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On hypergeometric functions in several variables II. The Wronskian of the hypergeometric functions of type $(n+1, m+1)$

By Michitake KITA

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Introduction.

Many specialists believe that the rank of the hypergeometric system $E(n+1, m+1; \lambda)$ is equal to $\binom{m-1}{n}$ for sufficiently generic complex parameters $\lambda=(\lambda_1, \cdots, \lambda_{m+1})$. (For the definition, see §2.1.) Although this fact is fundamental in studying the hypergeometric functions, no explicit statement with rigorous proof is known. In a paticular case $E(3, 6; \lambda)$, [**M-S-Y**, p. 64, Theorem 1.8.3] gave explicitly 6 solutions of the form $y^{\rho}\sum_{n\in Z^4_{\geqq 0}}A(n)y^n$ where $y=(y_1, y_2, y_3, y_4)$ and $\rho=(\rho_1, \rho_2, \rho_3, \rho_4)$. Also [**G-Gr**, p. 13] gave 6 solutions of $E(3, 6)$; but it is not clear under which conditions the solutions are linearly independent.

In this paper, we shall give a proof of the fact *under a very simple non-integral condition* $\lambda_j \in \boldsymbol{C}-\boldsymbol{Z}$. Our proof is based on the theory of twisted rational de Rham cohomologies and twisted homologies; we make use of the perfect pairing of those associated with the hypergeometric integral of type $(n+1, m+1)$. Since the theory of twisted cycles is very important in the study of hypergeometric integrals (see [**K**]), we shall give a concise introduction to the theory in §1.

Let $f_j(u)=z_{0j}+\sum_{i=1}^n z_{ij}u_i$ $(1\leqq j\leqq m)$ be m *real* linear polynomials of n variables; we consider a many-valued function $U=\prod_{j=1}^m f_j^{\lambda_j}$. Then the hypergeometric integral $F(\lambda, z)$ is defined as $F(\lambda, z)=\int_{\sigma} U du_1\wedge\cdots\wedge du_n$ where σ is a twisted n-cycle associated with U. We suppose the configuration determined by hyperplanes $f_j=0$ $(1\leqq j\leqq m)$ be *in general position*. Then it determines $r:=\binom{m-1}{n}$ relatively compact chambers in the real u-space $\boldsymbol{R}^n$. In §1.7~1.8, we construct

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r twisted cycles $\sigma_\nu(1\leqq\nu\leqq r)$ associated with the chambers, which form a basis of twisted n-dimensional homology associated with U, *provided that the non-integral condition stated above holds* (Theorem 2). This basis is crucial in writing the Wronskian of our hypergeometric integrals *in a closed form*. We show in §2 that r hypergeometric integrals $\int_{\sigma_\nu} U du_1\wedge\cdots\wedge du_n (1\leqq\nu\leqq r)$ are linearly independent solutions of the hypergeometric system $E(n+1, m+1; \lambda)$. In [**M-S-T-Y**], this fact is essentially used to determine the monodromy of the system $E(n+1, m+1; \lambda)$. By making a *proper choice of partial derivatives of the integrals,* we see that the Wronskian of the r integrals turns out to be a product of minors of the matrix (z_{ij}) and the determinant of the matrix

$$\left(\int_{\sigma_\nu} U\cdot\varphi\langle j_1\cdots j_n\rangle\right)$$

where we set

$$\varphi\langle j_1, \cdots, j_n\rangle=\frac{df_{j_1}}{f_{j_1}}\wedge\cdots\wedge\frac{df_{j_n}}{f_{j_n}}, \qquad (1\leqq j_1<\cdots<j_n\leqq m-1);$$

these n-forms from a basis of the twisted rational de Rham cohomology associated with the integral. Since the pairing is *perfect,* this determinant is non-zero and hence we can conclude that the Wronskian is non-zero *under the condition* $\lambda_j\in \boldsymbol{C}-\boldsymbol{Z}(1\leqq j\leqq m)$ *and* $\sum_{i=1}^m \lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$ (Theorem 3). On the other hand, this determinant turns out to be the one which A. N. Varchenko studied in [**V1, 2**]. He evaluates the determinant as the product of Γ-factors and critical values of the functions $f_j^{\lambda_j}$ on the compact chambers. Using this result, we write the Wronskian in a closed form (Theorem 4).

§1. Twisted rational de Rham cohomology and twisted cycles.

1.1. An intuitive explanation. Let X be an n-dimensional connected complex manifold and $\pi: \tilde{X}\to X$ be the universal covering of X. Let $\tilde{u}_0\in\tilde{X}$ and $u_0=\pi(\tilde{u}_0)\in X$ be base points. We shall consider a many-valued holomorphic function $U(u)$ on X such that $\omega:=dU/U$ is a *single-valued* holomorphic 1-form on X. Let $\mathcal{E}_X^p$ be the sheaf of smooth p-forms on X; then for any $\varphi\in\Gamma(X, \mathcal{E}_X^p)$, $U\varphi$ can be viewed, on one hand, as a single-valued p-form on the universal covering manifold $\tilde{X}$, on the other hand, as a many-valued p-form on X. Since $U\varphi$ is single-valued on $\tilde{X}$, the usual Stokes theorem holds for $U\varphi$ on $\tilde{X}$ but not on X. We explain our situation by illustrating some examples:

EXAMPLE 1. Let C be an oriented path in X with starting point p and end point q and let $\tilde{C}$ be a lift of C by the map $\pi: \tilde{X}\to X$. The Stokes theorem on $\tilde{X}$ reads

$$\int_{\tilde{C}} d(U(u)\varphi) = U(\tilde{q})\varphi(q) - U(\tilde{p})\varphi(p) \qquad \text{for} \quad \varphi \in \Gamma(X, \mathcal{E}_X^0). \tag{1.1}$$

We shall interpret (1.1) in terms of quantities defined on the manifold X: let $\tilde{v}_0 \in \tilde{C}$ such that $\pi(\tilde{v}_0) = v_0$ and let U_{v_0} be the function element of the many-valued function $U(u)$ corresponding to that of $\pi^* U$ at $\tilde{v}_0$. Then we see that

$U(\tilde{q})$ = the value at q of the function element obtained by analytic continuation of U_{v_0} along $v_0 q$.

Since $U\varphi$ is many-valued, $d(U\varphi)$ has no meaning on X. So we use a formula $d(U\varphi) = U\nabla_\omega \varphi$ on $\tilde{X}$, where the operator ∇_ω is defined on functions on X by $\nabla_\omega \varphi = d\varphi + \omega \wedge \varphi$. Then the integral $\int_{\tilde{C}} d(U\varphi)$ can be interpreted as the integral $\int_C U\nabla_\omega \varphi$ where U is the single-valued function on C obtained by analytic continuation of the function element U_{v_0} along C. In order to show up the assignment of the branch of U, we write the integral by

$$\int_{C \otimes U_{v_0}} U\nabla_\omega \varphi .$$

We do not explain here what is $C \otimes U_{v_0}$, since a rigorous definition might disturb the flow of thought; see the next subsection. Now (1.1) is rewritten as

$$\int_{C \otimes U_{v_0}} U\nabla_\omega \varphi = \begin{pmatrix} \text{the value at } q \\ \text{of the element} \\ \text{obtained by analytic} \\ \text{continuation of } U_{v_0} \\ \text{along } v_0 q \end{pmatrix} \varphi(q) - \begin{pmatrix} \text{the value at } p \\ \text{of the element} \\ \text{obtained by analytic} \\ \text{continuation of } U_{v_0} \\ \text{along } v_0 p \end{pmatrix} \varphi(p). \tag{1.2}$$

In view of (1.2), we define $\partial_\omega(C \otimes U_{v_0})$ by

$$\partial_\omega(C \otimes U_{v_0}) := q \otimes U_q - p \otimes U_p$$

where U_q is the element at q obtained by analytic continuation of U_{v_0} along $v_0 q$ and U_p the similar one at p. Then the Stokes theorem (1.1) on $\tilde{X}$ is rewritten as

$$\int_{C \otimes U_{v_0}} U\nabla_\omega \varphi = \int_{\partial_\omega (C \otimes U_{v_0})} U \cdot \varphi \qquad \text{for} \quad \varphi \in \Gamma(X, \mathcal{E}_X^0).$$

EXAMPLE 2. We suppose that X is triangulated and to each simplex Δ is assigned the barycenter v_Δ. Let $\tilde{\Delta}$ be the lift of Δ corresponding to the element U_{v_Δ}. For simplicity we suppose that Δ is a 2-simplex; then for $\varphi \in \Gamma(X, \mathcal{E}^1)$, we have

$$\iint_{\Delta\otimes U_{v_\Delta}} U\cdot\nabla_\omega\varphi=\iint_{\tilde\Delta} d(U\varphi)=\int_{\partial\tilde\Delta} U\varphi=\int_{\sigma1\otimes U_{v_{\sigma1}}+\sigma2\otimes U_{v_{\sigma2}}+\sigma3\otimes U_{v_{\sigma3}}} U\varphi$$

where we use the following notations: (i) integration over $\Delta\otimes U_{v_\Delta}$ means integration over Δ on which values of U are determined by the function element U_{v_Δ} and its analytic continuation. (ii) $U_{v_{\sigma i}}$ is the analytic continuation of U_{v_Δ} along a path in Δ.

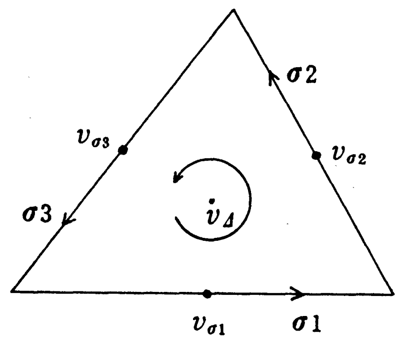


Figure 1.

