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Characters and character cycles

By Hiroyuki OCHIAI

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1. Introduction.

Let G_R be a connected real semisimple linear algebraic group. In [11], Kashiwara gave a conjecture that for a Harish-Chandra module V with a trivial infinitesimal character there is a G_R-equivariant object whose character cycle is the distribution character of V.

CONJECTURE 1.1 (Kashiwara, [11]). *For a Harish-Chandra module V with a trivial infinitesimal character, let Θ_V be the global character of V, and set $\mathcal{M}=\mathcal{D}_X\otimes_{\Gamma(X,\mathcal{D}_X)}V$, $\mathcal{F}=\mathrm{DR}(\mathcal{M})=R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ and $\mathcal{F}^a\in D_{G_R}(X)$ the corresponding object by the Matsuki correspondence for sheaves. Then the corresponding cycle $\mathrm{cc}(\Theta_V)$ equals to $\mathrm{ch}(\mathcal{F}^a)$ in $H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R})$;*

$$\mathrm{cc}(\Theta_V)=\mathrm{ch}(\mathcal{F}^a).$$

Several unexplained notations in this conjecture are given in § 2. The main theorem of this paper is the following.

THEOREM 1.2. *Conjecture 1.1 is true.*

We give a proof of Theorem 1.2 in § 5. To do this, we write both sides of the above formula in terms of local cohomologies with respect to Schubert cells on the flag variety. In § 3, we deal with the left hand side $\mathrm{cc}(\Theta)$ using the Osborne conjecture and the Beilinson-Bernstein intertwining operator. In § 4, we deal with the right hand side $\mathrm{ch}(\mathcal{F}^a)$ using the shrinking space introduced by Kashiwara-Schapira.

W. Schmid and K. Vilonen announced a different proof of the Kashiwara conjecture [17]. They formulate the conjecture for representations having an arbitrary infinitesimal character and do not use the Osborne conjecture to prove the Kashiwara conjecture.

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2. Preliminaries.

2.1. Groups and spaces. Let G_R be a connected real semisimple linear algebraic group, G_C a connected complexification of G_R, X the flag variety of G_C, $G_{C_{rs}}$ the set of regular semisimple elements of G_C, and for a subset A of G_C we will set $A_{rs}=A\cap G_{C_{rs}}$. Take a maximal compact subgroup K_R of G_R, and θ denote the corresponding Cartan involution. Let H_R be a θ-stable Cartan subgroup of G_R. For G_R, K_R, H_R, denote the corresponding Lie algebras by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$ and their complexifications by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ respectively. According to Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, we have $\mathfrak{h}_0=\mathfrak{t}_0\oplus\mathfrak{a}_0$.

Let $H_C=Z_{G_C}(\mathfrak{h})$ be the Cartan subgroup of G_C containing H_R. If $\nu\in\mathfrak{h}^*$ lifts to the character of H_C, we denote it by $e^\nu\in H_C\hat{}=\mathrm{Hom}(H_C, \boldsymbol{C}^\times)$. We use the same notation, e^ν, for the restriction of e^ν to H_R. Since $H_R=(H_R\cap K_R)\exp\mathfrak{a}_0$, we have $H_R/H_R{}^o=(H_R\cap K_R)/(H_R\cap K_R)^o$. Here for a Lie group M, denote the connected components of the identity by M^o. The group $H_R/H_R{}^o$ is known to be a 2-abelian group $(\boldsymbol{Z}/2\boldsymbol{Z})^l$ with some integer l. We regard a character $\varepsilon\in(H_R/H_R{}^o)\hat{}$ as an element of $H_R\hat{}$. Then any character of H_R is of the form $e^\nu\varepsilon\in H_R\hat{}$ for some $e^\nu\in H_C\hat{}$ and $\varepsilon\in(H_R/H_R{}^o)\hat{}$. For a semisimple H_R-module M and $e^\nu\varepsilon\in H_R\hat{}$, we denote the isotypic component by $M_{\nu,\varepsilon}=\mathrm{Hom}_{H_R}(e^\nu\varepsilon, M)$, we often write $M_\varepsilon=M_{0,\varepsilon}$.

Denote the root space by $\mathfrak{g}(\mathfrak{h}, \alpha)=\{x\in\mathfrak{g}\,;\,[h, x]=\alpha(h)x \text{ for all } h\in\mathfrak{h}\}$, the root system by $\Delta(\mathfrak{g}, \mathfrak{h})=\{\alpha\in\mathfrak{h}^*-\{0\}\,;\,\mathfrak{g}(\mathfrak{h}, \alpha)\neq 0\}$, and the set of real roots by $\Delta_R(\mathfrak{g}, \mathfrak{h})=\{\alpha\in\Delta(\mathfrak{g}, \mathfrak{h})\,;\,\alpha|_{\mathfrak{t}}=0\}$. For any $h_0\in H_{R_{rs}}$, there is a positive system $\Delta^+=\Delta^+(\mathfrak{g}, \mathfrak{h})\subset\Delta(\mathfrak{g}, \mathfrak{h})$ such that

$$e^\alpha(h_0)\notin\{x\in\boldsymbol{R}\,;\,x\geqq 1\} \qquad \text{for all} \quad \alpha\in\Delta^+.$$

We fix such a Δ^+. Define

$$H_R^{\,-\prime}=\{h\in H_R\,;\,e^\alpha(h_0)\notin\{x\in\boldsymbol{R}\,;\,x\geqq 1\} \text{ for all } \alpha\in\Delta^+\},$$
$$H_R^-=\{h\in H_R\,;\,e^\alpha(h)<1 \text{ for all } \alpha\in\Delta^+(\mathfrak{g}, \mathfrak{h})\cap\Delta_R(\mathfrak{g}, \mathfrak{h})\}.$$

Then we have $H_R^{\,-\prime}\subset H_R^-\cap H_{R_{rs}}$. Using this, we define

$$\mathfrak{n}=\oplus_{\alpha\in\Delta^+}\mathfrak{g}(\mathfrak{h}, \alpha), \qquad \bar{\mathfrak{n}}=\oplus_{\alpha\in\Delta^+}\mathfrak{g}(\mathfrak{h}, -\alpha),$$
$$\mathfrak{b}=\mathfrak{h}\oplus\mathfrak{n},$$
$$N=\exp\mathfrak{n}\subset G_C,$$
$$B=H_CN=N_{G_C}(\mathfrak{b})\,;\ \text{a Borel subgroup},$$
$$\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha,$$
$$W=W(G_C, H_C)=N_{G_C}(H_C)/H_C,$$
$$l\colon W\to\boldsymbol{Z}_{\geq 0}\,;\ \text{the length function with respect to } \Delta^+,$$

$w_0 \in W$; the longest element.

We realize the flag variety X as a quotient $X=G_C/B$ using this Borel subgroup B, and denote $x_0=eB/B\in G_C/B=X$ the origin of X under this realization.

