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On cohomology groups attached to towers of algebraic curves

To Professor Shoshichi Kobayashi on his 60th birthday

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Introduction.

Let $\boldsymbol{Q}^*$ be a subfield of $\bar{\boldsymbol{Q}}$, and suppose that we are given an open immersion $j^*: Y^* \hookrightarrow X^*$ of smooth and geometrically irreducible algebraic curves over $\boldsymbol{Q}^*$, where Y^* (resp. X^*) is affine (resp. proper) over $\boldsymbol{Q}^*$. Let $j: Y \hookrightarrow X$ be the base change of j^* to $\bar{\boldsymbol{Q}}$. Then for any abelian (or $\boldsymbol{Z}_l$-) sheaf F on the étale site of Y^*, we have three kinds of étale cohomology groups: $H^1(Y, F)$, $H_c^1(Y, F) = H^1(X, j_! F)$ and $H_P^1(Y, F) := H^1(X, j_* F) \cong \mathrm{Im}(H_c^1(Y, F) \to H^1(Y, F))$. Such cohomology groups, being equipped with the action of the Galois group $G_{\boldsymbol{Q}^*} := \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}^*)$, often come up as interesting objects when Y^*, X^* and F are suitably chosen. For instance, they naturally appear in the study of the elliptic modular forms, when Y^* and X^* are the canonical models of the modular curves (cf. [**D**]).

The purpose of this paper is to study the cohomology groups of the same type, not for a single pair $Y^* \hookrightarrow X^*$, but for a tower of algebraic curves. Namely, let $Y^* \hookrightarrow X^*$ be as above, and consider a tower $\{Y_n^*\}_{n \in N}$ of geometrically irreducible algebraic curves over $\boldsymbol{Q}^*$, all of which are étale coverings of Y^*. In the text, this tower will be subject to some simple "axioms", which include that all $Y_n := Y_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}}$ are Galois coverings of Y, and that $\mathfrak{G} := \varprojlim_{n \in N} \mathrm{Gal}(Y_n/Y)$ is an "almost pro-l group" with a prime number l. (See §1 for details, where two basic examples of such towers can be also found.) Let X_n^* be the normalization of X^* in Y_n^*, and put $X_n := X_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}}$. The group $\mathfrak{G}$ naturally acts on the various cohomology groups $H^1(Y_n, \boldsymbol{Z}_l)$, $H_c^1(Y_n, \boldsymbol{Z}_l)$ and $H_P^1(Y_n, \boldsymbol{Z}_l) \cong H^1(X_n, \boldsymbol{Z}_l)$; and hence we may consider them as modules over the completed group algebra $\mathcal{A} := \boldsymbol{Z}_l[[\mathfrak{G}]]$, as well as $G_{\boldsymbol{Q}^*}$-modules. Now we would like to put such cohomology groups together to get *single* cohomology theories corresponding to "H^1", "H_c^1" and "H_P^1", respectively attached to the given tower.

Natural candidates for such cohomology theories are simply the projective limits $\varprojlim_{n \in N} H^1(Y_n, \boldsymbol{Z}_l)$, $\varprojlim_{n \in N} H_c^1(Y_n, \boldsymbol{Z}_l)$ and $\varprojlim_{n \in N} H^1(X_n, \boldsymbol{Z}_l)$ relative to the trace mappings; and our aim is to study their structure. Let Z be the maximum connected pro-l étale Galois covering of the scheme $\varprojlim_{n \in N} Y_n$, and let $\mathfrak{F}$

be the Galois group of Z over Y. We can then consider Z as the pro-l universal covering of our tower. On the other hand, there is the canonical homomorphism: $\mathfrak{F}\twoheadrightarrow\mathfrak{G}$; and we hereafter consider $\mathcal{A}$ as an $\mathfrak{F}$-module through it. In § 2, we make succesive use of the Hochschild-Serre spectral sequence relative to Z and the "Shapiro isomorphism" for each member of the tower, and give the following isomorphism of (right) $\mathcal{A}$-modules:

$$\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l) \cong H^1(\mathfrak{F}, \mathcal{A}). \tag{1}$$

Here in the right hand side, H^1 means the continuous cochain cohomology group; and $H^1(\mathfrak{F}, \mathcal{A})$ is regarded as au $\mathcal{A}$-module via the obvious right $\mathcal{A}$-module structure of $\mathcal{A}$. In § 3, we look at the Leray spectral sequences for $Y_n \hookrightarrow X_n$ to describle the $\mathcal{A}$-submodule $\varprojlim_{n\in N}H^1(X_n, \boldsymbol{Z}_l)$ of $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ in terms of continuous cochain "parabolic" cohomology group, which is defined in a similar manner as in the classical case:

$$\varprojlim_{n\in N}H^1(X_n, \boldsymbol{Z}_l) \cong H^1_P(\mathfrak{F}, \mathcal{A}). \tag{2}$$

In the text, we also describe the action of G_{Q^*} on $H^1(\mathfrak{F}, \mathcal{A})$ and $H^1_P(\mathfrak{F}, \mathcal{A})$ so that (1) and (2) are G_{Q^*}-equivariant.

On the other hand, the isomorphism (1), when combined with a result of Ihara (4.1.4), which is based on his free differential calculus on the "free almost pro-l group" $\mathfrak{F}$, enables us to give the following finite presentation of $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ as an $\mathcal{A}$-module (§ 4):

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{\oplus r} \longrightarrow \varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l) \longrightarrow 0 \tag{3}$$

($r=\operatorname{rank}\mathfrak{F}$), explicitly in terms of the (topological) free generators of $\mathfrak{F}$ via (1). This then allows us to describe $\varprojlim_{n\in N}H^1(X_n, \boldsymbol{Z}_l)\cong H^1_P(\mathfrak{F}, \mathcal{A})\subset H^1(\mathfrak{F}, \mathcal{A})$ as an $\mathcal{A}$-module also.

The results above should be contrasted with Ihara's theory of Galois representations [**I1**], which we now briefly recall: Set $\mathfrak{T}:=\varprojlim_{n\in N}T_l(\mathrm{Jac}(X_n^*))$, and consider it as an $\mathcal{A}$-module in a natural manner. Then in [**I1**], when $Y^*=\boldsymbol{P}^1_{Q^*}-\{0, 1, \infty\}$, Ihara proved that there is an exact sequence:

$$0 \longrightarrow \mathfrak{T} \longrightarrow \mathcal{A}^{\oplus 2} \longrightarrow \mathcal{A}$$

of $\mathcal{A}$-modules, which can be explicitly written down using free differential calculus on $\mathfrak{F}$. Moreover, he defined certain anti 1-cocycles on G_{Q^*} with values in $\mathcal{A}^\times$ by means of which he described the action of G_{Q^*} on $\mathfrak{T}\hookrightarrow\mathcal{A}^{\oplus 2}$ ([**I1**] § 1 (C)). We note that, in an unpublished part of the first draft of [**I1**], Ihara has already shown how these results can be generalized to the case of general base curves Y^* rather than $\boldsymbol{P}^1_{Q^*}-\{0, 1, \infty\}$; and also discussed the related cohomology groups. Anyway, one of his starting point of view was to interpret

$\mathfrak{T}$ as the abelianization of $\mathrm{Gal}(Z/\varprojlim_{n\in N} Y_n)$; and in this sense we may consider his theory as the "one-dimensional l-adic homology theory" of the tower. Our motivation for the present work has been to seek the corresponding l-adic cohomology theory. In this direction, when $Y^*=\boldsymbol{P}^1_{\boldsymbol{Q}^*}-\{0, 1, \infty\}$, we describe the action of $G_{\boldsymbol{Q}^*}$ on $\varprojlim_{n\in N} H^1(Y_n, \boldsymbol{Z}_l)$ in terms of Ihara's anti 1-cocycles; and show that the kernel of this Galois representation coincides with that of Ihara's ψ. These results owe to the above mentioned unpublished results of Ihara in an essential way.

As for H^1_c, we show that there is the following canonical isomorphism of $\mathcal{A}$-modules (§ 5):

$$\varprojlim_{n\in N} H^1_c(Y_n, \boldsymbol{Z}_l) \cong \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}) \tag{4}$$

where $\mathcal{D}_0$ is certain "completed cuspidal divisor group of degree 0 over $\boldsymbol{Z}_l$" of Z; and $\mathrm{Hom}_{c,\mathfrak{F}}$ means the group of continuous homomorphisms of $\mathfrak{F}$-modules. We use (étale version of) Grothendieck's G-sheaf theory to establish (4); and also describe explicitly the canonical mapping: $\varprojlim_{n\in N} H^1_c(Y_n, \boldsymbol{Z}_l)\to\varprojlim_{n\in N} H^1(Y_n, \boldsymbol{Z}_l)$ through (1) and (4). Such an interpretation of H^1_c was motivated by the work of Ash and Stevens [**AS2**] on modular symbols; and it is the starting point of our p-adic theory of modular symbols to be developed in a subsequent paper; cf. below.

So far, we explained our results on cohomology groups for the "total tower" $\{Y^*_n\}_{n\in N}$ for simplicity. But in the text, we also prove similar results for its certain "subtower". For example, in the case of the elliptic modular tower (§ 1), the corresponding results can be formulated as follows: Fix an integer $N\geqq 4$ and a prime number l; and let $\mathfrak{G}$ (resp. $\mathfrak{U}$) be the closure of the usual congruence subgroup $\Gamma_1(N)$ in $SL_2(\boldsymbol{Z}_l)$ (resp. the subgroup consisting of upper triangular unipotent matrices in $SL_2(\boldsymbol{Z}_l)$). Also let $\mathfrak{F}$ be the completion of $\Gamma_1(N)$ with respect to the pro-l topology of $\Gamma_1(N)\cap\Gamma(l)$. Denoting by $Y_1(N)$ and $X_1(N)$ the usual modular curves over $\boldsymbol{Q}$ attached to $\Gamma_1(N)$, we then have the following isomorphisms:

$$\begin{cases} \varprojlim_{n\in N} H^1(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \cong H^1(\mathfrak{F}, \boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]) \\ \varprojlim_{n\in N} H^1(X_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \cong H^1_P(\mathfrak{F}, \boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]) \\ \varprojlim_{n\in N} H^1_c(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \cong \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]). \end{cases} \tag{5}$$

Here, $\boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]$ is the maximum separated left $\mathcal{A}$-module quotient of $\mathcal{A}=\boldsymbol{Z}_l[[\mathfrak{G}]]$ on which $\mathfrak{U}$ acts trivially from the right; and $\mathcal{D}_0$ is a certain completion of the "degree 0-part" of the free abelian group on the set of cusps for $\Gamma_1(N)$.

Now one of the interesting features of our cohomology theories with "generic" coefficients such as $\mathcal{A}$ or $\boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]$ lies in that they have many "specializations". Namely, they "specialize" not only to the étale cohomology groups of Y_n with coefficients in the constant $\boldsymbol{Z}_l$-sheaf (via the natural projection), but also to those with coefficients in certain twisted constant sheaves. In § 6, we discuss such "specialization mappings", and give sufficient conditions for the finiteness of their cokernels.

In the final § 7, we study the groups (5) more closely. In this case, applying the method of § 6, we can define the specialization mapping from $\varprojlim_{n\in N} H^1(X_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l)$ to Deligne's l-adic representation space attached to cusp forms of weight k with respect to $\Gamma_1(Nl^n)$ for each $k\geqq 2$ and $n\geqq 0$. In § 7, we firstly deduce from a result in § 6, combined with a result of Shimura, that the cokernel of such a mapping is always finite. We then discuss Hecke operators acting on the groups (5); and prove compatibility properties of Hecke operators with respect to the specialization mappings. As an application, we derive Deligne's congruence relations on his l-adic representation spaces from the classical Eichler-Shimura congruence relations. We note that this method of reducing the problem of cusp forms of higher weights to those of weight 2 is originally due to Shimura [**Sh2**] (cf. also [**O1**]), and later it was further developed by Hida [**H2**].

Finally, we would like to mention further aspect of our cohomology theory: As we noted above, the isomorphism (4) and the third one in (5) were suggested by [**AS2**], in which Ash and Stevens developed the theory of higher weight modular symbols. From this point of view, via the third isomorphism in (5), an element of $\varprojlim_{n\in N} H_c^1(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l)$ can be interpreted as a modular symbol with values in $\boldsymbol{Z}_l[[\mathfrak{G}/\mathfrak{U}]]$, the group of $\boldsymbol{Z}_l$-valued measures on $\mathfrak{G}/\mathfrak{U}$. In a subsequent paper (in which we replace "l" by "p"), we will study the measures obtained from such "p-adic modular symbols" in detail; and relate the integrals against them with the special values of L-functions of elliptic cusp forms. We will in particular give a generalization of congruences of Ash and Stevens between the special values of L-functions attached to two cusp forms ([**AS2**] Corollary 4.6) to certain p-adic families of ordinary cusp forms (in the sense of Hida; cf. [**H2**]). This paper in part includes preliminaries for this application.

To conclude this introduction, I would like to express my sincere gratitude to Professor Y. Ihara, for kindly allowing me to include his unpublished results in this paper; and also for introducing me to this subject. During the preparation of a part of this paper, I was partially supported by the Max-Planck-Institut für Mathematik in Bonn; and I would like to thank the MPI both for its financial support and its hospitality.

Notation and conventions.

For a Galois extension L/K of fields, we denote by $\mathrm{Gal}(L/K)$ its Galois group. We let $\mathrm{Gal}(L/K)$ act on L from the right; i.e. $x^{\sigma\tau}=(x^{\sigma})^{\tau}$ for all $x\in L$ and $\sigma, \tau\in\mathrm{Gal}(L/K)$. For a field K, we denote by $\bar{K}$ its separable closure, and write G_K for $\mathrm{Gal}(\bar{K}/K)$.

When we are given a right action of a group G on a set S by $s\mapsto s^{\sigma}$ ($s\in S$, $\sigma\in G$), we sometimes convert this action into a left action by: $s\mapsto\sigma\cdot s:=s^{\sigma^{-1}}$; and vice versa.

If R is a ring and S is a set, the symbol $R[S]$ stands for the free R-module generated by the elements of S.

For a scheme X, we denote by $X_{\acute{e}t}$ the small étale site of X. If $f: X\to Y$ is a morphism of schemes, and F is a sheaf (resp. an abelian sheaf) on $Y_{\acute{e}t}$, we often write F (resp. $H^i(X, F)$) for f^*F (resp. $H^i(X, f^*F)$), when there is no fear of confusion.

§1. Ihara's tower of algebraic curves.

1.1. The basic setting. First we fix our notation which will be used throughout this paper, following Ihara [**I1**].

We fix a prime number l once and for all.

We fix an algebraic extension $\boldsymbol{Q}^*$ of $\boldsymbol{Q}$, and an algebraic function field of one variable K^* whose constant field is $\boldsymbol{Q}^*$. Let L^* be an extension of K^* without constant field extension. We set $K:=K^*\cdot\bar{\boldsymbol{Q}}$ and $L:=L^*\cdot\bar{\boldsymbol{Q}}$. We also assume that we are given a finite set of prime divisors C^* of $K^*/\boldsymbol{Q}^*$, and denote by $C=\{Q_1, \cdots, Q_s\}$ the set of all prime divisors of K lying above those of C^*. The elements of C or C^* (or their extensions) will be called "cusps".

The "axioms" of our theory will be the following three conditions:

(A1) L/K is a Galois extension, and is unramified outside C.

(A2) L/K is an "almost pro-l extension" in the sense that there is a finite Galois subextension K'/K of L/K such that L/K' is a pro-l extension.

(A3) If $C\neq\emptyset$, then the ramification index of each Q_j ($1\leqq j\leqq s$) in L/K is infinite.

Let M be the maximum pro-l Galois extension of L unramified outside the prime divisors above C. Then the extensions L/K^*, M/K^* (and hence M/L^*, M/K, and M/L) are Galois extensions, while the extension L^*/K^* need not be Galois. We set:

$$(1.1.1)\qquad \begin{cases} \mathfrak{F} := \mathrm{Gal}(M/K) \\ \mathfrak{N} := \mathrm{Gal}(M/L) \\ \mathfrak{G} := \mathrm{Gal}(L/K) \\ \tilde{\mathfrak{G}} := \mathrm{Gal}(L/K^*)\,. \end{cases}$$

There is an obvious exact sequence

$$(1.1.2)\qquad 1 \longrightarrow \mathfrak{G} \longrightarrow \tilde{\mathfrak{G}} \longrightarrow G_{\boldsymbol{Q}^*} \longrightarrow 1\,,$$

where $G_{\boldsymbol{Q}^*}=\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}^*)$. If we start from the situation $L/K/K^*$, then to give an L^* is equivalent to give a splitting of the exact sequence (1.1.2). Identifying $G_{\boldsymbol{Q}^*}$ with $\mathrm{Gal}(L/L^*)$ via the obvious isomorphism, $\rho\in G_{\boldsymbol{Q}^*}$ then acts on $\mathfrak{G}$ from the left by the inner automorphism, which we denote by J_ρ:

$$(1.1.3)\qquad J_\rho(g) := \rho g \rho^{-1}\,.$$

By this action, $\tilde{\mathfrak{G}}$ is a semidirect product of $G_{\boldsymbol{Q}^*}$ and $\mathfrak{G}$.

Fix a family $\{\mathfrak{f}_n\}_{n\in N}$ of open normal subgroups of $\mathfrak{G}$, with a directed set N, satisfying: (i) $\mathfrak{f}_n\subseteqq\mathfrak{f}_m$ if and only if $n\geqq m$; (ii) $\{\mathfrak{f}_n\}_{n\in N}$ is a fundamental neighbourhood system of $1\in\mathfrak{G}$; and (iii) $J_\rho(\mathfrak{f}_n)=\mathfrak{f}_n$ for all $\rho\in G_{\boldsymbol{Q}^*}$ and $n\in N$. Since $\mathfrak{F}$ and $\mathfrak{G}$ are finitely generated topologically (cf. 3.1), we can in fact take $\mathfrak{f}_n$ to be characteristic subgroups of $\mathfrak{G}$.

DEFINITION (1.1.4). We denote by $\mathcal{A}=\boldsymbol{Z}_l[[\mathfrak{G}]]$ the completed group algebra of $\mathfrak{G}$ over $\boldsymbol{Z}_l$. Thus

$$\mathcal{A} = \varprojlim_{m\in\boldsymbol{N},\, n\in N}(\boldsymbol{Z}/l^m\boldsymbol{Z})[\mathfrak{G}_n] = \varprojlim_{n\in N}\boldsymbol{Z}_l[\mathfrak{G}_n]\,,$$

where $\mathfrak{G}_n=\mathfrak{G}/\mathfrak{f}_n$. $\mathcal{A}$ is a compact topological ring with respect to the projective limit topology. The action J_ρ on $\mathfrak{G}$ ($\rho\in G_{\boldsymbol{Q}^*}$) naturally extends to a $\boldsymbol{Z}_l$-algebra automorphism of $\mathcal{A}$, which we denote by the same letter J_ρ. We denote by I the augmentation ideal of $\mathcal{A}$.

For each $n\in N$, put

$$(1.1.5)\qquad K_n := L^{\mathfrak{f}_n}\,,\qquad K_n^* := K_n\cap L^*\,.$$

Then it is easy to see that $K_n=K_n^*\cdot\bar{\boldsymbol{Q}}$. We shall use the following notation:

$$(1.1.6)\quad \begin{cases} X_n^* \text{ (resp. } X^*) := \text{(the complete nonsingular curve over } \boldsymbol{Q}^* \\ \qquad \text{whose function field is } K_n^* \text{ (resp. } K^*)) \\ X_n := X_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}}; \qquad X := X^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}} \\ Y^* := X^* - C^*; \qquad Y := X - C \\ Y_n^* := \text{(the normalization of } Y^* \text{ in } K_n^*) \\ Y_n := Y_n^* \otimes_{\boldsymbol{Q}^*} \bar{\boldsymbol{Q}} \\ C_n := X_n - Y_n . \end{cases}$$

Here, we identified C^* (resp. C) with the set of corresponding closed points of X^* (resp. X).

1.2. Examples of Ihara's tower. Here, we give two examples.

EXAMPLE (M): The maximum pro-l tower of $\boldsymbol{P}_{\boldsymbol{Q}}^1 - \{0, 1, \infty\}$ (cf. [**I1**] § 1 (D)). Let us take $\boldsymbol{Q}^* = \boldsymbol{Q}$ and $K^* = \boldsymbol{Q}(t)$, the rational function field of one variable over $\boldsymbol{Q}$. Let C^* be the set of prime divisors of K^* corresponding to $t = 0, 1$, and ∞; and $L = M$ the maximum pro-l extension of $K = \bar{\boldsymbol{Q}}(t)$ unramified outside C. Then there is so-called "Belyi's model" L^* of L over $\boldsymbol{Q}$, so that the conditions (A1)-(A3) are satisfied. In this case, $\mathfrak{F} = \mathfrak{G}$ is a free pro-l group of rank 2, and the action J_ρ of $G_{\boldsymbol{Q}}$ on $\mathfrak{G}$ or $\mathcal{A}$ gives a very large Galois representation.

EXAMPLE (E): Elliptic modular tower. Let $\mathcal{L}$ be the field of all automorphic functions with respect to congruence subgroups of $SL_2(\boldsymbol{Z})$ whose coefficients of the Fourier expansions (at the cusp $i\infty$) all belong to $\boldsymbol{Q}_{ab}$, the maximal abelian extension of $\boldsymbol{Q}$. Then by Shimura's theory (Shimura [**Sh1**] 6.5), there is a canonical surjective homomorphism

$$(1.2.1)\qquad \tau : \prod_p GL_2(\boldsymbol{Z}_p) \longrightarrow \mathrm{Gal}(\mathcal{L}/\boldsymbol{Q}(j))$$

whose kernel is $\{\pm 1\}$, and such that

$$(1.2.2)\qquad \begin{cases} f^{\tau(\alpha)}(z) = f(\alpha(z)) & \text{if } \alpha \in SL_2(\boldsymbol{Z}) \\ \tau(x) = [\det(x)^{-1}, \boldsymbol{Q}] & \text{on } \boldsymbol{Q}_{ab}, \end{cases}$$

where j is the usual elliptic modular j-function, and $[-, \boldsymbol{Q}]$ is the Artin symbol for $\boldsymbol{Q}$.

In the following, we fix a positive integer $N \geqq 4$, and put

$$(1.2.3)\qquad \begin{cases} U := \prod_{p \nmid N} GL_2(\boldsymbol{Z}_p) \\ \qquad \times \prod_{p \mid N} \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\boldsymbol{Z}_p) \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \bmod N \cdot M_2(\boldsymbol{Z}_p) \right\} \\ U_\infty := \{ x \in U \mid \text{the } l\text{-component of } x \text{ is } 1 \}. \end{cases}$$

Let K^* and L' be the fixed fields of $\tau(U)$ and $\tau(U_\infty)$, respectively. Then it follows from [**Sh1**] Proposition 6.9 and (1.2.2) above that K^* consists of all modular functions with respect to $\Gamma_1(N)=U\cap SL_2(\boldsymbol{Z})$ whose coefficients of the Fourier expansions are rational, and that the constant field of L' is $\boldsymbol{Q}(\mu_{l^\infty})$, where μl^∞ denotes the set of all l-powerth roots of unity. Now τ obviously induces an isomorphism:

$$(1.2.4)\quad \mathfrak{G}^* := \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\boldsymbol{Z}_l) \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \bmod N\cdot M_2(\boldsymbol{Z}_l)\right\} \simeq \mathrm{Gal}(L'/K).$$

Identifying these two groups by τ, let L^* be the subfield of L' corresponding to $\left\{\begin{bmatrix} 1 & 0 \\ 0 & * \end{bmatrix} \in \mathfrak{G}^*\right\}$, and put $\mathfrak{G} := \mathfrak{G}^* \cap SL_2(\boldsymbol{Z}_l)$. Then we have an isomorphism:

$$(1.2.5)\qquad \mathfrak{G} = \mathrm{Gal}(L'/K^*(\mu_{l^\infty})) \cong \mathrm{Gal}(L/K),$$

where, as before, we put $L := L^*\cdot\bar{\boldsymbol{Q}}$ and $K := K^*\cdot\bar{\boldsymbol{Q}}$. Putting $\boldsymbol{Q}^* := \boldsymbol{Q}$ and letting C be the set of all prime divisors of $K^*/\boldsymbol{Q}$ corresponding to the cusps in the usual sense, these data satisfy the conditions (A1)-(A3).

