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Cohomology for groups of $\mathrm{rank}_p G=2$ and Brown-Peterson cohomology

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Introduction.

For a finite group G, we say that $\mathrm{rank}_p G=n$ if n is the maximal number of rank of elementary abelian p subgroups of G. Let $BP^*(BG)$ be the BP-theory and $H^*(BG)$ be the ordinary cohomology theory with coefficient $Z_{(p)}$ of the classifying space BG. Define the Chern subring $\mathrm{Ch}(G)$ (resp. $\mathrm{Ch}_{BP}(G)$) to be the subring of $H^{even}(BG)$ (resp. BP^*-subalgebra of $BP^*(BG)$) generated by Chern classes of complex representations. When G are metacyclic groups and some other $\mathrm{rank}_p G=2$ groups, we get [**T-Y 1, 2**], [**Th1, 2**], [**H 1, 2**] $BP^*(BG)=\mathrm{Ch}_{BP}(G)$ and $BP^*(BG)\otimes_{BP^*}Z_{(p)}=H^{even}(BG)$, and hence $\mathrm{Ch}(G)=H^{even}(BG)$. However we see in [**L-Y**] that $BP^*(BG)\neq\mathrm{Ch}_{BP}(G)$, $H^{even}(BG)\neq\mathrm{Ch}(G)$ for other $\mathrm{rank}_p G=2$ groups for $p\geqq 5$. These groups also give counter examples of Atiyah conjecture about filtrations on the complex representation ring $R(G)$. Let $\mathrm{Tr}e(G)$ be the subring of $H^*(BG)$ generated by the correstriction (transfer) of the Euler classes (top Chern classes) of complex representations. Hopkins-Kuhn-Ravenel define that G is good if $K(n)^*(BG)=\mathrm{Tr}e_{K(n)}(G)$ where $K(n)^*(-)$ is the Morava K-theory, and they show that if G is good, then so is $Z/p\wr G$ where $\wr$ is the wreath product [**H-K-R**]. In this paper we see;

THEOREM. *If G is a p-group of $p\geqq 5$ and $\mathrm{rank}_p G=2$, then* $BP^*(BG)=\mathrm{Tr}e_{BP}(G)$, $BP^*(BG)\otimes_{BP^*}K(n)^*\cong K(n)^*(BG)$, $BP^*(BG)\otimes_{BP^*}Z_{(p)}\cong H^{even}(BG)$, *and hence* $K(n)^*(BG)=\mathrm{Tr}e_{K(n)}(G)$, $H^{even}(BG)=\mathrm{Tr}e(G)$.

Moreover we know explicitly generators of $BP^*(BG)$ as a BP^*-algebra. Using these arguments we decide $H^*(BG)$ for $\mathrm{rank}_p G=2$ groups of class 3, i.e., $[G,[G,G]]\neq 1$.

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§ 2. Groups of rank$_p G=2$.

When $p \geqq 5$, Blackburn classified rank$_p G=2$ p-groups as one of the following (2.1)-(2.3) ([**Th 3**], Satz 11.2 in III [**Hp**]).

(2.1) metacyclic groups, those are groups such as

$$\langle a, b \mid a^{p^{\alpha}}=1,\ a^{p^{\alpha'}}=b^{p^{\beta}},\ [a, b]=a^{k-1},\ \text{with } k^{p^{\beta}} \equiv 1 \bmod p^{\alpha},\ p^{\alpha'}(k-1) \equiv 0 \bmod p^{\alpha} \rangle$$

(2.2) $C(r+2)=\langle a, b, c \mid a^p=b^p=c^{p^r}=1,\ c \in \text{Center},\ [a, b]=c^{p^{r-1}} \rangle$

(2.3) $G(r+3, e)=\langle a, b, c \mid a^p=b^p=c^{p^{r+1}}=[b, c]=1,\ [a, b^{-1}]=c^{p^r e},\ [a, c]=b,$ $e \not\equiv 0 \bmod p \rangle$

Remark. When $p \leqq 3$, the situation is quite different (12.5 Bemerkungen [**Hp**]).

Cohomology groups of (2.1)-(2.2) are studied by Huebschman, Lewis, Leary and Thomas [**H 1, 2**], [**Ls**], [**Ly**], [**Th 1, 3**]. In particular $\mathrm{Ch}(G)=H^{even}(BG)$ for these groups. However for the group (2.3) $\mathrm{Ch}(G) \neq H^{even}(BG)$ by Leary-Yagita [**L-Y**]. Brown-Peterson cohomology and Morava K-theory for groups (2.1)-(2.2) are studied by Tezuka-Yagita [**T-Y** 1, 2].

§ 3. Brown-Peterson cohomology of (2.1)-(2.2).

Let $BP^*(-)$ be the Brown-Peterson cohomology theory with the coefficient $BP^*=Z_{(p)}[v_1, v_2, \cdots]$, $|v_i|=-2(p^i-1)$. Recall that $BP\langle n\rangle^*(-)$ (resp. $P(n)^*(-)$, $K(n)^*(-)$) is the BP^*-module cohomology theory with the coefficient $BP\langle n\rangle^*=Z_{(p)}[v_1, \cdots, v_n]$ (resp. $P(n)^*=Z/p[v_n, v_{n+1}, \cdots]$, $K(n)^*=Z/p[v_n, v_n^{-1}]$).

It is well known $BP^*(BS^1)=BP^*(CP^\infty) \cong BP^*[[u]]$, $|u|=2$. The usual product map $m: S^1 \times S^1 \to S^1$ induces the map

$$m^*: BP^*(BS^1) \longrightarrow BP^*(BS^1 \times BS^1) \cong BP^*[[u_1, u_2]]$$

and defines the formal group laws $m^*(u)=u_1+_{BP}u_2$ of BP-theory. Therefore we get

$$BP^*(BZ/p^r)=BP^*[[u]]/[p^r](u) \tag{3.1}$$

where $[p^r](u)$ is the p^r-product $u+_{BP} \cdots +_{BP}u$, in particular $[p](u)=pu+v_1u^p + \cdots + v_n u^{p^n} + \cdots$. This shows $BP^*(BZ/p^r)$ satisfies the condition of Landweber exact functor theorem [**La 1**], that is,

(3.2) $BP^*(BZ/p^r)$ is p-torsion free,

$BP(BZ/p^r)/(p, v_1, \cdots, v_{n-1})$ is v_n-torsion free.

Hence $BP^*(\times BZ/p^{r_i})\cong\otimes_{BP^*}BP^*(BZ/p^{r_i})$ [**La 1**].

First we study metacyclic groups. Consider exact sequence

$$(3.3)\qquad 1\longrightarrow\langle a\rangle\longrightarrow G\longrightarrow\langle b\rangle/\langle b^{p^\beta}\rangle\longrightarrow 1$$

and induced spectral sequence

$$(3.4)\qquad E_2^{*,*}=H^*(BZ/p^\beta;\ BP^*(BZ/p^\alpha))\Longrightarrow BP^*(BG).$$

In [**T-Y 2**] we see $E_2^{odd,*}=0$ and hence $E_\infty=E_2$. Moreover $E_2^{0,*}=BP^*(BZ/p^\alpha)^{\langle b\rangle}$ is generated as a BP^*-module by elements $u(i)$ such that $u(i)=p^{\alpha-\gamma-s}u^i \bmod (v_1, v_2, \cdots)$ where $k-1=\lambda p^\gamma$, and $i=\lambda' p^s(\lambda, \lambda'\not\equiv 0 \bmod p)$. Let ρ_2: $G\to G/\langle a\rangle\to \boldsymbol{C}^\times$ be given by $b\mapsto\exp(2\pi i/p^\beta)$, and let $\xi:\langle a, b^{p^{\alpha-\gamma}}\rangle\to\boldsymbol{C}^\times$ be given by $a\mapsto\exp(2\pi i/p^\alpha)$, $b\mapsto 1$ and $\eta=\mathrm{Ind}\langle a, b^{p^{\alpha-\gamma}}\rangle^G(\xi)$. Then $BP^*(BG)$ is generated by $c_i(\eta)$, $1\leqq i\leqq p^{\alpha-\gamma}$ and $c_1(\rho_2)$ as a BP^*-algebra. Using Theorem 2.6 in [**T-Y 2**], we see

LEMMA 3.5([**T-Y 2**]). *If $k=BP$, $BP\langle n\rangle$, $P(n)$, $K(n)$, $n\geqq 1$, and G is a metacyclic group, then*

(i) $k^*(BC)\cong k^*\otimes_{BP^*}BP^*(BG)=\mathrm{Ch}_k(G)$

(ii) $H^{even}(BG)\cong Z_{(p)}\otimes_{BP^*}BP^*(BG)=\mathrm{Ch}(G)$.