We set $\tilde{G}_C=\{(g, x)\in G_C\times X;\ gx=x\}$, $p\colon \tilde{G}_C\to G_C$ the restriction of the first projection $G_C\times X\to G_C$. There is an isomorphism

$$G_C\times_B B \longrightarrow \tilde{G}_C, \qquad (g, b)\longmapsto (gbg^{-1}, gB),$$

where $G_C\times_B B$ is the fiber bundle associated to the B-principal bundle $G\to X$ with the adjoint action of B on B. We set $\tilde{G}_{C_{rs}}=p^{-1}(G_{C_{rs}})$, and $p_{rs}\colon \tilde{G}_{C_{rs}}\to G_{C_{rs}}$ the restriction of p. This is a finite map and each fiber is parametrized by the Weyl group W. In fact, there is an isomorphism

$$G_C/H_C\times H_{C_{rs}} \longrightarrow \tilde{G}_{C_{rs}}, \qquad (g, t)\longmapsto (gtg^{-1}, gB).$$

The Weyl group acts on $G_C/H_C\times H_{C_{rs}}$ by $(g, t)w=(g\dot{w}, \dot{w}^{-1}t\dot{w})$, here $\dot{w}\in N_{G_C}(H_C)$ is a representative of $w\in W$. Then the quotient map by W gives p_{rs}. For $h\in H_{C_{rs}}$, $p^{-1}(h)=\{(h, wx_0)\,|\,w\in W\}\cong W$.

Denote the restriction of p to $p^{-1}(G_R)$ by $p_R\colon p^{-1}(G_R)\to G_R$. Let $\mathrm{pr}_X\colon G_R\times X\to X$ be the second projection and $\mu\colon G_R\times X\to X$ the multiplication; $\mathrm{pr}_X(g, x)=x$, $\mu(g, x)=gx$.

For a manifold M, denote the orientation sheaf (with coefficients $\boldsymbol{C}$) be or_M. We denote the set of one point by $\{\mathrm{pt}\}$. For a topological space M, $a_M\colon M\to\{\mathrm{pt}\}$ be the adjunction, $\omega_M=a_M^!\boldsymbol{C}$ be the dualizing sheaf. Here $f^!$ is the twisted inverse of a morphism f in the derived category of sheaves. For an R-constructible sheaf $\mathcal{F}\in D^b_{R\text{-}c}(M)$, $D\mathcal{F}=D_M\mathcal{F}=\mathrm{Hom}(\mathcal{F}, \omega_M)$ denote the Verdier dual of $\mathcal{F}$.

2.2. Characters. We summarize the results on the characters and the character cycles by [7], [8], [**11**].

First we review the results for holomorphic solutions. In this paper we only use the analytic category. Let $\mathcal{M}_\rho$ denote the $\mathcal{D}_{G_C}$-module for invariant eigendistributions. Hence $\mathcal{M}_\rho$ is a (left-)$\mathcal{D}_{G_C}$-module generated by a section u with relations

$$(\mathrm{Ad}\,\mathfrak{g})u=0, \qquad Pu=0 \qquad \text{for}\quad P\in Z(\mathfrak{g})\cap U(\mathfrak{g})\mathfrak{g}.$$

Here $\mathrm{Ad}(\mathfrak{g})$ is the image of $\mathfrak{g}\to\Gamma(G_C, \mathcal{D}_{G_C})$ derived by the adjoint action of G_C on G_C and $Z(\mathfrak{g})$ is the center of the universal enveloping algebra $U(\mathfrak{g})$ considered as the space of biinvariant differential operators on G_C. Then we have

PROPOSITION 2.1. (i) *$\mathcal{M}_\rho$ is a regular holonomic $\mathcal{D}_{G_C}$-module.*

(ii) $R\,\mathrm{Hom}(\mathcal{M}_\rho, \mathcal{O}_{G_C})=Rp_*\boldsymbol{C}_{\tilde{G}_C}$.

(iii) *$\mathcal{M}_\rho$ is the minimal extension of $p_{rs*}(\mathcal{O}_{\tilde{G}_{C_{rs}}})$.*

Here $\mathcal{O}_{\widetilde{G}_{C_{sr}}}$ is the de Rham system (a left $\mathcal{D}$-module), and p_{rs*} is the direct image in the category of (left) $\mathcal{D}$-modules.

To be more precise, we take a holomorphic function ϕ on $\widetilde{G}_{C_{rs}}=p^{-1}(G_{C_{rs}})$;

$$\phi(g, x)=\det(1-g, T_x^*X)^{-1} \qquad \text{for} \quad (g, x)\in \widetilde{G}_{C_{rs}}.$$

This gives a $\mathcal{D}_{G_{C_{rs}}}$-isomorphism $\mathcal{M}_{\rho|G_{C_{rs}}}\to p_{rs*}(\mathcal{O}_{\widetilde{G}_{C_{rs}}})$ by $Pu\mapsto P\phi$, $P\in\mathcal{D}_{G_{C_{rs}}}$.

Next we deal with the distribution solution. Since $\mathcal{M}_\rho$ is regular holonomic, the distribution and hyperfunction solutions coincide:

$$R\operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \operatorname{Dist}_{G_R})=R\operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \mathcal{B}_{G_R}).$$

Here $\operatorname{Dist}_{G_R}$ denotes the sheaf of distributions on G_R, and $\mathcal{B}_{G_R}$ that of hyperfunctions.

We can transform this as follows.

$$\begin{aligned}
R\operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \mathcal{B}_{G_R}) &= R\operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, R\Gamma_{G_R}\mathcal{O}_{G_C}\otimes or_{G_R}\otimes or_{G_C})[\dim G_R]\\
&= R\Gamma_{G_R}R\operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \mathcal{O}_{G_C})\otimes or_{G_R}\otimes or_{G_C})[\dim G_R]\\
&= R\Gamma_{G_R}Rp_*\boldsymbol{C}_{\widetilde{G}_C}\otimes or_{G_R}\otimes or_{G_C}[\dim G_R]\\
&= R\Gamma_{G_R}Rp_*\omega_{\widetilde{G}_C}\otimes or_{G_R}[-\dim G_R]\\
&= Rp_{R*}\omega_{p^{-1}(G_R)}\otimes or_{G_R}[-\dim G_R]\\
&= Rp_{R*}R\Gamma_{p^{-1}(G_R)}\omega_{G_R\times X}\otimes or_{G_R}[-\dim G_R]\\
&= Rp_{R*}R\Gamma_{p^{-1}(G_R)}pr_X^{-1}\omega_X.
\end{aligned}$$

Then we have

$$\begin{aligned}
\Gamma(G_R, \operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \mathcal{B}_{G_R})) &= H^{-\dim G_R}(p^{-1}(G_R), \omega_{p^{-1}(G_R)}\otimes or_{G_R})\\
&= H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R}).
\end{aligned}$$

Here H_i^{BM} is the i-th Borel-Moore (or infinite) homology group. Remark that $p^{-1}(G_R)$ may not be smooth but is the union of the finite number of smooth submanifolds with dimensions $\dim G_R$. In fact, we decompose X into G_R-orbits, $X=\coprod_j S_j^a$, take a representative $x_j\in S_j^a$, and denote the isotropy by $G_{R_{x_j}}$, then $p^{-1}(G_R)\cap pr_X^{-1}(S_j^a)=G_R\times{}^{G_{R_{x_j}}}G_{R_{x_j}}$.