Hence we define $\partial_\omega(\Delta\otimes U_{v_\Delta})$ by

$$\partial_\omega(\Delta\otimes U_{v_\Delta})=\sigma1\otimes U_{v_{\sigma1}}+\sigma2\otimes U_{v_{\sigma2}}+\sigma3\otimes U_{v_{\sigma3}}$$

where σi's have the orientations indicated in Figure 1, and we paraphrase the Stokes theorem as follows:

$$\iint_{\Delta\otimes U_{v_\Delta}} U\cdot\nabla_\omega\varphi=\int_{\partial_\omega(\Delta\otimes U_{v_\Delta})} U\cdot\varphi.$$

1.2. Definition of twisted cycles. We call

$$C_q(X;U):=\left(\begin{array}{l}\text{finite sums of } c\Delta\otimes U_{v_\Delta}\\ \qquad \text{where } c\in \boldsymbol{C} \text{ and } \Delta \text{ are } q\text{-simplices}\end{array}\right)$$

the *q-dimensional twisted chain groups.* We define the boundary operator

$$\partial_\omega:\ C_q(X;U)\longrightarrow C_{q-1}(X,U),$$

by

$$\partial_\omega(\Delta\otimes U_{v_\Delta})=\sum_{i=0}^{q}(-1)^q\sigma i\otimes U_{v_{\sigma i}}$$

where (i) Δ is an oriented q-simplex $(p_0 p_1 \cdots p_q)$ and σi is the i-th oriented face $(p_0 \cdots \hat{p}_i \cdots p_q)$ of Δ; (ii) $U_{v_{\sigma i}}$ is the analytic continuation of U_Δ along a path in Δ. It satisfies $\partial_\omega \cdot \partial_\omega = 0$ and determines the twisted homology groups $H_q(X; U)$, $(q=0, 1, \cdots)$.

EXAMPLE. $X = \boldsymbol{C} - \{0, 1\}$, $U(u) = u^\alpha (1-u)^\beta$.

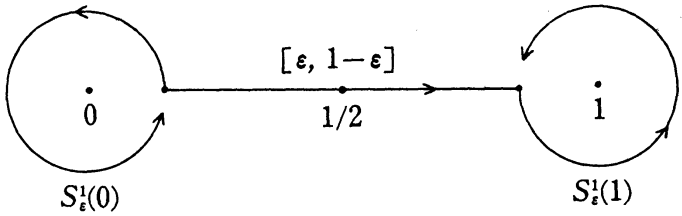


Figure 2.

For the following twisted 2-chains

$$S_\varepsilon^1(0) \otimes U_\varepsilon, \qquad [\varepsilon, 1-\varepsilon] \otimes U_{1/2}, \qquad S_\varepsilon^1(1) \otimes U_{1-\varepsilon},$$

we have

$$\partial_\omega (S_\varepsilon^1(0) \otimes U_\varepsilon) = (e^{2\pi i \alpha} - 1) \langle \varepsilon \rangle \otimes U_\varepsilon,$$
$$\partial_\omega ([\varepsilon, 1-\varepsilon] \otimes U_{1/2}) = \langle 1-\varepsilon \rangle \otimes U_{1-\varepsilon} - \langle \varepsilon \rangle \otimes U_\varepsilon,$$
$$\partial_\omega (S_\varepsilon^1(1) \otimes U_{1-\varepsilon}) = (e^{2\pi i \beta} - 1) \langle 1-\varepsilon \rangle \otimes U_{1-\varepsilon}.$$

Setting

$$\Delta^1(\omega) := \frac{1}{e^{2\pi i \alpha} - 1} S_\varepsilon^1(0) \otimes U_\varepsilon + [\varepsilon, 1-\varepsilon] \otimes U_{1/2} - \frac{1}{e^{2\pi i \beta} - 1} S_\varepsilon^1(1) \otimes U_{1-\varepsilon},$$

we get $\partial_\omega \Delta^1(\omega) = 0$ and hence $\Delta^1(\omega) \in H_1(X; U)$.

1.3. Let $\mathcal{S}_\omega$ be the complex local system of local solutions $\nabla_\omega \varphi = 0$ where $\varphi \in \mathcal{E}_X^0$. Since ∇_ω is integrable, the sequence

$$0 \longrightarrow \mathcal{S}_\omega \longrightarrow \mathcal{E}_X^0 \xrightarrow{\nabla_\omega} \mathcal{E}_X^1 \longrightarrow \cdots \xrightarrow{\nabla_\omega} \mathcal{E}_X^{2n} \longrightarrow 0$$

is exact; $\mathcal{E}_X^p$ being fine sheaves, we have

$$H^p(X; \mathcal{S}_\omega) \cong H^p(\Gamma(X, \mathcal{E}^\cdot), \nabla_\omega). \tag{1.3}$$

By the analogue of de Rham's theorem, we have a perfect pairing

$$H_p(X; U) \times H^p(\Gamma(X, \mathcal{E}^\cdot), \nabla_\omega) \longrightarrow \boldsymbol{C}$$
$$(\sigma, \varphi) \longmapsto \int_\sigma U \cdot \varphi.$$

In view of (1.3), henceforth $H_p(X; U)$ will be written as $H_p(X; \mathcal{S}_\omega^\vee)$ where $\mathcal{S}_\omega^\vee$

is the dual complex local system to $\mathcal{S}_{\omega}$.

1.4. From this subsection we shall suppose that

$$X = \boldsymbol{C}^n - \bigcup_{j=1}^{m} \{f_j=0\}$$

where $f_j(u)=a_{0j}+a_{1j}u_1+\cdots+a_{nj}u_n\,(1\leqq j\leqq m)$ are hyperplanes of $\boldsymbol{C}^n$ *in general position*. Let Ω_X^p be the sheaf of rational p-forms on $\boldsymbol{C}^n$ which are holomorphic in X; then by the Grothendieck-Deligne comparison theorem, we have

$$H^p(\Gamma(X, \mathcal{E}_X^{\cdot}), \nabla_{\omega}) \cong H^p(\Gamma(X, \Omega_X^{\cdot}), \nabla_{\omega}),$$

and hence we obtain a perfect pairing

$$H_p(X, \mathcal{S}_{\omega}^{\vee})\times H^p(\Gamma(X, \Omega_X^{\cdot}), \nabla_{\omega}) \longrightarrow \boldsymbol{C}$$

$$(\sigma, \varphi) \longmapsto \int_{\sigma} U\cdot\varphi\,.$$

We call the complex $(\Gamma(X, \Omega_X^{\cdot}), \nabla_{\omega})$ *twisted rational de Rham complex whose cohomology is denoted by* $H^p(X, \nabla_{\omega})$ $(0\leqq p\leqq n)$. Then we have the following fundamental theorem:

THEOREM 1 ([**A2**], [**KN**, p. 153, Theorem 6]). *Let* $U=\prod_{j=1}^{m} f_j(u)^{\lambda_j}$ *where* $f_j(u)$ $(1\leqq j\leqq m)$ *are hyperplanes in general position and set* $\omega=dU/U$. *Suppose that*

$$\lambda_j \in \boldsymbol{C}-\boldsymbol{Z}\ (1 \leqq j \leqq m) \quad and \quad \sum_{j-1}^{m} \lambda_j \in \boldsymbol{C}-\boldsymbol{Z}\,;$$

then we have

$$H^p(X, \nabla_{\omega}) = 0 \qquad for \quad p \neq n\,,$$

$$H^n(X, \nabla_{\omega}) \cong \frac{\left(\left(\dfrac{df_{j_1}}{f_{j_1}}\wedge\cdots\wedge\dfrac{df_{j_n}}{f_{j_n}}\,\middle|\,1\leqq j_1<\cdots<j_n\leqq m\right)\right)}{\omega\wedge\left(\left(\dfrac{df_{j_1}}{f_{j_1}}\wedge\cdots\wedge\dfrac{df_{j_{n-1}}}{f_{j_{n-1}}}\,\middle|\,1\leqq j_1<\cdots<j_{n-1}\leqq m\right)\right)}$$

and

$$\dim H^n(X, \nabla_{\omega}) = \binom{m-1}{n}.$$

Set

$I = \{1, \cdots, m\}$ and

$$\varphi(J) = \varphi\langle j_1, \cdots, j_n\rangle = \frac{df_{j_1}}{f_{j_1}}\wedge\cdots\wedge\frac{df_{j_n}}{f_{j_n}} \qquad \text{for} \quad J = \{j_1, \cdots, j_n\} \subset I\,.$$

Since $\omega=\sum_{j=1}^{m} \lambda_j(df_j/f_j)$, we have the following fundamental relations among $\varphi\langle j_1, \cdots, j_n\rangle$ in $H^n(X, \nabla_{\omega})$:

$$\lambda_m \varphi\langle j_1, \cdots, j_{n-1}, m\rangle = -\sum_{j\in I\setminus\{j_1,\cdots,j_{n-1},m\}} \lambda_j \varphi\langle j_1, \cdots, j_{n-1} j\rangle$$
$$\text{for any} \quad \{j_1, \cdots, j_{n-1}\} \subset I\setminus\{m\}.$$

Therefore we obtain the following

COROLLARY. *$\{\varphi\langle J\rangle \mid J\subset(I\setminus\{m\})$ and $\#J=n\}$ is a basis of $H^n(X, \nabla_\omega)$.*

1.5. It is known (see for example [**H**]) that

(i) By the Poincaré duality, the following pairing is perfect:

$$\begin{array}{ccc} H^p(X, \mathcal{S}_\omega)\times H_c^{2n-p}(X, \mathcal{S}_\omega^\vee) & \longrightarrow & \boldsymbol{C} \\ \wr\| \qquad\qquad \wr\| & & \\ H^p(\Gamma(X, \mathcal{E}_X^\cdot), \nabla_\omega)\times H^{2n-p}(\Gamma_c(X, \mathcal{E}_X^\cdot), \nabla_{-\omega}) & \longrightarrow & \boldsymbol{C} \end{array}$$

$$(\varphi, \psi) \longmapsto \int_X \varphi\wedge\psi$$

where $\Gamma_c(X, \mathcal{E}_X^\cdot)$ is the $\boldsymbol{C}$-vector space of global sections of $\mathcal{E}_X^\cdot$ with compact support and $H_c^q(X, \mathcal{S}_\omega^\vee)$ is the q-th twisted cohomology with compact support.