According to Hopkins-Kuhn-Ravenel [**H-K-R**], we define $\mathrm{Tr}e_k(G)$ to be a k^*-submodule of $k^*(BG)$ generated by transfer (correstriction) $\mathrm{Tr}(e(\rho))$ of the Euler class (top Chern class) of complex representations ρ. Note that $\mathrm{Tr}e_k(G)$ is k^*-subalgebra (Corollary 7.6 in [**H-K-R**]).

LEMMA 3.6. *If k and G are the same as Lemma 3.5, then*

(i) $k^*(BG)=\mathrm{Tr}e_k(G)$

(ii) $H^{even}(BG)=\mathrm{Tr}e(G)$.

PROOF. First note that, $\langle a^{p^{\alpha-\gamma}}, b\rangle$ is abelian. Let ξ': $\langle a^{p^{\alpha-\gamma}}, b\rangle\to\boldsymbol{C}^\times$ be the representation with $a^{p^{\alpha-\gamma}}\mapsto\exp(2\pi i/p^\gamma)$, $b\mapsto 1$, and let

$$\eta'_s=\mathrm{Ind}\langle a^{p^{\alpha-\gamma}}, b\rangle\langle a^{p^{\alpha-\gamma-s}}, b\rangle(\xi').$$

Then $e(\eta'_s)=c_{ps}(\eta'_s)=u^{p^s} \bmod (v_1, \cdots)$. By double coset formula

$$\mathrm{Cor}_{\langle a^{p^{\alpha-\gamma-s}}, c\rangle}{}^G(c_p{}^s(\eta'_s)|\langle a^{p^{\alpha-\gamma}}\rangle)=p^{\alpha-\gamma-s}u^{p^s} \bmod (v_1, \cdots).$$

Since $\eta_{\alpha-\gamma}'$, ρ_2 are representations of G, we get

$$\mathrm{Cor}_{\langle a^{p^{\alpha-\gamma-s}}, b\rangle}{}^G(e(l\eta'_s\oplus j\eta'_{\alpha-\gamma}\oplus k\rho_2))=\mathrm{Cor}^G(c_p{}^{ls}(\eta'_s))c_{p^{\alpha-\gamma}}(\eta'_{\alpha-\gamma})^j c_1(\rho_2)^k$$

make a BP^*-module generator. q. e. d.

Secondly consider the case $G=C(r+2)$. Consider the exact sequence

(3.7) $$1 \longrightarrow \langle b, c\rangle \longrightarrow G \longrightarrow \langle a\rangle \longrightarrow 1$$

and induced spectral sequence

(3.8) $$E_2^{*,*} = H^*(BZ/p;\ BP^*(B(Z/p\oplus Z/p^r))) \Longrightarrow BP^*(BG).$$

Then $E_2^{odd,*}=0$ and hence $E_\infty=E_2$ ([**T-Y 2**]). Let ρ_1: $\langle a\rangle \to \boldsymbol{C}^\times$, $a\mapsto \exp(2\pi i/p)$ and ρ_2 :$\langle b\rangle\to\boldsymbol{C}^\times$, $b\mapsto\exp(2\pi i/p)$. Let ξ: $\langle c, b\rangle\to\boldsymbol{C}^\times$, $c\mapsto\exp(2\pi i/p^r)$, $b\mapsto 1$ and $\eta=\mathrm{Ind}_{\langle c,b\rangle}{}^G(\xi)$. Then $BP^*(C(r+2))$ is generated as a BP^*-module by (§4 in [**T-Y 2**]) $c_i(\eta)c_p(\eta)^j$, $1\leqq i\leqq p-1$, $c_1(\rho_1)^k c_1(\rho_2)^l c_p(\eta)^j$.

LEMMA 3.9. *For* $G=C(r+2)$ (i), (ii) *in Lemma* 3.5 *and Lemma* 3.6 *also hold.*

PROOF. The statement in Lemma 3.5 is proved in [**T-Y 2**]. Let ξ': $\langle c^p, b\rangle\to\boldsymbol{C}^\times$, $c^p\mapsto\exp(2\pi i/p^{r-1})$, $b\mapsto 1$. Then we easily see that

$$\mathrm{Cor}_{\langle c^p,b\rangle}{}^G(e(i\xi'\oplus j\eta)) \quad\text{and}\quad e(k\rho_1\oplus l\rho_2\oplus j\eta)$$

are generators of $BP^*(BC(r+2))$ as a BP^*-module. q. e. d.

For the next section, we give here another expression of $BP^*(C(r+2))$.

LEMMA 3.10. (*For* $r=1$ [**T-Y 1**].) *Let* $p\geqq 3$ *and let us write* $y_i=c_1(\rho_i)$ *and* $c_j=c_j(\eta)$. *Then there is a* BP^**-algebra isomorphism*

$$\begin{aligned} BP^*(BC(r+2)) \cong (BP^*[[y_1, y_2]]\oplus BP^*\{c_1, \cdots, c_{p-1}\})[[c_p]] \\ /([p](y_1), [p](y_2), y_2\Pi_{\lambda\in z/p}(y_1+_{BP}[\lambda]y_2), \\ \text{other relations containing } c_i,\ 1\leqq i\leqq p-1). \end{aligned}$$

PROOF. (See [**T-Y 1**] for $r=1$ case, note that in that paper most arguments work with mod $(v_2, \cdots)$, while omitted there.) The central extension

$$1 \longrightarrow \langle c\rangle \longrightarrow G \longrightarrow \langle a, b\rangle \longrightarrow 1$$

induces the spectral sequence

$$E_2^{*,*} = H^*(\langle a, b\rangle;\ BP^*(\langle c\rangle)) \Longrightarrow BP^*(BG)$$

where

$$E_2^{*',*}\cong\begin{cases} Z/[y_1, y_2]\otimes\wedge(\alpha)\otimes BP^*[[u]]/[p^r](u) & \text{for } *'>0 \quad |\alpha|=3, \\ BP^*[[u]]/[p^r](u) & \text{for } *'=0. \end{cases}$$

The first differential is $d_2u=\alpha$ and the next differential is

$$d_{2p-1}(u^{p-1}\otimes\alpha) = \beta\wp^1(\alpha=x_1y_2-x_2y_1) = y_1^p y_2 - y_1 y_2^p \qquad (\text{here } \beta x_i = y_i)$$

by the Kudo's transgression theorem and the naturality of the spectral sequence. Hence we get

$$E_{2p}{}^{*,*} = BP^*(Z\{1, pu, \cdots, pu^{p-1}\} \oplus (Z/p[y_1, y_2] - \{1\})/(y_1{}^p y_2 - y_1 y_2{}^p))$$
$$\otimes Z[u^p]/([p^r](u))'$$

where $([p^r](u))'$ is the intersection of $[p^r](u)$ and its above BP^*-algebra. Since E_{2p} is generated by even dimensional elements, $E_{2p} \cong E_\infty$. The relation is invariant (except units) under the automorphism of G. The automorphism $a \mapsto ab^\lambda$, induces $y_2 \mapsto y_2 +_{BP} [\lambda] y_1$, which shows the relation contains y_2. q. e. d.