In this case, J_ρ is of elementary nature; i.e.,

$$(1.2.6)\qquad J_\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & \chi_l(\rho)^{-1}b \\ \chi_l(\rho)c & d \end{bmatrix}$$

for $\rho \in G_{\boldsymbol{Q}}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{G}$, where χ_l denotes the l-cyclotomic character.

We shall return to this situation in §7.

For other examples, see [**I1**].

§2. One dimensional cohomology groups attached to Ihara's tower.

In this section, we fix an Ihara's tower (1.1) satisfying (A1)-(A3).

2.1. Continuous cochain cohomology groups.

DEFINITION (2.1.1). When $\mathcal{M}$ is a pro-l abelian group, and $\mathfrak{G}$ acts on $\mathcal{M}$ continuously and $\boldsymbol{Z}_l$-linearly from the left, we call $\mathcal{M}$ a pro-l $\mathfrak{G}$-module. Pro-l $\mathfrak{F}$-modules and pro-l $\tilde{\mathfrak{G}}$-modules are defined similarly.

Note that a prol-l $\mathfrak{G}$-module can be considered as a left $\Lambda = \boldsymbol{Z}_l[[\mathfrak{G}]]$-module in a natural manner; and similarly for pro-l $\tilde{\mathfrak{G}}$- or pro-l $\mathfrak{F}$-modules. Now a pro-l $\mathfrak{G}$-module $\mathcal{M}$ can be considerd as a pro-l $\mathfrak{F}$-module via the natural projection $\mathfrak{F}\to\mathfrak{G}$, and we are interested in the continuous cochain cohomology groups $H^i(\mathfrak{F}, \mathcal{M})$; so let us recall their definitions. First, for any pro-l $\mathfrak{F}$-module $\mathcal{M}$, put

$$C^i(\mathfrak{F}, \mathcal{M}) := \{\alpha : \mathfrak{F}^i \to \mathcal{M} \mid \alpha \text{ is continuous}\}, \tag{2.1.2}$$

where $\mathfrak{F}^i$ is the product of i-copies of $\mathfrak{F}$ ($i \geqq 0$; we put $C^0(\mathfrak{F}, \mathcal{M}) = \mathcal{M}$). The differentiation operators $d^i : C^i(\mathfrak{F}, \mathcal{M}) \to C^{i+1}(\mathfrak{F}, \mathcal{M})$ are defined by:

$$(d^i\alpha)(x_1, \cdots, x_{i+1}) = x_1 \cdot \alpha(x_2, \cdots, x_{i+1}) \tag{2.1.3}$$

$$+\textstyle\sum_{j=1}^{i}(-1)^j \alpha(x_1, \cdots, x_j x_{j+1}, \cdots, x_{i+1}) + (-1)^{i+1}\alpha(x_1, \cdots, x_i).$$

We then put

$$\begin{cases} Z^i(\mathfrak{F}, \mathcal{M}) := \operatorname{Ker}(d^i) \\ B^i(\mathfrak{F}, \mathcal{M}) := \operatorname{Im}(d^{i-1}) \\ H^i(\mathfrak{F}, \mathcal{M}) := Z^i(\mathfrak{F}, \mathcal{M})/B^i(\mathfrak{F}, \mathcal{M}). \end{cases} \tag{2.1.4}$$

(Here we understand that $B^0(\mathfrak{F}, \mathcal{M}) = \{0\}$.) The cohomology class of $\alpha \in Z^i(\mathfrak{F}, \mathcal{M})$ will be denoted by $cl(\alpha)$. Recall that if $\mathcal{M} = \varprojlim_{i \in I} \mathcal{M}_i$ with pro-l $\mathfrak{F}$-modules $\mathcal{M}$ and $\mathcal{M}_i$, then we have a canonical isomorphism:

$$H^1(\mathfrak{F}, \mathcal{M}) \cong \varprojlim_{i \in I} H^1(\mathfrak{F}, \mathcal{M}_i) \tag{2.1.5}$$

(Serre [**Se3**] Proposition 7). Especially, since $\mathcal{M}$ is a pro-l group and $\mathfrak{F}$ is topologically finitely generated (cf. 3.1), $H^1(\mathfrak{F}, \mathcal{M})$ is naturally equipped with the structure of a pro-l abelian group.

Next assume that $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module. In view of the semidirect product expression: $\tilde{\mathfrak{G}} = G_{Q^*} \ltimes \mathfrak{G}$ (1.1), this is equivalent to saying that $\mathcal{M}$ is a pro-l $\mathfrak{G}$-module and moreover that either: i) G_{Q^*} acts on $\mathcal{M}$ continuously from the left, and $\rho(a \cdot m) = J_\rho(a) \cdot \rho(m)$ for all $\rho \in G_{Q^*}$, $a \in \mathfrak{G}$ and $m \in \mathcal{M}$; or ii) G_{Q^*} acts on $\mathcal{M}$ continuously from the right, and $(a \cdot m)^\rho = J_{\rho^{-1}}(a) \cdot m^\rho$ for all ρ, a and m as above. In this case, using the terminology i) above, we can consider $H^1(\mathfrak{F}, \mathcal{M})$ as a right G_{Q^*}-module as follows: For $\rho \in G_{Q^*} = \mathrm{Gal}(L/L^*)$, take $\tilde{\rho} \in \mathrm{Gal}(M/L^*)$ such that $\tilde{\rho}|_L = \rho$. For $\alpha \in Z^1(\mathfrak{F}, \mathcal{M})$, we put

$$\alpha^{\tilde{\rho}}(x) := \rho^{-1} \cdot \alpha(\tilde{\rho} x \tilde{\rho}^{-1}). \tag{2.1.6}$$

Then we can let ρ act on $H^1(\mathfrak{F}, \mathcal{M})$ by $cl(\alpha) \mapsto cl(\alpha^{\tilde{\rho}})$; it is easy to see that this is well-defined. It is also easy to see that this action of G_{Q^*} on $H^1(\mathfrak{F}, \mathcal{M})$ is continuous.

Let ${}_{\mathfrak{F}}C$ (resp. $C_{\mathfrak{F}}$) be the category of (discrete) abelian groups on which $\mathfrak{F}$ acts continuously from the left (resp. right). ${}_{\mathfrak{F}}C$ and $C_{\mathfrak{F}}$ are canonically equivalent (cf. **Notation and conventions**). When we restrict our attention to finite (pro-)l $\mathfrak{G}$-modules $\mathcal{M}$, it is well-known that the groups $H^i(\mathfrak{F}, \mathcal{M})$ (together with natural connecting homomorphisms) are isomorphic to the derived functor cohomology groups of $H^0(\mathfrak{F}, -)$ on ${}_{\mathfrak{F}}\mathcal{C}$. We shall always identify them. Also when $\mathcal{M}$ is finite, $H^1(\mathfrak{F}, \mathcal{M})$ may be considered as the set of isomorphism classes of (right) $\mathcal{M}$-torsors (=principal homogeneous spaces under $\mathcal{M}$) in the category of continuous left $\mathfrak{F}$-sets (Serre [**Se2**] I 5.2). From this point of view, an $\mathcal{M}$-torsor P corresponds to $cl(\alpha)\in H^1(\mathfrak{F}, \mathcal{M})$ via the formula:

(2.1.7) $$\sigma p = p\cdot\alpha(\sigma) \qquad \text{for a fixed } p\in P$$

(loc. cit.).

2.2. We give here a general remark which will be used repeatedly in the sequel. Suppose that we are given the following diagram of abelian categories and additive left exact functors between them:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "f"'] & \mathcal{B} \arrow[d, "g"] \arrow[dr, "G"] & \\ \mathcal{A}' \arrow[r, "F'"'] & \mathcal{B}' \arrow[r, "G'"'] & \mathcal{C} \end{tikzcd}$$

Assume that:

i) $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$, and $\mathcal{B}'$ have enough injective objects;

ii) F (resp. F') sends each injective object of $\mathcal{A}$ (resp. $\mathcal{A}'$) to a G-(resp. G'-) acyclic object;

iii) f and g are exact;

iv) We are given morphisms of functors:

$$\begin{cases} \iota : g\circ F \longrightarrow F'\circ f \\ \varepsilon : G \longrightarrow G'\circ g. \end{cases}$$

Then, for each $A\in\mathcal{A}$, we have two spectral sequences of composite functors by i) and ii):

$$S_1 : E_2^{p,q} = R^pG(R^qF(A)) \Longrightarrow R^{p+q}(G\circ F)(A)$$

$$S_2 : E_2^{p,q} = R^pG'(R^qF'(f(A)) \Longrightarrow R^{p+q}(G'\circ F')(f(A)),$$

and moreover a morphism $\xi : S_1 \to S_2$ of spectral sequences by iii) and iv), which can be described as follows: First note that there are unique morphisms of ∂-functors

$$\xi_1 : g\circ R^qF \longrightarrow R^qF'\circ f$$

which coincides with ι when $q=0$; and

$$\xi_2 : R^pG \longrightarrow R^pG'\circ g$$

which coincides with ε when $p=0$. Then the morphism of E_2-terms of ξ is the composite of:

$$R^pG(R^qF(A)) \xrightarrow{\xi_2} R^pG'(g\circ R^qF(A)) \xrightarrow{\xi_1} R^pG'(R^qF'(f(A)))\,.$$

On the other hand, the morphism of abutments of ξ is the unique morphism of ∂-functors

$$R^n(G\circ F) \longrightarrow R^n(G'\circ F')\circ f$$

which coincides with the composite of $G\circ F \xrightarrow{\varepsilon} G'\circ g\circ F \xrightarrow{\iota} G'\circ F'\circ f$ when $n=0$.

This fact must be well-known, and in fact can be proved directly, going back to the constructions of S_1 and S_2 through Cartan-Eilenberg resolutions (Grothendieck [**Gr**], [**EGAIII**]). The details are thus omitted.

2.3. Étale cohomology groups. Let $\mathcal{M}=\varprojlim_{i\in I}\mathcal{M}_i$ be a pro-l $\mathfrak{G}$-module with finite $\mathfrak{G}$-modules $\mathcal{M}_i$. Then, for each $i\in I$, there is a finite Galois subextension K'/K of L/K such that the action of $\mathfrak{G}$ on $\mathcal{M}_i$ factors through $\mathrm{Gal}(K'/K)$. Denote by W the normalization of Y in K'. Then taking the quotient of the constant group scheme $W\times\mathcal{M}_i$ over W by the diagonal left action of $\mathrm{Gal}(K'/K)$, we obtain a finite étale group scheme $F_{\mathcal{M}_i}$ over Y:

(2.3.1) $$F_{\mathcal{M}_i} := \mathrm{Gal}(K'/K)\backslash W\times\mathcal{M}_i\,.$$

Up to canonical isomorphisms, $F_{\mathcal{M}_i}$'s are independent of the choice of K', and we may consider them as a projective system of twisted constant sheaves on the étale site $Y_{\acute{e}t}$ of Y.

DEFINITION (2.3.2). We write $H^n(Y, F_{\mathcal{M}_i})$ $(n\geqq 0)$ for the étale cohomology groups, and put

$$H^n(Y, F_{\mathcal{M}}) := \varprojlim_{i\in I} H^n(Y, F_{\mathcal{M}_i})\,.$$

Note that, since $H^n(Y, F_{\mathcal{M}_i})$'s are finite, $H^n(Y, F_{\mathcal{M}})$ are pro-l abelian groups.

If $\mathcal{M}=\varprojlim_{i\in I}\mathcal{M}_i$ above is a pro-l $\tilde{\mathfrak{G}}$-module, then repeating the same argument as above replacing K by K^*, we get a projective system of twisted constant sheaves on $Y^*_{\acute{e}t}$ whose base change to $Y_{\acute{e}t}$ is $\{F_{\mathcal{M}_i}\}_{i\in I}$. In this case, $G_{\mathbf{Q}^*}$ naturally acts on $H^n(Y, F_{\mathcal{M}_i})$ and $H^n(Y, F_{\mathcal{M}})$ continuously from the right (i.e. contravariantly).

PROPOSITION (2.3.3). *For a pro-l $\mathfrak{G}$-module $\mathcal{M}$ as above, $H^1(\mathfrak{F}, \mathcal{M})$ and $H^1(Y, F_{\mathcal{M}})$ are canonically isomorphic. When $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, this isomorphism commutes with the right action of $G_{\mathbf{Q}^*}$.*

PROOF. For a finite subextension K'/K of M/K, let $Y_{K'}$ be the normalization of Y in K'. If $K'_1\supseteq K'_2$ are such subextensions, then the natural morphism

$Y_{K'_1} \to Y_{K'_2}$ is finite. Therefore the projective limit $\varprojlim_{K'/K} Y_{K'} =: Z$ exists, when K' runs through all finite subextensions of M/K. Z is a pro-étale Galois covering of Y, and hence we have the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(\mathfrak{F}, H^q(Z, F_{\mathcal{M}_i})) \Longrightarrow H^{p+q}(Y, F_{\mathcal{M}_i})$$

for each $i \in I$ (Artin [**A**] III 4.7, [**SGA4**] VIII Corollary 8.5). Since the base change of $F_{\mathcal{M}_i}$ to Z is isomorphic to the constant group scheme over Z corresponding to $\mathcal{M}_i$, we have $H^0(Z, F_{\mathcal{M}_i}) \cong \mathcal{M}_i$ as left $\mathfrak{F}$-modules, and $H^1(Z, F_{\mathcal{M}_i}) = \{0\}$, because this group is isomorphic to the group of continuous homomorphisms of $\pi_1^{alg}(Z)$ (the algebraic fundamental group of Z) to $\mathcal{M}_i$. Thus the edge homomorphism induces an isomorphism: $H^1(\mathfrak{F}, \mathcal{M}_i) \cong H^1(Y, F_{\mathcal{M}_i})$ for each $i \in I$. Taking projective limits, we obtain an isomorphism: $H^1(\mathfrak{F}, \mathcal{M}) \cong H^1(Y, F_{\mathcal{M}})$ of pro-l abelian groups.

To show that this is an isomorphism of G_{Q^*}-modules when $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, we may assume that $\mathcal{M}$ is finite. Fix $\rho \in G_{Q^*}$ and choose a lifting $\tilde{\rho} \in \mathrm{Gal}(M/L^*)$ as in 2.1. Thus $\tilde{\rho} = \mathrm{id} \otimes \rho$ on $K \cong K^* \otimes_{Q^*} \bar{\boldsymbol{Q}}$. Denote by q the automorphism $\mathrm{id} \times \mathrm{Spec}(\rho)$ of $Y = Y^* \otimes_{Q^*} \bar{\boldsymbol{Q}}$, and consider the diagram:

$$\begin{tikzcd}
Y_{\tilde{et}}^{ab} \arrow[r, "F"] \arrow[d, "q^*"'] & C_{\mathfrak{F}} \arrow[d, "g"] \arrow[dr, "G"] & \\
 & & (Ab) \\
Y_{\tilde{et}}^{ab} \arrow[r, "F"'] & C_{\mathfrak{F}} \arrow[ur, "G"'] &
\end{tikzcd}$$

where $Y_{\tilde{et}}^{ab}$ is the category of abelian sheaves on $Y_{\acute{e}t}$, $C_{\mathfrak{F}}$ is the category of continuous right $\mathfrak{F}$-modules as before, and (Ab) is the category of abelian groups, respectively. Define the functors F, G and g as follows: $F(A) := H^0(Z, A)$ for $A \in Y_{\tilde{et}}^{ab}$; $G(M) := M^{\mathfrak{F}} = (\mathfrak{F}$-invariant elements in $M)$; and $g(M) := (M$ as an abelian group on which $\sigma \in \mathfrak{F}$ acts as $\tilde{\rho}\sigma\tilde{\rho}^{-1})$, for $M \in C_{\mathfrak{F}}$. If we define morphisms of functors $\iota: g \circ F \to F' \circ q^*$ by $g(H^0(Z, A)) = H^0(Z, A) \xrightarrow{\text{can.}} H^0(Z, q^*A)$ (pull-back of sections), and $\varepsilon := \mathrm{id}: G \to G \circ g = G$, then these data satisfy the assumptions in 2.2. Recall that the spectral sequence of Hochshild-Serre is that of the composition of functors F and G above. We can therefore apply 2.2 to obtain, for each $A \in Y_{\tilde{et}}^{ab}$, a morphism ξ of Hochschild-Serre spectral sequences for A and q^*A as described in loc. cit.. The morphism of E_2-terms of ξ:

$$H^p(\mathfrak{F}, H^q(Z, A)) \longrightarrow H^p(\mathfrak{F}, H^q(Z, q^*A))$$

is easily seen to be induced from the change of groups: $\mathfrak{F} \leftarrow \mathfrak{F}$ which sends σ to $\tilde{\rho}\sigma\tilde{\rho}^{-1}$, and the map: $H^q(Z, A) \xrightarrow{\text{can.}} H^q(Z, q^*A)$ compatible with it (cf. Serre [**Se1**] VII §5). On the other hand, the morphism of abutments of ξ: $H^n(Y, A)$

$\to H^n(Y, q^*A)$ is the canonical one. Our conclusion now follows immediately. ■

2.4. One dimensional cohomology groups with "generic" coefficients. Recall that $\mathcal{A}=\boldsymbol{Z}_l[[\mathfrak{G}]]$ is the completed group algebra of $\mathfrak{G}$ over $\boldsymbol{Z}_l$ (1.1.4). This is in a natural manner a pro-l $\mathfrak{G}$-module. Moreover, letting $\rho\in G_{\boldsymbol{Q}^*}$ act on $\mathcal{A}$ by J_ρ from the left, $\mathcal{A}$ becomes a pro-l $\tilde{\mathfrak{G}}$-module (cf. 2.1). Our first main result is the following:

THEOREM (2.4.1). *$H^1(\mathfrak{F}, \mathcal{A})$ is canonically isomorphic to $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ as a right $G_{\boldsymbol{Q}^*}$-module, where the projective limit is taken relative to trace mappings of étale cohomology groups* ([**SGA4**] XVII 6.2.7, [**M**] V Lemma 1.12).

More generally, take and fix a closed subgroup $\mathfrak{U}$ of $\mathfrak{G}$ which is $G_{\boldsymbol{Q}^*}$-stable (under the action J_ρ). Put

$$(2.4.2)\qquad \begin{cases} \mathfrak{U}_n := (\text{the image of } \mathfrak{U} \text{ in } \mathfrak{G}_n=\mathrm{Gal}(K_n/K)) \\ K_{1,n} := K_n^{\mathfrak{U}_n} \\ K_{1,n}^* := K_{1,n}\cap K_n^* \\ X_{1,n}^* := (\text{the complete nonsingular model of } K_{1,n}^*/\boldsymbol{Q}^*) \\ Y_{1,n}^* := (\text{the normalization of } Y^* \text{ in } X_{1,n}^*) \\ X_{1,n} := X_{1,n}^*\otimes_{\boldsymbol{Q}^*}\bar{\boldsymbol{Q}} \\ Y_{1,n} := Y_{1,n}^*\otimes_{\boldsymbol{Q}^*}\bar{\boldsymbol{Q}}\,. \end{cases}$$

It is easy to see that $K_{1,n}^*\cdot\bar{\boldsymbol{Q}}=K_{1,n}$. Now the inclusion: $\mathfrak{U}\hookrightarrow\mathfrak{G}$ induces an injective ring homomorphism: $\boldsymbol{Z}_l[[\mathfrak{U}]]\hookrightarrow\boldsymbol{Z}_l[[\mathfrak{G}]]$. Denote by $I_{\mathfrak{U}}$ the augmentation ideal of $\boldsymbol{Z}_l[[\mathfrak{U}]]$. Thus $I_{\mathfrak{U}}$ is the closed $\boldsymbol{Z}_l$-submodule of $\boldsymbol{Z}_l[[\mathfrak{U}]]$ generated by $u-1$ $(u\in\mathfrak{U})$, and $\overline{\mathcal{A}I_{\mathfrak{U}}}$ (the closure of $\mathcal{A}I_{\mathfrak{U}}$ in $\mathcal{A}$) is the closed left ideal of $\mathcal{A}$ generated by them. Therefore $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}$ is the maximal separated left $\mathcal{A}$-module quotient of $\mathcal{A}$ on which $\mathfrak{U}$ acts trivially from the right; i.e.,

$$(2.4.3)\qquad \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}} \cong \varprojlim_{n\in N}\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]\,,$$

where $\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]$ is the free $\boldsymbol{Z}_l$-module on $\mathfrak{G}_n/\mathfrak{U}_n$. $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}$ is naturally equipped with the structure of a pro-l $\tilde{\mathfrak{G}}$-module in the same manner as $\mathcal{A}$. Note that if $\mathfrak{U}$ is generated by finitely many elements topologically, then $\mathcal{A}I_{\mathfrak{U}}$ itself is closed. With these terminologies, we have the following:

THEOREM (2.4.4). *$H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$ is canonically isomorphic to $\varprojlim_{n\in N}H^1(Y_{1,n}, \boldsymbol{Z}_l)$ as a right $G_{\boldsymbol{Q}^*}$-module, where the projective limit is taken, as before, relative to the trace mappings.*

2.5. Proof of Theorem (2.4.4). Put $\mathfrak{F}_1^n := \mathrm{Gal}(M/K_{1,n})$. Then we have an obvious isomorphism: $\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]\cong\boldsymbol{Z}_l[\mathfrak{F}/\mathfrak{F}_1^n]$ of left $\mathfrak{F}$-modules. Consider the mappings:

$$H^1(\mathfrak{F},\ \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) \xrightarrow{\text{Res}} H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) \xrightarrow{e_n} H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l)\,, \tag{2.5.1}$$

where the first arrow is the restriction mapping, and the second one is defined as follows: Let $\mathfrak{G}_n=\coprod_i g_i\mathfrak{U}_n$ be the disjoint decomposition, and write $\bar{g}_i$ for $g_i\mathfrak{U}_n$. Then every $\alpha\in Z^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n])$ can be written as $\alpha=\sum_i\alpha_{\bar{g}_i}\cdot\bar{g}_i$ with mappings $\alpha_{\bar{g}_i}$ from $\mathfrak{F}_1^n$ to $\boldsymbol{Z}_l$. We then put $e_n(cl(\alpha))=cl(\alpha_{\bar{1}})$. As is well-known, $s_n:=e_n\circ\text{Res}$ is an isomorphism. Indeed, if we replace $\boldsymbol{Z}_l$ by $\boldsymbol{Z}/l^k\boldsymbol{Z}$ $(k>0)$ in the argument above, the composite of the mappings in (2.5.1) is an isomorphism ("Shapiro isomorphism"; cf. Serre [**Se2**] I 2.5). Thus our claim follows by taking $\varprojlim_k$.

Since $L^*/K_{1,n}^*$ satisfies the conditions (A1)-(A3) in 1.1 with the obvious definition of the set of "cusps", we can consider the associated tower. Especially we can let G_{Q^*} act on $H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l)$ by (2.1.6). Then it is easy to see that the mapping s_n is compatible with the action of G_{Q^*}. Notice also that (2.3.3) applied to this tower yields an isomorphism: $H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l)\xrightarrow{\sim}H^1(Y_{1,n},\ \boldsymbol{Z}_l)$ of G_{Q^*}-modules.

