and $\lambda^*(e_j)=0$). Let $X=\{(L, W)\in CP^{n-1}\times F_{n,k} \mid W\subseteqq L^\perp\}$. Let $p_1: X\to CP^{n-1}$, $p_2: X\to F_{n,k}$ be the projection maps. Then we have $Y^\Phi=p_1(p_2^{-1}(\Phi(M)))$, so $\dim_C Y^\Phi\leqq\dim_C p_2^{-1}(\Phi(M))$. We claim that $\dim_C p_2^{-1}(\Phi(z))\leqq n-3$ for all $z\in M$. Since $\dim_C M=1$, we may then conclude that $\dim_C Y^\Phi<n-1$, as required. From the expression $\Phi=\sum_{\alpha=0}^m T_\alpha\lambda^\alpha$ we see that

$$p_2^{-1}(\Phi(z))=\{L \mid \Phi(z)\subseteqq L^\perp\}=\boldsymbol{P}(\operatorname{Ker} T_0^*(z)),$$

so the claim follows from the hypothesis. □

It is appropriate at this point to make some comments on the use of Morse theory in the proof of Theorem 6.2. The fact that $F_{n,k}$ is in general a singular variety (to which ordinary Morse theory does not apply) is irrelevant for our purposes, as we are concerned only with the given flow. However, to study this flow in practice, it is useful to regard it as the restriction of a flow on the Grassmannian $Gr_{kn-k}(C^{kn})$, where it is indeed the downwards gradient flow of a Morse-Bott function. This type of Morse-Bott function is well-understood: it is an example of a "height function" on an orbit of the adjoint representation of a compact Lie group. In the Appendix to this paper, we summarize the basic facts concerning such height functions. Briefly, the situation is as follows. Consider a finite dimensional generalized flag manifold of G, i.e. an orbit $\operatorname{Ad}(G)P$ of a point P of $\mathfrak{g}$ under the adjoint representation. Let Q be any element of $\mathfrak{g}$. Then one may define the height function $h^Q: \operatorname{Ad}(G)P\to \boldsymbol{R}$ by $h^Q(X)=\langle\langle X, Q\rangle\rangle$. This is a Morse-Bott function and its non-degenerate critical manifolds can be described explicitly in Lie theoretic terms. Let ∇h^Q be the gradient of h^Q with respect to the natural Kähler metric on $\operatorname{Ad}(G)P$. Then the flow line of $-\nabla h^Q$ which passes through a point X of $\operatorname{Ad}(G)P$ is given by $t\mapsto(\exp itQ)^\natural X$.

This can be used to obtain results analogous to Theroem 6.2 for harmonic maps $M\to G/K$, for various inner symmetric spaces G/K, because the total space of the corresponding twistor fibration is a generalized flag manifold. Although this is simply a special case of the discussion above, it is instructive to give a direct argument (avoiding the paraphernalia of extended solutions), and this we shall do for each of the three examples considered in § 3. This will, incidentally, provide some examples of extended solutions Φ which satisfy the hypotheses of Theorem 6.2.

EXAMPLE 6.3 (cf. Example 3.4). Let $\operatorname{Hol}_d(S^2, Gr_k(C^n))$ denote the space of holomorphic maps $\Phi: S^2\to Gr_k(C^n)$ which have degree d. It is well-known that this space is connected. However, we shall give a proof of this fact as an illustration of the technique introduced above.

We identify $Gr_k(\boldsymbol{C}^n)$ with the orbit $\mathrm{Ad}(U_n)P$ in $\mathfrak{u}_n$, where $P=i\pi_V$ for some k-plane V. Let $Q=i\pi_1$, where $\pi_1: \boldsymbol{C}^n \to \boldsymbol{C}$ denotes orthogonal projection onto the line spanned by the first standard basis vector. The action of the one-parameter subgroup $\{\exp itQ\}$ gives the downwards gradient flow of a Morse-Bott function $Gr_k(\boldsymbol{C}^n) \to \boldsymbol{R}$. (See the Appendix.) The critical points are those k-planes $W \in Gr_k(\boldsymbol{C}^n)$ for which $[i\pi_1, i\pi_W]=0$, i.e. for which $\boldsymbol{C} \subseteq W$ or $W \subseteq \boldsymbol{C}^\perp$. Thus there are two connected critical manifolds:

$$G^+ = \{W \mid \boldsymbol{C} \subseteq W\} \cong Gr_{k-1}(\boldsymbol{C}^{n-1})$$

$$G^- = \{W \mid W \subseteq \boldsymbol{C}^\perp\} \cong Gr_k(\boldsymbol{C}^{n-1}).$$