For $\Theta\in\Gamma(G_R, \operatorname{Hom}_{\mathcal{D}_{G_C}}(\mathcal{M}_\rho, \mathcal{D}'_{G_R}))$, we call the corresponding element $\operatorname{cc}(\Theta)$ $\in H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R})$ by the *character cycle*.

PROPOSITION 2.2 (Harish-Chandra, [8, **9**]).

(1) *There is no singular invariant eigendistribution. This corresponds to the fact that the restriction map*

$$H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R}) \longrightarrow H^{BM}_{\dim G_R}(p^{-1}(G_{R_{rs}}), or_{G_R})$$

is injective.

(2) *An invariant eigendistribution is real analytic on the set of regular semisimple elements* $G_{R_{rs}}$. *This corresponds to the fact that there is a canonical isomorphism*

$$H^0(p^{-1}(G_{R_{rs}}), C) \longrightarrow H^{BM}_{\dim G_R}(p^{-1}(G_{R_{rs}}), or_{G_R}).$$

For any $\zeta \in p^{-1}(G_{R_{rs}})$ and $\sigma \in H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R})$, $\sigma(\zeta)$ denotes the multiplicity of the component containing ζ in σ. In other words, $\sigma(\zeta)$ is the image of

$$\begin{array}{ccccc} H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R}) & \longrightarrow & H_0(p^{-1}(p(\zeta)), C) & \longrightarrow & H_0(\{\zeta\}, C) \\ \| & & \| & & \| \\ H^0(p^{-1}(G_{R_{rs}}), C) & \longrightarrow & H^0(p^{-1}(p(\zeta)), C) & \longrightarrow & H^0(\{\zeta\}, C). \end{array}$$

Remark that the element $\sigma \in H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_C})$ is determined by the values $\sigma(\zeta)$ for $\zeta \in p^{-1}(H_{R_{rs}})$, where $H_{R_{rs}}$ runs over all θ-stable Cartan subgroup of G_R.

For $g \in G_{R_{rs}}$,

$$\Theta(g) = \sum_{\zeta \in p^{-1}(g)} \mathrm{cc}(\Theta)(\zeta)\phi(\zeta).$$

For $h \in H_{R_{rs}}$, $w \in W$,

$$\phi(h, wx_0) = \prod_{\alpha \in w\Delta^-}(1-e^{-\alpha})^{-1}(h) = \frac{(-1)^{l(ww_0)}e^{ww_0\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}(h).$$

Then we have

$$\Theta(h) = \sum_{w\in W} \frac{(-1)^{l(w)}\mathrm{cc}(\Theta)(h, ww_0x_0)e^{w\rho}}{\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})}(h).$$

2.3. Trace. Let $D_{G_R}(X)$ (resp. $D_{K_C}(X)$) be the equivariant derived category on X introduced by Bernstein-Lunts [**16**], [**15**]. By the definition, for an $\mathcal{F}^a \in D_{G_R}(X)$ there is an element $\varphi \in \mathrm{Hom}(\mu^{-1}\mathcal{F}^a, \mathrm{pr}_X^{-1}\mathcal{F}^a)$ which gives a G_R-equivariance. Let $\delta: X \to X \times X$ be the diagonal embedding, $s=(\mu, \mathrm{pr}_X): G_R \times X \to X \times X$, and $\boxtimes$ denotes the exterior product. We have

$$\begin{aligned} &\mathrm{Hom}(\mathcal{F}^a, \mathcal{F}^a) \\ &= H^0(X, \delta^!(\mathcal{F}^a \boxtimes D\mathcal{F}^a)) \\ &\longrightarrow H^0(X, \delta^! Rs_* s^{-1}(\mathcal{F}^a \boxtimes D\mathcal{F}^a)) \\ &= H^0(R\Gamma_{p^{-1}(G_R)}(G_R \times X, \mu^{-1}\mathcal{F}^a \otimes \mathrm{pr}_X^{-1} D\mathcal{F}^a)) \\ &\xrightarrow{\varphi} H^0(R\Gamma_{p^{-1}(G_R)}(G_R \times X, \mathrm{pr}_X^{-1}\mathcal{F}^a \otimes \mathrm{pr}_X^{-1} D\mathcal{F}^a)) \\ &\longrightarrow H^0(R\Gamma_{p^{-1}(G_R)}(G_R \times X, \mathrm{pr}_X^{-1}\omega_X)) \\ &= H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R}). \end{aligned}$$

Here we use the following Cartesian product:

$$\begin{array}{ccc} p^{-1}(G_R) & \longrightarrow & G_R \times X \\ \downarrow & \square & \downarrow s \\ X & \xrightarrow{\delta} & X \times X . \end{array}$$

We denote the image of the identity $\mathrm{id}_{\mathcal{F}^a}$ by $\mathrm{ch}(\mathcal{F}^a) \in H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R})$ and call this the *character cycle* of $\mathcal{F}^a$ [**11**].

2.4. The Kashiwara conjecture. In [**11**], Kashiwara gave a conjecture that for a Harish-Chandra module V with trivial infinitesimal character there is a G_R-equivariant object $\mathcal{F}^a \in D_{G_R}(X)$ whose character cycle is the distribution character Θ_V of V in $H^{BM}_{\dim G_R}(p^{-1}(G_R), or_{G_R})$;

$$\mathrm{cc}(\Theta_V) = \mathrm{ch}(\mathcal{F}^a) . \tag{2.4}$$

He gave a proof for any discrete series V. He constructed $\mathcal{F}^a$ using the Beilinson-Bernstein correspondence, the Riemann-Hilbert correspondence, and another conjecture of his [**10**], so called the Matsuki correspondence for sheaves, proved by [**15**].

PROPOSITION 2.5.

(i) *The Beilinson-Bernstein correspondence* [**2**], [**4**].
The category of Harish-Chandra modules is equivalent to those of K_C-equivariant $\mathcal{D}_X$-modules on X. A Harish-Chandra module V corresponds to a K_C-equivariant $\mathcal{D}_X$-module $\mathcal{M}$ by

$$V = \Gamma(X, \mathcal{M}) \quad \text{and} \quad \mathcal{M} = \mathcal{D}_X \underset{\Gamma(X, \mathcal{D}_X)}{\otimes} V .$$

Such an $\mathcal{M}$ is always regular holonomic. Functors $\Gamma(X, \ )$ and $\mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)}$ are exact.

(ii) *The Riemann-Hilbert correspondence.*
The derived category of K_C-equivariant $\mathcal{D}_X$-modules is equivalent to $D_{K_C}(X)$. A K_C-equivariant $\mathcal{D}_X$-module $\mathcal{M}$ corresponds to an $\mathcal{F} \in D_{K_C}(X)$ by $\mathcal{F} = \mathrm{DR}(\mathcal{M}) = R\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$.