Now for $n\geqq m$ $(n,\ m\in N)$, the corestriction mappings $H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}/l^k\boldsymbol{Z})\to H^1(\mathfrak{F}_1^m,\ \boldsymbol{Z}/l^k\boldsymbol{Z})$ $(k>0)$ induce a mapping: $H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l)\to H^1(\mathfrak{F}_1^m,\ \boldsymbol{Z}_l)$ by taking $\varprojlim_k$, which we again call the corestriction mapping. Then it is also well-known that the following diagram commutes (cf. [**Se2**] loc. cit.):

$$\begin{array}{ccc} H^1(\mathfrak{F},\ \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) & \xrightarrow{s_n} & H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l) \\ \text{can.}\downarrow & & \downarrow\text{Cor} \\ H^1(\mathfrak{F},\ \boldsymbol{Z}_l[\mathfrak{G}_m/\mathfrak{U}_m]) & \xrightarrow[s_m]{} & H^1(\mathfrak{F}_1^m,\ \boldsymbol{Z}_l) \end{array} \tag{2.5.2}$$

where the left vertical arrow is induced from the canonical projection $\mathfrak{G}_n/\mathfrak{U}_n\to\mathfrak{G}_m/\mathfrak{U}_m$. We have therefore obtained an isomorphism of G_{Q^*}-modules:

$$H^1(\mathfrak{F},\ \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})\cong\varprojlim_{n\in N}H^1(\mathfrak{F}_1^n,\ \boldsymbol{Z}_l)\,, \tag{2.5.3}$$

the projective limit being taken relative to corestriction mappings above.

The proof of (2.4.4) is thus reduced to the commutativity of the following diagram, which holds for any finite (pro-)l $\mathfrak{G}$-module $\mathcal{M}$:

$$\begin{array}{ccc} H^1(\mathfrak{F}_1^n,\ \mathcal{M}) & \xrightarrow{(2.3.3)} & H^1(Y_{1,n},\ F_{\mathcal{M}}) \\ \text{Cor}\downarrow & & \downarrow\text{trace} \\ H^1(\mathfrak{F}_1^m,\ \mathcal{M}) & \xrightarrow{(2.3.3)} & H^1(Y_{1,m},\ F_{\mathcal{M}})\,. \end{array} \tag{2.5.4}$$

In fact, let $f:Y_{1,n}\to Y_{1,m}$ be the unique morphism corresponding to $K_{1,m}\hookrightarrow K_{1,n}$, and consider the diagram:

$$\begin{tikzcd} Y^{\sim, ab}_{1,m,\acute{e}t} \arrow[r, "F"] \arrow[d, "\mathrm{id}"'] & C_{\mathfrak{F}_1^n} \arrow[d, "g"] \arrow[dr, "G"] & \\ & & (Ab) \\ Y^{\sim, ab}_{1,m,\acute{e}t} \arrow[r, "F'"'] & C_{\mathfrak{F}_1^m} \arrow[ur, "G'"'] & \end{tikzcd}$$

where the functors are defined by: $F(A):=(H^0(Z, A)$ viewed as a right $\mathfrak{F}_1^n$-module); $F'(A):=(H^0(Z, A)$ viewed as a right $\mathfrak{F}_1^m$-module); $g(M):=M\otimes_{Z[\mathfrak{F}_1^n]}Z[\mathfrak{F}_1^m]$; $G(M):=M^{\mathfrak{F}_1^n}$; and $G'(M'):=M'^{\mathfrak{F}_1^m}$, respectively. Let $\iota: g\circ F\to F'$ be defined by: $H^0(Z, A)\otimes_{Z[\mathfrak{F}_1^n]}Z[\mathfrak{F}_1^m]\to H^0(Z, A)$ which sends $\sum_i m_i\otimes a_i(m_i\in H^0(Z, A)$, $a_i\in\mathfrak{F}_1^m)$ to $\sum_i m_i a_i$; and let $\varepsilon:=\mathrm{id}: G\to G'\circ g=G$. Applying 2.2 to this situation, we easily obtain the commutativity of (2.5.4), using the description of the trace mapping given in [**M**] loc. cit.. This completes the proof of (2.4.4).

REMARK (2.5.5). Fix an $m\in N$. Then as remarked above, we can consider the tower corresponding to $L^*/K^*_{1,m}$. If we set $\mathfrak{G}_1^m:=\mathrm{Gal}(L/K_{1,m})$ and $\mathcal{A}_{1,m}:=\boldsymbol{Z}_l[[\mathfrak{G}_1^m]]$, we therefore obtain an isomorphism:

$$H^1(\mathfrak{F}_1^m, \mathcal{A}_{1,m}/\overline{\mathcal{A}_{1,m}I_{\mathfrak{U}}})\cong\varprojlim_{n\geq m}H^1(Y_{1,n}, \boldsymbol{Z}_l),$$

by (2.4.4). Since the right hand side is identical with $\varprojlim_{n\in N}H^1(Y_{1,n}, \boldsymbol{Z}_l)$, this isomorphism, combined with (2.4.4), yields an isomorphism:

$$H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})\cong H^1(\mathfrak{F}_1^m, \mathcal{A}_{1,m}/\overline{\mathcal{A}_{1,m}I_{\mathfrak{U}}}).$$

Let us make this explicit: Let $\mathfrak{G}=\coprod_{i=0}^h g_i\mathfrak{G}_1^m$ be the disjoint decomposition with $g_0=1$. Then we see that $\mathcal{A}=\boldsymbol{Z}_l[[\mathfrak{G}]]=\bigoplus_{i=0}^h g_i\boldsymbol{Z}_l[[\mathfrak{G}_1^m]]$ with $g_0\boldsymbol{Z}_l[[\mathfrak{G}_1^m]]=\mathcal{A}_{1,m}$. From this, we may consider $\mathcal{A}_{1,m}/\overline{\mathcal{A}_{1,m}I_{\mathfrak{U}}}$ as a direct summand of $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}$ as a left $\mathfrak{F}_1^m$-module. The isomorphism above is then easily seen to be the composite of:

$$H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})\xrightarrow{\mathrm{Res}}H^1(\mathfrak{F}_1^m, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})\longrightarrow H^1(\mathfrak{F}_1^m, \mathcal{A}_{1,m}/\overline{\mathcal{A}_{1,m}I_{\mathfrak{U}}}),$$

where the second arrow corresponds to the projection $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}\to\mathcal{A}_{1,m}/\overline{\mathcal{A}_{1,m}I_{\mathfrak{U}}}$ to the direct factor.

2.6. Let the situation be as in 2.4, and fix a closed subgroup $\mathfrak{K}$ of $\mathfrak{G}$ which normalizes $\mathfrak{U}$. Then $\mathfrak{K}$ naturally acts on $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}$ from the right as left $\mathcal{A}$-module automorphisms; and hence we can consider $H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$ as a right $\mathfrak{K}$- or $\boldsymbol{Z}_l[[\mathfrak{K}]]$-module.

On the other hand, $K_{1,n}$ corresponds to the subgroup $\mathfrak{f}_n\mathfrak{U}$ of $\mathfrak{G}$, which is normalized by $\mathfrak{K}$. Therefore $\mathfrak{K}$ acts on $K_{1,n}$ (resp. $Y_{1,n}$ and $X_{1,n}$) from the right (resp. left); and so, $H^1(Y_{1,n}, \boldsymbol{Z}_l)$ and $\varprojlim_{n\in N} H^1(Y_{1,n}, \boldsymbol{Z}_l)$ may be also considered as right $\mathfrak{K}$- or $\boldsymbol{Z}_l[[\mathfrak{K}]]$-modules.

PROPOSITION (2.6.1). *The isomorphism in* (2.4.4) *is compatible with the right $\mathfrak{K}$-module structure described above.*

PROOF. Fix $k\in\mathfrak{K}$, and choose $\tilde{k}\in\mathfrak{F}$ such that $\tilde{k}|_L=k$. Via the isomorphism $s_n: H^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{S}_n/\mathfrak{U}_n])\simeq H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$ (2.5.1), the action of k on $H^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{S}_n/\mathfrak{U}_n])$ and the automorphism of $H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$ defined by: $cl(\beta)\to cl(\beta')$ with $\beta'(x)=\beta(\tilde{k}x\tilde{k}^{-1})$ $(\beta, \beta'\in Z^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l))$ commute. In fact, this follows easily from the formula: $\alpha(\tilde{k}x\tilde{k}^{-1})=(1-\tilde{k}x\tilde{k}^{-1})\alpha(\tilde{k})+\tilde{k}\alpha(x)$ for $\alpha\in Z^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{S}_n/\mathfrak{U}_n])$. Let q be the automorphism of $Y_{1,n}$ corresponding to k. Then, as in the proof of (2.4.4), we get our conclusion by applying 2.2 to the following situation:

$$\begin{tikzcd}
Y^{\sim,ab}_{1,n,\acute{e}t} \arrow[r, "F"] \arrow[dd, "q^*"'] & C_{\mathfrak{F}_1^n} \arrow[dr, "G"] \arrow[dd, "g"] & \\
 & & (Ab) \\
Y^{\sim,ab}_{1,n,\acute{e}t} \arrow[r, "F"'] & C_{\mathfrak{F}_1^n} \arrow[ur, "G"'] &
\end{tikzcd}$$

where $F(A):=H^0(Z, A)$; $g(M):=(M$ on which $\sigma\in\mathfrak{F}_1^n$ acts as $\tilde{k}\sigma\tilde{k}^{-1})$; $G(M):=M^{\mathfrak{F}_1^n}$; $\iota: g\circ F\to F\circ q^*$ is defined by $g(H^0(Z, A))=H^0(Z, A)\overset{\text{can.}}{\to} H^0(Z, q^*A)$; and $\varepsilon:=\mathrm{id}: G\to G'\circ g=G$. ■

§3. Parabolic cohomology groups.

Fix an Ihara's tower (1.1) satisfying (A1)-(A3). We hereafter assume that $C\neq\emptyset$, until the end of §6.

3.1. Remarks on the group $\mathfrak{F}$. As before, we identify $C=\{Q_1, \cdots, Q_s\}$ with a set of closed points of X; and also with a subset of $X(\bar{\boldsymbol{Q}})$ or $X(\boldsymbol{C})$. Thus $Y(\boldsymbol{C})=X(\boldsymbol{C})-C$. Let g be the genus of X. Then

$$\Gamma:=\pi_1^{top}(Y(\boldsymbol{C})), \tag{3.1.1}$$

the topological fundamental group of $Y(\boldsymbol{C})$ (with a fixed base point), has a well-known structure: It is generated by $2g+s$ elements $\{x_1, \cdots, x_g, y_1, \cdots, y_g, z_1, \cdots, z_s\}$ with the fundamental relation

$$[x_1, y_1]\cdots[x_g, y_g]z_1\cdots z_s=1. \tag{3.1.2}$$

Here, $[x, y]=xyx^{-1}y^{-1}$, and z_j corresponds to a certain closed path circulating around Q_j counterclockwise. Since we assumed that $s>0$, Γ is a free group of rank $r:=2g+s-1$. By our assumption (A3), r is strictly positive.

By the well-known theory of Riemann surfaces, the universal covering space $\mathcal{U}$ of $Y(\boldsymbol{C})$ is isomorphic to either $\boldsymbol{C}$ or the complex upper half plane H. $\mathcal{U}\cong\boldsymbol{C}$ if and only if $g=0$ and $s=2$; i.e., $Y(\boldsymbol{C})\cong\boldsymbol{P}^1(\boldsymbol{C})-\{\text{two points}\}$. When $\mathcal{U}\cong H$, we may consider Γ as a Fuchsian group of the first kind ($\hookrightarrow PSL_2(\boldsymbol{R})$). In this latter case, denoting by H^* the union of H and the set of cusps for Γ, $X(\boldsymbol{C})$ is isomorphic to $\Gamma\backslash H^*$, and the terminology "cusps" of X in the sense of 1.1 coincides with the usual one.

Now the algebraic fundamental group $\pi_1^{alg}(Y\otimes_{\bar{\boldsymbol{Q}}}\boldsymbol{C})\cong\pi_1^{alg}(Y)$ (with a fixed base point) is canonically isomorphic to the profinite completion of Γ. If we take K' as in (A2), and if Γ' is the normal subgroup of Γ corresponding to $K'\otimes_{\bar{\boldsymbol{Q}}}\boldsymbol{C}$, then $\mathfrak{F}$ is isomorphic to the completion of Γ with respect to the pro-l topology of Γ':

$$\mathfrak{F}\cong\varprojlim_N\Gamma/N, \tag{3.1.3}$$

where N runs through all normal subgroups of Γ which are contained in Γ' and such that $|\Gamma':N|$ is a power of l. Thus $\mathfrak{F}$ is a free almost pro-l group in the sense of [**I1**]. $\mathfrak{F}$ is generated by x_i, y_i $(1\leqq i\leqq g)$ and z_j $(1\leqq j\leqq s)$ topologically, and we may consider z_j a topological generator of the inertia group of an extension of Q_j to M.

3.2. Parabolic cohomology groups. Let the notation be as in 2.1. Following Shimura [**Sh1**] 8.1, we make the following:

DEFINITION (3.2.1). For a pro-l $\mathfrak{F}$-module $\mathcal{M}$, using the notation (2.1.4), we set:

$$\begin{cases} C_P^1(\mathfrak{F},\mathcal{M}):=\{\alpha\in C^1(\mathfrak{F},\mathcal{M})\,|\,\alpha(z_j)\in(z_j-1)\mathcal{M}\ (1\leqq j\leqq s)\} \\ Z_P^1(\mathfrak{F},\mathcal{M}):=Z^1(\mathfrak{F},\mathcal{M})\cap C_P^1(\mathfrak{F},\mathcal{M}) \\ B_P^2(\mathfrak{F},\mathcal{M}):=d^1(C_P^1(\mathfrak{F},\mathcal{M})) \\ H_P^1(\mathfrak{F},\mathcal{M}):=Z_P^1(\mathfrak{F},\mathcal{M})/B^1(\mathfrak{F},\mathcal{M}) \\ H_P^2(\mathfrak{F},\mathcal{M}):=Z^2(\mathfrak{F},\mathcal{M})/B_P^2(\mathfrak{F},\mathcal{M}). \end{cases}$$

LEMMA (3.2.2). (cf. [**Sh1**] (8.1.30)) *For $\alpha\in Z^1(\mathfrak{F},\mathcal{M})$, α belongs to $Z_P^1(\mathfrak{F},\mathcal{M})$ if and only if $\alpha(x)\in(x-1)\mathcal{M}$ holds for any $x\in\mathfrak{F}$ which is contained in an inertia group of some prime divisor of M.*

PROOF. The "if" part is obvious. Assume conversely that $\alpha\in Z_P^1(\mathfrak{F},\mathcal{M})$. The element x as above can be expressed as $x=tz_j^at^{-1}$ for some j $(1\leqq j\leqq s)$, $a\in\hat{\boldsymbol{Z}}$ (the profinite completion of $\boldsymbol{Z}$), and $t\in\mathfrak{F}$. By our assumption, $\alpha(z_j)=(z_j-1)m$ for some $m\in\mathcal{M}$. When a is an integer, we have $\alpha(x)=(x-1)(tm-\alpha(t))\in(x-1)\mathcal{M}$ ([**Sh1**] loc. cit.). But by continuity, we obtain the same formula for arbitrary $a\in\hat{\boldsymbol{Z}}$. ■

COROLLARY (3.2.3). *When $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, $H_P^1(\mathfrak{F}, \mathcal{M})$ is a G_{Q*}-stable subgroup of $H^1(\mathfrak{F}, \mathcal{M})$.*

PROOF. This follows immediately from the definition of the action of G_{Q*} on $H^1(\mathfrak{F}, \mathcal{M})$ (2.1), and the lemma above. ■

LEMMA (3.2.4). *For a pro-l $\mathfrak{F}$-module $\mathcal{M}$, we have the exact sequence:*

$$0 \longrightarrow H_P^1(\mathfrak{F}, \mathcal{M}) \longrightarrow H^1(\mathfrak{F}, \mathcal{M}) \xrightarrow{f} \bigoplus_{j=1}^{s}\mathcal{M}/(z_j-1)\mathcal{M}$$
$$\xrightarrow{g} H_P^2(\mathfrak{F}, \mathcal{M}) \longrightarrow H^2(\mathfrak{F}, \mathcal{M}) \longrightarrow 0\,,$$

where the unlabelled arrows are the natural ones, and $f(cl(\alpha)):=(\alpha(z_j) \bmod (z_j-1)\mathcal{M})_{1\leq j\leq s}$ for $\alpha\in Z^1(\mathfrak{F}, \mathcal{M})$. For $m=(m_j \bmod (z_j-1)\mathcal{M})_{1\leq j\leq s}\in\bigoplus_{j=1}^{s}\mathcal{M}/(z_j-1)\mathcal{M}$, there exists an $\alpha\in C^1(\mathfrak{F}, \mathcal{M})$ satisfying $\alpha(z_j)=m_j$ $(1\leqq j\leqq s)$. g is then defined by $g(m)=cl(d^1\alpha)$.

PROOF. Direct. ■

COROLLARY (3.2.5). *For $\mathcal{M}=\varprojlim_{i\in I}\mathcal{M}_i$ with pro-l $\mathfrak{F}$-modules $\mathcal{M}$ and $\mathcal{M}_i$, we have a canonical isomorphism:*

$$H_P^1(\mathfrak{F}, \mathcal{M}) \cong \varprojlim_{i\in I} H_P^1(\mathfrak{F}, \mathcal{M}_i)\,.$$

PROOF. This follows from (2.1.5), (3.2.4) and the fact that $\varprojlim$ is exact in the category of compact abelian groups. ■

In the first draft of [**I1**], Ihara proved the following:

LEMMA (3.2.6) (Ihara). *For any pro-l $\mathfrak{F}$-module $\mathcal{M}$, $H^2(\mathfrak{F}, \mathcal{M})$ vanishes.*

PROOF. As is well-known, to each element of $Z^2(\mathfrak{F}, \mathcal{M})$ corresponds an extension: $1\to\mathcal{M}\xrightarrow{p}G\xrightarrow{q}\mathfrak{F}\to1$, where G is a profinite group, and both p and q are continuous homomorphisms. Let $\bar{x}_i$, $\bar{y}_i$ and $\bar{z}_j$ be any liftings of x_i, y_i and z_j to G, respectively. Then, using the fact that $\mathfrak{F}$ is a free almost pro-l group and that $\mathcal{M}$ is a pro-l group, it is easy to see that the closed subgroup of G generated by $\bar{x}_i$, $\bar{y}_i$ and $\bar{z}_j$ is isomorphic to $\mathfrak{F}$ via q, and hence the extension above splits. ■

Next we want to relate parabolic cohomology groups, and the exact sequence (3.2.4) with étale cohomology groups. Let $j: Y\hookrightarrow X$ be the natural open immersion. Since we assumed that $C\neq\emptyset$, i.e., that Y is an affine scheme, the étale cohomology group $H^2(Y, F)$ vanishes for any abelian torsion sheaf F on $Y_{\acute{e}t}$ ([**M**] V Remark 2.4(a)). Leray spectral sequence for j then induces an exact sequence:

(3.2.7) $0 \longrightarrow H^1(X, j_*F) \longrightarrow H^1(Y, F) \longrightarrow H^0(X, R^1j_*F) \longrightarrow H^2(X, j_*F) \longrightarrow 0.$

PROPOSITION (3.2.8). *Let $\mathcal{M}$ be a finite (pro-)l $\mathfrak{G}$-module. Then we have the following commutative diagram:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_P(\mathfrak{F}, \mathcal{M}) & \longrightarrow & H^1(\mathfrak{F}, \mathcal{M}) & \longrightarrow & \bigoplus_{j=1}^{s}\mathcal{M}/(z_j-1)\mathcal{M} \\ & & \downarrow\wr & & \downarrow\wr\,(2.3.3) & & \downarrow\wr \\ 0 & \longrightarrow & H^1(X, j_*F_{\mathcal{M}}) & \longrightarrow & H^1(Y, F_{\mathcal{M}}) & \longrightarrow & H^0(X, R^1j_*F_{\mathcal{M}}) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H^2_P(\mathfrak{F}, \mathcal{M}) & \longrightarrow & 0 \\ & \downarrow\wr & & \\ \longrightarrow & H^2(X, j_*F_{\mathcal{M}}) & \longrightarrow & 0\,, \end{array}$$

where the upper (resp. lower) horizontal exact sequence is that of (3.2.4) *(cf.* (3.2.6)) *(resp.* (3.2.7)), *and the unlabelled vertical isomorphisms will be described in the course of the proof.*

PROOF. Let $\tilde{Q}_j$ be an extension of the prime divisor Q_j of K to $\bar{K}$, and $\tilde{I}_j$ the inertia group of $\tilde{Q}_j$ for each j $(1 \leqq j \leqq s)$. $\tilde{I}_j$ is generated by an element $\tilde{z}_j$ topologically. Taking $\tilde{Q}_j$ suitably, we may assume that $\tilde{z}_j|_M = z_j$. Then $R^1j_*F_{\mathcal{M}}$ is a skyscraper sheaf supported at C (considered as a set of closed points of X), and its stalk at Q_j is isomorphic to $H^1(\mathrm{Spec}(\bar{K}^{\tilde{I}_j}), F_{\mathcal{M}})$, because the base change by j of the strict localization of X at Q_j is isomorphic to $\mathrm{Spec}(\bar{K}^{\tilde{I}_j})$ (cf. [**M**] III Theorem 1.15). This cohomology group is canonically isomorphic to the Galois cohomology group $H^1(\tilde{I}_j, \mathcal{M})$, which is isomorphic to $\mathcal{M}/(\tilde{z}_j-1)\mathcal{M} = \mathcal{M}/(z_j-1)\mathcal{M}$ by the correspondence: $cl(\alpha) \mapsto \alpha(\tilde{z}_j) \bmod (z_j-1)\mathcal{M}$. Thus, if we denote by I_j the subgroup of $\mathfrak{F}$ generated by z_j topologically, the homomorphism $\tilde{I}_j \to I_j$ $(\tilde{z}_j \mapsto z_j)$ induces an isomorphism: $H^1(I_j, \mathcal{M}) \simeq H^1(\tilde{I}_j, \mathcal{M})$.

Let $T_{\mathfrak{F}}$ be the category of finite sets on which $\mathfrak{F}$ acts continuously from the right, endowed with natural Grothendieck topology ([**A**] I (0.6 bis)). Since $\mathfrak{F}$ can be naturally regarded as a quotient of $\pi_1^{alg}(Y)$, there is a morphism of sites $f: Y_{\acute{e}t} \to T_{\mathfrak{F}}$ by the theory of fundamental groups. The direct image functor for the category of set-valued sheaves $f_*: Y_{\acute{e}t}^{\sim} \to T_{\mathfrak{F}}^{\sim}$ is given by $f_*(F) = \Gamma(Z, F)$, where we identified an object of $T_{\mathfrak{F}}^{\sim}$ with a continuous right $\mathfrak{F}$-set (cf. loc. cit.). Now recall that the Hochschild-Serre spectral sequence appearing in the proof of (2.3.3) is nothing but the Leray spectral sequence for this morphism of sites ([**A**] III (4.7)).