§ 4. $BP^*(BG(r+3, e))$.

In this section we always assume $G = G(r+3, e)$ and $p \geqq 3$. The extension

(4.1) $$1 \longrightarrow \langle a, b, c^p \rangle \longrightarrow G \longrightarrow \langle c \rangle \longrightarrow 1$$

induces the spectral sequence

(4.2) $$E_2{}^{*,*} = H^*(BZ/p\,;\, BP^*(BC(r+2))) \Longrightarrow BP^*(BG)\,.$$

The action c^* induced from (4.1) is given by

(4.3) $$\begin{cases} c^* c_i = c_i, & c^* y_1 = y_1, \\ c^* y_2 = y_2 -_{BP} y_1\,. \end{cases}$$

Let us write $w = \prod_{\lambda \in Z/p}(y_2 +_{BP} [\lambda](y_1)) = y_2{}^p + \cdots$. Then note $y_2{}^i w$, $i \geqq 0$ are invariant under the action c^* since $y_1 w = 0$ in $BP^*(BC(r+1))$ from Lemma 3.10.

LEMMA 4.4. *The invariant $BP^*(BC(r+1))^{\langle c \rangle}$ is multiplicatively generated by elements*

$$1,\ y_1,\ c_1, \cdots, c_p,\ y_2{}^i w, \qquad 0 \leqq i \leqq p-1\,.$$

PROOF. Let $x \in BP^*(BC(r+1))$. Since $[p](y_2) = 0$, $p y_2{}^i = 0 \bmod (y_1, y_2{}^i w)$. By taking off the above invariant from x, we may consider the case

(4.5) $$x = y_2{}^i y_1{}^s c_p{}^t + \cdots, \qquad 1 \leqq i \leqq p-1\,.$$

However

(4.6) $$\begin{aligned}(1-c^*)x &= ((y_2 +_{BP} y_1)^i - y_1{}^i) y_1{}^s c_p{}^t + \cdots \\ &= i y_2{}^{i-1} y_1{}^{s+1} c_p{}^t + \cdots \neq 0\,.\end{aligned}$$

Therefore x is not invariant. q. e. d.

Let $N = 1 + c^* + \cdots + c^{*p-1}$. Then note $E_2{}^{2n-1,*} = \mathrm{Ker}\, N/\mathrm{Im}\,(1-c^*)$.

LEMMA 4.7. $\mathrm{Ker}\, N/\mathrm{Im}\,(1-c^*) = 0$.

PROOF. If $x \mid \langle b, c^p \rangle = \tilde{x} \neq 0$, then $Nx \mid \langle b, c^p \rangle = p\tilde{x}$. Since $BP^*(\langle b, c^p \rangle)$ has no p-torsion (3.3), $p\tilde{x}\neq 0$. Hence $x\notin \operatorname{Ker} N$. Suppose $x|\langle b, c^p\rangle=0$. Then from the fact that (3.8) collapses and from Lemma 3.10, x is expressed as (4.5) but $0\leqq i\leqq p-1$, $s\geqq 1$. From (4.6), if $i\neq p-1$, then $y_2^i y_1^s c_p^t+\cdots\in \operatorname{Im}(1-c^*)$. Taking of elements in $(1-c^*)$, we may only consider the case $i=p-1$. But this case

$$(4.8)\qquad \begin{aligned} Nx &= \textstyle\sum_{\lambda\in Z/p}(y_2+\lambda y_1)^{p-1}c_p^t \\ &= -y_1^{s+p-1}c_p^t+\cdots \neq 0 \qquad \bmod (p, v_1, \cdots). \end{aligned}$$

q. e. d.

COROLLARY 4.9. *$E_\infty=E_2$ and $BP^*(BG)=BP^{even}(BG)$ and all elements in $BP^*(C(r+1))^{\langle c\rangle}=E_2^{0,*}$ are permanent cycles.*

Let $\tilde{P}(n)^*(-)$ be the cohomology theory with the coefficient $\tilde{P}(n)^*=Z_{(p)}[v_n, v_{n+1}, \cdots]$, that is, $\tilde{P}(n)^*/p=P(n)^*$. If $k=\tilde{P}(n)$ $n\geqq 1$ or $BP\langle n\rangle$ $n\geqq 2$, then almost all arguments in their proof of Lemma 4.7 work and $E_\infty=E_2=E_2^{even,*}$ for the spectral sequence of k-theory. Hence we get

$$(4.10)\qquad k^*(BG)\cong k^*\otimes_{BP^*}BP^*(BG).$$

Moreover this formula also hold for $k=P(n)$ and $K(n)$ by Theorem 2.6 in [**T-Y 2**].

LEMMA 4.11. *For $k=BP\langle 1\rangle$, $E_2^{odd,*}\neq 0$. However $d_r E_2^{even,*}=0$ and $E_\infty^{odd,*}=0$ and* (4.10) *also holds for this case.*

PROOF. All arguments in the proof of Lemma 4.7 are still true except that $BP\langle 1\rangle^*(B\langle b, c^p\rangle)$ is p-torsion free. Hence we may assume $x=y_2^i c_p^t+\cdots$. Let $\tilde{y}\in E_2^{odd,*}$ be an element which corresponds $x|\langle b, c^p\rangle$ in $\operatorname{Ker} N/\operatorname{Im}(1-c^*)=\operatorname{Ker} N$ in the spectral sequence induced from

$$1\longrightarrow \langle b, c^p\rangle \longrightarrow \langle b, c\rangle \longrightarrow \langle c\rangle \longrightarrow 1.$$

But $BP^{odd}(\langle b, c\rangle)=0$. Hence $d_r\tilde{y}\neq 0$ for some r. By the naturality an element $y\in E_2^{odd,*}$ which corresponds $x\in \operatorname{Ker} N/\operatorname{Im}(1-c^*)$, has non zero differential d_r.

q. e. d.

COROLLARY 4.12. *$H^{even}(BG)=BP^*(BG)\otimes_{BP^*}Z_{(p)}$.*

PROOF. From Lemma 4.11 and the Sullivan-Bockstein exact sequence the corollary is immediate. q. e. d.

Now we consider $\mathrm{Ch}(G)$ and $\mathrm{Tr}e(G)$. $BP^*(BG)$ (resp. $H^{even}(BG)$) is BP^*-algebraically (resp. multiplicatively) generated by

$$(4.12)\qquad 1,\ y_1,\ y_2^i w,\ c_1, \cdots, c_p,\ \tilde{y}$$

where $\tilde{y}$ is an element corresponding at a non zero element in $E_2^{2,0}$. We can take as $\tilde{y}$, $c_1(\rho_2)$ where $\rho_2: G\to G/\langle a, b, c^p\rangle\to \boldsymbol{C}^\times$ by $c\mapsto \exp(2\pi i/p)$. Let $\xi: \langle b, c\rangle$

$\to C^\times$ with $c \mapsto \exp(2\pi i/p^{r+1})$, $b \mapsto 1$ and $\eta = \mathrm{Ind}_{\langle b, c\rangle}{}^G(\xi)$. Then we can take $c_i = c_i(\eta)$.