(iii) *The Matsuki correspondence for sheaves* [**10**], [**15**].
The categories $D_{K_C}(X)$ and $D_{G_R}(X)$ are equivalent. An $\mathcal{F} \in D_{K_C}(X)$ corresponds to an $\mathcal{F}^a \in D_{G_R}(X)$ as follows. Consider the following diagram.

$$X \xleftarrow{\mathrm{pr}_X} G_R \times X \xrightarrow{r} G_R \underset{K_R}{\times} X \xrightarrow{q} X .$$

Here $q(g, x) = gx$, and r is a natural projection. For any $\mathcal{F} \in D_{K_C}(X)$, there is an $\tilde{\mathcal{F}} \in D^b_{G_R}(G_R \times_{K_R} X)$ such that $p^{-1}\mathcal{F} = r^{-1}\tilde{\mathcal{F}}$. Define $\mathcal{F}^a = Rq_* \tilde{\mathcal{F}}$.

3. Formula for $\mathrm{cc}(\Theta)$.

In this section we give a formula for $\mathrm{cc}(\Theta)$ using the Euler-Poincaré characteristic of some geometric object.

3.1. Now we quote the Osborne conjecture for a maximal nilpotent subalgebra $\mathfrak{n}$.

LEMMA 3.1 (Osborne Conjecture, [6, Theorem 7.22]). *For a Harish-Chandra module V, the global character Θ of V satisfies the following formula on $H_{\bar{R}}\cap H_{R_{rs}}$.*

$$\Big(\prod_{\alpha\in\Delta^+}(1-e^{\alpha})\Big)\Theta=\sum_{i=0}^{\infty}(-1)^i\Theta_{H_R}(H_i(\mathfrak{n},V)).$$

Here $\Theta_{H_R}(H_i(\mathfrak{n},V))$ is the character of H_R on $H_i(\mathfrak{n},V)$. More explicitly, for a Harish-Chandra module V with a trivial infinitesimal character, we have

$$\Theta_{H_R}(H_i(\mathfrak{n},V))(h)=\sum_{\varepsilon\in(H_R/H_R^0)^\wedge,\,w\in W}\dim(H_i(\mathfrak{n},V)_{\rho+w\rho,\varepsilon})e^{\rho+w\rho}(h)\varepsilon(h).$$

3.2. Now we localize the $\mathfrak{n}$-homology. Let $\mathcal{D}_w$ be the sheaf of twisted differential operators acting on the invertible sheaf $\mathcal{O}_w=\{f\in\mathcal{O}_{G_C};\ f(ghn)=e^{w\rho-\rho}(h)f(g)$ for $h\in H_C,\ n\in N,\ g\in G_C\}$. When $w=e$, we have $\mathcal{O}_e=\mathcal{O}_X$ and $\mathcal{D}_e=\mathcal{D}_X$. Remark that $\Gamma(X,\mathcal{D}_w)=U(\mathfrak{g})/U(\mathfrak{g})(Z(\mathfrak{g})\cap U(\mathfrak{g})\mathfrak{g})$, then V is $\Gamma(X,\mathcal{D}_w)$-module.

LEMMA 3.2 ([19, Theorem 4.1]). *For a Harish-Chandra module V with a trivial infinitesimal character, we have*

$$\mathrm{Hom}_{\mathfrak{h}}(H_i(\mathfrak{n},V),\boldsymbol{C}_{\rho-w\rho})=\underline{\mathrm{Ext}}^i_{\mathcal{D}_w}(\mathcal{D}_w\overset{L}{\underset{\Gamma(X,\mathcal{D}_w)}{\otimes}}V,\mathcal{O}_w)_{x_0}$$

as $H_R/H_R{}^0$-modules.

3.3. For a fixed $w\in W$, let $X_w=BwB/B\subset X$ be a Schubert cell on X, $Y_w=\{(gB,gwB)\in X\times X;\ g\in G_C\}$ be a G_C-orbit on $X\times X$ through (eB,wB), and $X\overset{p_1}{\leftarrow}Y_w\overset{p_2}{\rightarrow}X$ projections $p_1(x,y)=x$, $p_2(x,y)=y$. The isotropy subgroup of G_C at $(eB,wB)\in Y_w$ is $(B\cap wBw^{-1})$. Let $\mathcal{L}=\{\tilde{f}\in\mathcal{O}_{G_C};\ \tilde{f}(ghn)=e^{w\rho-\rho}(h)\tilde{f}(g)$, for $hn\in B\cap wBw^{-1}\}$ be an invertible sheaf on $Y_w=G_C/(B\cap wBw^{-1})$. Denote $\mathcal{M}_w=\mathcal{D}_w\otimes^L_{\Gamma(X,\mathcal{D}_w)}V$.

LEMMA 3.3 ([3, Theorem 12]). *For a $w\in W$, we use the notations above. We have*

$$\mathcal{M}_w=p_{1*}(p_2^!\mathcal{M}_e\otimes\mathcal{L})[-l(w)].$$

Here p_{1*} or $p_2^!$ is a direct or inverse image in the derived category of (twisted) $\mathcal{D}$-modules.

3.4. Now we give a formula for $\mathrm{cc}(\Theta)$. For a bounded complex $C^{\cdot}$ of vector spaces, we denote the Euler-Poincaré characteristic of $C^{\cdot}$ by $\chi(C^{\cdot})=\sum_{i=0}^{\infty}(-1)^{i}\dim H^{i}(C^{\cdot})$.

THEOREM 3.4. *For a Harish-Chandra module V with a trivial infinitesimal character, let Θ be the global character of V, and set $\mathcal{M}=\mathcal{M}_{e}=\mathcal{D}_{X}\otimes_{\Gamma(X,\mathcal{D}_{X})}V$, $\mathcal{F}=\mathrm{DR}(\mathcal{M})=R\,\mathrm{Hom}_{\mathcal{D}_{X}}(\mathcal{O}_{X}, \mathcal{M})$. Then for a $h\in H_{\bar{R}}\cap H_{R_{rs}}$ and $w\in W$, we have*

$$\mathrm{cc}(\Theta)(h, wx_{0})=\sum_{\varepsilon\in(H_{R}/H_{R}^{0})^{\wedge}}\chi(R\Gamma_{X_{w}}(X, \mathcal{F})_{\varepsilon})\varepsilon(h).$$

PROOF. By §3.3, we have

$$\begin{aligned}
&\underline{\mathrm{Ext}}^{i}_{\mathcal{D}_{w}}(\mathcal{M}_{w}, \mathcal{O}_{w})_{x_{0}}\\
&=\underline{\mathrm{Ext}}^{i}_{\mathcal{D}_{X}}(\mathcal{O}_{w}^{\otimes -1}\underset{\mathcal{O}_{X}}{\otimes}\mathcal{M}_{w}, \mathcal{O}_{X})_{x_{0}}\\
&=H^{i}(R\,\mathrm{Hom}_{\mathcal{D}_{X}}(p_{1*}p_{2}^{!}\mathcal{M}_{e}[-l(w)], \mathcal{O}_{X})_{x_{0}}\\
&=H^{i}((D_{X}p_{1*}p_{2}^{!}\mathrm{DR}(\mathcal{M}_{e})[l(w)-\dim_{R}X])_{x_{0}})\\
&=H^{i}(R\Gamma_{X_{w}}(X, \mathcal{F})^{*}[l(w)-\dim_{R}X])\\
&=H^{\dim_{R}X-l(w)-i}(R\Gamma_{X_{w}}(X, \mathcal{F})).
\end{aligned}$$

$$\begin{tikzcd}
X_{w} \arrow[r, "i'"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{w} \arrow[d, "p_{1}"] \arrow[dr, "p_{2}"] & \\
\{x_{0}\} \arrow[r, "i_{x_{0}}"'] & X & X.
\end{tikzcd}$$

We have already known the following.