On the other hand, it is well-known that $H^1(Y, F)$ can be identified with the set of isomorphism classes of F-torsors in $Y_{\acute{e}t}^{\sim}$ for any abelian sheaf F on $Y_{\acute{e}t}$ (Giraud [**Gi**]). From this point of view, by [**Gi**] V 3.1 and the remark above, the inverse of the isomorphism: $H^1(\mathfrak{F}, \mathcal{M}) \simeq H^1(Y, F_{\mathcal{M}})$ (2.3.3) can be interpreted as the correspondence: (isomorphism class of an $F_{\mathcal{M}}$-torsor P)$\mapsto$ (isomorphism class of the $\mathcal{M}$-torsor $\Gamma(Z, P)$ in $T_{\tilde{\mathfrak{F}}}$) ($\leftrightarrow cl(\alpha)$, with α given by $xp = p^{x^{-1}} = p\cdot\alpha(x)$ for a fixed $p \in \Gamma(Z, P)$). Now in view of [**Gi**] V 2.1 and 3.1, we have the commutative diagram:

$$\begin{tikzcd} H^1(\mathfrak{F}, \mathcal{M}) \arrow[r, "\bigoplus\mathrm{Res}"] & \bigoplus_{j=1}^{s} H^1(I_j, \mathcal{M}) \arrow[r, "\sim"] & \bigoplus_{j=1}^{s} H^1(\tilde{I}_j, \mathcal{M}) \\ H^1(Y, F_{\mathcal{M}}) \arrow[u, "(2.3.3)^{-1}\ \wr"] \arrow[r, "\mathrm{edge}"'] & H^0(X, R^1 j_* F_{\mathcal{M}}) \arrow[r, "\sim"] & \bigoplus_{j=1}^{s} H^1(\mathrm{Spec}(K^{\tilde{I}_j}), \mathcal{M}), \arrow[u, "\wr"'] \end{tikzcd}$$

because both of the two mappings $H^1(Y, F_{\mathcal{M}}) \to \bigoplus_{j=1}^{s} H^1(\tilde{I}_j, \mathcal{M})$ obtained by composition are induced from the correspondence: (an $F_{\mathcal{M}}$-torsor P on Y)$\mapsto$ ($\Gamma(\mathrm{Spec}(\bar{K}), P)$ viewed as an $\mathcal{M}$-torsor in $T_{\tilde{I}_j})_{1\le j\le s}$. Our conclusion now follows immediately. ■

Remark (3.2.9). As is well-known, there is a canonical isomorphism:

$$H^1(X, j_*F) \cong \mathrm{Im}(H_c^1(Y, F) \longrightarrow H^1(Y, F)),$$

where H_c^1 denotes the cohomology group with compact support. In fact, this follows from the long exact sequence of cohomology groups deduced from:

$$0 \longrightarrow j_!F \longrightarrow j_*F \longrightarrow i_*i^*j_*F \longrightarrow 0 \qquad \text{(exact)},$$

where $i: C \hookrightarrow X$ is the natural closed immersion.

3.3. Parabolic cohomology groups with "generic" coefficients. Let the notation be as in 2.4. If we denote by $j_{1,n}: Y_{1,n} \hookrightarrow X_{1,n}$ the natural open immersion, we have the mapping:

$$H^1(X_{1,n}, \boldsymbol{Z}_l) \xrightarrow{\text{can.}} H^1(Y_{1,n}, j_{1,n}^* \boldsymbol{Z}_l) = H^1(Y_{1,n}, \boldsymbol{Z}_l). \tag{3.3.1}$$

Notice that this mapping is identical with the edge homomorphism: $H^1(X_{1,n}, \boldsymbol{Z}_l) = H^1(X_{1,n}, j_{1,n*}\boldsymbol{Z}_l) \to H^1(Y_{1,n}, \boldsymbol{Z}_l)$ induced from the Leray spectral sequence for $j_{1,n}$ (cf. the argument of [**EGA III**] Chapter 0 (12.1.7)). Especially the mapping above is injective. When n varies, these mappings are compatible with trace mappings ([**SGA4**] XVII 6.2.3), and hence we obtain:

$$\varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l) \hookrightarrow \varprojlim_{n\in N} H^1(Y_{1,n}, \boldsymbol{Z}_l), \tag{3.3.2}$$

where both projective limits are taken relative to trace mappings.

Theorem (3.3.3). *There is a unique isomorphism of G_{Q^*}-modules:*

$$H_P^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathrm{U}}}) \xrightarrow{\sim} \varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l)$$

which makes the following diagram commutative:

$$\begin{array}{ccccc}
0 & \longrightarrow & H^1_P(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}) & \longrightarrow & H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}) \\
 & & \wr\downarrow & & \downarrow\wr(2.4.4) \\
0 & \longrightarrow & \varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l) & \underset{(3.3.2)}{\longrightarrow} & \varprojlim_{n\in N} H^1(Y_{1,n}, \boldsymbol{Z}_l),
\end{array}$$

where the upper right horizontal arrow is the obvious inclusion.

PROOF. Recall the "Shapiro isomorphism" $s_n: H^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) \simeq H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l) = \mathrm{Hom}_c(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$ (2.5.1), where the right hand side is the group of continuous homomorphisms. We first claim that this induces an isomorphism: $H^1_P(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) \simeq H^1_P(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$. In fact, it is trivial that $H^1_P(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n])$ is sent to $H^1_P(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$ by s_n.

Conversely, suppose that $cl(\alpha)$ $(\alpha \in Z^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]))$ is sent to $H^1_P(\mathfrak{F}_1^n, \boldsymbol{Z}_l)$ by s_n. Thus if $\mathfrak{G}_n = \coprod_i g_i \mathfrak{U}_n = \coprod_i \bar{g}_i$ is disjoint and if $\alpha = \sum_i \alpha_{\bar{g}_i} \cdot \bar{g}_i$ as in 2.5, we have $\alpha_{\bar{1}}(x) = 0$ whenever x is contained in an inertia subgroup of $\mathfrak{F}_1^n$ for some place of M. Since $\alpha(gxg^{-1}) = g\alpha(x) + (1 - gxg^{-1})\alpha(g)$ for any $x \in \mathfrak{F}$, we see that, if x is contained in an inertia subgroup of $g^{-1}\mathfrak{F}_1^n g$, then we have: $\alpha_{\bar{1}}(gxg^{-1}) = \alpha_{\overline{g^{-1}}}(x) = 0$. Now suppose that $x \in \mathfrak{F}$ is contained in an inertia subgroup, and let d be the order of the image $\tilde{x}$ of x in $\mathfrak{G}_n = \mathfrak{F}/\mathfrak{F}^n$, where $\mathfrak{F}^n := \mathrm{Gal}(M/K_n)$. Then we clearly have $x^d \in \mathfrak{F}^n \subseteqq \bigcap_{g\in\mathfrak{F}} g^{-1}\mathfrak{F}_1^n g$, and hence $\alpha(x^d) = (1 + x + \cdots + x^{d-1})\alpha(x) = 0$.

Let $\mathfrak{G}_n = \coprod_k \langle\tilde{x}\rangle \xi_k \mathfrak{U}_n$ be the disjoint double coset decomposition. The number d_k of distinct $\tilde{x}^i \xi_k \mathfrak{U}_n =: \overline{x^i \xi_k}$, for a fixed k, is a divisor of d. If an element $a = \sum_k \sum_{i=0}^{d_k-1} a_{\overline{x^i\xi_k}} \cdot \overline{x^i \xi_k} \in \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]$ $(a_{\overline{x^i\xi_k}} \in \boldsymbol{Z}_l)$, is annihilated by $\sum_{j=0}^{d-1} x^j$, then it is easy to see that $\sum_{i=0}^{d_k-1} a_{\overline{x^i\xi_k}} = 0$ for any k. This implies that $a = \sum_k \sum_{i=0}^{d_k-1} a_{\overline{x^i\xi_k}}(x^i - 1) \cdot \bar{\xi}_k \in (x-1)\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]$, which proves our first claim.

On the other hand, in view of (3.2.8) and the remark after (3.3.1), we know that there is a unique isomorphism: $H^1_P(\mathfrak{F}_1^n, \boldsymbol{Z}_l) \simeq H^1(X_{1,n}, \boldsymbol{Z}_l)$ which makes the following diagram commutative:

$$\begin{array}{ccccc}
0 & \longrightarrow & H^1_P(\mathfrak{F}_1^n, \boldsymbol{Z}_l) & \longrightarrow & H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l) \\
 & & \wr\downarrow & & \downarrow\wr(2.3.3) \\
0 & \longrightarrow & H^1(X_{1,n}, \boldsymbol{Z}_l) & \underset{(3.3.1)}{\longrightarrow} & H^1(Y_{1,n}, \boldsymbol{Z}_l).
\end{array}$$

Summing up what we have said above and taking projective limits, we get our conclusion. ■

Let $\mathrm{Jac}(X^*_{1,n})$ be the Jacobian variety of $X^*_{1,n}$ defined over $\boldsymbol{Q}^*$. Then the Kummer theory gives an isomorphism: $H^1(X_{1,n}, \boldsymbol{Z}_l) \cong T_l(\mathrm{Jac}(X^*_{1,n}))(-1)$ of $G_{\boldsymbol{Q}^*}$-

modules, where, as usual, T_l (resp. (-1)) means the l-adic Tate module (resp. Tate twist). By **[SGA4]** XVII 6.3.18, we therefore obtain an isomorphism of G_{Q*}-modules:

$$\varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l) \cong \varprojlim_{n\in N} T_l(\mathrm{Jac}(X^*_{1,n}))(-1), \tag{3.3.4}$$

where the projective limit in the right hand side is taken relative to the Albanese mappings: $\mathrm{Jac}(X^*_{1,n}) \to \mathrm{Jac}(X^*_{1,m})$ for $n \geqq m$.

3.4. Let the situation be as in 2.6. Especially $\mathfrak{K}$ is a closed subgroup of $\mathfrak{G}$ which normalizes $\mathfrak{U}$. Then it is clear that the right action of $\mathfrak{K}$ on $H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$ (resp. $\varprojlim_{n\in N} H^1(Y_{1,n}, \boldsymbol{Z}_l)$) described there leaves $H^1_P(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$ (resp. $\varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l)$) stable. By (2.6.1), the isomorphism in (3.3.3) is an isomorphism of right $\mathfrak{K}$-modules. On the other hand, the action of $\mathfrak{K}$ on $\varprojlim_{n\in N} H^1(X_{1,n}, \boldsymbol{Z}_l)$ corresponds to the action via "Pic-functoriality" of $\mathfrak{K}$ on $\varprojlim_{n\in N} T_l(\mathrm{Jac}(X^*_{1,n}))(-1)$ as follows: Namely, for each $n \in N$, as explained in 2.6, $k \in \mathfrak{K}$ induces an automorphism of $X_{1,n}$, and thus an automorphism of $\mathrm{Jac}(X^*_{1,n}) \otimes_{Q*} \bar{\boldsymbol{Q}}$ viewed as the Picard variety of $X_{1,n}$. If we define the action of k on $T_l(\mathrm{Jac}(X^*_{1,n}))(-1) = T_l(\mathrm{Jac}(X^*_{1,n})) \otimes T_l(\boldsymbol{G}_m)^{\otimes(-1)}$ by T_l(the action above)$\otimes$id for each $n \in N$, $\varprojlim_{n\in N} T_l(\mathrm{Jac}(X^*_{1,n}))(-1)$ becomes a right $\mathfrak{K}$-module, and the isomorphism (3.3.4) is $\mathfrak{K}$-equivariant.

§ 4. Applications of Ihara's free differential calculus.

As in the previous section, we fix an Ihara's tower (1.1), and assume that $C \neq \emptyset$. Recall that $\mathfrak{F}$ is generated by $2g+s$ elements x_i, y_i $(1 \leqq i \leqq g)$, and z_j $(1 \leqq j \leqq s)$ topologically, and that $\mathfrak{F}$ is a free almost pro-l group of rank $r = 2g+s-1 > 0$ (3.1). In this section, we often write $\{x_1, \cdots, x_r\}$ for $\{x_1, \cdots, x_g, y_1, \cdots, y_g, z_1, \cdots, z_{s-1}\}$, which is a set of topological generators of $\mathfrak{F}$. The content of this section relies heavily on Ihara's (both published and unpublished) results (cf. **[I1]**).

4.1. Ihara's results and some consequences. First, we recall Ihara's free differential calculus. Put $\mathcal{B} = \boldsymbol{Z}_l[[\mathfrak{F}]]$. Then for each j $(1 \leqq j \leqq r)$, there is a continuous $\boldsymbol{Z}_l$-linear mapping

$$\frac{\partial}{\partial x_j} : \mathcal{B} \longrightarrow \mathcal{B} \tag{4.1.1}$$

so that every $\theta \in \mathcal{B}$ can be expressed uniquely as: $\theta = s(\theta) + \sum_{j=1}^r \partial\theta/\partial x_j (x_j - 1)$, where $s: \mathcal{B} \to \boldsymbol{Z}_l$ is the augmentation homomorphism (**[I1]** Theorem 2.1). The mappings $\partial/\partial x_j$ satisfy:

(4.1.2) $$\frac{\partial x_i}{\partial x_j}=\delta_{ij} \qquad (1\leqq i,\ j\leqq r)$$

(4.1.3) $$\frac{\partial(\alpha\beta)}{\partial x_j}=\frac{\partial\alpha}{\partial x_j}\cdot s(\beta)+\alpha\frac{\partial\beta}{\partial x_j} \qquad (\alpha,\ \beta\in\mathcal{B}).$$

THEOREM (4.1.4) (Ihara). *For a pro-l* $\mathfrak{F}$*-module* $\mathcal{M}$*, the correspondence:* $\alpha\mapsto(\alpha(x_j))_{1\leq j\leq r}$ *gives an isomorphism of* $\boldsymbol{Z}_l$*-modules*

$$I: Z^1(\mathfrak{F},\ \mathcal{M}) \xrightarrow{\sim} \mathcal{M}^{\oplus r}\,.$$

This mapping then induces an isomorphism:

$$H^1(\mathfrak{F},\ \mathcal{M}) \xrightarrow{\sim} \operatorname{Coker}(\mathcal{M} \xrightarrow{i} \mathcal{M}^{\oplus r}),$$

where i *sends* $m\in\mathcal{M}$ *to* $((x_j-1)m)_{1\leq j\leq r}$.

PROOF. I is clearly injective and $\boldsymbol{Z}_l$-linear. For any given $m=(m_j)_{1\leq j\leq r}\in\mathcal{M}^{\oplus r}$, define $\alpha:\mathfrak{F}\to\mathcal{M}$ by $\alpha(x)=\sum_{j=1}^{r}(\partial x/\partial x_j)m_j$. By (4.1.3), α belongs to $Z^1(\mathfrak{F},\ \mathcal{M})$, and by (4.1.2), we have $\alpha(x_j)=m_j$ $(1\leqq j\leqq r)$. This shows that $I(\alpha)=m$. The second assertion follows from the first one easily. ■

Let $\pi:\mathcal{B}\to\mathcal{A}$ be the algebra homomorphism induced from the natural projection: $\mathfrak{F}\to\mathcal{B}$.

COROLLARY (4.1.5). *Assume that* $\mathcal{M}$ *is a pro-l* $\tilde{\mathfrak{G}}$*-module. For* $\rho\in G_{Q^*}$*, let* $\tilde{\rho}\in\operatorname{Gal}(M/L^*)$ *be its lifting. Denote by* $M(\tilde{\rho})$ *the* $r\times r$ *matrix whose* $(i,\ j)$*-entry is* $\pi(\partial(\tilde{\rho}x_i\tilde{\rho}^{-1})/\partial x_j)\in\mathcal{A}$*. Then the action of* ρ *on* $H^1(\mathfrak{F},\ \mathcal{M})$ *corresponds, via* (4.1.4)*, to the automorphism:*

$$\begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} \bmod \operatorname{Im}(i) \longmapsto J_{\rho^{-1}}(M(\tilde{\rho}))\begin{bmatrix} \rho^{-1}m_1 \\ \vdots \\ \rho^{-1}m_r \end{bmatrix} \bmod \operatorname{Im}(i)$$

of $\operatorname{Coker}(i)$*, where we view an element of* $\mathcal{M}^{\oplus r}$ *as a column vector.*

PROOF. Recall that we defined the action of ρ on $H^1(\mathfrak{F},\ \mathcal{M})$ by $cl(\alpha)^{\rho}=cl(\alpha^{\tilde{\rho}})$ with $\alpha^{\tilde{\rho}}(x)=\rho^{-1}\alpha(\tilde{\rho}x\tilde{\rho}^{-1})$ (2.1.6). For each x_i $(1\leqq i\leqq r)$, in view of the proof of (4.1.4), we have

$$\alpha^{\tilde{\rho}}(x_i)=\rho^{-1}\Big(\sum_{j=1}^{r}\frac{\partial(\tilde{\rho}x_i\tilde{\rho}^{-1})}{\partial x_j}\alpha(x_j)\Big)=\sum_{j=1}^{r}J_{\rho^{-1}}\Big(\pi\Big(\frac{\partial(\tilde{\rho}x_i\tilde{\rho}^{-1})}{\partial x_j}\Big)\Big)\cdot\rho^{-1}\alpha(x_j).$$

■

Now let us specialize to the case where $\mathcal{M}=\mathcal{A}$. As above, we define $i:\mathcal{A}\to\mathcal{A}^{\oplus r}$ by $i(a)=((x_j-1)a)_{1\leq j\leq r}$. Then by (2.4.1) and (4.1.4) (cf. also (2.6.1)), we have an isomorphism of right $\mathcal{A}$-modules:

(4.1.6) $$\varprojlim_{n\in N}H^1(Y_n,\ \boldsymbol{Z}_l)\cong\operatorname{Coker}(i)\,.$$

Notice here that i is injective. In fact, it is easy to see that $a\in\mathrm{Ker}(i)$ implies $(g-1)a=0$ for all $g\in\mathfrak{G}$. But by the condition (A3), each $\pi(z_j)-1\in\mathcal{A}$ $(1\leqq j\leqq s)$ is not a left (or right) zero-divisor by [**I1**] Lemma 3.1; and hence we must have $a=0$. The isomorphism above therefore gives *a free resolution of* $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ *of length* 1 *as a right* $\mathcal{A}$*-module.* Also, it is clear from (3.2.4), (3.3.3) and (4.1.4) that we have an isomorphism of right $\mathcal{A}$-modules:

$$\text{(4.1.7)} \qquad \varprojlim_{n\in N}H^1(X_n, \boldsymbol{Z}_l)\cong \mathrm{Ker}(\mathrm{Coker}(i)\longrightarrow \oplus_{j=1}^{s}\mathcal{A}/(z_j-1)\mathcal{A}),$$

where the right arrow corresponds, via (4.1.4), to the mapping f in (3.2.4).

4.2. The case where $Y^*=\boldsymbol{P}^1_{\boldsymbol{Q}^*}-\{0, 1, \infty\}$. In this case, we write $x_1=x$, $x_2=y$ and $x_3=z$, respectively. For $\rho\in G_{\boldsymbol{Q}^*}$ and its lifting $\tilde{\rho}\in\mathrm{Gal}(M/L^*)$ as before, it is known that:

$$\text{(4.2.1)} \qquad \begin{cases} \tilde{\rho}x\tilde{\rho}^{-1}=sx^{\alpha}s^{-1} \\ \tilde{\rho}y\tilde{\rho}^{-1}=ty^{\alpha}t^{-1}, \end{cases}$$

for some $s, t\in\mathfrak{F}$, with $\alpha=\chi(\rho^{-1})$, where $\chi: G_{\boldsymbol{Q}^*}\to\hat{\boldsymbol{Z}}^{\times}$ is the cyclotomic character ([**I1**] Proposition 1.2; remember our convention on the composition law on Galois groups, which is inverse to that in [**I1**]). We then recall Ihara's anti 1-cocycles ([**I1**] p. 433, p. 435):

$$\text{(4.2.2)} \qquad \begin{cases} \psi(\rho):=\pi\Big(s-\dfrac{\partial(s-t)}{\partial x}(\pi-1)\Big)=\pi\Big(t-\dfrac{\partial(t-s)}{\partial y}(y-1)\Big) \\ \psi_x(\rho):=\pi\Big(s\dfrac{x^{\alpha}-1}{x-1}\Big)+(1-J_{\rho}(\pi(x)))\pi\Big(\dfrac{\partial(s-t)}{\partial x}\Big) \\ \psi_y(\rho):=\pi\Big(t\dfrac{y^{\alpha}-1}{y-1}\Big)+(1-J_{\rho}(\pi(y)))\pi\Big(\dfrac{\partial(t-s)}{\partial y}\Big)\,. \end{cases}$$

THEOREM (4.2.3) (Ihara). *Let $\mathcal{M}$ be a pro-l $\tilde{\mathfrak{G}}$-module. Then the action of $\rho\in G_{\boldsymbol{Q}^*}$ on $H^1(\mathfrak{F}, \mathcal{M})$ corresponds, via* (4.1.4), *to the automorphism:*

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \bmod \mathrm{Im}(i) \longmapsto \begin{bmatrix} J_{\rho^{-1}}(\psi_x(\rho)) & 0 \\ 0 & J_{\rho^{-1}}(\psi_y(\rho)) \end{bmatrix}\begin{bmatrix} \rho^{-1}m_1 \\ \rho^{-1}m_2 \end{bmatrix} \bmod \mathrm{Im}(i)$$

of $\mathrm{Coker}(\mathcal{M}\xrightarrow{i}\mathcal{M}^{\oplus 2})$.

PROOF. By [**I1**] Lemma 3.2, we see from (4.2.1) that:

$$M(\tilde{\rho})=\pi\left(\begin{bmatrix} s\cdot(x^{\alpha}-1)/(x-1)+(1-\tilde{\rho}x\tilde{\rho}^{-1})\partial s/\partial x & (1-\tilde{\rho}x\tilde{\rho}^{-1})\partial s/\partial y \\ (1-\tilde{\rho}y\tilde{\rho}^{-1})\partial t/\partial x & t\cdot(y^{\alpha}-1)/(y-1)+(1-\tilde{\rho}y\tilde{\rho}^{-1})\partial t/\partial y \end{bmatrix}\right).$$

A simple calculation then shows that $J_{\rho^{-1}}\left(M(\tilde{\rho})-\begin{bmatrix}\psi_x(\rho) & 0\\ 0 & \psi_y(\rho)\end{bmatrix}\right)$ annihilates Coker(i); and hence the conclusion follows from (4.1.5). ■

Note especially that the action of ρ on $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)\cong\mathrm{Coker}(\mathcal{A}\xrightarrow{i}\mathcal{A}^{\oplus 2})$ is induced from the automorphism:

$$\begin{bmatrix}a_1\\ a_2\end{bmatrix}\longmapsto\begin{bmatrix}J_{\rho^{-1}}(\psi_x(\rho)) & 0\\ 0 & J_{\rho^{-1}}(\psi_y(\rho))\end{bmatrix}\begin{bmatrix}J_{\rho^{-1}}(a_1)\\ J_{\rho^{-1}}(a_2)\end{bmatrix} \tag{4.2.4}$$

of $\mathcal{A}^{\oplus 2}$.

COROLLARY (4.2.5). *For $\rho\in G_{Q^*}$, ρ acts on $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ trivially if and only if $\rho\in\mathrm{Ker}(\psi)$; i.e., $\psi(\rho)=1$.*

PROOF. Assume that ρ acts on $\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)\cong H^1(\mathfrak{F}, \mathcal{A})$ trivially. The injection $i_x:\mathcal{A}\hookrightarrow\mathcal{A}^{\oplus 2}$ $\left(a\mapsto\begin{bmatrix}a\\ 0\end{bmatrix}\right)$ induces, via (4.1.4), an injective homomorphism $\iota_x:\mathcal{A}\hookrightarrow H^1(\mathfrak{F}, \mathcal{A})$ by Lemma 3.1 or Proposition 1.4 of [**I1**]. The description of the action of ρ above then implies that $a=J_{\rho^{-1}}(\psi_x(\rho)a)$; i.e., $\psi_x(\rho)a=J_\rho(a)$ for all $a\in\mathcal{A}$. Taking especially $a=1\in\mathfrak{G}$, we obtain: $\psi_x(\rho)=1$; and hence also $a=J_\rho(a)$ for all a. From the relations ([**I1**] p. 435):

$$\begin{cases}\psi_x(\rho)(\pi(x)-1)=(J_\rho(\pi(x))-1)\psi(\rho)\\ \psi_y(\rho)(\pi(y)-1)=(J_\rho(\pi(y))-1)\psi(\rho),\end{cases} \tag{4.2.6}$$

we conclude that $\psi(\rho)=1$.