LEMMA 4.13 (Lemma 1 in [**L-Y**]).

$$\mathrm{Cor}_{\langle b, c\rangle}{}^G(u^{p-1}y_2{}^i)\,|\,\langle b\rangle = -y_2{}^{p-1+i} \quad \mathrm{mod}\ (v_1, v_2, \cdots).$$

PROOF. By the double coset formula, the lefthand side above formula is

$$\begin{aligned}&\textstyle\sum_{\langle b, c\rangle g\langle b\rangle}\mathrm{Cor}_{g^{-1}\langle b, c\rangle g\cap\langle b\rangle}{}^{\langle b\rangle}g^*(u^{p-1}y_2{}^i\,|\,\langle b\rangle\cap g^{-1}\langle b, c\rangle g)\\ &=\textstyle\sum_j a^{j*}(u^{p-1}y_2{}^i)\,|\,\langle b\rangle\\ &\equiv\textstyle\sum_j(u+jey_2)^{p-1}(y_2+jp^ru)^i\,|\,\langle b\rangle \qquad \text{(see (4.12) in [T-Y2])}\\ &\equiv -y_2{}^{p-1+i} \qquad \mathrm{mod}\ (v_1, v_2, \cdots).\end{aligned}$$

COROLLARY 4.14. $\mathrm{Tr}e_{BP}(G) = BP^*(BG)$ *and* $\mathrm{Tr}e(G) = H^{even}(BG)$.

PROOF. $\mathrm{Cor}_{\langle b, c\rangle}{}^G(e(p-1)\xi + i\rho_2)$ represents $\tilde{y}^i w$ and $\mathrm{Cor}_{\langle b, c\rangle}{}^G(e(i\xi))$ represents c_i mod $(c_1, \cdots, c_{i-1})$ for $1 \leqq i \leqq p-1$. q. e. d.

Next we consider in $H^*(BG)$. Recall $H^*(B\langle b\rangle) \cong Z/p[y_2]$ for $* > 0$.

LEMMA 4.15. $H^*(BG)\,|\,\langle b\rangle = Z_{(p)}\{1\} \oplus Z/p\{y^{p-1}, y^p, y^{p+1}, \cdots\}$.

PROOF. Let $j\colon \langle b\rangle \subsetneq G$ be the inclusion. Then $j^*(y_1) = 0$, $j^*(y_2{}^i w) = y_2{}^{i+p}$ $j^*(c_i) = 0$; $1 \leqq i \leqq p-2$, $j^*(c_i) = y_2{}^i$; $i = p-1, p$. q. e. d.

Since $|G/\langle b, c\rangle| = p$ and $\langle b, c\rangle$ is an abelian group, the dimension of irreducible representations are 1 or p. Therefore $c_i(\rho)\,|\,\langle b\rangle = 0$ except for $i = p-1$, and $= p$ for all irreducible representations.

LEMMA 4.16. *Let us write* $t_i = \mathrm{Cor}_{\langle b, c\rangle}{}^G(e((p-1)\xi + (i+1)\rho_2))$. *Then* $t_i \notin \mathrm{Ch}(G)$ *for* $1 \leqq i \leqq p-3$.

PROOF. The restriction is $t_i\,|\,\langle b\rangle = -y^{p+i}$, which is not the products of $y_2{}^{p-1}$ and $y_2{}^p$. q. e. d.

COROLLARY 4.17. *Let* $A = \langle b, c^{p^r}\rangle \cong Z/p \oplus Z/p$. *Then* $H^*(BG)\,|\,A \neq H^*(BA)^{N_G(A)}$ *where* $N_G(A)$ *is the normalizer of* A *in* G.

PROOF. Let us write by $\sqrt{0}$ the ideal of nilpotent elements. Then

$$H^*(BA)^{N_G(A)}/\sqrt{0} = Z/p[y_2, u^p - y_2{}^{p-1}u] \subset Z/p[y_2, u].$$

But

$$H^*(BG)/\sqrt{0}\,|\,A = Z/p[y_2{}^{p+i}, u^p - y_2{}^{p-1}u;\ i \geqq 0]. \qquad \text{q. e. d.}$$

Here we recall the Atiyah conjecture [**A**]. Let $\lambda^i(x)$ be the i-th exterior power of representation $x \in R(G)$ and $\lambda_t(x) = \sum\lambda^i(x)t^i$. Let us denote $\lambda_{t/1-t}(x) =$

$\gamma_t(x)=\sum\gamma^i(x)t^i$. Grothendieck defined the γ-filtration on $R(G)$ by using this $\gamma^i(x)$. If this γ-filtration is equal to the filtration defined geometrically identifying $R(G)\hat{}=K^0(BG)$, then, by using splitting principle and multiplicative property of $\gamma_t(x)$, we can easily see that $[c_n(\rho)]=[\gamma^n(\rho-\dim\rho)]$ for each representation ρ, in the E_∞-term of the Atiyah-Hirzebruch spectral sequence

$$E_2^*=H^*(BG;\,Z)\Longrightarrow K^0(BG)=R(G)\hat{}\,.$$

Therefore $\mathrm{Ch}(G)$ maps epic to the E_∞-term if both filtrations are equal. Atiyah conjectured the equality. Leary-Yagita give the counter example for p-groups (for non-p groups examples are given by Weiss [**Th 2**]).

THEOREM 4.18 ([**L-Y**]). *For $G=G(r+3)$ and $p\geqq 5$, $\mathrm{Ch}(G)$ does not map epic to the E_∞-term, and hence Atiyah conjecture does not hold.*

PROOF. Let us consider the connected K-theory $\tilde{k}(n)^*(-)$ with the coefficient $\tilde{k}(n)^*=Z_{(p)}[v_n]$ and $\tilde{K}(n)^*(-)=\tilde{k}(n)^*(-)[v_n^{-1}]$. The complex K-theory localized at p is the direct sum of $\tilde{K}(1)$-theory. Since there are natural (Thom) maps $\rho: BP\to\tilde{k}(1)\to HZ_{(p)}$, we see t_i in $H^*(BG)$ is a permanent cycle in the spectral sequence $E_2^{*,*}=H^*(BG,\,\tilde{k}(1)^*)\Rightarrow\tilde{k}(1)^*(BG)$. Comparing another spectral sequence $E_2^{*,*}=H^*(B\langle b\rangle;\,\tilde{K}(1)^*)\Rightarrow\tilde{K}(1)^*(B\langle b\rangle)$, we know t_i is non zero after localized by v_1 and is not represented by element in $\mathrm{Ch}(G)=\rho(\mathrm{Ch}_{BP}G)$ in the E_∞-term for $1\leqq i\leqq p-3$. q.e.d.

REMARK 4.19. It is wellknown that $K(1)^*(BG)=\mathrm{Ch}_{K(1)}(BG)$ for all finite groups (Landweber [**La 2**]). Indeed, $[p](y_2)=py_2+v_1y_2{}^p=0$ in $\tilde{k}(1)^*(B\langle b\rangle)$, hence $y_2{}^{p+i}=v_1^{-1}(py_2^{i+1})\in\mathrm{Ch}_{K(1)}(G)|\langle b\rangle$. However for $\tilde{K}(n)$-theory, $[p](y_2)=py_2+v_ny_2{}^{p^n}$ and we know, if $n\geqq 2$, then $\mathrm{Ch}_{\tilde{K}(n)}(G)\subsetneqq\tilde{K}(n)^*(BG)$ and $\mathrm{Ch}_{K(n)}(G)\subsetneqq K(n)^*(BG)$.

§5. $H^*(BG(r+3,\,e))$.

In this section we always assume $p\geqq 5$ and compute $H^*(BG(r+3),\,e)$. For ease of notations, let us write $G=G(r+3,\,e)$ and $E=C(r+2)$, and $H^*(BG)$ by $H^*(G)$. We consider the exact sequence (4.1) and induced spectral sequence

(5.1) $$E_2^{*,*}=H^*(Z/p;\,H^*(E))\Longrightarrow H^*(G)\,.$$

REMARK. From Corollary 4.9 and Corollary 4.12, there is the epimorphism $((E_2 \text{ in } (4.2))\otimes_{BP*}Z_{(p)})\to(E_\infty{}^{total\ even} \text{ in } (5.1))$. In particular $E_\infty{}^{odd,odd}=0$ and $E_\infty{}^{even,0}=(E_2{}^{even,0} \text{ in } (4.2))\otimes_{BP*}Z_{(p)}$.