The corresponding stable manifolds are:

$$S^Q(G^+) = G^+$$

$$S^Q(G^-) = \{W \mid W \cap \boldsymbol{C} = \{0\}\}.$$

We claim that the inclusions

$$\mathrm{Hol}_d(S^2, Gr_k(\boldsymbol{C}^{n-1})) \cong \mathrm{Hol}_d(S^2, G^-) \longrightarrow \mathrm{Hol}_d(S^2, S^Q(G^-)) \longrightarrow \mathrm{Hol}_d(S^2, Gr_k(\boldsymbol{C}^n))$$

induce bijections on the sets of connected components. In the case of the first inclusion, this is so because, if $\Phi(S^2) \subseteq S^Q(G^-)$, then $\{(\exp itQ)^{\natural}\Phi\}_{0 \le t \le \infty}$ provides a continuous deformation of Φ into G^-. For the second inclusion, it is because $\mathrm{Hol}_d(S^2, S^Q(G^-))$ is obtained from the manifold $\mathrm{Hol}_d(S^2, Gr_k(\boldsymbol{C}^n))$ by removing a closed subvariety of complex codimension 1. By induction it follows that $\mathrm{Hol}_d(S^2, Gr_k(\boldsymbol{C}^n))$ has the same number of connected components as $\mathrm{Hol}_d(S^2, \boldsymbol{C}P^k)$. However, from the usual description of holomorphic maps $S^2 \to \boldsymbol{C}P^k$ in terms of polynomials, it follows that this space is connected.

By modifying this argument slightly (see the proof of Theorem 6.5 below), it can be shown that $\mathrm{Hol}_d(M, Gr_k(\boldsymbol{C}^n))$ is connected for any compact Riemann surface M, providing that $d \geqq 2g$, where g is the genus of M. The last restriction ensures that $\mathrm{Hol}_d(M, S^2)$ is connected (see Corollary 1.3.13 of [**Na**]). In fact, these conditions may be weakened; for example in [**To**] it is shown that $\mathrm{Hol}_d(M, S^2)$ is connected when $d \geqq g$, and it follows from [**FL**] that $\mathrm{Hol}_d(M, S^2)$ is connected for "generic" M when $d \geqq (g+3)/2$.

EXAMPLE 6.4 (cf. Example 3.5). Let $\mathrm{Harm}_d(S^2, \boldsymbol{C}P^n)$ denote the space of harmonic maps $\varphi: S^2 \to \boldsymbol{C}P^n$ which have degree d. If $\Phi: S^2 \to F_{r,r+1}(\boldsymbol{C}^{n+1})$ corresponds to a harmonic map φ as in Example 3.5, and if $\deg \Phi = (\deg W_r, \deg W_{r+1}) = (k, l)$, then we have

$$d = l-k, \qquad E = l+k$$

where E denotes the (suitably normalized) energy. If $n>1$, it is easy to construct examples of harmonic maps φ_1, φ_2 with $\deg \varphi_1 = \deg \varphi_2$ but $E(\varphi_1) \neq E(\varphi_2)$, so the space of harmonic maps of fixed degree cannot be connected. However, we can prove:

THEOREM 6.5. (i) *The inclusion* $\mathrm{Harm}_d(S^2, \boldsymbol{C}P^2) \to \mathrm{Harm}_d(S^2, \boldsymbol{C}P^n)$ *induces a bijection on the sets of connected components, if* $n \geqq 2$. (ii) *More generally, the same is true if harmonic maps from* S^2 *are replaced by complex isotropic harmonic maps from any compact Riemann surface* M.