(2.3), (3.1); $\mathrm{cc}(\Theta)(h, ww_{0}x_{0})(-1)^{l(w)}=(-1)^{|\Delta^{+}|}\sum_{i,\varepsilon}(-1)^{i}\dim(H_{i}(\mathfrak{n}, V)_{\rho+w\rho,\varepsilon})\varepsilon(h)$.

(3.2); $\dim(H_{i}(\mathfrak{n}, V)_{\rho-w\rho,\varepsilon})=\dim\underline{\mathrm{Ext}}^{i}_{\mathcal{D}_{w}}(\mathcal{M}_{w}, \mathcal{O}_{w})_{x_{0},\varepsilon}$.

(3.3); $\dim\underline{\mathrm{Ext}}^{i}_{\mathcal{D}_{w}}(\mathcal{M}_{w}, \mathcal{O}_{w})_{x_{0},\varepsilon}=\dim H^{\dim_{R}X-l(w)-i}(R\Gamma_{X_{w}}(X, \mathcal{F}))_{\varepsilon}$.

Summing up the above with $-\rho=w_{0}\rho$, we have the result. ■

REMARK 3.5. We use here the de Rham functor $\mathrm{DR}=R\,\mathrm{Hom}_{\mathcal{D}_{X}}(\mathcal{O}_{X}, \cdot)$ instead of the solution functor $R\,\mathrm{Hom}_{\mathcal{D}_{X}}(\cdot, \mathcal{O}_{X})=(\mathrm{DR}(\cdot)^{*})$. Here $(\cdot)^{*}=R\,\mathrm{Hom}_{C_{X}}(\cdot, C_{X})=D_{X}(\cdot)[-2\dim_{C}X]$. Even if we use the solution functor, we have the same results. In fact, for a K_{C}-orbit $S\subset X$, an embedding $j: S\hookrightarrow X$ and an irreducible K_{C}-equivariant local system E on S, we define $j_{!*}E$ the direct image of the DGM-extension of E; $j_{!+}E=\bar{j}_*({}^{\pi}E)$, $\bar{j}: \bar{S}\to X$. Remark that the elements of the form $j_{!+}E$ generate the Grothendieck group of $D_{K_C}(X)$. Since we have $(j_{!+}E)^*=D_X(j_{!+}E)[-2\dim_C X]=j_{!+}D_S E[-2\dim_C X]=j_{!+}E[-2\operatorname{codim}_C(S\subset X)]$, we conclude that $\mathcal{F}$ and $\mathcal{F}^*$ are equal in the Grothendieck group of $D_{K_C}(X)$ for any $\mathcal{F}\in D_{K_C}(X)$. Then we have $\chi(R\Gamma_{X_w}(X, \mathcal{F}^*)_\varepsilon)=\chi(R\Gamma_{X_w}(X, \mathcal{F})_\varepsilon)$, we have $\mathrm{Hom}_{C_S}(E, \boldsymbol{C}_S)\cong E$, then $\mathcal{F}=j_{!+}E$ is self dual, $\mathcal{F}=\mathcal{F}^*$.

4. Formula for $\mathrm{ch}(\mathcal{F}^a)$.

In this section, for an $\mathcal{F}^a\in D_{G_R}(X)$, we give a formula for $\mathrm{ch}(\mathcal{F}^a)$.

4.1. We quote a lemma from [**11**, Proposition 2.8.1]. For a complex $C^\cdot$ of vector spaces, and $\varphi\in\mathrm{Hom}(C^\cdot, C^\cdot)$, we set $\mathrm{tr}(\varphi; C^\cdot)=\sum_{i=0}^{\infty}(-1)^i \operatorname{trace}(\varphi; H^i(C^\cdot))$.

LEMMA 4.1. *For $\mathcal{F}^a\in D_{G_R}(X)$, $h\in H_R^-$ and $w\in W$, we have*

$$(\mathrm{ch}(\mathcal{F}^a))(h, wx_0)=\mathrm{tr}(h; R\Gamma_{X_w}(X, \mathcal{F}^a)).$$

We give a proof of Lemma 4.1 in Appendix, which uses the shrinking space as the proof of Proposition 2.8.1 in [**11**]. To be more precise, since $X_w=(N\cap w\bar{N}w^{-1})wB/B\subset X$, we have

$$T_{wx_0}X=w\bar{\mathfrak{n}}w^{-1}=\sum_{\alpha\in w\Delta^-}\mathfrak{g}(\mathfrak{h}, \alpha),$$
$$T_{wx_0}X_w=w\bar{\mathfrak{n}}w^{-1}\cap\mathfrak{n}=\sum_{\alpha\in w\Delta^-\cap\Delta^+}\mathfrak{g}(\mathfrak{h}, \alpha).$$

Then for $h\in H_R^-$,

$$e^\alpha(h)\in\boldsymbol{C}-\{x\in\boldsymbol{R};\ x\geqq 1\}\qquad\text{for}\quad \alpha\in w\Delta^-\cap\Delta^+,$$
$$e^\alpha(h)\in\boldsymbol{C}-\{x\in\boldsymbol{R};\ 0\leqq x\leqq 1\}\qquad\text{for}\quad \alpha\in w\Delta^-\cap\Delta^-.$$

Hence X_w seems to play the role of the shrinking space V_s in [**13**, Proposition 9.6.14].

4.2.

LEMMA 4.2. *For an $\mathcal{F}^a\in D_{G_R}(X)$, the function $\mathrm{tr}(h; R\Gamma_{X_w}(X, \mathcal{F}^a))$ is a locally constant function in $h\in H_R$. Especially, we have*

$$\mathrm{tr}(h; R\Gamma_{X_w}(X, \mathcal{F}^a))=\sum_{\varepsilon}\chi(R\Gamma_{X_w}(X, \mathcal{F}^a)_\varepsilon)\varepsilon(h).$$

PROOF. For an $\mathcal{F}^a\in D_{H_R}(X)$, we have $i_w^!\mathcal{F}^a\in D_{H_R}(X_w)$, $R\Gamma_{X_w}(X, \mathcal{F}^a)=Ra_*i_w^!\mathcal{F}^a\in D_{H_R}(\{\mathrm{pt}\})$, that is an H_R-equivariant constructible sheaf on a point. Here $X\xleftarrow{i_w}X_w\xrightarrow{a}\{\mathrm{pt}\}$. Hence the only component group H_R/H_R^o acts on $R\Gamma_{X_w}(X, \mathcal{F}^a)$. ∎

Hence we have the following corollary by the two lemmas above.