Using the universal coefficient theorem for twisted cohomology and the Poincaré duality, we have that

(ii) $H_c^p(X, \mathcal{S}_\omega)$ is dual to the homology $H_p^{lf}(X, \mathcal{S}_\omega^\vee)$ of locally finite chains with coefficients in $\mathcal{S}_\omega^\vee$.

In view of (i) and (ii), we see that

$$H_p^{lf}(X, \mathcal{S}_\omega) \cong H_c^p(X, \mathcal{S}_\omega^\vee)^\vee \cong H^{2n-p}(X, \mathcal{S}_\omega). \tag{1.4}$$

Under the assumption of Theorem 1, it follows from (1.4) that

$$H_p^{lf}(X, \mathcal{S}_\omega^\vee) = 0 \qquad \text{for} \quad p\neq n$$

and

$$H_n^{lf}(X, \mathcal{S}_\omega^\vee) \cong H^n(X, \mathcal{S}_\omega^\vee) \cong H_n(X, \mathcal{S}_\omega)^\vee ;$$

hence

$$\dim H_n^{lf}(X, \mathcal{S}_\omega^\vee) = \binom{m-1}{n}. \tag{1.5}$$

1.6. We suppose that $f_j\,(1\leqq j\leqq m)$ are *real* linear polynomials:

$$f_j(u) \in \boldsymbol{R}[u_1, \cdots, u_n] \qquad \text{for} \quad 1\leqq j\leqq m.$$

Let Δ_l's be the relatively compact connected components of

$$X\cap\boldsymbol{R}^n = \boldsymbol{R}^n - \bigcup_{j=1}^m \{f_j(u)=0\}.$$

We set

$$D := \bigcup_{j=1}^m \{f_j=0\} \cup \{\text{the hyperplane at infinity}\} ;$$

then D and $D\cup\coprod_l\Delta_l$ are closed subset of the n-dimensional complex projective space $\boldsymbol{CP}^n$. For the triple $(\boldsymbol{CP}^n, D\cup\coprod\Delta_l, D)$, we have a cohomology exact sequence with coefficient in $\mathcal{S}_\omega$ (see for example [**H**]):

$$\text{(1.6)} \qquad \longrightarrow H^{q-1}(D\cup\coprod\Delta_l, D) \longrightarrow H^q(\boldsymbol{CP}^n, D\cup\coprod\Delta_l) \longrightarrow H^q(\boldsymbol{CP}^n, D) \longrightarrow$$
$$\longrightarrow H^q(D\cup\coprod\Delta_l, D) \longrightarrow \cdots.$$

We have the following isomorphisms:

$$\begin{aligned} H_q^{lf}(\boldsymbol{CP}^n-D, \mathcal{S}_\omega^\vee) &\cong H_c^q(\boldsymbol{CP}^n-D, \mathcal{S}_\omega)^\vee \qquad \text{(by (1.4))} \\ &\cong H^q(\boldsymbol{CP}^n, \text{nbd. of } D; \mathcal{S}_\omega)^\vee \\ &\quad \text{(since } \boldsymbol{CP}^n \text{ is compact and } D \text{ is closed)} \\ &\cong H^q(\boldsymbol{CP}^n, D; \mathcal{S}_\omega)^\vee \\ &\quad \text{(since } D \text{ possesses fundamental system of} \\ &\quad \text{ neighbourhood of which } D \text{ is retract.)} \end{aligned}$$

Similary we have

$$H_q^{lf}(D\cup\coprod_l\Delta_l-D; \mathcal{S}_\omega^\vee) \cong H^q(D\cup\coprod_l\Delta_l, D; \mathcal{S}_\omega)^\vee,$$
$$H_q^{lf}(\boldsymbol{CP}^n-D\cup\coprod_l\Delta_l; \mathcal{S}_\omega^\vee) \cong H^q(\boldsymbol{CP}^n, D\cup\coprod_l\Delta_l; \mathcal{S}_\omega)^\vee.$$

Since we set $X=\boldsymbol{CP}^n-D$, (1.6) yields the following exact sequence:

$$\text{(1.7)} \qquad \longrightarrow H_{n+1}^{lf}(X-\coprod\Delta_l, \mathcal{S}_\omega^\vee) \longrightarrow H_n^{lf}(\coprod_l\Delta_l; \mathcal{S}_\omega^\vee) \longrightarrow$$
$$\longrightarrow H_n^{lf}(X; \mathcal{S}_\omega^\vee) \longrightarrow H_n^{lf}(X-\coprod\Delta_l, \mathcal{S}_\omega^\vee) \longrightarrow H_{n-1}^{lf}(\coprod_l\Delta_l; \mathcal{S}_\omega^\vee) \longrightarrow \cdots.$$

By (1.4), we see that

$$H_q^{lf}(X-\coprod_l\Delta_l, \mathcal{S}_\omega^\vee) \cong H^{2n-q}(X-\coprod_l\Delta_l, \mathcal{S}_\omega^\vee).$$

Since $X-\coprod_l\Delta_l$ is homotopic to $\bar{X}=\boldsymbol{C}^n-\cup_{j=1}^m\{\bar{f}_j(u)=0\}$ where $\bar{f}_j(u)=\sum_{i=1}^n a_{ij}u_i$, we get

$$\text{(1.8)} \qquad H^{2n-q}(X-\coprod_l\Delta_l, \mathcal{S}_\omega^\vee) \cong H^{2n-q}(\bar{X}, \mathcal{S}_{\bar{\omega}}^\vee)$$

where $\mathcal{S}_{\bar{\omega}}$ is the local system determined by the connection form $\bar{\omega}=\sum\lambda_j d\bar{f}_j/\bar{f}_j$. On the other hand, we have

LEMMA 1 ([**K-N** Theorem 2, p. 138]). *Suppose that*

$$\sum_{j=1}^m \lambda_j \in \boldsymbol{C}-\boldsymbol{Z};$$

then

$$H^q(\bar{X}, \mathcal{S}_{\bar{\omega}}) = 0 \qquad \textit{for all } q.$$

By virtue of (1.7), (1.8) and Lemma 1, we have

$$H_{n-1}^{lf}(X-\coprod_{l}\Delta_l,\ \check{\mathcal{S}_\omega})=0 \quad \text{and} \quad H_n^{lf}(X-\coprod_{l}\Delta_l,\ \check{\mathcal{S}_\omega})=0$$

and hence we obtain

$$H_n^{lf}(\coprod_{l}\Delta_l,\ \check{\mathcal{S}_\omega})\cong H_n^{lf}(X,\ \check{\mathcal{S}_\omega}). \tag{1.9}$$

Since each Δ_l is homeomorphic to $\boldsymbol{R}^n$, we have

$$H_n^{lf}(\coprod_{l}\Delta_l,\ \check{\mathcal{S}_\omega})\cong \sum_{l}[\Delta_l]\otimes\boldsymbol{C}.$$

Using (1.5) and (1.9), we know that *the number of relatively compact chambers Δ_l is equal to* $\binom{m-1}{n}$. Summing up, we obtain

LEMMA 2. *If $\sum_{j=1}^{m}\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$, then*

$$H_n^{lf}(X,\ \check{\mathcal{S}_\omega})\cong \sum_{l=1}^{\binom{m-1}{n}}[\Delta_l]\otimes\boldsymbol{C}:$$

is words, $\{[\Delta_l]\,|\,1\leqq l\leqq\binom{m-1}{n}\}$ *is a basis of* $H_n^{lf}(X,\ \check{\mathcal{S}_\omega})$.

1.7. Construction of a twisted cycle $\Delta(\omega)$ associated with a compact chamber Δ. For notational simplicity we assume that a compact chamber Δ is determined by r hyperplanes $H_1, \cdots, H_r$. We assume also $\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$, $(1\leqq j\leqq r)$. Put

$$H_I=\bigcap_{i\in I}H_i \quad \text{for} \quad I\subset\{1, \cdots, r\},\quad \Delta_I=\Delta\cap H_I,$$

$$U_\varepsilon(I)=\varepsilon\text{-neighbourhood of } \Delta_I \text{ for a sufficiently small } \varepsilon>0.$$

After giving $\boldsymbol{R}^n$ the standard orientation induced by the ordering of the coordinate system $(u_1, \cdots, u_n)$, on Δ we assign the orientation induced by that of $\boldsymbol{R}^n$.

We set

$$c(0)=\Delta\setminus\bigcup_{I}U_\varepsilon(I)$$

with the orientation on Δ. For each point $p\in\partial\Delta$, there exists a subset $I\subset\{1, \cdots, r\}$ such that

$$p\in\Delta_I\setminus\bigcup_{J\supsetneqq I}\Delta_J.$$

Then we can choose a local coordinate system $(w_1, \cdots, w_n)$ on a neighbourhood of $W(I, p)$ of Δ_I satisfying the following conditions:

1) The standard orientation of $\boldsymbol{R}^n$ coincides with the orientation on $W\cap\boldsymbol{R}^n$ induced by the ordering of the local coordinates $(w_1, \cdots, w_n)$.

2) $W(I,\ p)=\{w\in \boldsymbol{C}^n \mid |w_j|\leqq 1\quad (1\leqq j\leqq n)\}$,

$W(I,\ p)\cap\Delta=\{w\in \boldsymbol{R}^n \mid 0\leqq w_i\leqq 1(1\leqq i\leqq k),\ |w_j|\leqq 1(k+1\leqq j\leqq n)\}$,

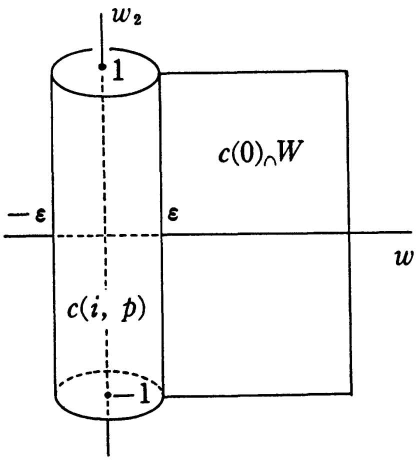
$W(I,\ p)\cap\Delta_I=\{w\in \boldsymbol{R}^n \mid w_i=0(1\leqq i\leqq k),\ |w_i|\leqq 1(k+1\leqq j\leqq n)\}$,

$W(I,\ p)\cap U_\varepsilon(I)=\{w\in W(I,\ p) \mid |w_i|<\varepsilon(1\leqq i\leqq k)\}$.