We recall the cohomology of E by Theorem 3 in [**Ly**]. Let us write $\tilde{Z}/p^r[a]=Z[a]/(p^ra)$ and write

$$(5.2)\qquad \begin{cases} Y = \tilde{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \\ C = Z/p^{r-1}\{c_1\} \oplus Z/p^r\{c_2, \cdots, c_{r-1}\} \\ C_p = \tilde{Z}/p^{r+1}[c_p]. \end{cases}$$

Then $H^{even}(E) \cong (Y \oplus C) \otimes C_p$ with the multiplication

$$\begin{cases} c_{p-1}{}^2 = y_1^{2p-2} + y_2^{2p-2} - y_1^{p-1} y_2^{p-1}, \quad c_{p-1} y_2 = y_2^p, \quad c_{p-1} y_1 = y_1^p \\ c_i c_j = 0 \mod p, \qquad 1 \leqq i, j \leqq p-2. \end{cases}$$

(For details see Lewis or Leary [**Ls**], [**Ly**].) The odd dimensional part is

$$(5.3)\qquad H^{odd}(E) = (Y\{d_1, d_2\}/(y_1 d_2 - y_2 d_1, y_1^p d_2 - y_2^p d_1)) \otimes C_p, \quad |d_i| = 3.$$

The action induced from (4.1) is (same as (4.3))

$$(5.4)\qquad \begin{cases} c^* c_i = c_i, \qquad c^* y_1 = y_1 \\ c^* y_2 = y_2 - y_1. \end{cases}$$

By the arguments similar to (4.4) we can know the invariant parts. Let

$$\begin{cases} Y_i = Z/p[y_i] \\ Y_w = Y_2\{w\} = Z/p[y_2]\{w\} \qquad \text{where} \quad w = y_2^p - y_1^{p-1} y_2. \end{cases}$$

Then we get

$$(5.5)\qquad H^{even}(E)^{\langle c\rangle} = (Y_1 \oplus Y_w \oplus C) \otimes C_p.$$

Next consider image N. By the reason similar to (4.8),

$$NC = pC, \quad NC_p = pC_p \quad \text{and}$$

$$N y_2^i y_1^s = \begin{cases} 0 & i < p-1 \\ y_1^{s+p-1} & i = p-1. \end{cases}$$

Therefore we have

$$(5.6)\qquad H^{even}(E)^{\langle c\rangle}/N = (Y_1/y_1^{p-1} \oplus Y_w \oplus C/p) \otimes C_p/p.$$

The $\operatorname{Ker} N$ is computed

$$\operatorname{Ker} N | C \otimes C_p = {}_p(C \otimes C_p) = Z/p\{(p^{r-2}c_1, p^{r-1}c_2, \cdots, p^{r-1}c_{p-1})c_p^j, p^r c_p^j\}$$

$$\operatorname{Ker} N | Y \otimes C_p = (Y - Y_1\{y_2^{p-1}\}) \otimes C_p.$$

The image $(1-c^*)$ is computed as

$$(1-c^*) y_2^i y_1^s = i y_2^{i-1} y_1^{s+1} + \cdots$$

and hence $(1-c^*)Y = \{y_2^i y_1^{s+1};\ i \neq p-1\}$. Therefore

(5.7) $\quad \operatorname{Ker} N/(1-c^*)^{even} = {}_p(C\otimes C_p)\oplus(Z/p\{y_2, \cdots, y_2^{p-2}\}\oplus Y_w)\otimes C_p$.

Now we will consider odd dimensional case. The c^*-action is given

(5.8)
$$c^*d_1 = d_1 \quad \text{since} \quad d_1|\langle b, c^p\rangle = 0$$
$$c^*d_2 = d_2 - d_1 \quad \text{since} \quad y_2d_1 = y_1d_2.$$

First we study the invariant under c^*. Since

$$(1-c^*)(y_2^iy_1^s+\cdots)d_2 = ((y_2-y_1)^iy_1+\cdots)(d_2-d_1)-(y_2^iy_1^s+\cdots)d_2$$
$$= (i+1)y_2^{i-1}y_1^{s+1}d_2+\cdots,$$

we see for $1\leqq i\leqq p-2$, $y_2^iy_1^s+\cdots$ is not invariant under c^*. It is easily checked that $d_w=(y_2^{p-1}-y_1^{p-1})d_2$ is invariant;

$$c^*d_w = ((y_2-y_1)^{p-1}-y_1^{p-1})(d_2-d_1) = (y_2^{p-1}+y_2^{p-2}y_1+\cdots+y_2y_1^{p-2})(d_2-d_1)$$
$$= y_2^{p-1}d_2-y_2y_1^{p-2}d_1 = d_w, \quad \text{since} \quad y_2d_1 = y_1d_2.$$

By $y_2^pd_1=y_1^pd_2$, $y_1d_w=0$ and hence Y_2d_w is also invariant.

(5.9)
$$H^{odd}(E)^{\langle c\rangle} = ((Y_w\oplus Y_1)d_1\oplus Y_2d_w)\otimes C_p$$
$$= (Y_1d_1\oplus Y_2d_w)\otimes C_p, \quad \text{since} \quad Y_wd_1 = 0.$$

The image N is computed as

$$Ny_2^iy_1^sd_1 = \begin{cases} 0 & i \neq p-1 \\ y_1^{s+i}d_1 & i = p-1 \end{cases}$$

$$Ny_2^id_2 = \textstyle\sum_k(y_2-ky_1)^i(d_2-kd_1)$$
$$= \textstyle\sum_s\binom{i}{s}\sum_k(k^sd_2-k^{s+1}d_1)y_2^{i-s}y_1^s = \begin{cases} y_1^{p-2}d_1 & i = p-2 \\ 0 & 1\leqq i\leqq p-3. \end{cases}$$

Therefore

(5.10)
$$H^{odd}(E)^{\langle c\rangle}/N = (Y_1/y_1^{p-2}d_1\oplus Y_2d_w)\otimes C_p.$$

The $\operatorname{Ker} N$ is given from above computation

$$\operatorname{Ker} N = (Z/p\{1, y_2, \cdots, y_2^{p-2}\}Y_1d_1\oplus Z/p\{1, y_2, \cdots, y_2^{p-3}\}d_2\oplus Y_2d_w)\otimes C_p.$$

The image $(1-c^*)Yd_1=\{y_2^iy_1^{s+1}|i\neq p-1\}d_1$. Hence

(5.11) $\quad \operatorname{Ker} N/(1-c^*)^{odd} = (Z/p\{1, y_2, \cdots, y_2^{p-3}\}d_2\oplus Y_2d_w)\otimes C_p$.

To study differential of this spectral sequence (5.1), we study differential of other ones. Consider exact sequences

$$
(5.12)\qquad
\begin{array}{ll}
(1) & 1 \longrightarrow \langle c^p\rangle \longrightarrow \langle b,\, c\rangle \longrightarrow \langle b,\, c\rangle/\langle c^p\rangle \longrightarrow 1 \\
 & \qquad\ \ \Vert \qquad\qquad\quad \cap \qquad\qquad\quad \cap \\
(2) & 1 \longrightarrow \langle c^p\rangle \longrightarrow \quad G \quad \longrightarrow \langle a,\, b,\, c\rangle/\langle c^p\rangle \longrightarrow 1 \\
 & \qquad\ \ \Vert \qquad\qquad\quad \cup \qquad\qquad\quad \cup \\
(3) & 1 \longrightarrow \langle c^p\rangle \longrightarrow \langle a,\, b,\, c^p\rangle \longrightarrow \langle a,\, b\rangle \longrightarrow 1
\end{array}
$$

and induced spectral sequences, in particular

$$
(5.13)\qquad E_2^{*,*'} = H^*(C(3);\, H^*(Z/p^r)) \Longrightarrow H^*(G)
$$

$$
= \begin{cases} Z/p[u]\otimes H^*(C(3);\, Z/p^r) & *'>0 \\ H^*(C(3)) & *'=0. \end{cases}
$$