Conversely, suppose that $\psi(\rho)=1$. Then this implies that $J_\rho=\mathrm{id}$ (cf. Ihara [**I2**] § 3). Therefore we obtain $\psi_x(\rho)=1$ from (4.2.6); and also $\psi_y(\rho)=1$ for the same reason. ■

In [**I2**] loc. cit., Ihara relates ψ with a certain representation $G_{Q^*}\to\Psi$ (which is called "ψ" there), whose kernel coincides with Ker(ψ). Especially in the case of Example (M) (1.2), we see from the corollary above that the kernels of the two Galois representations: $G_Q\to\mathrm{Aut}(\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l))$ and $G_Q\to\mathrm{Out}(\pi_1^{pro\text{-}l}(\boldsymbol{P}^1_{\boldsymbol{Q}}-\{0, 1, \infty\}))$ coincide.

Finally, let us describe $H^1_P(\mathfrak{F}, \mathcal{A})$. For $\alpha\in Z^1(\mathfrak{F}, \mathcal{A})$, α belongs to $Z^1_P(\mathfrak{F}, \mathcal{A})$ if and only if $\alpha(x)\in(x-1)\mathcal{A}$, $\alpha(y)\in(y-1)\mathcal{A}$, and $\alpha(z)\in(z-1)\mathcal{A}$, by definition. The last condition is equivalent to $\alpha(z^{-1})\in(z-1)\mathcal{A}$; and by the relation $xyz=1$, this in turn is equivalent to $x\alpha(y)+\alpha(x)\in(z-1)\mathcal{A}$. Therefore I (4.1.4) induces as isomorphism:

$$Z^1_P(\mathfrak{F}, \mathcal{A})\xrightarrow{\sim}\left\{\begin{bmatrix}a_1\\ a_2\end{bmatrix}\in(x-1)\mathcal{A}\oplus(y-1)\mathcal{A}\mid a_1+xa_2\in(z-1)\mathcal{A}\right\}.$$

It is then easy to see that ι_x above gives an isomorphism:

$$(x-1)\mathcal{A}\cap(z-1)\mathcal{A} \xrightarrow{\sim} H_P^1(\mathfrak{F}, \mathcal{A}) \tag{4.2.7}$$

of right $\mathcal{A}$-modules. The action of $\rho\in G_{Q^*}$ on $H_P^1(\mathfrak{F}, \mathcal{A})$ corresponds to $(x-1)\mathcal{A}\cap(z-1)\mathcal{A}\ni a\mapsto J_{\rho^{-1}}(\psi_x(\rho)a)$ via this isomorphism.

4.3. We add here another application of (4.1.4). Let the notation be as in 3.1. For a Γ-module M, we denote by $H^i(\Gamma, M)$ the usual group cohomology. Replacing "continuous cochains" by the usual cochains, we can also define $H_P^i(\Gamma, M)$ for $i=1, 2$ in the same manner as in (3.2.1), which are nothing but the classical parabolic cohomology groups ([**Sh1**] 8.1) when $\mathcal{V}\cong H$.

PROPOSITION (4.3.1). *Let $\mathcal{M}$ be a pro-l $\mathfrak{F}$-module. Then the homomorphism:*

$$H^1(\mathfrak{F}, \mathcal{M}) \longrightarrow H^1(\Gamma, \mathcal{M})$$

obtained by restricting (continuous) 1-cocycles on $\mathfrak{F}$ to Γ, is an isomorphism. Also, this induces an isomorphism:

$$H_P^1(\mathfrak{F}, \mathcal{M}) \xrightarrow{\sim} H_P^1(\Gamma, \mathcal{M}).$$

PROOF. The classical free differential calculus (Fox [**F**]) applied to Γ yields, exactly in the same manner as in the proof of (4.1.4), an isomorphism: $H^1(\Gamma, M)\to\mathrm{Coker}(M\xrightarrow{i} M^{\oplus r})$ for any Γ-module M. The first assertion follows from this and (4.1.4). The second assertion follows from the first one and the fact that the exact sequence (3.2.4) also holds if one replaces $\mathfrak{F}$ by Γ. ∎

§5. Cohomology groups with compact support.

5.1. Preliminaries on G-sheaves. In [**Gr**] Chapter V, Grothendieck developed a general cohomology theory of G-sheaves on topological spaces. As he noticed later (SGA 5X1), a similar theory holds for étale cohomology groups. In the following, we shall freely use this language for étale cohomology groups, referring to [**Gr**] for the corresponding statements in the classical case.

We first recall some definitions. Suppose that a group G acts on a scheme T from the left. Then a (set-valued) sheaf F on $T_{\acute{e}t}$ is called a G-sheaf if we are given an isomorphism $\varphi_F(g)\colon F\simeq g^*F$ for each $g\in G$, and the diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi_F(g_1g_2)} & (g_1g_2)^*F \\ {\scriptstyle\varphi_F(g_2)}\downarrow & & \downarrow\wr \\ g_2^*F & \xrightarrow[g_2^*\varphi_F(g_1)]{} & g_2^*g_1^*F \end{array} \tag{5.1.1}$$

commutes for each $g_1, g_2 \in G$. If we denote by $\psi_F(g): F \to g_* F$ the morphism corresponding to $\varphi_F(g)^{-1}: g^* F \to F$ by adjunction, this is equivalent to the commutativity of:

$$\begin{tikzcd} F \arrow[r, "\psi_F(g_1 g_2)"] \arrow[d, "\psi_F(g_1)"'] & (g_1 g_2)_* F \arrow[d, "\wr"] \\ g_{1*} F \arrow[r, "g_{1*}\psi_F(g_2)"'] & g_{1*} g_{2*} F . \end{tikzcd} \tag{5.1.2}$$

Call $T^{\sim}_{\acute{e}t, G}$ the category of (set-valued) G-sheaves on $T_{\acute{e}t}$, with the obvious definition of morphisms. The category $T^{\sim\, ab}_{\acute{e}t, G}$ of abelian G-sheaves on $T_{\acute{e}t}$ can be defined similarly. For $F \in T^{\sim}_{\acute{e}t, G}$ and any $U \to T$ étale, the composites of

$$\begin{cases} F(U) \xrightarrow{\text{can.}} g^* F(g^* U) \xrightarrow{\varphi_F(g)^{-1}} F(g^* U), \quad \text{and} \\ F(U) \xrightarrow{\psi_F(g)} g_* F(U) = F(g^* U) \end{cases} \tag{5.1.3}$$

coincide, where we denote by $g^* U$ the base change of U by g. Next assume that T is an S-scheme, and let $f: T \to S$ be the structure morphism. Suppose that G acts on T as S-automorphisms. Then, for any $U \to S$ étale, G acts on $f_* F(U) = F(f^* U) = F(g^* f^* U)$ by (5.1.3). Let $f^G_* F$ be the sheaf on $S_{\acute{e}t}$ defined by

$$f^G_* F(U) := f_* F(U)^G \tag{5.1.4}$$

for $U \to S$ étale. Thus on $T^{\sim\, ab}_{\acute{e}t, G}$, $f^G_* F$ is defined as the kernel of

$$\Pi_{g \in G}(f_* \psi_F(g) - \mathrm{id}) : f_* F \longrightarrow \Pi_{g \in G} f_* F . \tag{5.1.5}$$

Now let X' be a complete variety defined over a separably closed field k. We assume that a finite group G acts on X' as k-automorphisms admissibly from the left (in the sense of SGA 1 V 1.7). Let $f: X' \to X := G\backslash X'$ be the quotient morphism. Also assume that we are given an open immersion $j: Y \hookrightarrow X$ defined over k. Then G acts on $Y' := Y \times_X X'$ admissibly, and we may identify Y with $G\backslash Y'$. Call h the restriction of f to Y', and j' the open immersion: $Y' \hookrightarrow X'$:

$$\begin{tikzcd} Y' \arrow[r, "j'"] \arrow[d, "h"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "j"'] & X . \end{tikzcd} \tag{5.1.6}$$

Recall that the base change morphism induces a canonical isomorphism of functors on $Y^{\sim\, ab}_{\acute{e}t}$:

$$f^* j_! \cong j'_! h^* \tag{5.1.7}$$

([**SGA4**] XVII 5.1.2). On the other hand, there is a canonical isomorphism: $f_* j'_! \cong j_! h_*$ on $Y'^{\sim, ab}_{\acute{e}t}$ because f and h are proper ([**SGA4**] XVII 6.1). From this and the description (5.1.5) above, it is easy to see that $j'_!$ naturally defines a functor: $Y'^{\sim, ab}_{\acute{e}t, G} \to X'^{\sim, ab}_{\acute{e}t, G}$, and that the isomorphism above gives rise to a canonical isomorphism of functors:

$$f^G_* j'_! \cong j_! h^G_* \tag{5.1.8}$$

on $Y'^{\sim, ab}_{\acute{e}t, G}$. From the adjunction morphisms, we also have monomorphisms:

$$\begin{cases} h^* h^G_* F \hookrightarrow F \\ f^* f^G_* j'_! F \hookrightarrow j'_! F \end{cases} \tag{5.1.9}$$

for each $F \in Y'^{\sim, ab}_{\acute{e}t, G}$, which are isomorphisms when h is étale (use [**Gr**] Theorem 5.3.1, which holds without the assumption (D) there in our present case (cf. [**SGA4**] VIII 5.5), to look at stalks).

For any abelian sheaf F on $Y_{\acute{e}t}$, we have the canonical homomorphism of étale cohomology groups with compact support:

$$H^i_c(Y, F) \longrightarrow H^i_c(Y', F), \tag{5.1.10}$$

which is by definition the composite of: $H^i_c(Y, F) = H^i(X, j_! F) \xrightarrow{\text{can.}} H^i(X', f^* j_! F) \overset{(5.1.7)}{\cong} H^i(X', j'_! h^* F) = H^i_c(Y', F)$. G acts on $H^i_c(Y', F)$ in a natural manner, and the image of (5.1.10) is G-invariant. For $F \in Y'^{\sim, ab}_{\acute{e}t, G}$, we can also define a homomorphism:

$$H^1_c(Y, h^G_* F) \longrightarrow H^1_c(Y', F) \tag{5.1.11}$$

as the composite of: $H^1_c(Y, h^G_* F) \xrightarrow{(5.1.10)} H^1_c(Y', h^* h^G_* F) \xrightarrow{(5.1.9)} H^1_c(Y', F)$. G again acts on $H^1_c(Y', F)$ in a natural manner, and the image of (5.1.11) is G-invariant. It can be easily seen that the mapping (5.1.11) coincides with the composite of: $H^1_c(Y, h^G_* F) = H^1(X, j_! h^G_* F) \overset{(5.1.8)}{\cong} H^1(X, f^G_* j'_! F) \xrightarrow{\text{can.}} H^1(X', f^* f^G_* j_! F) \xrightarrow{(5.1.9)} H^1(X', j'_! F) = H^1_c(Y', F)$.

PROPOSITION (5.1.12). *The notation being as above, assume that h is étale. Then for any $F \in Y'^{\sim, ab}_{\acute{e}t, G}$ satisfying $H^0_c(Y', F) = \{0\}$, the mapping* (5.1.11) *above induces an isomorphism*

$$H^1_c(Y, h^G_* F) \xrightarrow{\sim} H^1_c(Y', F)^G .$$

PROOF. Denoting by $H^n(X'; G, -)$ the n-th right derived functor of $H^0(X, f^G_*(-)) = H^0(X', -)^G$ on $X'^{\sim, ab}_{\acute{e}t, G}$, we have two spectral sequences:

$$E^{p,q}_2 = H^p(X, R^q f^G_*(j'_! F)) \Longrightarrow H^{p+q}(X'; G, j'_! F)$$

$$E_2^{p,q} = H^p(G, H^q(X', j'_!F)) \Longrightarrow H^{p+q}(X'; G, j'_!F)$$

by [**Gr**] Theorem 5.2.1. Under our assumption on h, we see by [**Gr**] Theorem 5.3.1 that the stalk of $R^q f_*^G(j'_!F)$ at any geometric point of X vanishes for $q>0$; and hence the first spectral sequence degenerates. We therefore obtain two isomorphisms: $H^1(X, f_*^G j'_!F) \simeq H^1(X'; G, j'_!F)$, and $H^1(X'; G, j'_!F) \simeq H^1(X', j'_!F)^G$. Composing them with the isomorphism: $H_c^1(Y, h_*^G F) \overset{(5.1.8)}{\cong} H^1(X, f_*^G j'_!F)$, we obtain an isomorphism: $H_c^1(Y, h_*^G F) \simeq H_c^1(Y', F)^G$, which is easily seen to be induced from the mapping (5.1.11), in view of the remark made before the proposition (cf. the remark after (5.2.7) in [**Gr**]). ■

5.2. The group of "modular symbols". Let us return to the situation of §3. We fix a set $\{K_\nu\}_{\nu \in N'}$ of finite Galois subextensions of M/K such that the composite of all K_ν is M. We define the partial order in N' by setting $\nu \geqq \mu$ if and only if $K_\nu \supseteq K_\mu$. We may (and do) assume that: i) N' is a directed set with respect to this partial order; and ii) $\mathrm{Gal}(M/K_\nu)$ is a normal subgroup of $\mathrm{Gal}(M/K^*)$ for each $\nu \in N'$ (cf. 1.1). For $\nu \in N'$, we denote by X_ν (resp. Y_ν) the normalization of X (resp. Y) in K_ν, and set $C_\nu := X_\nu - Y_\nu$. As before, C_ν will be often identified with a subset of $X_\nu(\bar{\boldsymbol{Q}})$. If $\nu \geqq \mu$ ($\mu, \nu \in N'$), the morphism $X_\nu \to X_\mu$ induces a mapping $C_\nu \to C_\mu$; and hence a $\boldsymbol{Z}_l$-homomorphism $\boldsymbol{Z}_l[C_\nu] \to \boldsymbol{Z}_l[C_\mu]$. We denote by $\deg_\nu : \boldsymbol{Z}_l[C_\nu] \to \boldsymbol{Z}_l$ the mapping which sends an element of $\boldsymbol{Z}_l[C_\nu]$ to the sum of its coefficients, and by $\boldsymbol{Z}_l[C_\nu]_0 := \mathrm{Ker}(\deg_\nu)$ the "degree 0-part" of $\boldsymbol{Z}_l[C_\nu]$. It is obvious that $\deg_\nu$'s are compatible with $\boldsymbol{Z}_l[C_\nu] \to \boldsymbol{Z}_l[C_\mu]$.

DEFINITION (5.2.1). The notation being as above, we set

$$\begin{cases} \mathcal{D} := \varprojlim_{\nu \in N'} \boldsymbol{Z}_l[C_\nu] \\ \deg := \varprojlim_{\nu \in N'} \deg_\nu : \mathcal{D} \longrightarrow \boldsymbol{Z}_l \\ \mathcal{D}_0 := \mathrm{Ker}(\deg) = \varprojlim_{\nu \in N'} \boldsymbol{Z}_l[C_\nu]_0 . \end{cases}$$

By our assumption ii) above, $\mathrm{Gal}(M/K^*)$ acts on $C_\nu (\subset X_\nu)$ from the left, and hence $\mathcal{D}$ and $\mathcal{D}_0$ are continuous left $\mathrm{Gal}(M/K^*)$-modules.

For a pro-l $\mathfrak{G}$-module $\mathcal{M}$, let $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ be the group of continuous homomorphisms of $\mathfrak{F}$-modules. If $\mathcal{M} = \varprojlim_{i \in I} \mathcal{M}_i$ with pro-l $\mathfrak{G}$-modules $\mathcal{M}$ and $\mathcal{M}_i$, then it is clear that the natural mapping: $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}) \to \varprojlim_{i \in I} \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}_i)$ is an isomorphism. On the other hand, when $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, we can consider $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ as a right $G_{\boldsymbol{Q}^*}$-module as follows: For $\rho \in G_{\boldsymbol{Q}^*}$, take its lifting $\tilde{\rho} \in \mathrm{Gal}(M/L^*)$. Then for $f \in \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$, we define

$$f^\rho(d) := \rho^{-1} f(\tilde{\rho}(d)) \quad (d \in \mathcal{D}_0). \tag{5.2.2}$$

It is easy to see that this is well-defined.

Recall that $\mathfrak{F}$ is a completion of Γ (3.1). When the universal covering space $\mathcal{U}$ of $Y(\boldsymbol{C})$ is isomorphic to H, let $\mathcal{C}$ be the set of cusps of Γ viewed as a Fuchsian group of the first kind (loc. cit.), and denote by $D:=\boldsymbol{Z}[\mathcal{C}]$ the free abelian group on $\mathcal{C}$. When $\mathcal{U}\cong\boldsymbol{C}$, each C_ν consists of just two elements, and hence so is $\mathcal{C}:=\varprojlim_{\nu\in N'}C_\nu$. We again put $D:=\boldsymbol{Z}[\mathcal{C}]$. In either case, we can define the "degree 0-part" D_0 of D in an obvious way. The natural mappings: $\mathcal{C}\to C_\nu$ induce injective homomorphisms $D\to\mathcal{D}$ and $D_0\to\mathcal{D}_0$ whose image is dense.

PROPOSITION (5.2.3). *For a pro-l $\mathfrak{G}$-module $\mathcal{M}$, the natural mappings:*

$$\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D},\mathcal{M})\longrightarrow\mathrm{Hom}_{\Gamma}(D,\mathcal{M}),\quad \textit{and}$$

$$\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0,\mathcal{M})\longrightarrow\mathrm{Hom}_{\Gamma}(D_0,\mathcal{M})$$

are isomorphisms.

PROOF. Take a set $\{c_1,\cdots,c_s\}$ of representatives of $\Gamma\backslash\mathcal{C}$. We may assume that $z_j\in\Gamma$ generates the stabilizer subgroup Γ_j of c_j in Γ $(1\leqq j\leqq s)$. Then D is isomorphic to $\bigoplus_{j=1}^s\boldsymbol{Z}[\Gamma/\Gamma_j]$ as a left Γ-module; and hence $\mathrm{Hom}_\Gamma(D,\mathcal{M})\cong\bigoplus_{j=1}^s\mathcal{M}^{\Gamma_j}$. On the other hand, the inertia group $I_j(\subseteq\mathfrak{F})$ of the image of c_j in $\varprojlim_{\nu\in N'}C_\nu=$("cuspidal" prime divisors of M) is topologically generated by z_j. Since $\varprojlim_{\nu\in N'}C_\nu\cong\bigoplus_{j=1}^s\mathfrak{F}/I_j$, we have an isomorphism $\mathcal{D}\cong\bigoplus_{j=1}^s\boldsymbol{Z}_l[[\mathfrak{F}]]/\boldsymbol{Z}_l[[\mathfrak{F}]](z_j-1)$ of $\mathfrak{F}$-modules. This implies that $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D},\mathcal{M})\cong\bigoplus_{j=1}^s\mathcal{M}^{I_j}=\bigoplus_{j=1}^s\mathcal{M}^{\Gamma_j}$; and hence the first mapping is an isomorphism.

Next, from the following obvious morphism of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & D_0 \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & \boldsymbol{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}_0 \arrow[r] & \mathcal{D} \arrow[r] & \boldsymbol{Z}_l \arrow[r] & 0,
\end{tikzcd}$$

we obtain a morphism of well-known exact sequences:

(5.2.4)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{c,\mathfrak{F}}(\boldsymbol{Z}_l,\mathcal{M}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D},\mathcal{M}) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathrm{Hom}_\Gamma(\boldsymbol{Z},\mathcal{M}) \arrow[r] & \mathrm{Hom}_\Gamma(D,\mathcal{M})
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r] & \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Ext}_{c,\mathfrak{F}}(\boldsymbol{Z}_l,\mathcal{M}) \arrow[r] \arrow[d] & \mathrm{Ext}_{c,\mathfrak{F}}(\mathcal{D},\mathcal{M}) \arrow[d] \\
\arrow[r] & \mathrm{Hom}_\Gamma(D_0,\mathcal{M}) \arrow[r] & \mathrm{Ext}_\Gamma(\boldsymbol{Z},\mathcal{M}) \arrow[r] & \mathrm{Ext}_\Gamma(D,\mathcal{M}),
\end{tikzcd}$$

where $\mathrm{Ext}_{c,\mathfrak{F}}$ (resp. Ext_Γ) means the group of isomorphism classes of extensions in the category of profinite $\mathfrak{F}$-modules (i.e. profinite abelian groups with continuous left $\mathfrak{F}$-action) (resp. usual Γ-modules); and the middle right (resp. right) vertical arrow is induced from the correspondence: (an extension of $\mathcal{M}$ by $\boldsymbol{Z}_l$ (resp. $\mathcal{D}$) as profinite $\mathfrak{F}$-modules)$\mapsto$(its "pull-back" by $\boldsymbol{Z}\to\boldsymbol{Z}_l$ (resp. $D\to\mathcal{D}$) viewed as an extension of Γ-modules).

Now for $\alpha\in Z^1(\mathfrak{F}, \mathcal{M})$, let M_α be $\mathcal{M}\times\boldsymbol{Z}_l$ as a topological group on which $x\in\mathfrak{F}$ acts as: $x\cdot(m, a):=(xm+a\alpha(x), a)$. Via the obvious injection $\mathcal{M}\hookrightarrow M_\alpha$ and the surjection $M_\alpha\twoheadrightarrow\boldsymbol{Z}_l$, we can consider M_α as an extension of $\mathcal{M}$ by $\boldsymbol{Z}_l$ in the category of profinite $\mathfrak{F}$-modules. By a direct computation (as is perhaps well-known), we easily see that this correspondence gives an isomorphism: $H^1(\mathfrak{F}, \mathcal{M})\simeq\mathrm{Ext}_{c,\mathfrak{F}}(\boldsymbol{Z}_l, \mathcal{M})$. In a similar fashion, we obtain an isomorphism: $H^1(\Gamma, \mathcal{M})\simeq\mathrm{Ext}_\Gamma(\boldsymbol{Z}, \mathcal{M})$. From this and (4.3.1), we see that the mapping: $\mathrm{Ext}_{c,\mathfrak{F}}(\boldsymbol{Z}_l, \mathcal{M})\to\mathrm{Ext}_\Gamma(\boldsymbol{Z}, \mathcal{M})$ is an isomorphism; and therefore it is enough to show the injectivity of the right vertical arrow to complete the proof. For this, first note that the natural inclusion: $\boldsymbol{Z}_l\hookrightarrow\boldsymbol{Z}_l[[\mathfrak{F}]]$ induces an injective homomorphism of I_j-modules: $\boldsymbol{Z}_l\hookrightarrow\boldsymbol{Z}_l[[\mathfrak{F}]]/\boldsymbol{Z}_l[[\mathfrak{F}]](z_j-1)$ for each j $(1\leqq j\leqq s)$. Thus the correspondence: (an extension of $\mathcal{M}$ by $\boldsymbol{Z}_l[[\mathfrak{F}]]/\boldsymbol{Z}_l[[\mathfrak{F}]](z_j-1)$ as profinite $\mathfrak{F}$-modules)$\mapsto$(its pull-back by the mapping above, viewed as an extension of profinite I_j-modules) induces a homomorphism: $\mathrm{Ext}_{c,\mathfrak{F}}(\boldsymbol{Z}_l[[\mathfrak{F}]]/\boldsymbol{Z}_l[[\mathfrak{F}]](z_j-1), \mathcal{M})\to\mathrm{Ext}_{c,I_j}(\boldsymbol{Z}_l, \mathcal{M})$ for each j $(1\leqq j\leqq s)$, which is easily seen to be injective. On the other hand, by the same reasoning as above, we have an isomorphism: $\mathrm{Ext}_{c,I_j}(\boldsymbol{Z}_l, \mathcal{M})\simeq H^1(I_j, \mathcal{M})$. Arguing in the same manner for (discrete) Γ-modules, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ext}_{c,\mathfrak{F}}(\mathcal{D}, \mathcal{M}) & \longrightarrow & \bigoplus_{j=1}^s\mathrm{Ext}_{c,I_j}(\boldsymbol{Z}_l, \mathcal{M}) & \xrightarrow{\sim} & \bigoplus_{j=1}^s H^1(I_j, \mathcal{M}) \\
\downarrow & & \downarrow & & \downarrow\mathrm{Res} \\
\mathrm{Ext}_\Gamma(D, \mathcal{M}) & \longrightarrow & \bigoplus_{j=1}^s\mathrm{Ext}_{\Gamma_j}(\boldsymbol{Z}, \mathcal{M}) & \xrightarrow{\sim} & \bigoplus_{j=1}^s H^1(\Gamma_j, \mathcal{M})
\end{array}$$

with injective left horizontal arrows. Since $H^1(I_j, \mathcal{M})$ and $H^1(\Gamma_j, \mathcal{M})$ are isomorphic to $\mathcal{M}/(z_j-1)\mathcal{M}$ by the correspondence: $cl(\alpha)\mapsto\alpha(z_j) \bmod (z_j-1)\mathcal{M}$, the mapping "Res" above is an isomorphism; and hence our conclusion follows. ∎

As the argument above shows, $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ is finite whenever $\mathcal{M}$ is finite. Thus for any pro-l $\mathfrak{G}$-module $\mathcal{M}$, $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ is naturally equipped with the structure of a pro-l abelian group; and the action of G_{Q*} defined by (5.2.2) is continuous.