PROOF. It suffices to give the proof of (ii). Let $\Phi=(W_r, W_{r+1}): M \to F_{r,r+1}(\boldsymbol{C}^{n+1})$ be a holomorphic horizontal map associated to φ. We shall use the method of Example 6.3 to show that Φ may be deformed into $F_{r,r+1}(\boldsymbol{C}^n)$, if $r<n-1$. Hence, by induction, we obtain a map (also denoted by Φ) whose image lies in $F_{r,r+1}(\boldsymbol{C}^{r+2})$. By repeating this argument with $\Phi^*=(W_{r+1}^{\perp}, W_r^{\perp})$, we can similarly deform Φ into $\{(E_r, E_{r+1}) \in F_{r,r+1}(\boldsymbol{C}^{r+2}) \mid \boldsymbol{C}^{r-1} \subseteqq E_r\}$. Thus we obtain a deformation of φ into $P(\boldsymbol{C}^{r+2}/\boldsymbol{C}^{r-1})$, and hence (by applying a projective transformation) into $\boldsymbol{C}P^2$.

We identify $F_{r,r+1}(\boldsymbol{C}^{n+1})$ with the orbit $\mathrm{Ad}(U_{n+1})(i\pi_{V_r}+i\pi_{V_{r+1}})$, where (V_r, V_{r+1}) is a fixed element of $F_{r,r+1}(\boldsymbol{C}^{n+1})$. Let $\pi_n^{\perp}$ denote orthogonal projection onto the line $(\boldsymbol{C}^n)^{\perp}$ in $\boldsymbol{C}^{n+1}$ spanned by the last standard basis vector, and set $Q=i\pi_n^{\perp}$. We shall use the Morse-Bott function on $F_{r,r+1}(\boldsymbol{C}^{n+1})$ whose downwards gradient flow is given by the action of $\{\exp itQ\}$. A point (E_r, E_{r+1}) is a critical point if and only if $[i\pi_n^{\perp}, i\pi_{E_r}+i\pi_{E_{r+1}}]=0$, i.e. the line $(\boldsymbol{C}^n)^{\perp}$ is contained in E_r, $E_r^{\perp} \cap E_{r+1}$, or $E_{r+1}^{\perp}$. The three connected critical manifolds are:

$$F^+ = \{(E_r, E_{r+1}) \mid (\boldsymbol{C}^n)^{\perp} \subseteqq E_r\} = F_{r-1,r}(\boldsymbol{C}^n)$$

$$F^0 = \{(E_r, E_{r+1}) \mid (\boldsymbol{C}^n)^{\perp} = E_r^{\perp} \cap E_{r+1}\} \cong Gr_r(\boldsymbol{C}^n)$$

$$F^- = \{(E_r, E_{r+1}) \mid (\boldsymbol{C}^n)^{\perp} \subseteqq E_{r+1}^{\perp}\} = F_{r,r+1}(\boldsymbol{C}^n).$$

The corresponding stable manifolds are:

$$S^Q(F^+) = F^+$$

$$S^Q(F^0) = \{(E_r, E_{r+1}) \mid (\boldsymbol{C}^n)^{\perp} \subseteqq E_{r+1},\ (\boldsymbol{C}^n)^{\perp} \cap E_r = \{0\}\}$$

$$S^Q(F^-) = \{(E_r, E_{r+1}) \mid (\boldsymbol{C}^n)^{\perp} \cap E_{r+1} = \{0\}\}.$$

If $\Phi(S^2) \subseteqq S^Q(F^-)$, then $\{(\exp itQ)^{\#}\Phi\}_{0 \leq t \leq \infty}$ provides a continuous deformation of Φ into $F^-=F_{r,r+1}(\boldsymbol{C}^n)$. So it suffices to show that Φ can be deformed into $S^Q(F^-)$. In Example 6.3, the corresponding fact was true for dimensional reasons, but a different argument is necessary in the present situation as the space of holomorphic horizontal maps is not in general a manifold. (The argument we are about to give is also needed in Example 6.3, in the case of a Riemann surface.)