Let us use notations with $\tilde{y}$, $\tilde{c}_i$, $\tilde{d}$ in $H^*(C(3))$ instead for y_2, c_i, d_2, for example, $\tilde{y}\in H^2(\langle c\rangle/\langle c^p\rangle)$. Moreover $\tilde{x}\in H^1(\langle c\rangle;\, Z/p^r)$, $x_1\in H^1(\langle a\rangle;\, Z/p)$ are the non zero elements corresponding $\tilde{y}$ and y_1 respectively. Then we have

$$
(5.14)\qquad d_2u = d_1, \qquad \text{since} \quad du\,|\,(3) = x_1y_2 - x_2y_1.
$$

Since $d(\tilde{x}u)\,|\,(1)=\tilde{y}^2$ and $d(\tilde{x}u)\,|\,(3)=0$, we have

$$
(5.15)\qquad d(\tilde{x}u) = \tilde{y}^2 + \lambda\tilde{c}_2.
$$

Let $z=\tilde{y}x_1 - y_1\tilde{x}$ in $H^*(C(3);\, Z/p^r)$. Then $pz=0$ and $z\in H^*(C(3);\, Z/p)$, moreover $\beta(z)=0$ and $z\,|\,\langle a\rangle = z\,|\,\langle c\rangle=0$. Hence $z=0$ in $H^*(C(3))$. Therefore $y_1d(\tilde{x}u)=\tilde{y}d(x_1u)$. Hence

$$
(5.16)\qquad d(x_1u) = y_1\tilde{y} + \lambda'\tilde{c}_2.
$$

We will consider another exact sequence

$$
1 \longrightarrow \langle b,\, c^p\rangle \longrightarrow \langle b,\, c\rangle \longrightarrow \langle c\rangle \longrightarrow 1
$$

and induced spectral sequence

$$
(5.17)\qquad E_2^{*,*'} = H^*(Z/p;\, H^*(Z/p^r\oplus Z/p)) \Longrightarrow H^*(Z/p\oplus Z/p^{r+1})
$$

$$
\begin{cases} Z/p[u]\otimes Z/p[y_2]\otimes \Lambda(d_2)\otimes Z/p[\tilde{y}]\otimes\wedge(\tilde{x}) & *'>0 \\ Z/p[\tilde{y}] & *'=0. \end{cases}
$$

For this spectral sequence

$$
(5.18)\qquad d_2(d_2) = y_2\tilde{y}, \qquad d_2(\tilde{x}d_2) = y_2\tilde{y}\tilde{x},
$$

$$
d_r(y_2) = 0 \qquad \text{for all } r.
$$

Now we return to the first spectral sequence (5.1). Let us write by $\tilde{y}$ the element in $E_2^{2,0}$ corresponding the same letter $\tilde{y}\in H^2(\langle c\rangle/\langle c^p\rangle)$ and $\tilde{x}a$ the element in $E_2^{1,*}$ for $a\in\operatorname{Ker} N/\operatorname{Im}(1-c^*)$. Note that $\tilde{x}a$ does not mean the product of a and $\tilde{x}$, while some restrictions may be its product for some subgroups.

First assume $r\geqq 2$, that is the cases $c_1\neq 0$. From (5.14)-(5.16) $H^3(G)\cong Z/p\{\tilde{d}\}$. From (5.7) and (5.9), 3-dimensional elements in E_2 are in $Z/p\{d_1, \tilde{x}y_2, \tilde{x}_p c_1\}$. Since $\tilde{x}y_2|\langle b, c\rangle=\tilde{d}\,|\langle b, c\rangle$, $\tilde{x}y_2$ is the only permanent cycle. Hence

(5.19) $$d_3(\tilde{x}_p c_1)=\tilde{y}^2$$

(5.20) $$d_2(d_1)=y_1\tilde{y}+\lambda c_1\tilde{y} \qquad \text{since} \quad c_1|\langle a, b, c^p\rangle=0.$$

By the naturality and (5.18), we get

(5.21) $$d_2(y_2^s d_w)=y_2^s w\tilde{y}, \quad d_2(y_2^s\tilde{x}d_2)=y_2^{s+1}\tilde{y}\tilde{x}, \quad d(y_2^s\tilde{x}d_w)=y_2^s w\tilde{y}.$$

From (5.19), we see $c_i\tilde{y}^2=0$ in $E_4^{*,*}$, and we know

(5.22) $$d_3({}_pc_i\tilde{x})=c_{i-1}\tilde{y}^2.$$

We compute E_3-terms. From (5.10) and (5.6)

$$\begin{array}{lll} d_2:\ E_2^{even,odd} & \longrightarrow & E_2^{even,even} \\ \quad\ \wr\Vert & & \quad \wr\Vert \\ \left\{\begin{array}{ll}(Y_1/y_1^{p-2}d_1\oplus Y_2d_w)\otimes C_p & \text{even}>0 \\ (Y_1d_1+Y_2d_w)\otimes C_p & \text{even}=0.\end{array}\right. & & (Y_1/y_1^{p-1}\oplus Y_w\oplus C)\otimes C_p/p \end{array}$$

Hence from (5.20) and (5.21)

(5.23) $$\left\{\begin{array}{ll} E_3^{even,odd}=0 & even>0 \\ E_3^{0,odd}=Y_1\{y_1^{p-2}d_1\}\otimes C_p \end{array}\right. \qquad \left\{\begin{array}{ll} E_3^{even,even}=(C\otimes C_p)/p & even>0 \\ E_3^{0,even}=(Y_1\oplus Y_w\oplus C)\otimes C_p. \end{array}\right.$$

From (5.11) and (5.7)

$$\begin{array}{ccc} d_2:\ E_2^{odd,odd} & \longrightarrow & E_2^{odd,even} \\ \wr\Vert & & \wr\Vert \\ (Z/p\{1, y_2, \cdots, y_2^{p-3}\}d_2\oplus Y_2d_w)\otimes C_p & & ({}_p(C\otimes C_p)\oplus(Z/p\{y_2, \cdots, y_2^{p-2}\}\oplus Y_w))\otimes C_p. \end{array}$$

Hence from (5.21)

(5.24) $E_3^{odd,odd}=0$,

$$E_3^{odd,even}=\left\{\begin{array}{ll} {}_p(C\otimes C_p) & odd=3 \\ ({}_p(C\otimes C_p)\oplus(Z/p\{y_2, \cdots, y_2^{p-2}\}\oplus Y_w))\otimes C_p & \text{for } odd=1. \end{array}\right.$$

The differential (5.22) shows

(5.25) $$\begin{aligned} E_4^{0,*}&=(Y_1\oplus Y_w\oplus C\oplus Y_1\{y_1^{p-2}d_1\})\otimes C_p \\ E_4^{1,*}&=(Z/p\{y_2, \cdots, y_2^{p-2}\}\oplus Y_w)\otimes C_p\cdot\tilde{x} \\ E_4^{2,*}&=(\{1\}\oplus C)\otimes(C_p/p)\cdot\tilde{y} \end{aligned}$$

$$E_4^{i,*}=0 \quad \text{for} \quad i \geqq 3.$$

Therefore we get

PROPOSITION 5.26. $E_4 \cong E_\infty$.