REMARK (5.2.5). The groups $\mathrm{Hom}_\Gamma(D_0, \mathcal{M})$ (in the elliptic modular case (cf. § 7)) were first introduced by Ash and Stevens **[AS2]** in their theory of higher weight modular symbols. Our definition of the groups $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ and the Theorem (5.3.2) below were directly motivated by their work (cf. especially **[AS2]** p. 862). Indeed, the above groups (again in the elliptic modular case) for "large" $\mathcal{M}$ will play a central role in the "p-adic analytic theory of modular symbols" to be developed in a subsequent paper.

5.3. H_c^1 **and "modular symbols".** Let $\mathcal{M}=\varprojlim_{i\in I}\mathcal{M}_i$ be a pro-l $\mathfrak{G}$-module with finite $\mathcal{M}_i$. Then we defined in 2.3 a projective system $\{F_{\mathcal{M}_i}\}_{i\in I}$ of twisted constant sheaves on $Y_{\acute{e}t}$. We put

$$H_c^1(Y,\ F_{\mathcal{M}}):=\varprojlim_{i\in I}H_c^1(Y,\ F_{\mathcal{M}_i})\,. \tag{5.3.1}$$

If $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, G_{Q*} acts on this group continuously from the right.

THEOREM (5.3.2). *The notation being as above and as in* 5.2, $H_c^1(Y,\ F_{\mathcal{M}})$ *is canonically isomorphic to* $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0,\ \mathcal{M})$ *for any pro-l* $\mathfrak{G}$*-module* $\mathcal{M}$. *When* $\mathcal{M}$ *is a pro-l* $\tilde{\mathfrak{G}}$*-module, this isomorphism commutes with the action of* G_{Q*}.

PROOF. We may assume that $\mathcal{M}$ is finite. In this case, there is a $\nu\in N'$ such that $\mathcal{M}$ is a trivial $\mathrm{Gal}(M/K_\nu)$-module. For the moment, we fix such a ν, and put $\mathfrak{F}_\nu:=\mathrm{Gal}(K_\nu/K)$. For $\sigma\in\mathfrak{F}_\nu$ or $\mathfrak{F}$, we denote by $[\sigma]$ the automorphism of Y_ν or X_ν corresponding to $\sigma: K_\nu\xrightarrow{\sim}K_\nu$. Then there is an isomorphism of group schemes $\iota: F_{\mathcal{M}}\times_Y Y_\nu\xrightarrow{\sim}Y_\nu\times\mathcal{M}=:\underline{\mathcal{M}}_{Y_\nu}$ (the constant group scheme) over Y_ν, and the following diagram commutes for all $\sigma\in\mathfrak{F}_\nu$ or $\mathfrak{F}$:

$$\begin{array}{ccc} F_{\mathcal{M}}\times_Y Y_\nu & \xrightarrow{\mathrm{id}\times[\sigma]} & F_{\mathcal{M}}\times_Y Y_\nu \\ {\scriptstyle\iota\wr}\downarrow & & \downarrow{\scriptstyle\wr\iota} \\ Y_\nu\times\mathcal{M} & \xrightarrow[{[\sigma]\times\sigma}]{} & Y_\nu\times\mathcal{M}\,. \end{array} \tag{5.3.3}$$

In other words, if we define $\varphi(\sigma)$: $Y_\nu\times\mathcal{M}\to[\sigma]^*(Y_\nu\times\mathcal{M})=Y_\nu\times\mathcal{M}$ by $\mathrm{id}\times\sigma$, $\{\varphi(\sigma)\}_{\sigma\in\mathfrak{F}_\nu}$ gives an $\mathfrak{F}_\nu$-sheaf structure on $\underline{\mathcal{M}}_{Y_\nu}$. If we denote by $h_\nu: Y_\nu\to Y$ the natural morphism, it is easy to see that $h_{\nu*}^{\mathfrak{F}_\nu}(\underline{\mathcal{M}}_{Y_\nu})\cong F_{\mathcal{M}}$.

Let $j_\nu: Y_\nu\hookrightarrow X_\nu$ (resp. $i_\nu: C_\nu\hookrightarrow X_\nu$) be the natural open (resp. closed) immersion. Then we have the exact sequence in $X_{\nu,\acute{e}t}^{\sim\,ab}$:

$$0\longrightarrow j_{\nu!}F_{\mathcal{M}}\longrightarrow j_{\nu*}F_{\mathcal{M}}\longrightarrow i_{\nu*}i_\nu^*j_{\nu*}F_{\mathcal{M}}\longrightarrow 0\,. \tag{5.3.4}$$

Notice that $j_{\nu*}F_{\mathcal{M}}$ is isomorphic to the constant sheaf on X_ν defined by $\mathcal{M}$, which we denote by $\underline{\mathcal{M}}_{X_\nu}$, and also that $i_{\nu*}i_\nu^*j_{\nu*}F_{\mathcal{M}}$ is isomorphic to the direct image by i_ν of the constant sheaf defined by $\mathcal{M}$ on C_ν. We therefore have isomorphisms:

$$H^0(X_\nu,\ j_{\nu*}F_{\mathcal{M}})\cong\mathcal{M} \tag{5.3.5}$$

$$H^0(X_\nu,\ i_{\nu*}i_\nu^*j_{\nu*}F_{\mathcal{M}})\cong\bigoplus_{x\in C_\nu}\mathcal{M}\,. \tag{5.3.6}$$

If we let $\sigma\in\mathfrak{F}$ act on $\mathcal{M}$ (resp. $\bigoplus_{x\in C_\nu}\mathcal{M}$) by $m\mapsto\sigma^{-1}m$ (resp. $(m_x)_{x\in C_\nu}\mapsto(m'_x)_{x\in C_\nu}$

with $m'_x = \sigma^{-1} m_{\sigma x}$) from the right, it is easily seen from (5.3.3) that (5.3.5) (resp. (5.3.6)) is $\mathfrak{F}$-equivariant. Thus from the long exact sequence of cohomology groups deduced from (5.3.4), we obtain the exact sequence of $\mathfrak{F}$-modules:

$$(5.3.7) \qquad 0 \longrightarrow \mathcal{M} \longrightarrow \bigoplus_{x \in C_\nu} \mathcal{M} \longrightarrow H^1_c(Y_\nu, F_{\mathcal{M}}) \longrightarrow H^1(X_\nu, j_{\nu *} F_{\mathcal{M}}),$$

where the second arrow is the diagonal injection. Now we have a canonical isomorphism:

$$(5.3.8) \qquad \bigoplus_{x \in C_\nu} \mathcal{M} \xrightarrow{\sim} \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu], \mathcal{M})$$

by sending $(m_x)_{x \in C_\nu}$ to f defined by $f(x) = m_x$ $(x \in C_\nu)$. If we define the right action of $\mathfrak{F}$ on $\mathrm{Hom}(\boldsymbol{Z}_l[C_\nu], \mathcal{M})$ by $f^\sigma(x) := \sigma^{-1} f(\sigma x)$, this isomorphism is $\mathfrak{F}$-equivariant. From the split exact sequence:

$$0 \longrightarrow \boldsymbol{Z}_l[C_\nu]_0 \longrightarrow \boldsymbol{Z}_l[C_\nu] \xrightarrow{\deg_\nu} \boldsymbol{Z}_l \longrightarrow 0,$$

we obtain an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\boldsymbol{Z}_l, \mathcal{M}) \longrightarrow \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu], \mathcal{M}) \longrightarrow \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M}) \longrightarrow 0.$$

Thus from (5.3.7) and (5.3.8), we obtain the exact sequence of right $\mathfrak{F}$-modules:

$$(5.3.9) \quad 0 \to \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M}) \to H^1_c(Y_\nu, F_{\mathcal{M}}) \to H^1(X_\nu, j_{\nu *} F_{\mathcal{M}}) \ (\cong H^1(X_\nu, \underline{\mathcal{M}}_{X_\nu})),$$

where we let $\mathfrak{F}$ act on $\mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M})$ by the same formula as above. When $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, by the same reasoning as above replacing $\mathfrak{F}$ by $\mathrm{Gal}(M/K^*)$, we see that (5.3.9) is $\mathrm{Gal}(M/K^*)$-equivariant.

Now fix a $\mu \in N'$ such that $\mathcal{M}$ is a trivial $\mathrm{Gal}(M/K_\mu)$-module. For any $\nu \geqq \mu$, we have a homomorphism: $\mathrm{Hom}(\boldsymbol{Z}_l[C_\mu]_0, \mathcal{M}) \to \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M})$ induced from the obvious mapping: $C_\nu \to C_\mu$, and it is easy to see that the following diagram commutes:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(\boldsymbol{Z}_l[C_\mu]_0, \mathcal{M}) & \longrightarrow & H^1_c(Y_\mu, F_{\mathcal{M}}) & \longrightarrow & H^1(X_\mu, \underline{\mathcal{M}}_{X_\mu}) \\
 & \downarrow & & \downarrow {\scriptstyle (5.1.10)} & & \downarrow {\scriptstyle \mathrm{can.}} \\
0 \longrightarrow & \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M}) & \longrightarrow & H^1_c(Y_\nu, F_{\mathcal{M}}) & \longrightarrow & H^1(X_\nu, \underline{\mathcal{M}}_{X_\nu}).
\end{array}$$

Since $\varinjlim_{\nu \geqq \mu} H^1(X_\nu, \underline{\mathcal{M}}_{X_\nu})$ is isomorphic to the H^1 of $\varprojlim_{\nu \geqq \mu} X_\nu$ with values in the constant sheaf defined by $\mathcal{M}$ ([**SGA4**] VII 5.8), this group vanishes by the definition of the field M (cf. the proof of (2.3.3)). Therefore we have, by taking the inductive limit, an isomorphism of right $\mathfrak{F}$-modules:

$$\varinjlim_{\nu \geqq \mu} \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M}) \cong \varinjlim_{\nu \geqq \mu} H^1_c(Y_\nu, F_{\mathcal{M}}).$$

It is easy to see that the left hand side is isomorphic to the group of continuous homomorphisms $\mathrm{Hom}_c(\mathcal{D}_0, \mathcal{M})$. Therefore taking $\mathfrak{F}$-invariants, we obtain the desired isomorphism:

$$\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}) \cong H_c^1(Y, F_{\mathcal{M}})$$

by applying (5.1.12).

Finally, when $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, this isomorphism is compatible with the action of G_{Q^*}, as is easily seen from the remark after (5.3.9). ■

5.4. Let the notation be as in 5.3. For a pro-l $\mathfrak{G}$-module $\mathcal{M}=\varprojlim_{i\in I}\mathcal{M}_i$ with finite $\mathcal{M}_i$, natural mappings: $H_c^1(Y, F_{\mathcal{M}_i})\to H^1(Y, F_{\mathcal{M}_i})$ induce

$$(5.4.1) \qquad H_c^1(Y, F_{\mathcal{M}}) \longrightarrow H^1(Y, F_{\mathcal{M}}).$$

On the other hand, we define a homomorphism

$$(5.4.2) \qquad \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}) \longrightarrow H^1(\mathfrak{F}, \mathcal{M})$$

as follows: Fix an element $d\in\varprojlim_{\nu\in N'}C_\nu$, and for $f\in\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$, put $\alpha(\sigma):=f(\sigma d-d)$ $(\sigma\in\mathfrak{F})$. It is then easy to see that $\alpha\in Z^1(\mathfrak{F}, \mathcal{M})$, and that we can define (5.4.2) by the correspondence: $f\mapsto cl(\alpha)$, which is independent of the choice of d.

THEOREM (5.4.3). *For each pro-l $\mathfrak{G}$-module $\mathcal{M}$, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}) \arrow[r, "(5.3.2)", "\sim"'] \arrow[d, "(5.4.2)"'] & H_c^1(Y, F_{\mathcal{M}}) \arrow[d, "(5.4.1)"] \\
H^1(\mathfrak{F}, \mathcal{M}) \arrow[r, "\sim", "(2.3.3)"'] & H^1(Y, F_{\mathcal{M}}).
\end{tikzcd}$$

PROOF. We may assume that $\mathcal{M}$ is finite. The mapping (5.4.1) is, by definition, given by: $H_c^1(Y, F_{\mathcal{M}}) = H^1(X, j_!F_{\mathcal{M}}) \xrightarrow{\text{can.}} H^1(Y, j^*j_!F_{\mathcal{M}}) \cong H^1(Y, F_{\mathcal{M}})$. In terms of torsors, it is therefore induced from the correspondence: (a $j_!F_{\mathcal{M}}$-torsor P on X)$\mapsto$($j^*j_!F_{\mathcal{M}}\cong F_{\mathcal{M}}$-torsor j^*P on Y). Also, in view of the proof of (3.2.8), the inverse of the isomorphism (2.3.3) may be interpreted as the correspondence: (isomorphism class of an $F_{\mathcal{M}}$-torsor P)$\mapsto cl(\alpha)$ with $\alpha\in Z^1(\mathfrak{F}, \mathcal{M})$ defined by: $\sigma p=p^{\sigma^{-1}}=p\cdot\alpha(\sigma)$ $(\sigma\in\mathfrak{F})$ for a fixed $p\in\Gamma(Z, P)$.

Now suppose that we are given an element $f\in\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})\cong\varinjlim_{\nu\in N'}\mathrm{Hom}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M})$, and assume that it is represented by an element of $\mathrm{Hom}_{\mathfrak{F}}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M})$, which we denote by the same letter f. We may assume that $\mathcal{M}$ is a trivial $\mathrm{Gal}(M/K_\nu)$-module. Let P_ν (resp. P) be a $j_{\nu!}F_{\mathcal{M}}$-torsor (resp. $j_!F_{\mathcal{M}}$-torsor) on X_ν (resp. X) corresponding to f via:

$$\mathrm{Hom}_{\mathfrak{F}}(\boldsymbol{Z}_l[C_\nu]_0, \mathcal{M}) \xrightarrow{(5.3.9)} H_c^1(Y_\nu, F_{\mathcal{M}})^{\mathfrak{F}} \xleftarrow[\sim]{(5.1.12)} H_c^1(Y, F_{\mathcal{M}}).$$

Then, denoting by $h_\nu: Y_\nu \to Y$ and $f_\nu: X_\nu \to X$ the natural morphisms, $f_\nu^* P$ is isomorphic to P_ν as a $j_{\nu !}F_{\mathcal{M}}$ $(=j_{\nu !}h_\nu^* F_{\mathcal{M}} \cong f_\nu^* j_! F_{\mathcal{M}})$-torsor, and by (5.1.12), P is uniquely characterized by this property up to isomorphisms.

On the other hand, take $\tilde{f} \in \mathrm{Hom}(\boldsymbol{Z}_l[C_\nu], \mathcal{M})$ such that $\tilde{f}|_{\boldsymbol{Z}_l[C_\nu]_0} = f$. Considering $(\tilde{f}(x))_{x \in C_\nu} \in \bigoplus_{x \in C_\nu} \mathcal{M}$ as a section of $i_{\nu *} i_\nu^* j_{\nu *} F_{\mathcal{M}}$ on X_ν by (5.3.6), we can take P_ν to be its inverse image by $j_{\nu *} F_{\mathcal{M}} \to i_{\nu *} i_\nu^* j_{\nu *} F_{\mathcal{M}}$, viewed as a $j_{\nu !}F_{\mathcal{M}}$-torsor via (5.3.4) (cf. [**Gi**] III 3.1, 3.5). Via the canonical isomorphism: $j_{\nu *} F_{\mathcal{M}} \cong \underline{\mathcal{M}}_{X_\nu}$, we may consider P_ν as a subsheaf of $\underline{\mathcal{M}}_{X_\nu}$.

As before, we define $\phi(\sigma): \underline{\mathcal{M}}_{X_\nu} (= X_\nu \times \mathcal{M}) \to [\sigma]^* \underline{\mathcal{M}}_{X_\nu} = \underline{\mathcal{M}}_{X_\nu}$ by $\phi(\sigma) := \mathrm{id} \times \sigma$ $(\sigma \in \mathfrak{F}_\nu)$, and consider $\underline{\mathcal{M}}_{X_\nu}$ as an $\mathfrak{F}_\nu$-sheaf on $X_{\nu, \acute{e}t}$ by means of this. Now it is easy to see that the inverse image by $\phi(\sigma)$ of $[\sigma]^* P_\nu \subset [\sigma]^* \underline{\mathcal{M}}_{X_\nu} = \underline{\mathcal{M}}_{X_\nu}$ is the subsheaf of $\underline{\mathcal{M}}_{X_\nu}$ corresponding to $\tilde{f}^\sigma$ exactly in the same manner as P_ν corresponded to $\tilde{f}$. Denote this sheaf by P_ν^σ. For any $x, y \in C_\nu$, $f^\sigma(x-y) = f(x-y)$ implies that $\tilde{f}^\sigma(x) - \tilde{f}(x) =: m_\sigma \in \mathcal{M} = \Gamma(X_\nu, \underline{\mathcal{M}}_{X_\nu})$ is independent of x, and hence we can define an isomorphism $a(\sigma): P_\nu \to P_\nu^\sigma$ of torsors by: (local section $s) \mapsto s +$ (the pull-back of)m_σ. Denote by $b(\sigma): P_\nu \to [\sigma]^* P_\nu$ the composite of $a(\sigma)$ and $\phi(\sigma)$. Then it is easy to see that $\{b(\sigma)\}_{\sigma \in \mathfrak{F}_\nu}$ defines an $\mathfrak{F}_\nu$-sheaf structure on P_ν, and that the action $P_\nu \times j_{\nu !}F_{\mathcal{M}} \to P_\nu$ is compatible with the $\mathfrak{F}_\nu$-sheaf structures. Applying the functor $f_{\nu *}^{\mathfrak{F}_\nu}$, we get an $f_{\nu *}^{\mathfrak{F}_\nu} j_{\nu !} F_{\mathcal{M}} \cong j_! F_{\mathcal{M}}$-torsor $f_{\nu *}^{\mathfrak{F}_\nu} P_\nu$ on X. Since $f_\nu^*(f_{\nu *}^{\mathfrak{F}_\nu} P_\nu)$ is isomorphic to P_ν as a $j_{\nu !}F_{\mathcal{M}}$-torsor, we may take P to be the torsor above. For any $\sigma \in \mathfrak{F}_\nu$, the right action of σ on $\Gamma(Y_\nu, f_\nu^* P) \cong \Gamma(Y_\nu, P_\nu) = \Gamma(Y_\nu, [\sigma]^* P_\nu) \cong \mathcal{M}$ is then given by $\Gamma(Y_\nu, b(\sigma)^{-1})$. We therefore see that the cohomology class $cl(\alpha) \in H^1(\mathfrak{F}, \mathcal{M})$ corresponding to P via the composite of (5.3.2), (5.4.1) and the inverse of (2.3.3) is given by:

$$\alpha(\sigma) = (\sigma m - m_{\sigma^{-1}}) - m = (\sigma m - (\tilde{f}^{\sigma^{-1}}(x) - \tilde{f}(x))) - m$$

for any fixed $m \in \mathcal{M}$ and $x \in C_\nu$. Fixing $x \in C_\nu$ and putting $m = \tilde{f}(x)$, we conclude that

$$\alpha(\sigma) = \sigma \tilde{f}(x) - \sigma \tilde{f}(\sigma^{-1} x) = \sigma f(x - \sigma^{-1} x) = f(\sigma x - x)$$

for all $\sigma \in \mathfrak{F}$. From this, our conclusion follows. ■

COROLLARY (5.4.4). *For any pro-l $\mathfrak{G}$-module $\mathcal{M}$, the image of the mapping* (5.4.2) *is presicely $H_P^1(\mathfrak{F}, \mathcal{M})$.*

PROOF. In view of (3.2.8) and (3.2.9), this follows immediately from the theorem above. ■

5.5. "Modular symbols" with "generic" values. We now return to the situation considered in (2.4.2), and propose to describe $\varprojlim_{n \in N} H_c^1(Y_{1,n}, \boldsymbol{Z}_l)$ in terms of "modular symbols", where the projective limit is taken relative to the trace mappings.

THEOREM (5.5.1). *There is an isomorphism:*

$$\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}) \xrightarrow{\sim} \varprojlim_{n\in N} H_c^1(Y_{1,n}, \boldsymbol{Z}_l)$$

of right $G_{\boldsymbol{Q}*}$-modules. Moreover this isomorphism makes the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}) \arrow[r, "\sim"] \arrow[d, "(5.4.2)"'] & \varprojlim_{n\in N} H_c^1(Y_{1,n}, \boldsymbol{Z}_l) \arrow[d, "\mathrm{can.}"] \\
H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}) \arrow[r, "\sim", "(2.4.4)"'] & \varprojlim_{n\in N} H^1(Y_{1,n}, \boldsymbol{Z}_l).
\end{tikzcd}$$

PROOF. In general, let $\mathcal{M}$ be a pro-l $\mathfrak{G}$-module, which is a free $\boldsymbol{Z}_l$-module of finite rank. For $n, m\in N$ such that $n\geqq m$, we have an isomorphism: $H_c^1(Y_{1,m}, F_{\mathcal{M}})\cong H_c^1(Y_n, F_{\mathcal{M}})^G$ by (5.1.12), where $G=\mathrm{Gal}(K_n/K_{1,m})$. Identifying these two groups, we first claim that the trace mapping: $H_c^1(Y_n, F_{\mathcal{M}})\rightarrow H_c^1(Y_{1,m}, F_{\mathcal{M}})$ is given by the mapping: $x\mapsto\sum_{g\in G}x^g$, which we call t. In fact, since the groups above are free $\boldsymbol{Z}_l$-modules of finite rank (cf. (5.3.2)), it is enough to show this after tensoring $\boldsymbol{Q}_l$ over $\boldsymbol{Z}_l$. As a G-module, we have a direct sum decomposition: $H_c^1(Y_n, F_{\mathcal{M}})\otimes_{Z_l}\boldsymbol{Q}_l = A\oplus B$, where $A:= (H_c^1(Y_n, F_{\mathcal{M}})\otimes_{Z_l}\boldsymbol{Q}_l)^G$ and B is a direct sum of nontrivial irreducible G-modules. Since the trace mapping is G-equivariant, it is identically zero on B; while t is also zero on B. On the other hand, the composite of: $H_c^1(Y_{1,m}, F_{\mathcal{M}})\simeq H_c^1(Y_n, F_{\mathcal{M}})^G \xrightarrow{\text{trace}} H_c^1(Y_{1,m}, F_{\mathcal{M}})$ is multiplication by $|G|$ by [**SGA4**] XVII 6.2.3. Therefore the trace mapping and t agree also on A, which shows our claim.

From the remark above, it is easy to see that the trace mapping: $H_c^1(Y_{1,n}, F_{\mathcal{M}})\rightarrow H_c^1(Y_{1,m}, F_{\mathcal{M}})$ is given by: $x\mapsto\sum_{g\in\mathfrak{F}_1^m/\mathfrak{F}_1^n}x^{g^{-1}}$. Thus we see that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \mathcal{M}) \arrow[r, "(5.3.2)"] \arrow[d] & H_c^1(Y_{1,n}, F_{\mathcal{M}}) \arrow[d, "\mathrm{trace}"] \\
\mathrm{Hom}_{c,\mathfrak{F}_1^m}(\mathcal{D}_0, \mathcal{M}) \arrow[r, "(5.3.2)"] & H_c^1(Y_{1,m}, F_{\mathcal{M}}),
\end{tikzcd} \tag{5.5.2}$$

if we define the left arrow by: $f\mapsto\sum_{g\in\mathfrak{F}_1^m/\mathfrak{F}_1^n}f^{g^{-1}}$.