COROLLARY 4.3. *For an* $\mathcal{F}^a \in D_{G_R}(X)$, $ah \in H_R^-$ *and* $w \in W$, *we have*

$$\mathrm{ch}(\mathcal{F}^a)(h, wx_0) = \sum_{\varepsilon \in (H_R/H_R^0)^\wedge} \chi(R\Gamma_{X_w}(X, \mathcal{F}^a)_\varepsilon)\varepsilon(h).$$

5. Proof of Theorem 2.

The purpose of this section is to prove the following theorem.

THEOREM 5.1. *If* $\mathcal{F} \in D_{K_C}(X)$ *and* $\mathcal{F}^a \in D_{G_R}(X)$ *correspond to each other by the Matsuki correspondence for sheaves, then we have*

$$\chi(R\Gamma_{X_w}(X, \mathcal{F})_\varepsilon) = \chi(R\Gamma_{X_w}(X, \mathcal{F}^a)_\varepsilon).$$

Remark that this theorem implies Theorem 1.2 by the help of Theorem 3.4 and Corollary 4.3.

5.1. In this subsection, we prove Theorem 5.1 in the case $w=e$, that is, $X_e = \{x_0\}$. Since both sides of the formula are additive, so we may prove only for an element $\mathcal{F} \in D_{K_C}(X)$ of a basis of Grothendieck group of $D_{K_C}(X)$. Hence we can restrict the situation as follows.

Let S or S^a be a K_C- or G_R- orbit on X, and assume that S and S^a correspond by the Matsuki correspondence. We write the embedding $j: S \to X$, $j^a: S^a \to X$. Let E or E^a be a K_C- or G_R-equivariant local system on S or S^a, and assume that $E|_{S \cap S^a} = E^a|_{S \cap S^a}$. We set $\mathcal{F} = Rj_! E[\mathrm{codim}_C S] \in D_{K_C}(X)$, then the corresponding $\mathcal{F}^a \in D_{G_R}(X)$ is given by $\mathcal{F}^a = Rj^a_*(E^a \otimes j^{a!} \boldsymbol{C}_X)[\mathrm{codim}_C S]$ (see [**15**]).

The left hand side of the formula:

$$\begin{aligned} \chi(R\Gamma_{X_e}(X, \mathcal{F})_\varepsilon) &= \chi((i_{x_0}^! \mathcal{F})_\varepsilon) \\ &= \chi((i_{x_0}^{-1} \mathcal{F})_\varepsilon) \qquad \text{(Remark 3.5)} \\ &= \chi((i_{x_0}^{-1} Rj_! E[\mathrm{codim}_C S])_\varepsilon) \\ &= \chi((Rj'_! i'^{-1}_{x_0} E[\mathrm{codim}_C S])_\varepsilon). \end{aligned}$$

Here

$$\begin{tikzcd} \{x_0\} \cap S \arrow[r, "i'_{x_0}"] \arrow[d, "j'"'] \arrow[dr, phantom, "\square"] & S \arrow[d, "j"] \\ \{x_0\} \arrow[r, "i_{x_0}"] & X. \end{tikzcd}$$

If $x_0 \notin S$, then $\chi(R\Gamma_{X_e}(X, \mathcal{F})_\varepsilon) = 0$. If $x_0 \in S$, then

$$\begin{aligned} \chi(R\Gamma_{X_e}(X, \mathcal{F})_\varepsilon) &= \chi((i'^{-1}_{x_0} E[\mathrm{codim}_C S])_\varepsilon) \\ &= \chi((E|_{x_0})_\varepsilon[\mathrm{codim}_C S]). \end{aligned}$$

The right hand side of the formula:

$$\begin{aligned}\chi(R\Gamma_{X_e}(X, \mathcal{F}^a)_\varepsilon) &= \chi((i^!_{x_0}Rj^a_*(E^a\otimes j^{a!}\boldsymbol{C}_X)[\mathrm{codim}_{\boldsymbol{C}}S])_\varepsilon)\\ &= \chi((Rj^{a\prime}_* i''^!_{x_0}(E^a\otimes j^{a!}\boldsymbol{C}_X)[\mathrm{codim}_{\boldsymbol{C}}S])_\varepsilon).\end{aligned}$$

Here

$$\begin{tikzcd}\{x_0\}\cap S^a \arrow[r, "i''_{x_0}"] \arrow[d, "j^{a\prime}"'] \arrow[dr, phantom, "\square"] & S^a \arrow[d, "j^a"] \\ \{x_0\} \arrow[r, "i_{x_0}"] & X.\end{tikzcd}$$

As before, if $x_0 \notin S^a$, then $\chi(R\Gamma_{X_e}(X, \mathcal{F}^a)_\varepsilon)=0$. If $x_0 \in S^a$, then

$$\begin{aligned}\chi(R\Gamma_{X_e}(X, \mathcal{F}^a)_\varepsilon) &= \chi((i''^!_{x_0}(E^a\otimes j^{a!}\boldsymbol{C}_X)[\mathrm{codim}_{\boldsymbol{C}}S])_\varepsilon)\\ &= \chi((i''^{-1}_{x_0}E^a\otimes i''^!_{x_0}j^{a!}\boldsymbol{C}_X)[\mathrm{codim}_{\boldsymbol{C}}S])_\varepsilon)\\ &= \chi((E^a\,|_{x_0}[\mathrm{codim}_{\boldsymbol{C}}S-\dim_{\boldsymbol{R}}X])_\varepsilon)\\ &= \chi((E^a\,|_{x_0}[\mathrm{codim}_{\boldsymbol{C}}S])_\varepsilon).\end{aligned}$$

Since $x_0\in S$ and $x_0 \in S^a$ are equivalent, we have $\chi(R\Gamma_{X_e}(X, \mathcal{F})_\varepsilon)=\chi(R\Gamma_{X_e}(X, \mathcal{F}^a)_\varepsilon)$.

5.2. General case.

LEMMA 5.2. *For a fixed $w\in W$, we recall the notation in § 3.3. Let Y_w be a G_C-orbit on $X\times X$ through w, denote projections $X \overset{p_1}{\leftarrow} Y_w \overset{p_2}{\rightarrow} X$, given by $p_1(x, y)=x$, $p_2(x, y)=y$.*

If $\mathcal{F}\in D_{K_C}(X)$ and $\mathcal{F}^a \in D_{G_R}(X)$ correspond to each other by the Matsuki correspondence for sheaves, then $Rp_{1*}p_2^!\mathcal{F}[-l(w)]\in D_{K_C}(X)$ and $Rp_{1*}p_2^!\mathcal{F}^a[-l(w)] \in D_{G_R}(X)$ correspond to each other by the Matsuki correspondence for sheaves.