As in §1.2 let $S_\varepsilon^1(0)$ be the positively oriented circle about the origin with radius ε and with the fixed starting point ε in $\boldsymbol{C}$. We define an n-dimensional chain $c(I,\ p)$ in $W(I,\ p)$ as

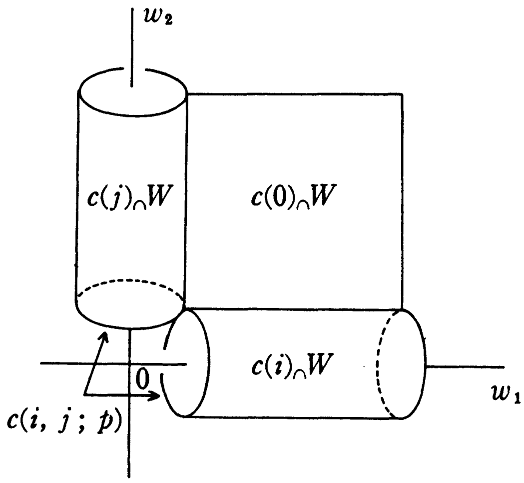
$$c(I,\ p)=(e^{2\pi i\lambda_{i_1}}-1)^{-1}S_\varepsilon^1(0)\times\cdots\times(e^{2\pi i\lambda_{i_k}}-1)^{-1}S_\varepsilon^1(0)\times[-1,\ 1]\times\cdots\times[-1,\ 1]$$

with product orientation where $\{w_l=0 \text{ in } W(I,\ p)\}=H_{i_l}\cap W(I,\ p)$ and $H_{i_l}=\{f_{i_l}=0\}$. Using the uniqueness of tubular neighbourhood (see [**M**, p. 21]), we can patch together $c(I,\ p)$ at each point $p\in\Delta_I\setminus\bigcup_{J\supsetneq I}U_\varepsilon(J)$ to get a twisted chain $c(I)$.



$W=W(i;\ p)$

Figure 3.



$W=W(i,\ j;\ p)$

Figure 4.

Set

$$\Delta(\omega)=c(0)+\sum_{|I|=1}^{n}\sum_I c(I).$$

LEMMA 3. $\partial_\omega\Delta(\omega)=0$.

PROOF. We compute $\partial_\omega\Delta(\omega)=0$ on each neighbourhood $W(I,\ p)$. We proceed inductively on the number $|I|$. In case $|I|=1$, we may suppose $I=\{1\}$. For simplicity set $W=W(1,\ p)$; then we have (see Figure 3)

$$c(0)\cap W=[\varepsilon, 1]\times[-1, 1]\times\cdots\times[-1, 1]$$

and

$$c(1)\cap W=\frac{1}{e^{2\pi i\alpha_1}-1}S_\varepsilon^1(0)\times[-1, 1]\times\cdots\times[-1, 1].$$

In W, we have the following boundaries of $c(0)$ and $c(1)$.

$$\partial_\omega(c(0)\cap W)=-\langle\varepsilon\rangle\times[-1, 1]\times\cdots\times[-1, 1],$$
$$\partial_\omega(c(1)\cap W)=\langle\varepsilon\rangle\times[-1, 1]\times\cdots\times[-1, 1].$$

Hence $\partial_\omega\Delta(\omega)\equiv 0$ modulo a chain with support in ∂W.

In case $|I|=2$, for simplicity we may suppose $I=\{1, 2\}$ and set $W(1, 2; p)=W$. Then we have (see Figure 4)

$$c(0)\cap W=[\varepsilon, 1]\times[\varepsilon, 1]\times[-1, 1]\times\cdots\times[-1, 1]$$

$$c(1)\cap W=\frac{1}{e^{2\pi i\alpha_1}-1}S_\varepsilon^1(0)\times[\varepsilon, 1]\times\cdots\times[-1, 1]$$

$$c(2)\cap W=[\varepsilon, 1]\times\frac{1}{e^{2\pi i\alpha_2}-1}S_\varepsilon^1(0)\times[-1, 1]\times\cdots\times[-1, 1] \tag{1.10}$$

$$c(1, 2)\cap W=\frac{1}{e^{2\pi i\alpha_1}-1}S_\varepsilon^1(0)\times\frac{1}{e^{2\pi i\alpha_2}-1}S_\varepsilon^1(0)\times[-1, 1]\times\cdots\times[-1, 1].$$

Here notice that, by definition, $c(2)\cap W$ is expressible as

$$\frac{1}{e^{2\pi i\alpha_2}-1}S_\varepsilon^1(0)\times[-1, 1]\times\cdots\times[-1, 1]$$

in the coordinate system $(w_2, w_1, \cdots, w_n)$ and hence in the coordinate system $(w_1, w_2, \cdots, w_n)$, $c(2)\cap W$ is written as (1.10). In W, we have the following boundaries of $c(0)\sim c(1, 2)$ where for simplicity we set $I^{n-2}=[-1, 1]\times\cdots\times[-1, 1]$:

$$\partial_\omega(c(0)\cap W)=-\langle\varepsilon\rangle\times[\varepsilon, 1]\times I^{n-2}+[\varepsilon, 1]\times\langle\varepsilon\rangle\times I^{n-2},$$

$$\partial_\omega(c(1)\cap W)=\langle\varepsilon\rangle\times[\varepsilon, 1]\times I^{n-2}-\frac{1}{e^{2\pi i\alpha_1}-1}S_\varepsilon^1(0)\times(-\langle\varepsilon\rangle)\times I^{n-2},$$

$$\partial_\omega(c(2)\cap W)=-\langle\varepsilon\rangle\times\frac{1}{e^{2\pi i\alpha_2}-1}S_\varepsilon^1(0)\times I^{n-2}-[\varepsilon, 1]\times\langle\varepsilon\rangle\times I^{n-2},$$

$$\partial_\omega(c(12)\cap W)=\langle\varepsilon\rangle\times\frac{1}{e^{2\pi i\alpha_2}-1}S_\varepsilon^1(0)\times I^{n-2}$$
$$-\frac{1}{e^{2\pi i\alpha_1}-1}S_\varepsilon^1(0)\times\langle\varepsilon\rangle\times I^{n-2}.$$

Hence $\partial_\omega\Delta(\omega)\equiv 0$ modulo a chain with support in ∂W. Doing such computation for $\partial_\omega \Delta(\omega)$ on all the W's and patching their boundaries, we can show $\partial_\omega \Delta(\omega)=0$.

1.8. Since $\{[\Delta_\nu] \mid 1 \leqq \nu \leqq \binom{m-1}{n}\}$ is a basis of $H_n^{lf}(X, \mathcal{S}_\omega^\vee)$, the following mapping

$$\tau:\ H_n^{lf}(X, \mathcal{S}_\omega^\vee) \longrightarrow H_n(X, \mathcal{S}_\omega^\vee);$$
$$[\Delta_\nu] \longrightarrow \Delta_\nu(\omega)$$

determines a well-defined homomorphism from $H_n^{lf}(X, \mathcal{S}_\omega^\vee)$ to $H_n(X, \mathcal{S}_\omega^\vee)$. On the other hand, there is a natural map

$$\nu:\ H_n(X, \mathcal{S}_\omega^\vee) \longrightarrow H_n^{lf}(X, \mathcal{S}_\omega^\vee).$$

From the construction of twisted cycle $\Delta_\nu(\omega)$ it follows that the support of $\nu\tau(\Delta_\nu)-\Delta_\nu$ is contained in a union of ε-neighbourhood of $f_j=0$:

$$\mathrm{Supp}[\nu\tau(\Delta_\nu)-\Delta_\nu] \subset \sum_{j=1}^{m} U_\varepsilon(j) \qquad \text{for arbitrary } \varepsilon>0;$$

hence

$$\int_{\nu\tau(\Delta_\nu)-\Delta_\nu} U\cdot\varphi = 0 \qquad \text{for any} \quad \varphi \in \Gamma_c(X, \mathcal{E}_X^n).$$

By 1.5, (ii) we have $\nu\tau(\Delta_\nu)-\Delta_\nu=0$ in $H_n^{lf}(X, \mathcal{S}_\omega^\vee)$; thus $\nu\tau=1$ on $H_n^{lf}(X, \mathcal{S}_\omega^\vee)$. Since $\dim H_n(X, \mathcal{S}_\omega^\vee)=\dim H_n^{lf}(X, \mathcal{S}_\omega^\vee)$, the homomorphism τ and ν are *isomorphisms*. Putting these results together, we obtain the following.

THEOREM 2. *If $\lambda_j \in \boldsymbol{C}-\boldsymbol{Z}$ $(1 \leqq j \leqq m)$ and $\sum \lambda_j \in \boldsymbol{C}-\boldsymbol{Z}$, then the relatively compact chambers $\{\Delta_l \mid 1 \leqq l \leqq \binom{m-1}{n}\}$ form a basis of $H_n^{lf}(X, \mathcal{S}_\omega^\vee)$ and the homomorphism*

$$H_n^{lf}(X, \mathcal{S}_\omega^\vee) \longrightarrow H_n(X, \mathcal{S}_\omega^\vee)$$
$$[\Delta_\nu] \longrightarrow \Delta_\nu(\omega)$$

is an isomorphism. Hence $\{\Delta_\nu(\omega) \mid 1 \leqq \nu \leqq \binom{m-1}{n}\}$ is a basis of $H_n(X, \mathcal{S}_\omega^\vee)$. Moreover it holds that for arbitrary $J \subset \{1, \cdots, m\}$ with $|J|=n$,

$$\int_{\Delta_\nu} U\cdot\varphi\langle J\rangle = \int_{\Delta_\nu(\omega)} U\cdot\varphi\langle J\rangle \qquad \textit{if} \quad \mathrm{Re}\,\lambda_j>0 \ (1 \leqq j \leqq m).$$

COROLLARY.

$$\det\Big(\int_{\Delta_\nu(\omega)} U\cdot\varphi\langle J\rangle\Big) \neq 0$$

where $1 \leqq \nu \leqq \binom{m-1}{n}$ and $J \subset \{1, 2, \cdots, m-1\}$ with $|J|=n$.