Next recall that $\mathfrak{F}/\mathfrak{F}_1^n\cong\mathfrak{G}_n/\mathfrak{U}_n$ canonically as left $\mathfrak{F}$-sets, and let $\mathfrak{G}_n=\coprod_i g_i\mathfrak{U}_n =\coprod_i\bar{g}_i$ be the disjoint decomposition. In analogy with (2.5.1), we define the mapping:

$$\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) \longrightarrow \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l) \tag{5.5.3}$$

by sending $f=\sum_i f_{\overline{g_i}}\cdot\bar{g}_i$, with continuous mappings $f_{\overline{g_i}}$ from $\mathcal{D}_0$ to $\boldsymbol{Z}_l$, to $f_{\overline{1}}$. Then a simple computation shows that this mapping is an isomorphism of G_{Q*}-modules, and that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) & \xrightarrow{\sim} & \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\mathfrak{G}_m/\mathfrak{U}_m]) & \xrightarrow{\sim} & \mathrm{Hom}_{c,\mathfrak{F}_1^m}(\mathcal{D}_0, \boldsymbol{Z}_l)\,, \end{array} \tag{5.5.4}$$

where the left (resp. right) arrow is induced from the natural projection: $\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]\to\boldsymbol{Z}_l[\mathfrak{G}_m/\mathfrak{U}_m]$ (resp. the previous one). Combining (5.5.2) and (5.5.4), and taking projective limits, we obtain our first assertion.

The second assertion follows from (5.4.3) and the commutativity of:

$$\begin{array}{ccc} \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) & \xrightarrow{(5.5.3)} & \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l) \\ {\scriptstyle (5.4.2)}\downarrow & & \downarrow{\scriptstyle (5.4.2)} \\ H^1(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]) & \xrightarrow[(2.5.1)]{} & H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l)\,, \end{array} \tag{5.5.5}$$

which can be checked easily. ■

Suppose finally that we are in the situation of 2.6. Then the group $\mathfrak{K}$ naturally acts on $\varprojlim_{n\in N} H_c^1(Y_{1,n}, \boldsymbol{Z}_l)$ from the right. On the other hand, the obvious right action of $\mathfrak{K}$ on $\mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}}$ induces the action of $\mathfrak{K}$ on $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$. We claim that the isomorphism (5.5.1) is $\mathfrak{K}$-equivariant. In fact, it is clear from the proof of (5.3.2) that the isomorphism: $\mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l)\xrightarrow{\sim} H_c^1(Y_{1,n}, \boldsymbol{Z}_l)$ is $\mathfrak{K}$-equivariant if we let $k\in\mathfrak{K}$ act on the right hand side by: $f\mapsto f^k$ with $f^k(x):=f(\tilde{k}x)$ for any $\tilde{k}\in\mathfrak{F}$ such that $\tilde{k}|_L=k$. It is then easy to see that this action corresponds, via (5.5.3), to the natural right action of k on $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n])$, which shows the desired compatibility.

§ 6. Specialization mappings.

6.1. Let us keep the notation and assumptions of the previous sections. Let $\mathcal{M}$ be a pro-l $\mathfrak{G}$-module, and fix an element $m_0\in\mathcal{M}$. Since $\mathcal{M}$ is an $\mathcal{A}$-module in a natural manner, we obtain a morphism of pro-l $\mathfrak{G}$- (and hence pro-l $\mathfrak{F}$-) modules:

$$\mathcal{A} \longrightarrow \mathcal{M} \qquad (a\mapsto a\cdot m_0)\,. \tag{6.1.1}$$

From this, we obtain

$$(6.1.2)\qquad \begin{cases} sp_{m_0}: H^1(\mathfrak{F}, \mathcal{A}) \longrightarrow H^1(\mathfrak{F}, \mathcal{M}) \\ sp_{P,m_0}: H^1_P(\mathfrak{F}, \mathcal{A}) \longrightarrow H^1_P(\mathfrak{F}, \mathcal{M}) \\ sp_{c,m_0}: \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}) \longrightarrow \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}). \end{cases}$$

We call these mappings "specialization mappings", and write them simply sp, sp_P and sp_c, respectively, when there is no fear of confusion.

If a closed subgroup $\mathfrak{U}$ of $\mathfrak{G}$ stabilizes m_0, then it is easy to see that sp, sp_P, and sp_c factor through $H^1(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$, $H^1_P(\mathfrak{F}, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$ and $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\overline{\mathcal{A}I_{\mathfrak{U}}})$, respectively. The induced mappings will be also called "specialization mappings".

Suppose next that $\mathcal{M}$ is a pro-l $\tilde{\mathfrak{G}}$-module, and hence especially a G_{Q^*}-module (cf. 2.1). If $m_0 \in \mathcal{M}$ above is fixed under G_{Q^*}, then the mappings in (6.1.2) commute with the action of G_{Q^*}, as can be seen easily from the definitions of the Galois action (2.1, 5.2).

PROPOSITION (6.1.3). *Let $\mathcal{M}$ be a pro-l $\mathfrak{G}$-module. If the cokernel of* (6.1.1) *is finite* (*for example, if $\mathcal{M}$ is a free $\boldsymbol{Z}_l$-module of finite rank, $\mathcal{M}\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l$ is an irreducible $\mathfrak{G}$-module, and $m_0\neq 0$*), *then the cokernel of sp: $H^1(\mathfrak{F}, \mathcal{A})\to H^1(\mathfrak{F}, \mathcal{M})$ is finite.*

PROOF. Let $\mathcal{K}$ (resp. $\mathcal{M}'$) be the kernel (resp. the image) of $\mathcal{A}\to\mathcal{M}$ ($a\mapsto a\cdot m_0$). Then from the short exact sequences:

$$\begin{cases} 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \longrightarrow \mathcal{M}' \longrightarrow 0 \\ 0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0 \qquad (\mathcal{N} := \mathcal{M}/\mathcal{M}'), \end{cases}$$

we obtain, via the long exact sequences of cohomology groups, the following diagram:

$$\begin{array}{ccccc} H^1(\mathfrak{F}, \mathcal{A}) & \longrightarrow & H^1(\mathfrak{F}, \mathcal{M}') & \longrightarrow & H^2(\mathfrak{F}, \mathcal{K}) \\ & & \downarrow & & \\ & & H^1(\mathfrak{F}, \mathcal{M}) & & \\ & & \downarrow & & \\ & & H^1(\mathfrak{F}, \mathcal{N}) & & \end{array}$$

in which the mapping $H^1(\mathfrak{F}, \mathcal{A})\to H^1(\mathfrak{F}, \mathcal{M})$ obtained by composition is sp. Now $H^2(\mathfrak{F}, \mathcal{K})=0$ by (3.2.6), and $H^1(\mathfrak{F}, \mathcal{N})$ is finite by our assumption, because $\mathfrak{F}$ is topologically finitely generated. Thus our conclusion follows. ■

The proposition above especially means that the Galois representation on $H^1(\mathfrak{F}, \mathcal{A})\cong\varprojlim_{n\in N}H^1(Y_n, \boldsymbol{Z}_l)$ (2.4.1) essentially contains all information for those on $H^1(\mathfrak{F}, \mathcal{M})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l\cong H^1(Y, F_{\mathcal{M}})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l$ (2.3.3), for pro-l $\tilde{\mathfrak{G}}$-modules $\mathcal{M}$ with $m_0\in\mathcal{M}^{G_{Q^*}}$, satisfying the condition in (6.1.3) (see 7.2 for explicit examples). On the other hand, we remark that the conclusion of (6.1.3) does not hold unconditionally for sp_P and sp_c. For example, consider the case of Example (M) (1.2). In this case, $H^1_P(\mathfrak{F}, \mathcal{A})$ vanishes: This follows from the isomorphism $H^1_P(\mathfrak{F}, \mathcal{A}) \cong \mathfrak{N}^{ab}(-1)$ (combine (3.3.3), (3.3.4) and **[I1]** Proposition 1.3) because $\mathfrak{F}=\mathfrak{G}$; or also from (4.2.7). Nevertheless, "its specializations" $H^1_P(\mathfrak{F}, \boldsymbol{Z}_l[\mathfrak{G}_n]) \cong H^1(X_n, \boldsymbol{Z}_l)$ (cf. 3.3) can have arbitrarily large rank. Also, since we have the commutative diagram:

$$(6.1.4) \qquad \begin{array}{ccccc} \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}) & \xrightarrow{(5.4.2)} & H^1_P(\mathfrak{F}, \mathcal{A}) & \longrightarrow & 0 \\ \downarrow{\scriptstyle sp_c} & & \downarrow{\scriptstyle sp_P} & & \\ \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M}) & \xrightarrow[(5.4.2)]{} & H^1_P(\mathfrak{F}, \mathcal{M}) & \longrightarrow & 0 \end{array}$$

with exact horizontal lines by (5.4.4), (6.1.3) cannot hold for sp_c neither. (We can actually prove that $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A})=\{0\}$ in this case.)

The purpose of the rest of this section is to prove the following:

THEOREM (6.1.5). *The notation and the assumptions being as in* (6.1.3), *we put* $\tilde{\mathcal{M}} := \mathrm{Hom}_c(\mathcal{M}, \boldsymbol{Q}_l/\boldsymbol{Z}_l)$ *(the Pontrjagin dual of* $\mathcal{M}$*). We consider* $\tilde{\mathcal{M}}$ *as a discrete abelian group on which* $\mathfrak{G}$ *(and hence* $\mathfrak{F}$*) acts continuously from the left by:* $(g\cdot f)(m) := f(g^{-1}m)$ *for* $f\in\tilde{\mathcal{M}}$, $g\in\mathfrak{G}$ *and* $m\in\mathcal{M}$. *If* $H^1(\mathfrak{G}, \tilde{\mathcal{M}})$ *is finite, then the cokernels of* $sp_P: H^1_P(\mathfrak{F}, \mathcal{A})\to H^1_P(\mathfrak{F}, \mathcal{M})$ *and* $sp_c: \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A})\to \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{M})$ *are finite.*

6.2. Proof of (6.1.5). Let the notation be as in (6.1.5). In view of (6.1.4), it is enough to prove the assertion only for sp_c. For this, first note that we may replace $\mathcal{M}$ by $\mathrm{Im}(\mathcal{A}\to\mathcal{M})$; and hence we may assume that $\mathcal{A}\to\mathcal{M}$ is surjective. This indeed follows from a similar argument as in the proof of (6.1.3), noting that $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{N}) \cong H^1_c(Y, F_{\mathcal{N}})$ is finite whenever $\mathcal{N}$ is finite. Next, by (5.2.3), we may identify sp_c with

$$\mathrm{Hom}_\Gamma(D_0, \mathcal{A}) \longrightarrow \mathrm{Hom}_\Gamma(D_0, \mathcal{M})$$

induced from (6.1.1), and want to show that its cokernel is finite. Let $\mathcal{K} := \mathrm{Ker}(\mathcal{A}\to\mathcal{M})$; i.e.,

$$(6.2.1) \qquad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \longrightarrow \mathcal{M} \longrightarrow 0 \qquad \text{(exact)}.$$

Then by the definition of D_0 (5.2), we obtain a commutative diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{Ext}^1_\Gamma(\boldsymbol{Z}, \mathcal{K}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(\boldsymbol{Z}, \mathcal{A}) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathrm{Ext}^1_\Gamma(D, \mathcal{K}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(D, \mathcal{A}) \\ & & & & \downarrow & & \downarrow \\ \mathrm{Hom}_\Gamma(D_0, \mathcal{A}) & \xrightarrow{sp_c} & \mathrm{Hom}_\Gamma(D_0, \mathcal{M}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(D_0, \mathcal{K}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(D_0, \mathcal{A}) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathrm{Ext}^2_\Gamma(\boldsymbol{Z}, \mathcal{K}) & \longrightarrow & \mathrm{Ext}^2_\Gamma(\boldsymbol{Z}, \mathcal{A}), \end{array}$$

where the third horizontal line and the two vertical lines are exact. Note here that $\mathrm{Ext}^i_\Gamma(\boldsymbol{Z}, -)\cong H^i(\Gamma, -)$; and hence these groups vanish for $i\geqq 2$, because Γ is a free group (cf. the proof of (3.2.6)). Also note that, since $\mathrm{Hom}_\Gamma(D, -)\cong\bigoplus_{j=1}^s(-)^{\Gamma_j}$ ($\Gamma_j=\langle z_j\rangle$) as we saw in the course of the proof of (5.2.3), its right derived functors $\mathrm{Ext}^i_\Gamma(D, -)$ are canonically isomorphic to $\bigoplus_{j=1}^s H^i(\Gamma_j, -)$, and $\mathrm{Ext}^i_\Gamma(\boldsymbol{Z}, -)\to\mathrm{Ext}^i_\Gamma(D, -)$ may be identified with the direct sum of restrictions: $H^i(\Gamma, -)\xrightarrow{\oplus\mathrm{Res}}\bigoplus_{j=1}^s H^i(\Gamma_j, -)$. From this, and the exact sequence (3.2.4) with $\mathfrak{F}$ replaced by Γ, we are reduced to prove the finiteness of $\mathrm{Ker}(H^2_P(\Gamma, \mathcal{K})\to H^2_P(\Gamma, \mathcal{A}))$.

LEMMA (6.2.2). *Let I' be the augmentation ideal of $\boldsymbol{Z}[\Gamma]$. For any Γ-module $\mathcal{M}$, $H^2_P(\Gamma, \mathcal{M})$ is canonically isomorphic to $\mathcal{M}/I'\mathcal{M}$.*

Admitting this, the proof of (6.1.5) can be completed as follows: It now remains to prove the finiteness of $\mathrm{Ker}(\mathcal{K}/I'\mathcal{K}\to\mathcal{A}/I'\mathcal{A})=\mathrm{Ker}(\mathcal{K}/I\mathcal{K}\to\mathcal{A}/I\mathcal{A})$, where I is the augmentation ideal of $\mathcal{A}$ (1.1). From the Pontrjagin dual of the exact sequence (6.2.1):

$$0\longrightarrow\tilde{\mathcal{M}}\longrightarrow\tilde{\mathcal{A}}\longrightarrow\tilde{\mathcal{K}}\longrightarrow 0 \qquad \text{(exact)},$$

we obtain a long exact sequence of cohomology groups:

$$\cdots\longrightarrow\tilde{\mathcal{A}}^{\mathfrak{G}}\longrightarrow\tilde{\mathcal{K}}^{\mathfrak{G}}\longrightarrow H^1(\mathfrak{G}, \tilde{\mathcal{M}})\longrightarrow\cdots$$

of discrete abelian groups. Taking again the Pontrjagin dual of this exact sequence, we obtain an exact seqence:

$$\cdots\longleftarrow\mathcal{A}/I\mathcal{A}\longleftarrow\mathcal{K}/I\mathcal{K}\longleftarrow H^1(\mathfrak{G}, \tilde{\mathcal{M}})^\sim\longleftarrow\cdots.$$

Thus the finiteness of $H^1(\mathfrak{G}, \tilde{\mathcal{M}})$ implies the desired conclusion.

PROOF OF (6.2.2). We first assume that $\mathcal{V}\cong\boldsymbol{C}$; i.e. that Γ is generated by two elements x and y with the fundamental relation $xy=1$. In this case, $H^1(\Gamma, \mathcal{M})\cong\mathcal{M}/(x-1)\mathcal{M}$ by the correspondence: $cl(\alpha)\mapsto\alpha(x) \bmod (x-1)\mathcal{M}$. Noting that $\alpha(y)=-x^{-1}\alpha(x)$ for any $\alpha\in Z^1(\Gamma, \mathcal{M})$, $H^2_P(\Gamma, \mathcal{M})$ is isomorphic to the cokernel of: $\mathcal{M}/(x-1)\mathcal{M}\to\mathcal{M}/(x-1)\mathcal{M}\oplus\mathcal{M}/(y-1)\mathcal{M}$ which sends $m \bmod (x-1)\mathcal{M}$ to $(m, -x^{-1}m) \bmod (x-1)\mathcal{M}\oplus(y-1)\mathcal{M}$. We obtain our conclusion immediately from this.

In the other case, Γ may be considered as a Fuchsian group of the first kind; and hence our claim is a special case of [**Sh1**] Propositions 8.1 and 8.2. (We can also give a direct proof along the same line as above, after a somewhat tedious computation). ■

§7. Elliptic modular tower.

7.1. Summary of what we know. We now focus our attention to the case of Example (E) (1.2). Thus we fix a positive integer $N\geqq 4$, and use the notation of loc. cit.. Let l^e ($e\geqq 0$) be the largest power of l dividing N. As for the index set "N" in 1.1, we take the set $\boldsymbol{N}$ of natural numbers, and for each $n\in\boldsymbol{N}$ we put

$$\mathfrak{f}_n := \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{G} \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod l^{n+e}\cdot M_2(\boldsymbol{Z}_l)\right\}.$$

The curve X_n^* is then the canonical model of the modular curve $\Gamma_1(N)\cap\Gamma(l^{n+e})\backslash H^*$ defined over $\boldsymbol{Q}$, for which the cusp $i\infty$ is $\boldsymbol{Q}$-rational. In the following, we set

$$\mathfrak{U} := \left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \in SL_2(\boldsymbol{Z}_l)\right\}. \tag{7.1.1}$$

This subgroup of $\mathfrak{G}$ is $G_{\boldsymbol{Q}}$-stable (cf. (1.2.6)); and hence we are in the situation of 2.4. The curve $X_{1,n}^*$ is the canonical model of $\Gamma_1(Nl^n)\backslash H^*$ defined over $\boldsymbol{Q}$ for which the cusp $i\infty$ is $\boldsymbol{Q}$-rational. We henceforce write it $X_1(Nl^n)$, following the usual terminology. We also write $Y_1(Nl^n)$ for $Y_{1,n}^*$, which is the open subscheme of $X_1(Nl^n)$ corresponding to $\Gamma_1(Nl^n)\backslash H$.

Let $\mathcal{A}$ be, as before, the completed group algebra of $\mathfrak{G}$ over $\boldsymbol{Z}_l$. Since the group $\mathfrak{U}$ above is topologically cyclic, $\mathcal{A}I_{\mathfrak{U}}$ is a closed left ideal of $\mathcal{A}$. Thus by (2.4.4), (3.3.3) and (5.5.1), we have the following isomorphisms of right $G_{\boldsymbol{Q}}$-modules:

$$\begin{cases} H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}}) \cong \varprojlim_{n\in\boldsymbol{N}} H^1(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \\ H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}}) \cong \varprojlim_{n\in\boldsymbol{N}} H^1(X_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \\ \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}}) \cong \varprojlim_{n\in\boldsymbol{N}} H_c^1(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l). \end{cases} \tag{7.1.2}$$

On the other hand, for each non-negative integer d, let $S^d(\boldsymbol{Z}_l)$ (resp. $S^d(\boldsymbol{Q}_l)$) be the set of column vectors of size $d+1$ with entries in $\boldsymbol{Z}_l$ (resp. $\boldsymbol{Q}_l$). We denote by

$$(7.1.3)\qquad \rho_d : GL_2(\boldsymbol{Q}_l) \longrightarrow GL(S^d(\boldsymbol{Q}_l))$$

the symmetric tensor representation of degree d, which is realized as in [**Sh1**] 8.2: Namely, writing $\begin{bmatrix} x \\ y \end{bmatrix}^d$ for the element of $S^d(\boldsymbol{Q}_l)$ whose i-th entry is $x^{d+1-i}y^{i-1}$ $(x, y \in \boldsymbol{Q}_l)$, ρ_d is the unique representation satisfying:

$$(7.1.4)\qquad \rho_d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}^d = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}^d$$

for all $x, y \in \boldsymbol{Q}_l$. (Here we understand that $\begin{bmatrix} x \\ y \end{bmatrix}^0 = 1$, and that ρ_0 is the trivial representation.) Of course ρ_d induces a representation of $GL_2(\boldsymbol{Z}_l)$ on $S^d(\boldsymbol{Z}_l)$, and hence we may consider $S^d(\boldsymbol{Z}_l)$ as a pro-l $\mathfrak{G}$-module via ρ_d. Moreover letting $\rho \in G_{\boldsymbol{Q}}$ act on $S^d(\boldsymbol{Z}_l)$ by $\rho_d\left(\begin{bmatrix} 1 & 0 \\ 0 & \chi_l(\rho) \end{bmatrix}\right)$, then $S^d(\boldsymbol{Z}_l)$ becomes a pro-l $\tilde{\mathfrak{G}}$-module by (1.2.6) (cf. 2.1).

By our assumption that $N \geqq 4$, $Y_1(N)$ can be considered as the fine moduli scheme classifying elliptic curves together with certain level structure over $\boldsymbol{Q}$-schemes. Let $f : E \to Y_1(N)$ be the universal family of such an elliptic curve. Then the twisted constant $\boldsymbol{Z}_l$-sheaf on $Y_1(N)_{\acute{e}t}$ defined by $S^1(\boldsymbol{Z}_l)$ (cf. 2.3) is isomorphic to $R^1 f_*(\boldsymbol{Z}_l)$. In fact, this can be proved by the same method as in the proof of [**O2**] (3.3.3), looking at the corresponding representations of the algebraic fundamental group of $Y_1(N)$ at the generic point. Therefore by (2.3.3), (3.2.8) and (3.2.9), we see that $H^1_P(\mathfrak{F}, S^d(\boldsymbol{Z}_l))(\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l)$ *is isomorphic to the space of l-adic representation of* $G_{\boldsymbol{Q}}$ *attached by Deligne* (cf. [**D**]) *to the space* $S_{d+2}(\Gamma_1(N))$ *of cusp forms of weight* $d+2$ *with respect to* $\Gamma_1(N)$.

7.2. Specialization mappings. The notation being as above, we fix a non-negative integer d. We first note that $S^d(\boldsymbol{Z}_l)^{G_{\boldsymbol{Q}}}$ is the free $\boldsymbol{Z}_l$-module generated by $m_0 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}^d$. Then the mapping (6.1.1), which is induced from: $\mathfrak{G} \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \rho_d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)m_0 = \begin{bmatrix} a \\ c \end{bmatrix}^d \in S^d(\boldsymbol{Z}_l)$, gives rise to "specialization mappings" (cf. (6.1.2) and the remark after it):

$$(7.2.1)\qquad \begin{cases} sp : H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{u}}) \longrightarrow H^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l)) \\ sp_P : H^1_P(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{u}}) \longrightarrow H^1_P(\mathfrak{F}, S^d(\boldsymbol{Z}_l)) \\ sp_c : \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\mathcal{A}I_{\mathfrak{u}}) \longrightarrow \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, S^d(\boldsymbol{Z}_l)). \end{cases}$$

They all commute with the action of $G_{\boldsymbol{Q}}$ (6.1). We already know that the cokernel of sp is finite (6.1.3).