We will consider the extention problem of E_∞. First we see odd parts. Since Y_1 is p-torsion, we only need to see $\{y_1^{p-2}d_1\}$, but there is no other element in E_∞ of degree $2p-1$, and hence all odd dimensional elements are exactly p-torsion. Next we consider even degree parts. Since Y_w is the image of corestriction of p-torsion (4.13), Y_w is also p-torsion. To see the exponent of c_i, we use the restriction to $BP^*(B\langle c\rangle)\cong BP^*[[u]]/[p^{r+1}](u)$. Since $[p](u)=pu+v_1u^p \bmod (v_1p)$ we have

$$[p^{r+1}](u)=p^{r+1}u+v_1^{1+p+\cdots+p^r}u^{p^{r+1}} \quad \bmod (v_1p). \tag{5.27}$$

First note that we can take

$$c_i|\langle c\rangle = pu^i \quad \bmod [p^r](u) \qquad \text{for} \quad 1 \leqq i \leqq p-1$$
$$c_p|\langle c\rangle = u^p \quad \bmod [p^r](u).$$

Since $p^{r-1}c_1|\langle c\rangle = p^r u \neq 0 \bmod (v_1)$, hence $p^{r-1}c_1 \neq 0$. Therefore $p^{r-1}c_1$ must be $\tilde{y}$ by dimensional reason. For $2 \leqq i \leqq p-1$, we see

$$p^r c_i|\langle c\rangle = p^{r+1}u^i = -v_1^{1+\cdots+p^r}u^{p^{r+i}+i-1}$$
$$\neq 0 \bmod ((v_1p) \text{ and } u^{np}|n \geqq 1).$$

Therefore $p^r c_i \neq 0$ and $p^r c_i = {}_p c_{i-1}\tilde{y}$ by dimensional reasons. The similar fact $p^{r+1}c_p = {}_p c_{p-1}\tilde{y}$ also holds.

At last we consider the case $r=1$. By the similar arguments, we get

$$d_5 c_2 \tilde{x} = \tilde{y}^3 \quad \text{instead of (5.19).} \tag{5.27$'$}$$

Using this, we can show

$$\begin{cases} E_6^{i,*}=E_4^{i,*} \text{ in (5.25)} & \text{for } 0 \leqq i \leqq 2 \\ E_6^{3,*}=0 & \\ E_6^{4,*}=(C_p/p)\cdot \tilde{y}^2 & \\ E_6^{i,*}=0 & \text{for } i \geqq 5. \end{cases} \tag{5.28}$$

This case we write $\tilde{y}$ by c_1. Then we get the following theorem except for the extension problem of E_∞ for $r=1$.

THEOREM 5.29. *Let $G=G(r+3, e)$ and $p \geqq 5$. Then there is an additive isomorphism*

$$H^{even}(BG) \cong (Y_1 \oplus Y_w \oplus C') \otimes C'_p$$
$$H^{odd}(BG) \cong (Y_1 d_1 \oplus E) \otimes C'_p$$

where

(1) $C' = Z/p^r\{c_1\} \oplus Z/p^{r+1}\{c_2, \cdots, c_{p-1}\}$

$C'_p = Z/p^{r+2}\{c_p\} \qquad |c_i| = 2i$

(2) $Y_1 = Z/p[y_1] \qquad |y_1| = 2$

$Y_w = \{y_2{}^s w \mid s \geqq 0\} = Z/p[y_2]\{w\} \qquad |y_2{}^s w| = 2(s+p)$

(3) $|d_1| = 2p-1$

$E = Z/p\{e_1, \cdots, \hat{e}_{p-1}, e_p\} \oplus Y_w e_1 \qquad |e_i| = 2i+1$

with multiplications $we_k = y_2{}^{k-1} we_1$.

We can also decide the multiplicative structure of $H^*(BG)$, by using arguments of Leary [**Ly 2**].

Take generators such as

(5.30)
$$c'_i = \mathrm{Cor}_{\langle b,c\rangle}{}^G(u^i) \qquad 1 \leqq i \leqq p-1$$
$$t_i = \mathrm{Cor}_{\langle b,c\rangle}{}^G(y_2{}^{i+1} u^{p-1}) \qquad 0 \leqq i.$$

LEMMA 5.31.

(1) $c_i' | \langle b, c\rangle = \begin{cases} pu^i & 1 \leqq i < p-1 \\ pu^{p-1} - y_2{}^{p-1} & i = p-1 \end{cases}$

(2) $t_i | \langle b, c\rangle = -y_2{}^{p+i}$.

PROOF. (Compare 4.13.) The conjugation action a^* in $H^*(\langle b, c\rangle)$ is given by

$$a^* \quad \begin{cases} u \longmapsto u + e y_2 \\ y_2 \longmapsto y_2 + p^r u \end{cases}$$

and hence

$$a^{*k} \quad \begin{cases} u \longmapsto u + k e y_2 + (k(k-1)/2) e p^r u \\ y_2 \longmapsto y_2 + k p^r u. \end{cases}$$

Therefore

$$\begin{aligned} \sum a^{*k}(u^i) &= \textstyle\sum_{k=0}^{p-1}(u+key_2)^i + \sum k(k-1)/2 \cdot iep^r u^i \\ &= \textstyle\sum_{s=0}^{i} \binom{i}{s} u^s \sum_{k=0}^{p-1}(key_2)^{i-s} + \sum k(k-1)/2 \cdot iep^r u^i \\ &= \begin{cases} pu^i + (i/2)(p(p-1)(2p-1)/6 - p(p-1)/2) p^r u^i & i < p-1 \\ pu^{p-1} - y_2{}^{p-1} & i = p-1. \end{cases} \end{aligned}$$

By the double coset formula and $p\geqq 5$, we have (1)

$$c_i'|\langle b, c\rangle = \mathrm{Cor}\,(u^i)|\langle b, c\rangle = \sum a^{*k}u^i .$$

Using similar arguments (compare Lemma 4.13) we get (2). q. e. d.

Now we see the multiplication. Let $i\neq p-1$. Then

$$\begin{aligned} c_i'c_j' &= \mathrm{Cor}\,(u^i)\,\mathrm{Cor}\,(u^j) = \mathrm{Cor}\,(\mathrm{Cor}\,u^i|\langle b, c\rangle\cdot u^j) \\ &= \mathrm{Cor}\,(pu^{i+j}) = p\,\mathrm{Cor}\,(u^{i+j}) \end{aligned}$$

(5.32) $$= \begin{cases} pc_{i+j} & i+j < p \\ pc_{i+j-p}c_p & i+j \geqq p, \quad \text{since} \quad c_p|\langle b, c\rangle = u^p . \end{cases}$$

Next consider the case $i=j=p-1$.

(5.33) $$c'_{p-1}c'_{p-1} = \mathrm{Cor}\,(pu^{2p-2}+y^{p-1}u^{p-1}) = pc'_{p-2}c_p+t_{p-2} .$$

Thus the extension of E_∞ for $r=1$, that is (1) in Theorem 5.29 has also proved, since $c_1c_i\neq 0$ from (5.28).

Since $y_1|\langle b, c\rangle=0$, we get

(5.34) $$y_1c_i = 0 \quad \text{and} \quad y_1t_i = 0 .$$

We easily see that

(5.35) $$t_it_j = \mathrm{Cor}\,(-y_2{}^{p+i}y_2{}^{j+1}u^{p-1}) = -t_{i+j+p} .$$

If $i\neq p-1$, then $c_it_j=\mathrm{Cor}\,(py_2{}^{j+1}u^{i+p-1})$. By the arguments similar as the proof of Lemma 5.31, we see

$$\mathrm{Cor}\,(y_2{}^{j+1}u^{i+p-1})|\langle b, c\rangle = \sum a^{*k}(y_2{}^{j+1}u^{i+p-1}) = 0 .$$

Since each element in Ker Res $(H^*(\langle b, c\rangle)\to H^*(\langle b, c^p\rangle))$ is p-tosion, we see

(5.36) $$c_it_j = 0 \qquad \text{for} \quad i\neq p-1 .$$

$$c_{p-1}t_j = \mathrm{Cor}\,(py_2{}^{j+1}u^{2p-2}-y_2{}^{p+j}u^{p-1}) = -t_{p+j-1} .$$

At last we consider odd dimensional elements. Define

(5.37) $$d_1' = \mathrm{Cor}_{\langle a, b, c^p\rangle}{}^G(y_2{}^{p-2}d_2) ,$$

since $d_1'|\langle a, b, c^p\rangle=\sum c^{*k}(y_2{}^{p-2}d_2)=y_1{}^{p-2}d_1$ from (5.9). In the E_∞-term (5.25), (5.28), odd dimensional elements in $E_\infty{}^{i,*}$, $i\geqq 2$ are all zero. Hence we get