§2. The Wronskian of the hypergeometric function of type $(n+1, m+1)$.

2.1. The hypergeometric function of type $(n+1, m+1)$. We recall the definition of the hypergeometric function of type $(n+1, m+1)$ following [G], [GG] and [S]. Let $W_{n+1,m+1}(n<m)$ denote the space of $(n+1)\times(m+1)$ complex matrices

$$w=\begin{pmatrix} w_{00} & w_{01} & \cdots\cdots & w_{0m} \\ w_{10} & w_{11} & & \vdots \\ \vdots & \vdots & & \vdots \\ w_{n0} & w_{n1} & \cdots\cdots & w_{nm} \end{pmatrix} \in M(n+1, m+1; \boldsymbol{C})$$

such that rank of w is $n+1$ and each column vector of w is non-zero. Let $[t_0: t_1: \cdots : t_n]$ be homogeneous coordinates of the n-dimensional complex projective space $\boldsymbol{P}^n$ and define an n-form τ on $\boldsymbol{C}^{n+1}$ by

$$\tau=\sum_{i=0}^{n}(-1)^i t_i dt_0\wedge\cdots\wedge\widehat{dt_i}\wedge\cdots\wedge dt_n .$$

For a set of complex numbers $\tilde{\lambda}=(\lambda_0, \lambda_1, \cdots, \lambda_m)\in(\boldsymbol{C}-\boldsymbol{Z})^{m+1}$ with the property

$$\sum_{j=0}^{m}\lambda_j+n+1=0 \tag{2.1}$$

and for a point $w\in W_{n+1,m+1}$, the n-form

$$\prod_{j=0}^{m}\Big(\sum_{i=0}^{n} w_{ij}t_i\Big)^{\lambda_j}\cdot\tau \tag{2.2}$$

can be seen as a many-valued n-form on $\boldsymbol{P}^n$ by the condition (2.1). Then we take a twisted n-cycle σ associated with n-form (2.2) and define a function by the integral

$$\Phi(\tilde{\lambda}, w)=\int_\sigma\prod_{j=0}^{m}\Big(\sum_{i=0}^{n} w_{ij}t_i\Big)^{\lambda_j}\cdot\tau \tag{2.3}$$

which will be called the *hypergeometric integral of type* $(n+1, m+1)$. (In a previous paper [**K**], it was called the *Aomoto-Gelfand hypergeometric function.*) The integral (2.3) is homogeneous under two kinds of group action. $W_{n+1,m+1}$ admits the left $G(n+1, \boldsymbol{C})$-action: $w\to g\cdot w\,(g\in GL(n+1, \boldsymbol{C}))$, under which the integral $\Phi(\tilde{\lambda}, w)$ changes as

$$\Phi(\tilde{\lambda}, g\cdot w)=(\det g)^{-1}\Phi(\tilde{\lambda}, w).$$

On the other hand, the Cartan subgroup H_{m+1}, consisting of diagonal matrices of $GL(m+1, \boldsymbol{C})$, acts on $W_{n+1,m+1}$ on the right. Under the action, $\Phi(\tilde{\lambda}, w)$ transforms as

$$\Phi(\tilde{\lambda}, wh)=\sum_{j=0}^{m} h_j{}^{\lambda_j}\cdot\Phi(\tilde{\lambda}, w)$$

where $h=\mathrm{diag}[h_0, h_1, \cdots, h_m]$. It is easy to see that our integral $\Phi(\tilde{\lambda}, w)$, viewed as a function on $W_{n+1,m+1}$, satisfies the following system $E(n+1, m+1; \lambda)$ of differential equations:

$$E(n+1, m+1)\quad \begin{cases} \sum_{i=0}^{n} w_{ip}\dfrac{\partial\Phi}{\partial w_{ip}} = \lambda_p\Phi, & 0 \leqq p \leqq m \quad (H_{m+1} \text{ homogeneity}); \\ \sum_{p=0}^{m} w_{ip}\dfrac{\partial\Phi}{\partial w_{jp}} = -\delta_{ij}\Phi, & 0 \leqq i, j \leqq n \\ \qquad (GL(n+1, \boldsymbol{C})\text{-homogeneity}) & \\ \dfrac{\partial^2\Phi}{\partial w_{ip}\partial w_{jq}} = \dfrac{\partial^2\Phi}{\partial w_{iq}\partial w_{jp}}, & 0 \leqq i, j \leqq n, \quad 0 \leqq p, q \leqq m. \end{cases}$$

Suppose that $w \in W_{n+1,m+1}$ satisfies the condition that any $(n+1)$-minor is nonzero. By using both the left $GL(n+1, \boldsymbol{C})$ and the right H_{m+1}-actions, we can reduce w into the following form:

$$\text{(2.4)}\qquad \begin{pmatrix} 1 & & & 1 & \cdots\cdots & 1 & 1 \\ & 1 & & z_{1,n+1} & & z_{1,m-1} & -1 \\ & & \ddots & \vdots & & & \vdots \\ & & 1 & z_{n,n+1} & \cdots\cdots & z_{n,m-1} & -1 \end{pmatrix} \in M(n+1, m+1; \boldsymbol{C}).$$

We introduce the non-homogeneous coordinate system $(u_1, \cdots, u_n)$ by setting $u_i=t_i/t_0 (1\leqq i\leqq n)$ and put

$$f_i(u) = u_i, \qquad (1 \leqq i \leqq n)$$

$$f_j(u) = 1+\sum_{j=1}^{n} z_{ij}u_i, \qquad (n+1 \leqq j \leqq m-1)$$

$$f_m(u) = 1-\sum_{i=1}^{n} u_i .$$

Then the integral $\Phi(\tilde{\lambda}, w)$ can be written as

$$\text{(2.5)}\qquad \Psi(\lambda;\, z) = \int_\sigma \prod_{j=1}^{m} f_j(u)^{\lambda_j} du_1\wedge \cdots \wedge du_n$$

where $\lambda=(\lambda_1, \cdots, \lambda_m)\in(\boldsymbol{C}-\boldsymbol{Z})^m$. In case σ is the twisted cycle $\Delta^n(\omega)$ associated with the n-simplex $\Delta=\{u\in\boldsymbol{R}^n\,|\,0\leqq u_i\,(1\leqq i\leqq n),\ \sum u_i\leqq 1\}$ where $\omega=d\log\{\prod_{i=1}^n u_i^{\lambda_i}\cdot(1-\sum_{i=1}^n u_i)^{\lambda_m}\}$, we obtain the following power series expansion of $\Psi(\lambda, z)$:

$$\Psi(\lambda, z) = c\sum_\nu \frac{\sum_{i=1}^n(\lambda_i+1\,;\ \sum_{j=n+1}^{m-1}\nu_{ij})\sum_{j=n+1}^{m-1}(-\lambda_j\,;\ \sum_{j=1}^n\nu_{ij})}{(-\sum_{i=1}^n\lambda_i-\lambda_m-n\,;\ \sum_{i,j}\nu_{ij})\nu\,!}z^\nu$$

where summation is taken over $\nu\in M(n, m-n-1, \boldsymbol{Z}_{\geqq 0})$, and

$$c = \frac{\prod_{i=1}^n\Gamma(\lambda_i+1)\cdot\Gamma(\lambda_m+1)}{\Gamma(\sum_{i=1}^n\lambda_i+\lambda_m+n+1)}.$$

This series is a generalization of the Gauss, Appell F_1 and Lauricella F_D hypergeometric series.

In the following subsections, we shall use the reduced matrix (2.4) and the integral expression (2.5), in order to plug into the results of § 1, which are written in terms of inhomogeneous coordinates $u_1, \cdots, u_n$.

2.2. To make the idea of this subsection clear, we begin by illustrating some important examples. In the following, instead of (2.5), we shall consider the hypergeometric integral

$$F(\lambda, z)=\int \prod_{j=1}^{m} f_j(u)^{\lambda_j} \varphi\langle 1 \cdots n\rangle$$

where we set, as before,

$$\varphi\langle 1 \cdots n\rangle=\frac{d u_1}{u_1} \wedge \cdots \wedge \frac{d u_n}{u_n}.$$

EXAMPLE 1. $E(2, 4)$.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & z & -1 \end{pmatrix} \in M(2, 4, \boldsymbol{R}).$$

Suppose $1<-1/z$ and set

$$f_1=u, \quad f_2=1+zu, \quad f_3=1-u, \quad U(u):=f_1{}^{\lambda_1} f_2{}^{\lambda_2} f_3{}^{\lambda_3},$$
$$\omega=d \log U, \quad \varphi\langle i\rangle=d f_i / f_i .$$

We consider the hypergeometric integral

$$F(z)=\int_\sigma U \cdot \varphi\langle 1\rangle$$

where $\sigma \in H_1(X, \mathcal{S}_\omega^\vee)$ is a twisted cycle. Notice that

$$\frac{d}{d z} F(z)=\frac{\lambda_2}{z} \int_\sigma U \cdot \varphi\langle 2\rangle .$$

Let $\sigma_1, \sigma_2 \in H_1(X, \mathcal{S}_\omega^\vee)$ be the twisted cycles associated with the segments $[0, 1]$ and $[1, -1/z]$, respectively. We take two solutions

$$w_{11}=\int_{\sigma_1} U \varphi\langle 1\rangle, \quad w_{12}=\int_{\sigma_2} U \varphi\langle 1\rangle$$

of the Gauss hypergeometric system $E(2, 4 ; \lambda)$; the Wronskian of the two solutions

$$W=\begin{vmatrix} w_{11} & w_{12} \\ \dfrac{d}{d z} w_{11} & \dfrac{d}{d z} w_{12} \end{vmatrix}$$

is, by the above formula, equal to $(\lambda_2/2)\det(w_{ij})$ where

$$w_{ij}=\int_{\sigma_j}U\cdot\varphi\langle i\rangle,\qquad (i,\ j=1,\ 2).$$

By Corollary to Theorem 1, $\{\varphi\langle 1\rangle, \varphi\langle 2\rangle\}$ is a basis of $H^1(X, \nabla_\omega)$; Corollary to Theorem 2 asserts that the above determinant $\det(w_{ij})$ is non-zero. Therefore the Wronskian $W\neq 0$ and hence the two solutions w_{11} and w_{12} of $E(2, 4; \lambda)$ are linearly independent if $\lambda_j\in \boldsymbol{C}-\boldsymbol{Z}$, $(j=1, 2)$ and $\sum\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$.