PROOF. Consider the following diagram.

$$\begin{tikzcd} X & G_R\times X \arrow[l, "p"'] \arrow[r, "r"] & G_R \underset{K_R}{\times} X \arrow[r, "q"] & X \\ Y_w \arrow[u, "p_2"] \arrow[d, "p_1"'] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & G_R\times Y_w \arrow[l, "p"'] \arrow[r, "r"] \arrow[u, "p_2'"] \arrow[d, "p_1'"'] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & G_R \underset{K_R}{\times} Y_w \arrow[r, "q"] \arrow[u, "p_2''"] \arrow[d, "p_1''"'] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & Y_w \arrow[u, "p_2"] \arrow[d, "p_1"] \\ X & G_R\times X \arrow[l, "p"'] \arrow[r, "r"] & G_R \underset{K_R}{\times} X \arrow[r, "q"] & X \end{tikzcd}$$

From the definition of the Matsuki correspondence for sheaves, there is an $\tilde{\mathcal{F}}\in D^b_{G_R}(G_R \underset{K_R}{\times} X)$ such that $p^{-1}\mathcal{F}=r^{-1}\tilde{\mathcal{F}}$, $\mathcal{F}^a=Rq_*\tilde{\mathcal{F}}$. We have

$$\begin{aligned} p^{-1}Rp_{1*}p_2^!\mathcal{F} &= p^!Rp_{1*}p_2^!\mathcal{F}\otimes or_{G_R}[-\dim_R G_R] \\ &= Rp'_{1*}p_2'^!p^!\mathcal{F}\otimes or_{G_R}[-\dim_R G_R] \\ &= Rp'_{1*}p_2'^!p^{-1}\mathcal{F} \\ &= Rp'_{1*}p_2'^!r^{-1}\tilde{\mathcal{F}} \\ &= r^{-1}Rp''_{1*}p_2''^!\tilde{\mathcal{F}}\,. \end{aligned}$$

Hence $Rp_{1*}p_2^!\mathcal{F}$ corresponds to $Rq_*Rp''_{1*}p_2''^!\tilde{\mathcal{F}}$, which is nothing but $Rp_{1*}p_2^!Rq_*\tilde{\mathcal{F}}$ $=Rp_{1*}p_2^!\mathcal{F}^a$. ∎

We have the action of the Weyl group W on the Grothendieck group $K(D_{K_C}(X))$ (or $K(D_{G_R}(X))$) of the equivariant derived category. Lemma 5.2 shows that the Matsuki correspondence for sheaves is W-equivariant between $K(D_{K_C}(X))$ and $K(D_{G_R}(X))$ ([**15**, Remark 6.7]).

Now we finish the proof of Theorem 5.1 for a general $w\in W$. For an $w\in W$, we use the notation in the proof of Theorem 3.4. We have

$$\begin{aligned} R\Gamma_{X_w}(X, \mathcal{F}) &= Ra_*i^!_{X_w}\mathcal{F} \\ &= Ra_*i'^!p_2^!\mathcal{F} \\ &= i^!_{x_0}Rp_{1*}p_2^!\mathcal{F} \\ &= R\Gamma_{X_e}(X, Rp_{1*}p_2^!\mathcal{F})\,. \end{aligned}$$

$$\begin{tikzcd} \{x_0\} \arrow[d, "i_{x_0}"'] & X_w \arrow[l, "a"'] \arrow[d, "i'"] \arrow[dr, "i_{X_w}"] & \\ X \arrow[ur, phantom, "\square"] & Y_w \arrow[l, "p_1"] \arrow[r, "p_2"'] & X. \end{tikzcd}$$

The same holds for $\mathcal{F}^a$. Then Lemma 5.2 and the proof in §5.1 imply Theorem 5.1.

Appendix. A proof of Lemma 4.1.

Since the object $\mathcal{F}$ is not conic, we cannot apply Proposition 9.6.14 in [**13**] directly, but almost all the arguments are repetitions of those in [**13**].

(a) Let $V=T_{wx_0}X$ and $V'_s=T_{wx_0}X_w$. We have an isomorphism ι

$$\begin{array}{ccccc} \iota\colon V = w\bar{\mathfrak{n}}w^{-1} \ni A & \longmapsto & (\exp A)wx_0 \in w\bar{N}B/B & \subset & X \\ \cup & & & & \cup \\ V'_s = w\bar{\mathfrak{n}}w^{-1}\cap\mathfrak{n} & & \longrightarrow & & X_w\,, \end{array}$$

and denote u the linear endomorphism on V corresponding to $\mathfrak{g}$ by ι. Then u is a semisimple operator without zero eigenvalue. We set the pull back $\mathcal{F}=\iota^{-1}R\Gamma_{w\bar{N}x_0}(\mathcal{F}^a)$, then $\mathcal{F}\in D^b_{R\text{-}c}(V)$, the number of strata is finite and each stratum is an algebraic set. We denote the corresponding action of h on $\mathcal{F}$ by $\varphi\in \mathrm{Hom}(u^{-1}\mathcal{F}, \mathcal{F})$. By [**13**, Remark 9.6.7],

$$(\mathrm{ch}(\mathcal{F}^a))(h, wx_0)=(\mathrm{ch}(R\Gamma_{w\bar{N}x_0}(\mathcal{F}^a)))(h, wx_0)=\mathrm{ch}(\mathcal{F})(u, 0)=C_0(\varphi),$$

where we use the notations in [**13**, Chapter 9].

Let V_C denote the complexified space $V\otimes_R C$, and for $\lambda\in C$, denote by V_C^λ the eigenspace of V_C with eigenvalue λ. Consider the decomposition $V=V_+\oplus V_-$ with:

$$V_+=\Big(\bigoplus_{|\lambda|\geqq 1}V_C^\lambda\Big)\cap V,\qquad V_-=\Big(\bigoplus_{|\lambda|<1}V_C^\lambda\Big)\cap V$$

and choose a metric on V such that $|u(x)|\geqq|x|$ on V_+, $|u(x)|\leqq|x|$ on V_-. Let $q\colon V\to V_+$ be the projection and $i_{V_-}\colon V_-\to V$ the inclusion. Set $Z_{ab}=\{x\in V_+;\ |x|\leqq a\}\times\{x\in V_-;\ |x|<b\}$, then $u^{-1}(Z_{ab})\cap Z_{ab}$ is closed in Z_{ab} and open in $u^{-1}(Z_{ab})$. Applying [**13**, Proposition 9.6.9], we find

$$C_0(\varphi)=\mathrm{tr}(\Gamma_{Z_{ab}}(\varphi), R\Gamma_{Z_{ab}}(V, \mathcal{F})).$$

Now we want to take limits $a\to 0$ and $b\to\infty$. We apply [**13**, Lemma 8.4.7] for $R\Gamma_{\{|x_-|<b\}}\mathcal{F}\in D^b_{R\text{-}c}(V)$ and q, then for $\forall b>0$, $\exists a_0>0$, $0<\forall a<a_0$,

$$R\Gamma_{q^{-1}(\{|x_+|\leqq a\})}(V, R\Gamma_{\{|x_-|<b\}}\mathcal{F})=R\Gamma_{q^{-1}(\{|x_+|=0\})}(V, R\Gamma_{\{|x_-|<b\}}\mathcal{F}).$$

Hence we have

$$\begin{aligned} R\Gamma_{Z_{ab}}(V, \mathcal{F}) &= R\Gamma_{V_-}(V, R\Gamma_{\{|x_-|<b\}}\mathcal{F})\\ &= R\Gamma(\{x\in V_-;\ |x|<b\}, i^!_{V_-}\mathcal{F}).\end{aligned}$$

Set $\hat{V}_-=\{(x, t)\in V_-\times \boldsymbol{R};\ |x|^2+t^2=1\}$ and embed V_- into $\hat{V}_-$ by $j\colon x\mapsto (x/\sqrt{1+|x|^2}, 1/\sqrt{1+|x|^2})$. Then V_- is an open subset of $\hat{V}_-$ defined by $t>0$.