We claim that there exists some $A\in U_{n+1}$ such that $A^{\natural}\Phi(M)\subseteq S^{Q}(F^{-})$, i.e. $AW_{r+1}(z)\not\supseteq(C^{n})^{\perp}$ for all $z\in M$; from this one can construct the required deformation, as U_{n+1} is connected. If suffices to find some line L such that $W_{r+1}(z)\not\supseteq L$ for all $z\in M$. Let

$$Y^{\Phi}=\{L\in CP^{n}\mid L\subseteq W_{r+1}(z) \text{ for some } z\in M\}.$$

Then our claim is that $Y^{\Phi}\neq CP^{n}$. Let $X=\{(L, E_{r}, E_{r+1})\in CP^{n}\times F_{r,r+1}(C^{n+1})\mid L\subseteq E_{r+1}\}$. Let p_{1} and p_{2} be the projections to CP^{n} and $F_{r,r+1}(C^{n+1})$. Then $Y^{\Phi}=p_{1}(p_{2}^{-1}(\Phi(M)))$. We have $\dim_{C}Y^{\Phi}\leqq\dim_{C}p_{2}^{-1}(\Phi(M))\leqq r+\dim_{C}\Phi(M)$ (as the fibre of p_{2} is $CP^{r})\leqq r+1$. Hence Y^{Φ} cannot be equal to CP^{n} if $r<n-1$. This completes the proof. □

REMARK. We have extended solutions of the form $\pi_{f}+\lambda\pi_{f}^{\perp}$ in Example 6.3, and $(\pi_{f_{r}}+\lambda\pi_{f_{r}}^{\perp})(\pi_{f_{r+1}}+\lambda\pi_{f_{r+1}}^{\perp})$ in Example 6.4. It follows that the deformations used in these examples could have been obtained by applying Theorem 6.2, because the deformation of Theorem 6.2 preserves the relevant Grassmannian or flag manifold and the hypotheses of that theorem are satisfied.

EXAMPLE 6.6 (cf. Example 3.6). Let $\mathrm{Harm}_{d}(S^{2}, S^{n})$ be the space of harmonic maps $\varphi: S^{2}\rightarrow S^{n}$ of energy d, with a similar definition for $\mathrm{Harm}_{d}(S^{2}, RP^{n})$.

THEOREM 6.7. (i) $\mathrm{Harm}_{d}(S^{2}, S^{n})$ *and* $\mathrm{Harm}_{d}(S^{2}, RP^{n})$ *are connected, if* $n\geqq 3$. (ii) *More generally, the space of isotropic harmonic maps of energy d of any compact Riemann surface M into* S^{n} *(or* RP^{n}*) is connected, if* $n\geqq 3$ *and if* $d\geqq 2g$, *where g is the genus of M.*

REMARK. This result is elementary if $n=3$. Part (i) was proved by Loo ([**Lo**]) and by Verdier ([**Ve3**]) for $n=4$, and extended to $n\geqq 4$ by Kotani ([**Kt**]).

PROOF. It suffices to give the proof of (ii). The result for S^{n} follows from that for RP^{n}, as the natural map $S^{n}\rightarrow RP^{n}$ induces a non-trivial double covering $\mathrm{Harm}_{d}^{iso}(M, S^{n})\rightarrow\mathrm{Harm}_{d}^{iso}(M, RP^{n})$, where Harm_{d}^{iso} denotes isotropic harmonic maps of energy d. By [**Ca1**], [**Ca2**] it suffices to take n even, say $n=2m$, and it suffices to show that the space $\mathrm{HH}_{d}(S^{2}, Z_{m})$ of holomorphic horizontal maps $\Phi: M\rightarrow Z_{m}$ of degree d is connected, as the map $\pi: Z_{m}\rightarrow RP^{2m}$ induces a surjection $\mathrm{HH}_{d}(M, Z_{m})\rightarrow\mathrm{Harm}_{d}^{iso}(M, RP^{2m})$. (The degree of Φ is equal to the energy of $\varphi=\pi\circ\Phi$, if the energy is normalized suitably.)

We shall prove that $\mathrm{HH}_{d}(M, Z_{m})$ is connected by induction on m. For $m=1$, the horizontality condition is vacuous, so $\mathrm{HH}_{d}(M, Z_{m})$ may be identified with the space $\mathrm{Hol}_{d}(M, S^{2})$. This is known to be connected if $d\geqq 2g$ (see Corollary 1.3.13 of [**Na**] and also the comments in Example 6.3), and so the induction begins.