EXAMPLE 2. $E(2, 3+l)$.

$$\begin{pmatrix}1 & 0 & 1 & \cdots & 1 & 1\\ 0 & 1 & z_1 & \cdots & z_l & -1\end{pmatrix}\in M(2, 3+l, \boldsymbol{R}),\qquad z=(z_1, \cdots, z_l).$$

Suppose z_i's satisfy the condition $1<-1/z_1<\cdots<-1/z_l$ and set

$$f_1=u,\qquad f_2=1+z_1u,\quad \cdots,\quad f_{l+1}=1+z_lu,\qquad f_{l+2}=1-u,$$

$$U=\prod_{j=1}^{l+2}f_j^{\lambda_j},\qquad \omega=d\log U,\qquad \varphi\langle i\rangle=df_i/f_i\quad (i=1, \cdots, l+2),$$

$$\partial_k=\partial/\partial z_k,\qquad (1\leqq k\leqq l).$$

We consider the hypergeometric integral

$$F(z)=\int_\sigma U\varphi\langle 1\rangle$$

where $\sigma\in H_1(X, \mathcal{S}_\omega^\vee)$ is a twisted cycle. Notice that

$$\partial_1F=\frac{\lambda_2}{z_1}\int_\sigma U\cdot\varphi\langle 2\rangle,\quad \cdots,\quad \partial_lF=\frac{\lambda_{l+1}}{z_l}\int_\sigma U\cdot\varphi\langle l+1\rangle.$$

Let $\sigma_1, \sigma_2, \cdots, \sigma_{l+1}\in H_1(X, \mathcal{S}_\omega^\vee)$ be the twisted cycles associated with the segments $[0, 1]$, $[1, -1/z_1]$, $\cdots$, $[-1/z_{l-1}, -1/z_l]$, respectively. We take $l+1$ solutions

$$w_{1j}=\int_{\sigma_j}U\cdot\varphi\langle 1\rangle,\qquad (1\leqq j\leqq l+1)$$

of the hypergeometric system $E(2, 3+l; \lambda)$; the Wronskian of the $l+1$ solutions

$$W=\begin{vmatrix} w_{11} & \cdots\cdots\cdots & w_{1,l+1}\\ \partial_1w_{11} & & \partial_1w_{1,l+1}\\ & \cdots\cdots\cdots & \\ \partial_lw_{11} & \cdots\cdots & \partial_lw_{1,l+1}\end{vmatrix}$$

is, by the above formula, equal to

$$\prod_{j=1}^{l}\frac{\lambda_{1+j}}{z_j}\cdot\det(w_{ij})$$

where we set

$$w_{ij}=\int_{\sigma_j} U\cdot\varphi\langle i\rangle, \qquad (1\leqq i,\, j\leqq l+1).$$

By Corollary to Theorem 1, $\{\varphi\langle 1\rangle, \varphi\langle 2\rangle, \cdots, \varphi\langle l+1\rangle\}$ is a basis of $H^1(X, \nabla_\omega)$; Corollary to Theorem 2 asserts that the above determinant is non-zero. Therefore the Wronskian $W\neq 0$ and hence $w_{11}, \cdots, w_{1,l+1}$ are *linearly independent solutions of* $E(2, 3+l;\lambda)$ *under the condition* $\lambda_j\in \boldsymbol{C}-\boldsymbol{Z}$, $(1\leqq j\leqq l+2)$ *and* $\sum\lambda_j\in \boldsymbol{C}-\boldsymbol{Z}$.

EXAMPLE 3. $E(3, 6)$.

$$\begin{pmatrix} 1 & & & 1 & 1 & 1 \\ & 1 & & z_{11} & z_{12} & -1 \\ & & 1 & z_{21} & z_{22} & -1 \end{pmatrix}\in M(3, 6, \boldsymbol{R}), \qquad z=\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix}.$$

Set

$$f_1=u_1,\quad f_2=u_2,\quad f_3=1+u_1z_{11}+u_2z_{21},\quad f_4=1+u_1z_{12}+u_2z_{22},\quad f_5=1-u_1-u_2,$$

$$U=\prod_{j=1}^{5} f_j^{\lambda_j}, \qquad \omega=d\log U,$$

$$\varphi\langle ij\rangle=\frac{df_i}{f_i}\wedge\frac{df_j}{f_j} \qquad (1\leqq i<j\leqq 6),$$

$$\partial_{kl}=\partial/\partial z_{kl}, \qquad (k, l=1, 2).$$

We consider the hypergeometric integral

$$F(z)=\int_\sigma U\varphi\langle 12\rangle$$

where $\sigma\in H_2(X, \mathcal{S}_\omega^\vee)$ is a twisted cycle. Notice that

$$\partial_{11}F=-\frac{\lambda_3}{z_{11}}\int_\sigma U\cdot\langle 23\rangle, \qquad \partial_{12}F=-\frac{\lambda_4}{z_{12}}\int_\sigma U\cdot\varphi\langle 24\rangle,$$

$$\partial_{21}F=\frac{\lambda_3}{z_{21}}\int_\sigma U\cdot\varphi\langle 13\rangle, \qquad \partial_{22}F=\frac{\lambda_4}{z_{22}}\int_\sigma U\cdot\varphi\langle 14\rangle,$$

$$\partial_{11}\partial_{22}F=\frac{\lambda_3\lambda_4}{\det z}\int_\sigma U\cdot\varphi\langle 34\rangle.$$

Since z_{ij} are real, the 5 real lines $f_j=0$ determine 6 relatively compact connected components $\Delta_j(1\leqq j\leqq 6)$ of $\boldsymbol{R}^2-\bigcup_{j=1}^5\{f_j=0\}$. Let $\sigma_j\in H_2(X, \mathcal{S}_\omega^\vee)$ be the twisted cycles associated with the chambers Δ_j.

We take six solutions

$$w_{1j}=\int_{\sigma_j} U\cdot\varphi\langle 12\rangle, \qquad (1\leqq j\leqq 6)$$

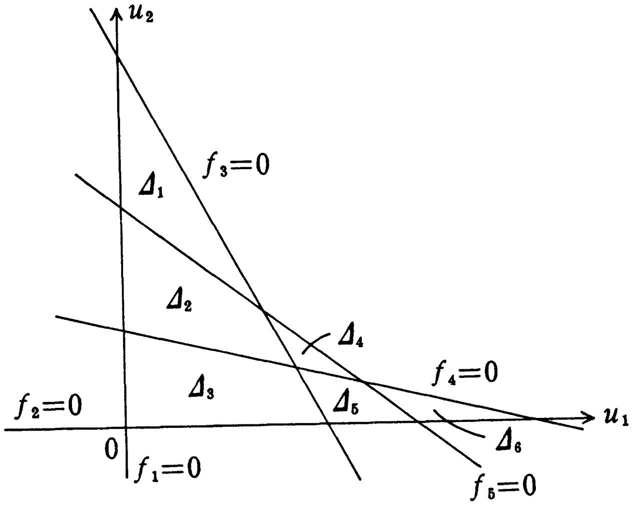


Figure 5.

of the hypergeometric system $E(3, 6; \lambda)$; the Wronskian of the six solutions

$$W = \begin{vmatrix} w_{11} & \cdots\cdots\cdots\cdots & w_{16} \\ \partial_{11}w_{11} & \cdots\cdots\cdots & \partial_{11}w_{16} \\ \cdots & \cdots\cdots\cdots & \cdots \\ \partial_{22}w_{11} & \cdots\cdots\cdots & \partial_{22}w_{16} \\ \partial_{11}\partial_{22}w_{11} & \cdots\cdots\cdots & \partial_{11}\partial_{22}w_{16} \end{vmatrix}$$

is, by the above formula, equal to

$$\frac{(\lambda_3\lambda_4)^3}{\prod_{i,j=1}^{2} z_{ij}\cdot\det z}\times\det(w_{ij})$$

where we set

$$w_{1j}=\int_{\sigma_j}U\cdot\varphi\langle 12\rangle, \qquad w_{2j}=\int_{\sigma_j}U\cdot\varphi\langle 13\rangle, \qquad w_{3j}=\int_{\sigma_j}U\cdot\varphi\langle 14\rangle,$$

$$w_{4j}=\int_{\sigma_j}U\cdot\varphi\langle 23\rangle, \qquad w_{5j}=\int_{\sigma_j}U\cdot\varphi\langle 24\rangle, \qquad w_{6j}=\int_{\sigma_j}U\cdot\varphi\langle 34\rangle.$$

By Corollary to Theorem 1, $\{\varphi\langle 12\rangle, \varphi\langle 13\rangle, \varphi\langle 14\rangle, \varphi\langle 23\rangle, \varphi\langle 24\rangle, \varphi\langle 34\rangle\}$ is a basis of $H^2(X, \nabla_\omega)$; Corollary to Theorem 2 asserts that the above determinant is non-zero. Therefore the Wronskian $W\neq 0$ and hence the 6 solutions $w_{1j}(1\leqq j\leqq 6)$ of $E(3, 6; \lambda)$ are *linealy independent if* $z_{ij}\neq 0\,(1\leqq i,\ j\leqq 2)$, $\det z\neq 0$, $\lambda_j\in \boldsymbol{C}-\boldsymbol{Z}\,(1\leqq j\leqq 5)$ *and* $\sum\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$.