CLAIM. *$Rj_*i^!_{V_-}\mathcal{F}$ is $\boldsymbol{R}$-constructible.*

PROOF OF CLAIM. There exists a finite covering $V_-=\bigcup_{i\in I}X_i$ by algebraic sets such that for all $j\in\boldsymbol{Z}$, all $i\in I$, the sheaves $H^j(i_{V_-}{}^{-1}D_V\mathcal{F})|_{X_i}$ are locally constant. Remark that $j(X_i)$ is also an algebraic set in $\hat{V}_-$. If we set $X_{-1}=\hat{V}_--j(V_-)$, $\hat{V}_-=\bigcup_{i\in I}j(X_i)\cup X_{-1}$ is a finite covering by algebraic sets such that for all $j\in\boldsymbol{Z}$, all $i\in I\cap\{-1\}$, the sheaves $H^j(Rj_!i_{V_-}{}^{-1}D_V\mathcal{F})|_{X_i}$ are locally constant. That is $Rj_!i_{V_-}{}^{-1}D_V\mathcal{F}\in D^b_{R\text{-}c}(\hat{V}_-)$. Since $i_{V_-}{}^{-1}D_V\mathcal{F}=D_{V_-}i^!_{V_-}\mathcal{F}$ by [**13**, Exercise VIII. 3(ii)], we can apply [**13**, Exercise VIII. 3(iii)] for $G=i^!_{V_-}\mathcal{F}\in D^b_{R\text{-}c}(V_-)$, we have $Rj_*i^!_{V_-}\mathcal{F}\in D^b_{R\text{-}c}(\hat{V}_-)$. ∎

We can apply [**13**, Proposition 8.4.3(i)], for $\exists b_0>0$, $\forall b>b_0$,

$$
\begin{aligned}
&R\Gamma(\{|x_-|<b\},\ i^!_{V_-}\mathcal{F})\\
&=R\Gamma\Big(\Big\{t>\frac{1}{\sqrt{1+b^2}}\Big\},\ Rj_*i^!_{V_-}\mathcal{F}\Big)\\
&=R\Gamma(\{t>0\},\ Rj_*i^!_{V_-}\mathcal{F})\\
&=R\Gamma_{V_-}(V,\ \mathcal{F}).
\end{aligned}
$$

Then we have $(\mathrm{ch}(\mathcal{F}^a))(h,\ wx_0)=\mathrm{tr}(\Gamma_{V_-}(\varphi),\ R\Gamma_{V_-}(V,\ \mathcal{F}))$.

(b) Let $W=\bigoplus_{0<\lambda<1}V^{\lambda}_{C}\cap V$ the minimum shrinking space, and V_s be a shrinking space. By the previous step, it is enough to prove

$$\mathrm{tr}(R\Gamma_W(\varphi),\ R\Gamma_W(V,\ \mathcal{F}))=\mathrm{tr}(R\Gamma_{V_s}(\varphi),\ R\Gamma_{V_s}(V,\ \mathcal{F}))$$

since it will in particular imply $\mathrm{tr}(R\Gamma_W(\varphi))=\mathrm{tr}(R\Gamma_{V_-}(\varphi))$. By distinguished triangle, it is enough to prove

$$\mathrm{tr}(R\Gamma_{V_s\text{-}W}(\varphi),\ R\Gamma_{V_s\text{-}W}(V,\ \mathcal{F}))=0.$$

Take a u-stable subspace W' of V_s such that $V_s=W\oplus W'$. Remark that no eigenvalue of u on W' is contained in $[0,\ \infty)$.

Next we define compact manifolds $\widetilde{W}$, $\widetilde{W}'$ and maps constructing the following diagram.

$$
\begin{tikzcd}
V & V_s-W \arrow[l, "i"'] \arrow[r, hook] \arrow[d] & \widetilde{W}\times\widetilde{W}' \arrow[d] \\
 & W'-\{0\} \arrow[r, hook] \arrow[d] & \widetilde{W}' \arrow[dl] \\
 & S_{W'}. &
\end{tikzcd}
$$

Set $\widetilde{W}=\{(x,\ t)\in W\times\boldsymbol{R};\ |x|^2+t^2=1\}$ and embed W into $\widetilde{W}$ by $x\mapsto(x/\sqrt{1+|x|^2},\ 1/\sqrt{1+|x|^2})$. Set $\widetilde{W}'=\{(x,\ y,\ s,\ t)\in W\times W\times\boldsymbol{R}\times\boldsymbol{R};\ |x|^2=1-|y|^2=s^2=1-t^2,\ tx=sy,\ x\cdot y=st\}$ and embed $W'-\{0\}$ into $\widetilde{W}'$ by $x\mapsto(x/\sqrt{1+|x|^2},\ x/(|x|\sqrt{1+|x|^2}),\ |x|/\sqrt{1+|x|^2},\ 1/\sqrt{1+|x|^2})$. Set $S_{W'}=\{x\in W';\ |x|=1\}$ and define a map $\widetilde{W}'$ onto $S_{W'}$ by $(x,\ y,\ s,\ t)\mapsto y/t$ if $t\neq0$, x/s if $s\neq0$. Then for

$$
\begin{tikzcd}
V & V_s-W \arrow[l, "i"'] \arrow[r, "j"] \arrow[d, "p"'] & \widetilde{W}\times\widetilde{W}' \arrow[dl, "p'"] \\
 & S_{W'}, &
\end{tikzcd}
$$

we have $Rp'_*Rj_*i^!\mathcal{F}\in D^b_{\mathbf{R}-c}(S_{W'})$ as the step (a). Hence we have

$$\begin{aligned}&\mathrm{tr}(R\Gamma_{V_s-W}(\varphi), R\Gamma_{V_s-W}(V, \mathcal{F}))\\&= \mathrm{tr}(p_*\Gamma_{V_s-W}(\varphi), R\Gamma(S_{W'}, Rp'_*Rj_*i^!\mathcal{F}))=0\end{aligned}$$

by [**13**, Proposition 9.6.2] since u has no fixed point on $S_{W'}$. ■

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Hiroyuki OCHIAI
Department of Mathematics
Rikkyo University
Nishi-ikebukuro Tokyo 171
Japan