For the inductive step, we shall identify Z_{m} with the orbit $\mathrm{Ad}(SO_{2m+1})\times$

$(i\pi_V - i\pi_{\bar{V}})$, where V is a fixed element of Z_m. Let L be an isotropic line in C^{2m+1}, and set $Q = i\pi_L - i\pi_{\bar{L}}$. The critical points $W \in Z_m$ of the Morse-Bott function whose downwards gradient flow is given by the action of $\{\exp itQ\}$ are given by $[i\pi_W - i\pi_{\bar{W}}, i\pi_L - i\pi_{\bar{L}}] = 0$, i.e. $W = W_1 \oplus W_2 \oplus W_3$ with $W_1 \subseteqq L$, $W_2 \subseteqq \bar{L}$, $W_3 \subseteqq (L \oplus \bar{L})^\perp$. There are two connected critical manifolds, namely:

$$Z^+ = \{W \mid L \subseteqq W \subseteqq \bar{L}^\perp\} \cong Z_{m-1}$$

$$Z^- = \{W \mid \bar{L} \subseteqq W \subseteqq L^\perp\} \cong Z_{m-1}.$$

The corresponding stable manifolds are

$$S^Q(Z^+) = Z^+$$

$$S^Q(Z^-) = \{W \mid W \cap L = \{0\}\}.$$

The embeddings $I^\pm : Z_{m-1} \to Z_m$ defined by the inclusions of $Z^\pm$ in Z_m are holomorphic. They also respect the horizontality condition $(\partial/\partial z)C^\infty\Phi \perp C^\infty\bar{\Phi}$, in the sense that a map $\Phi : M \to Z_{m-1}$ is horizontal if and only if either of the maps $I^\pm \circ \Phi : M \to Z_m$ are horizontal. We shall accomplish the inductive step by showing that any element Φ of $\mathrm{HH}_d(M, Z_m)$ may be deformed into Z^-.

If $\Phi(M) \subseteqq S^Q(Z^-)$, then $\{(\exp itQ)^\natural \Phi\}_{0 \le t \le \infty}$ provides a continuous deformation of Φ into Z^-. So it suffices to show that Φ can be deformed into $S^Q(Z^-)$. We claim that there exists some $A \in SO_{2m+1}$ such that $A^\natural \Phi(M) \subseteqq S^Q(Z^-)$, i.e. $A\Phi(z) \not\supseteqq L$ for all $z \in M$; this will give the required deformation, as SO_{2m+1} is connected. Since SO_{2m+1} acts transitively on the space Y_m of all isotropic lines in C^{2m+1}, it suffices to find some isotropic line L' such that $\Phi(z) \not\supseteqq L'$ for all $z \in M$. Let

$$Y_m^\Phi = \{L' \in Y_m \mid L' \subseteqq \Phi(z) \text{ for some } z \in M\}.$$

Then our claim is that $Y_m^\Phi \neq Y_m$. Let $X_m = \{(L', W) \in Y_m \times Z_m \mid L' \subseteqq W\}$. Let p_1, p_2 be the projections to Y_m, Z_m. Then $Y_m^\Phi = p_1(p_2^{-1}(\Phi(M)))$. We have $\dim_C Y_m^\Phi \leqq \dim_C p_2^{-1}(\Phi(M)) \leqq m - 1 + \dim_C \Phi(M)$ (as the fibre of p_2 is CP^{m-1}) $\leqq m$. Since $\dim_C Y_m = 2m-1$, Y_m^Φ cannot be equal to Y_m if $m \geqq 2$. This completes the proof. □

It should be clear from these examples that a similar method applies to those harmonic maps $\varphi : M \to G/K$ which are of the form $\varphi = \pi \circ \Phi$, where Φ is holomorphic and superhorizontal with respect to a twistor fibration $\pi : G/H \to G/K$. That is, for a height function $h^Q : G/H \to \boldsymbol{R}$ (where $G/H = \mathrm{Ad}(G)P$), we obtain deformations Φ_t of Φ such that Φ_∞ takes values (generically) in a critical manifold $C(T_Q)/C(T_Q)_X = \mathrm{Ad}\, C(T_Q)X$ of h^Q. To obtain a continuous deformation, one must ensure that the image of Φ lies entirely in the stable manifold of this critical manifold.