(5.38) $$y_1e_i = 0 \qquad t_ie_j = t_{i+j-1}e_1$$

$$c_k'e_i = \begin{cases} 0 & 1\leqq k < p-1 \\ t_{i-1}e_1 & k = p-1, \quad 2\leqq i \\ -e_p & k = p-1, \quad i = 1 . \end{cases}$$

Now we study the product by d_1'. From (5.25), (5.28), we know that if odd dimensional element x satifies $x|\langle a, b, c^p\rangle = x|\langle b, c\rangle = 0$ then $x=0$. Note

$$t_i d_i'|\langle a, b, c^p\rangle = (t_i y_1^{p-2})|\langle a, b, c^p\rangle d_1' = 0.$$

By the double coset formula

$$d_1|\langle b, c\rangle = \mathrm{Cor}_{\langle b, c^p\rangle}^{\langle b, c\rangle}(y_2^{p-2} d_2|\langle b, c^p\rangle).$$

Here

$$0 \neq d_2|\langle b, c^p\rangle = d \in H^3(Z/p \oplus Z/p^r) \cong H^1(Z/p^r;\ H^2(Z/p)) = H^1(Z/p^r;\ Z/p)$$

identifying from the spectral sequence induced by

$$1 \longrightarrow Z/p \longrightarrow \langle b, c^p\rangle \longrightarrow Z/p^r \longrightarrow 1.$$

We know that

$$\text{(5.39)} \qquad \mathrm{Cor}_{\langle c^p\rangle}^{\langle c\rangle}(d) = e \neq 0 \ \text{ in } H^1(Z/p^{r+1};\ Z/p)$$

by the definition of the corestriction, that is, identifying $d \in \mathrm{Hom}(Z/p^r;\ Z/p)$

$$\mathrm{Cor}_{\langle c^p\rangle}^{\langle c\rangle} d(n) = \textstyle\sum_{i=0}^{p-1} h(n, i) \equiv -n \mod p$$

where $n+i = j + h(n, i)p \bmod p^2$ with $0 \leqq i,\ j \leqq p-1$ (See Lewis [**L**] 1.2). Therefore

$$d_1'|\langle b, c\rangle = -y_2^{p-2} e, \qquad \text{where} \quad 0 \neq e \in H^3(\langle b, c\rangle).$$

Hence we get

$$\text{(5.40)} \qquad t_i d_1' = -t_{i+p-2} e_1.$$

Since $c_i|\langle a, b, c^p\rangle = \mathrm{Cor}_{\langle b, c^p\rangle}^{\langle a, b, c^p\rangle}(u^i)$, we know

$$c_i' d_1'|\langle a, b, c^p\rangle = \mathrm{Cor}_{\langle b, c^p\rangle}^{\langle a, b, c^p\rangle}(u^i y_1^{p-2}|\langle b, c^p\rangle) d_1 = 0 \quad \text{and}$$

$$c_i' d_1'|\langle b, c\rangle = \begin{cases} 0 & 1 \leqq i \leqq p-2 \\ -y_2^{p-1} y_2^{p-2} e & i = p-1. \end{cases}$$

Hence we get

$$\text{(5.41)} \qquad c_i' d_1' = \begin{cases} 0 & 1 \leqq i \leqq p-2 \\ t_{p-3} e_1 & i = p-1. \end{cases}$$

THEOREM 5.42. *$H^*(BG(r+3, e))$ is multiplicatively generated by y_1, t_j, c_i; $1 \leqq i \leqq p-1$, c_p and d_1', $e_1, \cdots, e_{p-2}$, e_p with the relation* (5.32)-(5.36), (5.38), (5.40)-(5.41). *In particular it is independent of e.*

REMARK 5.43. When $p=3$, $H(BG(r+3, e))$ is completely computed by Leary [**Ly 2**] and it is independent on choice of e for $r \geqq 2$. Theorem 5.29 also holds this case but for Theorem 5.42, we need some changes for relations.

Therefore we know $H^*(G)$ for $\mathrm{rank}_p G=2$ groups except for metacyclic groups. For metacyclic groups cases, see Huebschmann [**H 2**]. From Lemma 3.5, $H^{even}(G)$ is multiplicatively generated by $c_i(\eta)$ $1\leqq i\leqq p^{\alpha-\gamma}$ and $c_1(\rho_2)=\tilde{y}$. Here we only note about $H^*(G)/\sqrt{0}$. Quillen's main theorem for the mod p ordinary cohomology is;

THEOREM 5.44 ([**Q**]). *The induced map from restrictions*

$$r: H^*(BG;\ Z/p) \longrightarrow \operatorname*{Lim\,inv}_{A\in IE} H^*(BA;\ Z/p)$$

is an F-isomorphism (i.e., injective modulo $\sqrt{0}$ and for all x in the righthand side module, there is m such that $x^{p^m}\in \mathrm{Im}\, r$) where IE is the set of conjugacy classes of elementary abelian p-subgroups of G.

THEOREM 5.45. *Let G be a metacyclic group* (2.1) *with $k-1=\lambda p^{\gamma}$ ($\lambda\not\equiv 0$* mod p). *Then there is a ring isomorphism*

$$H^*(BG)/\sqrt{0} \cong Z/p[c_{p^{\alpha-\gamma}}(\eta),\ \tilde{y}]$$

PROOF. First note that $|G|H^*(BG)=0$ and hence $pH^*(BG)\subset\sqrt{0}$. The conjugacy classes of maximal elementary abelian p-groups of G is only one

$$M=\langle C,\ AB\rangle \quad \text{where} \quad C=a^{p^{\alpha-1}},\ A=a^{p^{\alpha'-1}},\ B^{-1}=b^{p^{\beta-1}}, \quad \text{and}$$

$$H^*(BM;\ Z/p)/\sqrt{0}=H^*(BM)/\sqrt{0}=Z/p[u,\ y].$$

We want to know the Chern class $c(\eta)|M$ for the representation given for Lemma 3.5. By the double coset formula

$$\eta|M=\mathrm{Ind}_{\langle a,b^{p^{\alpha-\gamma}}\rangle}{}^{G}(\xi)|M=\textstyle\bigoplus_{0\leq i\leq p^{\alpha-\gamma}}(b^{i*}(\xi|M)).$$

Since $b^{-1}ab=a^{p^{\gamma}+1}$, $b^{-1}Ab=A^{p^{\gamma}+1}=CA$ or $=A$ if $\alpha'+\gamma=\alpha$ or $\alpha'+\gamma>\alpha$ respectively. If $\alpha'+\gamma>\alpha$, then $\eta|M=p^{\alpha-\gamma}\xi|M$ and $c(\eta)|M=(1+u)^{p^{\alpha-\gamma}}$, hence $c_i(\eta)|M=0$ for $1\leqq i\leqq p^{\alpha-\gamma}-1$ and $c_{p^{\alpha-\gamma}}(\eta)|M=u^{p^{\alpha-\gamma}}$. Therefore theorem holds this case. Suppose $\alpha'+\gamma=\alpha$. Then $\eta|M=\bigoplus_i\rho^i$ where ρ is a 1-dimensional nonzero representation of $\langle AB\rangle$. Hence

$$\begin{aligned} c(\eta|M) &= \textstyle\prod_{0\leq i\leq p^{\alpha-\gamma}}(1+u+iy) \\ &= ((1+u)^p-y^{p-1}(1+u))^{p^{\alpha-\gamma-1}} \\ &= 1+y^{(p-1)p}+(u^p-y^{p-1}u)^{p^{\alpha-\gamma-1}}. \end{aligned} \tag{5.3}$$

Since $\tilde{y}\in H^*(BG)|M$, we also have the theorem. q.e.d.

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