EXAMPLE 4. $E(4, 8)$.

$$\begin{vmatrix} 1 & & & & 1 & 1 & 1 & 1 \\ & 1 & & & z_{11} & z_{12} & z_{13} & -1 \\ & & 1 & & z_{21} & z_{22} & z_{23} & -1 \\ & & & 1 & z_{31} & z_{32} & z_{33} & -1 \end{vmatrix} \in M(4, 8, \boldsymbol{R}), \qquad z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}.$$

Set

$$f_j(u) = u_j \ (1 \leqq j \leqq 3), \qquad f_{3+j}(u) = 1 + u_1 z_{1j} + u_2 z_{2j} + u_3 z_{3j} \ (1 \leqq j \leqq 3)$$

$$f_7(u) = 1 - u_1 - u_2 - u_3, \qquad U = \prod_{j=1}^{7} f_j^{\lambda_j}(u), \qquad \omega = d \log U,$$

$$\varphi\langle ijk \rangle = \frac{df_i}{f_i} \wedge \frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \qquad (1 \leqq i < j < k \leqq 20),$$

$$z\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \det\begin{pmatrix} z_{ik} & z_{il} \\ z_{jk} & z_{jl} \end{pmatrix}, \qquad \partial_{ij} = \partial/\partial z_{ij} \qquad (1 \leqq i, j \leqq 3).$$

We consider the hypergeometric integral

$$F(z) = \int_\sigma U \varphi\langle 123 \rangle$$

where $\sigma \in H_3(X, \check{\mathcal{S}}_\omega)$ is a twisted cycle. Notice that

$$\partial_{1j} F = \frac{\lambda_{3+j}}{z_{1j}} \int_\sigma U \cdot \varphi\langle 2, 3, 3+j \rangle \qquad (1 \leqq j \leqq 3),$$

$$\partial_{2j} F = -\frac{\lambda_{3+j}}{z_{2j}} \int_\sigma U \cdot \varphi\langle 1, 3, 3+j \rangle \qquad (1 \leqq j \leqq 3),$$

$$\partial_{3j} F = \frac{\lambda_{3+j}}{z_{3j}} \int_\sigma U \cdot \varphi\langle 1, 2, 3+j \rangle \qquad (1 \leqq j \leqq 3),$$

$$\partial_{1j} \partial_{2l} F = \frac{\lambda_{3+j} \lambda_{3+l}}{z\begin{pmatrix} 1 & 2 \\ j & l \end{pmatrix}} \int_\sigma U \cdot \varphi\langle 3, 3+j, 3+l \rangle \qquad (1 \leqq j < l \leqq 3),$$

$$\partial_{1j} \partial_{3l} F = -\frac{\lambda_{3+j} \lambda_{3+l}}{z\begin{pmatrix} 1 & 3 \\ j & l \end{pmatrix}} \int_\sigma U \cdot \varphi\langle 2, 3+j, 3+l \rangle \qquad (1 \leqq j < l \leqq 3),$$

$$\partial_{2j} \partial_{3l} F = \frac{\lambda_{3+j} \lambda_{3+l}}{z\begin{pmatrix} 2 & 3 \\ j & l \end{pmatrix}} \int_\sigma U \cdot \varphi\langle 1, 3+j, 3+l \rangle \qquad (1 \leqq j < l \leqq 3),$$

$$\partial_{11} \partial_{22} \partial_{33} F = \frac{\lambda_4 \lambda_5 \lambda_6}{\det z} \int_\sigma U \cdot \varphi\langle 456 \rangle.$$

Since z_{ij} are real, the 7 real lines $f_j = 0$ determine 20 relatively compact connected components $\Delta_j(1\leqq j\leqq 20)$ of $\boldsymbol{R}^3-\bigcup_{j=1}^7\{f_j=0\}$. Let $\sigma_j\in H_3(X, \mathcal{S}_\omega^\vee)$ be the twisted cycles associated with the chambers Δ_j; then by Theorem 2, $\{\sigma_j(1\leqq j\leqq 20)\}$ forms a basis of $H_2(X, \mathcal{S}_\omega^\vee)$. We take 20 solutions $\int_{\sigma_j} U\varphi\langle 123\rangle$ $(1\leqq j\leqq 20)$ of the hypergeometric system $E(4, 8; \lambda)$; let

$$W:=\begin{pmatrix} \int_{\sigma_1} U\varphi\langle 123\rangle \cdots\cdots & \int_{\sigma_{20}} U\varphi\langle 123\rangle \\ \partial_{ij}\int_{\sigma_1} U\varphi\langle 123\rangle \cdots\cdots & \partial_{ij}\int_{\sigma_{20}} U\varphi\langle 123\rangle \\ \cdots\cdots & \cdots\cdots \\ \partial_{pk}\partial_{ql}\int_{\sigma_1} U\varphi\langle 123\rangle \cdots\cdots & \partial_{pk}\partial_{ql}\int_{\sigma_{20}} U\varphi\langle 123\rangle \\ \cdots\cdots & \cdots\cdots \\ \partial_{11}\partial_{22}\partial_{33}\int_{\sigma_1} U\varphi\langle 123\rangle \cdots\cdots & \partial_{11}\partial_{22}\partial_{33}\int_{\sigma_{20}} U\varphi\langle 123\rangle \end{pmatrix}$$

be the Wronskian of the 20 solutions where $1\leqq i, j\leqq 3$, $1\leqq p<q\leqq 3$, $1\leqq k<l\leqq 3$. Then we see that

$$W=(\lambda_4\lambda_5\lambda_6)^{10}\frac{1}{\prod_{i,j=1}^{3} z_{ij}\cdot \prod_{\substack{1\leqq p<q\leqq 3\\ 1\leqq k<l\leqq 3}} z\begin{pmatrix} p & q\\ k & l\end{pmatrix}\cdot \det z}\det\left(\int_{\sigma_\nu} U\cdot\varphi\langle ijk\rangle\right) \tag{2.6}$$

where $1\leqq\nu\leqq 20$ and $\varphi\langle ijk\rangle$ runs over the basis $\{\varphi\langle ijk\rangle\,|\,1\leqq i<j<k\leqq 6\}$ in some order, which is easily seen from the Wronskian by using the above formula. Corollary to Theorem 2 asserts that the determinant is non-zero. Therefore the Wronskian $W\neq 0$ and hence 20 solutions $\int_{\sigma_j} U\varphi\langle 123\rangle$ $(1\leqq j\leqq 20)$ of $E(4, 8; \lambda)$ are *linearly independent if all minors of order* 1, 2 *and* 3 *in the matrix* $z=(z_{ij})$ *are not zero and* $\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$ $(1\leqq j\leqq 7)$, $\sum\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$.

2.3. General case $E(n+1, m+1)$.

$$\begin{pmatrix} 1 & & & 1 & \cdots 1 & 1\\ & 1 & 0 & z_{1,n+1} & z_{1,m-1} & -1\\ & & \ddots & \vdots & & \\ 0 & & & \vdots & & \\ & & 1 & z_{n,n+1} & \cdots z_{n,m-1} & -1\end{pmatrix}\in M(n+1, m+1, \boldsymbol{R}).$$

We set

$$z=\begin{pmatrix} z_{1,n+1} & z_{1,m-1}\\ \vdots & \\ z_{n,n+1} & \cdots z_{n,m-1}\end{pmatrix}\in M(n, m-n-1, \boldsymbol{R})$$

and for simplicity we write minors of order p as follows:

$$z\begin{pmatrix} i_1 & i_2 & \cdots i_p\\ k_1 & k_2 & \cdots k_p\end{pmatrix}=\det\begin{pmatrix} z_{i_1k_1} & z_{i_1k_2} & \cdots\cdots z_{i_1k_p}\\ \cdots\cdots & \cdots\cdots & \cdots\cdots\\ z_{i_pk_1} & z_{i_pk_2} & \cdots\cdots z_{i_pk_p}\end{pmatrix}.$$

Set $f_i(u)=u_i\,(1\leqq i\leqq n)$, $f_j(u)=1+\sum_{i=1}^{n}z_{ij}u_i\,(n+1\leqq j\leqq m-1)$,

$f_m(u)=1-u_1-\cdots-u_n$, $X=\boldsymbol{C}^n-\bigcup_{j=1}^{m}\{f_j=0\}$, $U(u)=\prod_{j=1}^{m}f_j{}^{\lambda_j}(u)$, $\omega=d\log U$.