Without loss of generality we may assume that $X=P$. A calculation similar to that of Lemma 3.7 then shows that the bundle $C(T_Q)/C(T_Q)_X \to C(T_Q)/C(T_Q)_X \cap K$ is a "twistor sub-fibration" of $G/H \to G/K$. Lemma 3.7 provides an infinite dimensional version of this phenomenon; namely, that the bundle $G/H \to G/K$ may be regarded as a twistor sub-fibration of the fibration $\Omega G \to G$. As explained in § 3. ΩG can be realized as the orbit of the point $\alpha=(i, 0) \in i\boldsymbol{R} \ltimes \Lambda\mathfrak{g}$, under the action of $S^1 \ltimes \Lambda G$. The theory described in the Appendix for a finite dimensional adjoint orbit extends almost entirely to ΩG (cf. [**AP**], § 8.9 of [**PS**], and [**Ko**]), although there are some new features. For example, the inner product $\langle , \rangle$ is not bi-invariant with respect to the action of $S^1 \ltimes \Lambda G$. From our point of view, the main difference is that it is not in general possible to integrate the gradient vector field on the infinite dimensional manifold ΩG. In fact (Theorem 8.9.9 of [**PS**]), every point $\gamma \in \Omega G$ admits "downwards" flow line, but only points of $\Omega_{\mathrm{alg}} G$ admit "upwards" flow lines. The asymmetrical nature of the flow reflects the fact that the action of $\boldsymbol{C}^*$ on $\Omega_{\mathrm{alg}} G$ extends to an action of $\boldsymbol{C}_1^* = \{\lambda \in \boldsymbol{C}^* \mid |\lambda| < 1\}$ on ΩG, but not to an action of $\boldsymbol{C}^*$.

Appendix. Height functions on generalized flag manifolds.

Let G be a compact connected Lie group. The orbit $M_P = \mathrm{Ad}(G)P$ of a point $P \in \mathfrak{g}$ under the adjoint representation is called a generalized flag manifold. It is known that the isotropy subgroup of P is the centralizer, $C(T_P)$, of that torus T_P which is the closure of the one-parameter subgroup $\{\exp tP\}$. The complex group G^C also acts transitively on M_P, and the isotropy subgroup of P is a parabolic subgroup G_P of G^C. Thus, we have natural diffeomorphisms

$$M_P \cong G/C(T_P) \cong G^C/G_P .$$

We denote the natural action of G^C on M_P by $(g, X) \mapsto g^\natural X$. (If $g \in G$, then $g^\natural X = \mathrm{Ad}(g)X$.) The standard example of this is given by $G=U_n$ and $P=i\pi_V \in \mathfrak{u}_n$, where V is a k-dimensional subspace of $\boldsymbol{C}^n$ and π_V denotes orthogonal projection from $\boldsymbol{C}^n$ to V with respect to the Hermitian inner product of $\boldsymbol{C}^n$. Then

$$M_P \cong U_n/U_k \times U_{n-k} \cong GL_n(\boldsymbol{C})/G_P ,$$

where $G_P = \{A \in GL_n(\boldsymbol{C}) \mid AV \subseteqq V\}$. This can be identified with the Grassmanian $Gr_k(\boldsymbol{C}^n)$, by identifying $\mathrm{Ad}(A)P$ with the k-plane AV. The action of $GL_n(\boldsymbol{C})$ on $Gr_k(\boldsymbol{C}^n)$ is then given by the formula $A^\natural V = AV$. The homogeneous space M_P has a natural Kähler structure, which is determined by the choice of P and a choice of an $\mathrm{Ad}(G)$-invariant inner product $\langle\!\langle , \rangle\!\rangle$ on $\mathfrak{g}$.

For any $Q \in \mathfrak{g}$, we define the "height function" $h^Q : M_P \to \boldsymbol{R}$ by

$$h^Q(X) = \langle\!\langle X, Q \rangle\!\rangle .$$

A point $X \in M_P$ is a critical point of h^Q if and only if $[Q, X]=0$. It follows from this that the critical points of h^Q form a finite number of orbits of the group $C(T_Q)$, say

$$N_1 = \mathrm{Ad}(C(T_Q))X_1, \cdots, N_r = \mathrm{Ad}(C(T_Q))X_r .$$

These critical manifolds are non-degenerate; in other words, h^Q is a "Morse-Bott function". In the standard example, where $M_P \cong Gr_k(\boldsymbol{C}^n)$, let us choose $Q=i\pi_l$ where $\pi_l: \boldsymbol{C}^n \to \boldsymbol{C}^l$ is orthogonal projection onto the span of the first l standard basis vectors. A point $\mathrm{Ad}(A)P=i\pi_W$ is a critical point of h^Q if and only if $[\pi_l, \pi_W]=0$, i.e. $W=W_0 \oplus W_1$ where $W_0 \subseteqq \boldsymbol{C}^l$, $W_1 \subseteqq (\boldsymbol{C}^l)^\perp$. The critical manifold N containing $W=W_0 \oplus W_1$ is the set of k-planes U such that $U=U_0 \oplus U_1$, where $U_0 \subseteqq \boldsymbol{C}^l$, $U_1 \subseteqq (\boldsymbol{C}^l)^\perp$, and $\dim U_i = \dim W_i$ for $i=0, 1$. It is the orbit of $i\pi_W$ under the group $C(T_Q)=U_l \times U_{n-l}$, and hence is a copy of $Gr_{w_0}(\boldsymbol{C}^l) \times Gr_{w_1}(\boldsymbol{C}^{n-l})$, where $w_i = \dim W_i$. The index of a critical manifold may be computed using the Stiefel diagram of G. This theory is due to Bott ([**Bo**]).

Let ∇h^Q be the gradient of h^Q with respect to the Kähler metric. The integral curves of ∇h^Q may be calculated explicitly, since

$$-\nabla h^Q = JQ^*$$

where Q^* is the vector field on M_P associated to the one parameter subgroup $\{\exp tQ\}$. This observation is due to Frankel ([**Fr**]). It follows that the flow line of $-\nabla h^Q$ which passes through a non-critical point X is

$$t \longmapsto (\exp itQ)^\natural X .$$

In the standard example, the flow line of $-\nabla h^Q$ passing through a non-critical point $i\pi_W$ is given by

$$t \longmapsto \pi_{W_t}, \qquad W_t = e^{-t\pi_l}W$$

where $e^{-t\pi_l}$ is the $n \times n$ diagonal matrix with diagonal terms $e^{-t}, \cdots, e^{-t}$ (l times) $1, \cdots, 1$ ($n-l$ times).

The stable (or unstable) manifold $S^Q(X)$ (or $U^Q(X)$) of a critical point X is by definition the union of the flow lines of $-\nabla h^Q$ which converge to X as $t \to \infty$ (or as $t \to -\infty$). The stable manifold of the critical manifold N is defined by $S^Q(N) = \bigcup_{Y \in N} S^Q(Y)$, with a similar definition of the unstable manifold $U^Q(N)$. Using the above description of the flow lines, it can be shown that

$$S^Q(N) = (G_Q)^\natural X$$

i.e. the orbit of X under the (complex) group G_Q. Similarly,

$$U^Q(N) = (G_Q^{opp})^\natural X$$

where G_Q^{opp} is the "opposite" parabolic subgroup to G_Q. In the standard example, the stable manifold of the critical manifold N is the set of k-planes U such that $\dim U\cap \boldsymbol{C}^l=w_0$. This is the orbit of W under the group $G_Q=\{A\in GL_n(\boldsymbol{C})\,|\,A(\boldsymbol{C}^l)\subseteqq \boldsymbol{C}^l\}$. The unstable manifold is the set of k-planes U such that $\dim U\cap(\boldsymbol{C}^l)^{\perp}=w_1$, i.e. the orbit of W under the group $G_Q^{opp}=\{A\in GL_n(\boldsymbol{C})\,|\,A(\boldsymbol{C}^l)^{\perp}\subseteqq(\boldsymbol{C}^l)^{\perp}\}$.

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