For simplicity we write an n-form $df_{j_1}/f_{j_1}\wedge\cdots\wedge df_{j_n}/f_{j_n}$ by $\varphi\langle j_1,\cdots,j_n\rangle$. Let $\lambda=(\lambda_1,\cdots,\lambda_m)\in(\boldsymbol{C}-\boldsymbol{Z})^m$ and $e_i\,(1\leqq i\leqq m)$ be the vector $(0,\cdots,1,\cdots,0)$ where 1 is in the i-th entry. We set $F(\lambda,z)=F(\lambda,z,1\cdots n):=\int_\sigma U\cdot\varphi\langle 12\cdots n\rangle$ where $\sigma\in H_n(X,\mathcal{S}_\omega^\vee)$; then by simple calculations we have

$$\partial F/\partial z_{ik}=\lambda_k F(\lambda+e_i-e_k,z)\quad\text{where } 1\leqq i\leqq n,\ n+1\leqq k\leqq m-1,$$

......

$$\frac{\partial^p F}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}}=\lambda_{k_1}\cdots\lambda_{k_p}F(\lambda+e_{i_1}+\cdots+e_{i_p}-e_{k_1}\cdots-e_{k_p},z)$$
$$\text{where }\ 1\leqq i_1<\cdots<i_p\leqq n,\quad n+1\leqq k_1<\cdots<k_p\leqq m-1.$$

Set

$$I=\{i_1,\cdots,i_p\},\qquad K=\{k_1,\cdots,k_p\},\qquad H=\{n+1,\cdots,m\}\setminus K,$$
$$L=\{1,\cdots,n\}\setminus I=\{l_1,\cdots,l_{n-p}\}$$

where we suppose $1\leqq l_1\leqq\cdots\leqq l_{n-p}\leqq n$. Then we get

$$\begin{aligned}\frac{\partial^p F}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}}&=\lambda_{k_1}\cdots\lambda_{k_p}\int_\sigma\prod_{i\in I}u_i^{\lambda_i+1}\cdot\prod_{l\in L}u_l{}^{\lambda_l}\cdot\prod_{k\in K}f_k{}^{\lambda_k-1}\cdot\prod_{h\in H}f_h{}^{\lambda_h}\\&\qquad\times\frac{du_1}{u_1}\wedge\cdots\wedge\frac{du_n}{u_n}\\&=\lambda_{k_1}\cdots\lambda_{k_p}\int_\sigma\prod_{i\in I}u_i{}^{\lambda_i}\prod_{l\in L}u_l{}^{\lambda_l-1}\prod_{k\in K}f_k{}^{\lambda_k-1}\prod_{h\in H}f_h{}^{\lambda_h}d^nu\\&=\lambda_{k_1}\cdots\lambda_{k_p}\int_\sigma U\cdot\frac{1}{\prod_{l\in L}u_l\cdot\prod_{k\in K}f_k}d^nu.\end{aligned}$$

Since

$$\begin{aligned}&du_{l_1}\wedge\cdots\wedge du_{l_{n-p}}\wedge df_{k_1}\wedge\cdots\wedge df_{k_p}\\&\quad=du_{l_1}\wedge\cdots\wedge du_{l_{n-p}}\wedge z\begin{pmatrix}i_1\cdots i_p\\k_1\cdots k_p\end{pmatrix}du_{i_1}\wedge\cdots\wedge du_{i_p}\\&\quad=z\begin{pmatrix}I\\K\end{pmatrix}\mathrm{sgn}\begin{pmatrix}1&2&\cdots&n\\&L&I&\end{pmatrix}d^nu,\end{aligned}$$

we have

$$\text{(2.7)}\qquad\frac{\partial^p F}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}}=\frac{\mathrm{sgn}\begin{pmatrix}1&2&\cdots&n\\&L&I&\end{pmatrix}}{z\begin{pmatrix}I\\K\end{pmatrix}}\prod_{k\in K}\lambda_k\int_\sigma U\cdot\varphi\langle LK\rangle.$$

On the other hand, since

$$1 \leqq l_1 < \cdots < l_{n-p} \leqq n, \qquad n+1 \leqq k_1 < \cdots < k_p \leqq m-1,$$

$\{\varphi\langle LK\rangle\}$ is a subset of the basis $\{\varphi\langle j_1 \cdots j_n\rangle | 1 \leqq j_1 < \cdots < j_n \leqq m-1\}$; by Corollary to Theorem 1 and the formula

$$\sum_{p=0}^{n} \binom{n}{n-p} \binom{m-n-1}{p} = \binom{m-1}{n},$$

we conclude that the set $\{\varphi\langle LK\rangle\}$ coincides with the basis $\{\varphi\langle j_1 \cdots j_n\rangle\}$. Let $\sigma_1, \cdots, \sigma_r$ be the twisted n-cycles associated with the $r=\binom{m-1}{n}$ relatively compact chambers, which form a basis of $H_n(X, \check{\mathcal{S}_\omega})$ by Theorem 2. Set

$$F_\nu(\lambda, x) = \int_{\sigma_\nu} U \cdot \varphi\langle 1 \cdots n\rangle \qquad (1 \leqq \nu \leqq r);$$

then by (2.7),

$$\text{(2.8)} \qquad \det\left(\frac{\partial^p F_\nu}{\partial z_{i_1 k_1} \cdots \partial z_{i_p k_p}}\right)_{\substack{\nu = 1, \cdots, r \\ I, K}} = \prod_{I, K} \left(\frac{\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ & L & I & \end{pmatrix}}{z\begin{pmatrix} I \\ K \end{pmatrix}} \lambda_{k_1} \cdots \lambda_{k_p} \right)$$

$$\times \det\left(\int_{\sigma_\nu} U \cdot \varphi\langle LK\rangle\right).$$

Using a formula

$$\sum_{p=0}^{n} p \binom{n}{n-p} \binom{m-n-1}{p} = (m-n-1)\binom{m-2}{n-1},$$

we can rewrite (2.8) as

$$\text{(2.9)} \qquad \det\left(\frac{\partial^p F_\nu}{\partial z_{i_1 k_1} \cdots \partial z_{i_p k_p}}\right)_{\substack{\nu = 1, \cdots, r \\ I, K}}$$

$$= (\lambda_{n+1} \cdots \lambda_{m-1})^{\binom{m-2}{n-1}} \prod_{I, K} \left(\frac{\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ & L & I & \end{pmatrix}}{z\begin{pmatrix} I \\ K \end{pmatrix}} \right) \cdot \det\left(\int_{\sigma_\nu} U \cdot \varphi\langle LK\rangle\right).$$

Since $\det\left(\int_{\sigma_\nu} U \cdot \varphi\langle LK\rangle\right) \neq 0$ by Corollary to Theorem 2, we obtain the following.

THEOREM 3. *Let* σ_ν, $(1 \leqq \nu \leqq \binom{m-1}{n})$ *be the twisted cycles associated with the* $\binom{m-1}{n}$ *relatively compact chambers of* $X \cap \boldsymbol{R}^n$ *and set*

$$F_\nu(\lambda, x) = \int_{\sigma_\nu} U \cdot \varphi\langle 1 \cdots n\rangle, \quad (1 \leqq \nu \leqq \binom{m-1}{n}) \text{ where } U = \prod_{j=1}^{m} f_j^{\lambda_j}.$$

We suppose that $\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$, $(1\leqq j\leqq m)$ *and* $\sum_{j=1}^{m}\lambda_j\in\boldsymbol{C}-\boldsymbol{Z}$. *If all minors of* z *of order* $1, \cdots, n$ *are not zero, then the Wronskian of the hypergeometric system* $E(n+1, m+1; \lambda)$ *is not zero*:

$$\det\Big(\frac{\partial^p F_\nu}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}}\Big)\neq 0$$

where $1\leqq i_1<\cdots<i_p\leqq n$, $n+1\leqq k_1<\cdots<k_p\leqq m-1$ *and* $0\leqq p\leqq n$. *Hence the* $\binom{m-1}{n}$ *solutions* $F_\nu(\lambda, z)$ *of* $E(n+1, m+1; \lambda)$ *are linearly independent and we have*

$$\operatorname{rank} E(n+1, m+1; \lambda)=\binom{m-1}{n}.$$

2.4. In [**V1, 2**] A. N. Varchenko evaluated the determinant of the hypergeometric integrals $\det\Big(\int_{\sigma_\nu}U\cdot\varphi\langle LK\rangle\Big)$. His result is stated as follows: Let $\Delta_\nu(1\leqq\nu\leqq\binom{m-1}{n})$ be the relatively compact chambers of $X\cap\boldsymbol{R}^n$; for each f_j we fix an argument of f_j on Δ_ν, which determines a branch of the many-valued function $f_j^{\lambda_j}$ on Δ_ν. Let $c(f_j^{\lambda_j}, \sigma_\nu)$ denote the value of $f_j^{\lambda_j}$ which is maximum in absolute value on the compact chamber Δ_ν corresponding to σ_ν. Then the determinant is written as follows:

LEMMA 4 ([**V1, 2**]). *If each of the numbers* $\lambda_j(1\leqq j\leqq m)$ *has positive real part, then*

$$\det\Big(\int_{\sigma_\nu}U\cdot\varphi\langle LK\rangle\Big)=\pm\frac{1}{(\lambda_1\cdots\lambda_{m-1})^{\binom{m-2}{n-1}}}B\prod_{j=1}^{m}\prod_{\nu=1}^{\binom{m-1}{n}}c(f_j^{\lambda_j}, \sigma_\nu)$$

where

$$B=\Big(\prod_{j=1}^{m}\Gamma(\lambda_j+1)/\Gamma\Big(\sum_{j=1}^{m}\lambda_j+1\Big)\Big)^{\binom{m-2}{n-1}}.$$

Using (2.9) and Lemma 4, we can write the Wronskian in the following closed form:

THEOREM 4. *Let* $F_\nu(\lambda, z)=\int_{\sigma_\nu}U\cdot\varphi\langle 1\cdots n\rangle$ $(1\leqq\nu\leqq\binom{m-1}{n})$. *If* $\mathrm{Re}(\lambda_j)>0$ $(1\leqq j\leqq m)$, *then the Wronskian of the hypergeometric integrals* F_ν *is written as*

$$\det\Big(\frac{\partial^p F_\nu}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}}\Big)$$
$$=\pm\frac{1}{(\lambda_1\cdots\lambda_n)^{\binom{m-2}{n-1}}\prod_{I,K}z\binom{I}{K}}\cdot B\cdot\prod_{j=1}^{m}\prod_{\nu=1}^{\binom{m-1}{n}}c(f_j^{\lambda_j}, \sigma_\nu)$$

where the product is taken over the I *and* K *such that* $I=\{i_1, \cdots, i_p\}$, $K=\{k_1, \cdots, k_p\}$, $1\leqq i_1<\cdots<i_p\leqq n$, $n+1\leqq k_1<\cdots<k_p\leqq m-1$, $0\leqq p\leqq n$ *and we set*

$$B=\Big(\prod_{j=1}^{m}\Gamma(\lambda_j+1)/\Gamma\Big(\sum_{j=1}^{m}\lambda_j+1\Big)\Big)^{\binom{m-2}{n-1}}.$$

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Michitake KITA
College of Liberal Arts
Kanazawa University
Kakuma-Machi
Kanazawa 920-11
Japan