THEOREM (7.2.2). *The cokernels of sp_P and sp_c above are finite.*

PROOF. Recall that there is a non-degenerate bilinear form $(\ ,\ )_d$ on $S^d(\boldsymbol{Q}_l)$ satisfying:

$$\begin{cases} (m, m')_d = (-1)^d(m', m)_d \\ (\rho_d(x)m, \rho_d(x)m')_d = \det(x)^d(m, m')_d \end{cases}$$

for all $x \in GL_2(\boldsymbol{Q}_l)$, and $m, m' \in S^d(\boldsymbol{Q}_l)$ ([**Sh1**] 8.2). Thus if we denote by $\mathcal{M}$ the dual $\boldsymbol{Z}_l$-lattice of $S^d(\boldsymbol{Z}_l)$ with respect to $(\ ,\ )_d$, the Pontrjagin dual $\widetilde{S^d(\boldsymbol{Z}_l)}$ of $S^d(\boldsymbol{Z}_l)$ is isomorphic to $\mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l$ as a $\mathfrak{G}$-module. By (6.1.5), the proof is therefore reduced to the following:

PROPOSITION (7.2.3) (Shimura; cf. [**Sh2**]). *Let $\mathfrak{G}$ be any open subgroup of $SL_2(\boldsymbol{Z}_l)$, and $\mathcal{M}$ a $\mathfrak{G}$-stable $\boldsymbol{Z}_l$-lattice in $S^d(\boldsymbol{Q}_l)$. Then $H^1(\mathfrak{G}, \mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l)$ is finite.*

PROOF. Although one can give a completely elementary proof for this fact (Ihara), we give here a short-cut proof. If $d=0$, the group $H^1(\mathfrak{G}, \mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l) \cong \mathrm{Hom}_c(\mathfrak{G}, \boldsymbol{Q}_l/\boldsymbol{Z}_l)$ is finite; and hence we hereafter assume that $d \geqq 1$.

If $\mathfrak{H}$ is an open normal subgroup of $\mathfrak{G}$, then there is the well-known exact sequence:

$$0 \longrightarrow H^1(\mathfrak{G}/\mathfrak{H}, (\mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l)^{\mathfrak{H}}) \xrightarrow{\mathrm{Inf}} H^1(\mathfrak{G}, \mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l) \xrightarrow{\mathrm{Res}} H^1(\mathfrak{H}, \mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l).$$

Since $(\mathcal{M} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l/\boldsymbol{Z}_l)^{\mathfrak{H}}$ is finite by the assumption above, it is enough to prove our assertion when $\mathfrak{G}$ is a principal congruence subgroup of $SL_2(\boldsymbol{Z}_l)$. On the other hand, it is easy to see that if our conclusion is true for one $\mathcal{M}$, then it also holds for all $\mathcal{M}$.

Let us now take an indefinite division quaternion algebra B over $\boldsymbol{Q}$ whose discriminant is prime to l, and its maximal order $\mathfrak{o}$. We fix an isomorphism: $B \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \cong M_2(\boldsymbol{Q}_l)$ which induces an isomorphism: $\mathfrak{o} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}_l \cong M_2(\boldsymbol{Z}_l)$. For each d, in [**O1**] §5, we have constructed a homomorphism

$$\rho : B^\times \longrightarrow GL_{\varepsilon(d+1)}(\boldsymbol{Q})$$

of algebraic groups over $\boldsymbol{Q}$ with $\varepsilon=1$ or 2 such that $\rho_{\boldsymbol{Q}_l} : (B \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l)^\times \cong GL_2(\boldsymbol{Q}_l) \to GL_{\varepsilon(d+1)}(\boldsymbol{Q}_l)$ is equivalent to $\rho_d^{\oplus \varepsilon}$. Also there is a $\rho(\mathfrak{o} \cap B^\times)$-stable $\boldsymbol{Z}$-lattice X in $\boldsymbol{Q}^{\varepsilon(d+1)}$ satisfying the condition (ρ3) of [**O1**]. Let $\nu : B \to \boldsymbol{Q}$ be the reduced norm, and put

$$\Gamma_m := \{\gamma \in \mathfrak{o} \mid \nu(\gamma)=1, \gamma-1 \in l^m \mathfrak{o}\}$$

for each $m \geqq 0$. We fix $k \geqq 0$ and put $\Gamma := \Gamma_k$. Then for any $n \geqq 0$ and $m \geqq$

$\mathrm{Max}\{k, n\}$, we may consider X/l^nX as a Γ/Γ_m-module by ρ. We claim that there is a constant $C>0$ such that

$$|H^1(\Gamma/\Gamma_m, X/l^nX)| < C$$

holds independently of m and n. Indeed, using the terminology of [**O1**] 3.3, we have an isomorphism: $H^1(\Gamma/\Gamma_m, X/l^nX) \cong H^1(\Gamma'/\Gamma'_m, X''/\mathfrak{l}^nX'')$ by the same reasoning as [**O1**] Corollary (3.3.8). But the inflation mapping: $H^1(\Gamma'/\Gamma'_m, X''/\mathfrak{l}^nX'') \to H^1(\Gamma', X''/\mathfrak{l}^nX'')$ is injective, and the argument in [**O1**] pp. 26-27 implies that the order of this latter group is bounded independently of n. This proves our claim, and consequently we obtain the finiteness of $\varinjlim_{m,n} H^1(\Gamma/\Gamma_m, X/l^nX) \cong H^1(\varprojlim_m \Gamma/\Gamma_m, X\otimes_{\boldsymbol{Z}} \boldsymbol{Q}_l/\boldsymbol{Z}_l)$. Since $\varprojlim_m \Gamma/\Gamma_m$ is isomorphic to the principal congruence subgroup of level l^k of $SL_2(\boldsymbol{Z}_l)$ via $\mathfrak{o}\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l \cong M_2(\boldsymbol{Z}_l)$, our conclusion follows. ■

REMARK (7.2.4). We have thus shown that $H^1_P(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}}) \cong \varprojlim_{n\in\boldsymbol{N}} H^1(X_1(Nl^n) \otimes_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}, \boldsymbol{Z}_l)$ contains all information about Deligne's l-adic representations attached to $S_k(\Gamma_1(Nl^n))$ for all $k\geqq 2$ and $n\geqq 0$. In [**Sh2**], Shimura interpreted the Eichler-Shimura cohomology groups in algebro-geometric terms to obtain l-adic representations attached to more general automorphic forms of one variable (cf. also [**O1**]). One of his principle was that one can obtain information about automorphic forms of higher weights from the knowledge of forms of weight 2 if one grows the level. Our theorem above and the results in 7.6 below may be thus considered as a variation of his principle.

7.3. Hecke operators. In this subsection, we mainly review formal properties of Hecke operators acting on the spaces of automorphic forms and the related cohomology groups (cf. Shimura [**Sh1**], [**Sh2**], Ash-Stevens [**AS1**], Hida [**H1**], [**O1**]). Fix $N\in\boldsymbol{N}$ and write Γ for $\Gamma_1(N)$ for simplicity. Fix also a non-negative integer d, and put $k := d+2$. To begin with, we recall the Eichler-Shimura isomorphism. Let $S^d(\boldsymbol{C}) = \boldsymbol{C}^{d+1}$ on which $GL_2(\boldsymbol{C})$ acts via the symmetric tensor representation of degree d (7.1.4). For $f\in S_k(\Gamma)$, define $S^d(\boldsymbol{C})$-valued 1-forms on H by:

$$\text{(7.3.1)} \qquad \begin{cases} d(f) := f(z)\begin{bmatrix} z \\ 1 \end{bmatrix}^d dz \\ \bar{d}(\bar{f}) := \bar{f}(z)\begin{bmatrix} \bar{z} \\ 1 \end{bmatrix}^d d\bar{z}, \end{cases}$$

where z is a variable on H, and the bar means the complex conjugation. Then we have the Eichler-Shimura isomorphism of $\boldsymbol{C}$-vector spaces:

$$\text{(7.3.2)} \qquad S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \xrightarrow{\sim} H^1_P(\Gamma, S^d(\boldsymbol{C}))$$

which is obtained by sending $(f, \bar{g})\in S_k(\Gamma)\oplus\overline{S_k(\Gamma)}$ to the cohomology class $cl(u)$ of

$$u(\gamma):=\int_{z_0}^{\gamma(z_0)} d(f)+\bar{d}(\bar{g}) \tag{7.3.3}$$

for any fixed $z_0\in H^*$. (Notice here that both sides of (7.3.2) vanish when $\Gamma\ni -1$ and d is odd.) Let $M_k(\Gamma)$ be the space of modular forms of weight k with respect to Γ. It is also known that the isomorphism above extends to an isomorphism:

$$M_k(\Gamma)\oplus\overline{S_k(\Gamma)}\xrightarrow{\sim} H^1(\Gamma, S^d(\boldsymbol{C})) \tag{7.3.4}$$

defined by the same formula as above for a fixed $z_0\in H$ (cf. Hida [**H2**] §5).

Now put

$$\Delta_1(N)=\Delta:=\left\{\alpha\in M_2(\boldsymbol{Z})\,|\det(\alpha)>0,\ \alpha\equiv\begin{bmatrix}1 & *\\ 0 & *\end{bmatrix}\bmod N\cdot M_2(\boldsymbol{Z})\right\}, \tag{7.3.5}$$

and denote by $R(\Gamma, \Delta)$ the Hecke ring with respect to Γ and Δ ([**Sh1**] 3.1). This ring acts on $S_k(\Gamma)$, $\overline{S_k(\Gamma)}$, and $M_k(\Gamma)$ in a well-known manner. Let ι denote the main involution of $M_2(\boldsymbol{Q})$. For any $\Delta^{\iota}:=\{\alpha^{\iota}\,|\,\alpha\in\Delta\}$-module $\mathcal{M}$, we can define the action of the ring $R(\Gamma, \Delta)$ on $H^1(\Gamma, \mathcal{M})$ and $H^1_P(\Gamma, \mathcal{M})$ by the formula [**Sh1**] (8.3.2); i.e. $R(\Gamma, \Delta)\ni\Gamma\alpha\Gamma=\coprod_i\Gamma\beta_i$ sends $cl(u)$ to $cl(v)$ ($u, v\in Z^1(\Gamma, \mathcal{M})$ or $Z^1_P(\Gamma, \mathcal{M})$) with

$$v(\gamma):=\textstyle\sum_i\beta_i^{\iota}u(\gamma_i) \tag{7.3.6}$$

if $\beta_i\gamma=\gamma_i\beta_j$ for some j and $\gamma_i\in\Gamma$. The isomorphisms (7.3.2) and (7.3.4) are then isomorphisms of $R(\Gamma, \Delta)$-modules (cf. [**Sh1**] Proposition 8.5). The action above of $A\in R(\Gamma, \Delta)$ on $S_k(\Gamma)$, $\overline{S_k(\Gamma)}$, $M_k(\Gamma)$, $H^1(\Gamma, \mathcal{M})$ and $H^1_P(\Gamma, \mathcal{M})$ will be denoted by $[A]$ indifferently. By definition, the Hecke operator $T(r)$ (resp. $T(q, q)$) is the operator $[\{\alpha\in\Delta\,|\det(\alpha)=r\}]$ (resp. $[\Gamma(q\cdot\sigma_q)\Gamma]$). Here, r is any positive integer, q is a positive integer prime to N, and σ_q is an element of $SL_2(\boldsymbol{Z})$ satisfying $q\cdot\sigma_q\equiv\begin{bmatrix}1 & *\\ 0 & q^2\end{bmatrix}\bmod N\cdot M_2(\boldsymbol{Z})$.

On the other hand, take an element $\Gamma\alpha\Gamma=\coprod_i\alpha_i\Gamma$ ($\Leftrightarrow\Gamma\alpha^{\iota}\Gamma=\coprod_i\Gamma\alpha_i^{\iota}$) of $R(\Gamma, \Delta)$. For $f\in M_k(\Gamma)$, we set

$$([\Gamma\alpha\Gamma]^*f)(z):=\textstyle\sum_i\det(\alpha)^{-1}(c_iz+d_i)^{-k}f\left(\dfrac{a_iz+b_i}{c_iz+d_i}\right) \tag{7.3.7}$$

with $\alpha_i^{-1}=\begin{bmatrix}a_i & b_i\\ c_i & d_i\end{bmatrix}$. (This is nothing but the operator $[\Gamma\alpha^{\iota}\Gamma]$). Then it is known that $[\Gamma\alpha\Gamma]^*$ is adjoint to $[\Gamma\alpha\Gamma]$ with respect to the Petersson inner product on $S_k(\Gamma)$ ([**Sh1**] (3.4.5)). For a Δ-module $\mathcal{M}$, we define the operator

$[\Gamma\alpha\Gamma]^*=[\coprod_i \alpha_i\Gamma]^*$ on $H^1(\Gamma, \mathcal{M})$ or $H^1_P(\Gamma, \mathcal{M})$ by sending $cl(u)$ to $cl(v)$ with

$$v(\gamma) = \sum_i \alpha_i u(\gamma_i) \tag{7.3.8}$$

if $\gamma^{-1}\alpha_i=\alpha_j\gamma_i^{-1}$ for some j and $\gamma_i\in\Gamma$. Then (7.3.2) and (7.3.4) are again isomorphisms of $R(\Gamma, \Delta)$-modules with respect to the action $[\Gamma\alpha\Gamma]^*$ ([**Sh2**]). By definition, $T^*(r):=[\{\alpha\in\Delta \mid \det(\alpha)=r\}]^*$ and $T^*(q, q):=[\Gamma(q\cdot\sigma_q)\Gamma]^*$ with the same notation as above.

In the following, we assume that $N\geqq 4$ for simplicity. To a Γ-module $\mathcal{M}$, we can associate a locally constant sheaf on $\Gamma\backslash H$ whose étale space is $\Gamma\backslash H \times\mathcal{M}$. We denote this sheaf by $\mathcal{F}_{\mathcal{M}}$ in the following. When $\mathcal{M}$ is a Δ^ι-module, we know how to define the action (again denoted by) $[A]$ of $A\in R(\Gamma, \Delta)$ on $H^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$, $H^1_c(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$ and $H^1_P(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$ $(:=\mathrm{Im}(H^1_c(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\to H^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}))$ so that the canonical isomorphisms: $H^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\cong H^1(\Gamma, \mathcal{M})$ and $H^1_P(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\cong H^1_P(\Gamma, \mathcal{M})$ are $R(\Gamma, \Delta)$-equivariant ([**H1**] § 3). Assume moreover that $\mathcal{M}$ is a finite abelian group on which $\mathfrak{G}$ acts continuously, and that the action of $\mathfrak{G}$ and that of Δ^ι coincide on $\mathfrak{G}\cap\Delta^\iota=\Gamma$. Then we can define a twisted constant sheaf on $Y_1(N)_{\acute{e}t}$ in a similar manner as above (cf. 2.3); and $R(\Gamma, \Delta)$ acts on the one-dimensional étale cohomology groups of the same kind as above of $Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}$ with values in that sheaf compatibly with the action above via the comparison isomorphisms.

Suppose next that $\mathcal{M}$ is a Δ-module, and fix $\Gamma\alpha\Gamma\in R(\Gamma, \Delta)$. Setting $\Gamma^\alpha:=\Gamma\cap\alpha^{-1}\Gamma\alpha$, we define the operator $[\Gamma\alpha\Gamma]^*$ on $H^1_\dagger(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$ as the composite of:

$$H^1_\dagger(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}) \xrightarrow{\text{can.}} H^1_\dagger(\Gamma^\alpha\backslash H, \mathcal{F}_{\mathcal{M}}) \longrightarrow H^1_\dagger(\Gamma^{\alpha^{-1}}\backslash H, \mathcal{F}_{\mathcal{M}}) \xrightarrow{\text{trace}} H^1_\dagger(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}). \tag{7.3.9}$$

Here, $H^1_\dagger$ means either one of H^1, H^1_P or H^1_c; and the middle arrow is induced from the following morphism of étale spaces:

$$\begin{array}{ccc} \Gamma^\alpha\backslash H\times\mathcal{M} & \xrightarrow{\alpha\times\alpha} & \Gamma^{\alpha^{-1}}\backslash H\times\mathcal{M} \\ \downarrow & & \downarrow \\ \Gamma^\alpha\backslash H & \underset{\alpha}{\xrightarrow{\sim}} & \Gamma^{\alpha^{-1}}\backslash H. \end{array}$$

This action of $R(\Gamma, \Delta)$ again corresponds to the action $[\Gamma\alpha\Gamma]^*$ above for the respective group cohomologies $H^1_\dagger(\Gamma, \mathcal{M})$ for $\dagger=\emptyset$ (empty) or P. A similar remark as in the last paragraph also applies for étale cohomology groups.

Let Λ be one of the rings $\boldsymbol{Z}/n\boldsymbol{Z}$ $(n\in\boldsymbol{N})$, $\boldsymbol{Q}$, $\boldsymbol{R}$ or $\boldsymbol{C}$; and let $\mathcal{M}$ be a $\Lambda[\Delta]$-module which is of finite type over Λ. We consider $\check{\mathcal{M}}:=\mathrm{Hom}_\Lambda(\mathcal{M}, \Lambda)$ as a $\Lambda[\Delta^\iota]$-module by: $(\alpha^\iota f)(x):=f(\alpha x)$ for $\alpha\in\Delta$. Then the natural pairing $\mathcal{M}\times\check{\mathcal{M}}\to\Lambda$ is Γ-equivariant, and hence it induces an isomorphism: $\mathcal{F}_{\check{\mathcal{M}}}\cong$

$\underline{\mathrm{Hom}}_{\Lambda}(\mathcal{F}_{\mathcal{M}}, \underline{\Lambda})$ of sheaves on $\Gamma\backslash H$, where $\underline{\Lambda}$ is the constant sheaf defined by Λ. The Poincaré duality theorem assures us that the cup product pairings:

$$(7.3.10)\qquad \begin{cases} H^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\times H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}^{\vee}) \longrightarrow H_c^2(\Gamma\backslash H, \underline{\Lambda}) \cong \Lambda \\ H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\times H^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}^{\vee}) \longrightarrow H_c^2(\Gamma\backslash H, \underline{\Lambda}) \cong \Lambda \\ H_P^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\times H_P^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}^{\vee}) \longrightarrow H_c^2(\Gamma\backslash H, \underline{\Lambda}) \cong \Lambda \end{cases}$$

are perfect. Denoting by $\langle\,,\rangle$ any one of these pairings, we have:

$$(7.3.11)\qquad \langle [A]^*x, y\rangle = \langle x, [A]y\rangle$$

for all $A\in R(\Gamma, \Delta)$ (cf. **[H1]** Proposition 3.3, **[AS1]** Lemma 1.4.3). When $\Lambda=\boldsymbol{Z}/n\boldsymbol{Z}$, and $\mathcal{M}$ is a finite continuous $\mathfrak{G}$-module, similar results hold for étale cohomology groups.

Now let $\{\mathcal{M}_i\}_{i\in I}$ be a projective system of finite $\boldsymbol{Z}_l[\Delta]$-modules as well as pro-l $\mathfrak{G}$-modules; and put $\mathcal{M}:=\varprojlim_{i\in I}\mathcal{M}_i$. Then $H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\overline{\boldsymbol{Q}}, F_{\mathcal{M}})=\varprojlim_{i\in I}H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\overline{\boldsymbol{Q}}, F_{\mathcal{M}_i})$ can be considered as an $R(\Gamma, \Delta)$-module via $[\ ]^*$. On the other hand, by (5.3.2) and (5.2.3), we have an isomorphism:

$$(7.3.12)\qquad H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\overline{\boldsymbol{Q}}, F_{\mathcal{M}}) \cong \mathrm{Hom}_{\Gamma}(D_0, \mathcal{M}),$$

where D_0 is the free abelian group on $\boldsymbol{P}^1(\boldsymbol{Q})$, the set of cusps for Γ.

PROPOSITION (7.3.13). *For $\Gamma\alpha\Gamma\in R(\Gamma, \Delta)$, let $\Gamma\alpha\Gamma=\coprod_i\alpha_i\Gamma$ be the disjoint decomposition. If we set*

$$([\Gamma\alpha\Gamma]^*f)(x):=\textstyle\sum_i\alpha_i f(\alpha_i^{-1}x) \qquad (x\in D_0)$$

for $f\in\mathrm{Hom}_{\Gamma}(D_0, \mathcal{M})$, this operator commutes with $[\Gamma\alpha\Gamma]^*$ on $H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\overline{\boldsymbol{Q}}, F_{\mathcal{M}})$ via (7.3.12).

PROOF. We may assume that $\mathcal{M}$ is finite. Let $\tilde{H}$ be the Borel-Serre completion of H, and $\boldsymbol{j}: H\hookrightarrow\tilde{H}$ the natural immersion. We can consider the constant sheaf $\underline{\mathcal{M}}_H$ on H (resp. $\boldsymbol{j}_!\underline{\mathcal{M}}_H$) as a Γ-sheaf on H (resp. $\tilde{H}$) in an obvious manner; and then by the same reason as in the étale case (cf. (5.1.12)), we have an isomorphism: $H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})\cong H^1(\tilde{H}, \boldsymbol{j}_!\underline{\mathcal{M}}_H)^{\Gamma}$. Since $\partial\tilde{H}=\tilde{H}-H$ is the disjoint union of $\boldsymbol{R}$ indexed by $\boldsymbol{P}^1(\boldsymbol{Q})$, arguing in the same manner as in the proof of (5.3.2), we have an exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\tilde{H}, \underline{\mathcal{M}}_{\tilde{H}}) & \longrightarrow & H^0(\partial H, \underline{\mathcal{M}}_{\partial H}) & \longrightarrow & H^1(\tilde{H}, \boldsymbol{j}_!\underline{\mathcal{M}}_H) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \longrightarrow & \mathcal{M} & \longrightarrow & \bigoplus_{x\in\boldsymbol{P}^1(\boldsymbol{Q})}\mathcal{M} & \longrightarrow & \mathrm{Hom}\,(D_0, \mathcal{M}) & \longrightarrow & 0\,. \end{array}$$

Combining the right vertical isomorphism and the remark above, we have an isomorphism: $\mathrm{Hom}_\Gamma(D_0, \mathcal{M}) \cong H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$, which is due to Ash and Stevens [**AS2**]. It is easy to see that this corresponds to (7.3.12) via the comparison isomorphism. We are thus reduced to show that the operator $[\Gamma\alpha\Gamma]^*$ on $H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}})$ (7.3.9) commutes with the one stated in the proposition, via the isomorphism above. For this, it is enough to show the commutativity of the following diagram:

$$\begin{array}{ccccccc}
H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}) & \longrightarrow & H_c^1(\Gamma^{\alpha}\backslash H, \mathcal{F}_{\mathcal{M}}) & \longrightarrow & H_c^1(\Gamma^{\alpha^{-1}}\backslash H, \mathcal{F}_{\mathcal{M}}) & \longrightarrow & H_c^1(\Gamma\backslash H, \mathcal{F}_{\mathcal{M}}) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathrm{Hom}_\Gamma(D_0, \mathcal{M}) & \underset{\text{can.}}{\longrightarrow} & \mathrm{Hom}_{\Gamma^\alpha}(D_0, \mathcal{M}) & \underset{(i)}{\longrightarrow} & \mathrm{Hom}_{\Gamma^{\alpha^{-1}}}(D_0, \mathcal{M}) & \underset{(ii)}{\longrightarrow} & \mathrm{Hom}_\Gamma(D_0, \mathcal{M}),
\end{array}$$

where the upper horizontal line is (7.3.9), and (i) (resp. (ii)) above sends $f \in \mathrm{Hom}_{\Gamma^\alpha}(D_0, \mathcal{M})$ (resp. $\mathrm{Hom}_{\Gamma^{\alpha^{-1}}}(D_0, \mathcal{M})$) to $f' \in \mathrm{Hom}_{\Gamma^{\alpha^{-1}}}(D_0, \mathcal{M})$ (resp. $\mathrm{Hom}_\Gamma(D_0, \mathcal{M})$) defined by $f'(x) = \alpha f(\alpha^{-1}x)$ (resp. $f'(x) = \sum_i \gamma_i f(\gamma_i^{-1}x)$ if $\Gamma = \coprod_i \gamma_i \Gamma^{\alpha^{-1}}$). This can be in fact proved directly applying 2.2; and the details are omitted. ■

REMARK (7.3.14). In the argument above, if we instead assume that $\{\mathcal{M}_i\}_{i\in I}$ is a projective system of finite $\boldsymbol{Z}_l[\Delta^\iota]$- and $\mathfrak{G}$-modules, we can define the operator $[\Gamma\alpha\Gamma]$ on $H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}})$ for $\alpha\in\Delta$. By the same reasoning as above, if we set

$$(f\,|\,[\Gamma\alpha\Gamma])(x) := \sum_i \beta_i^\iota f(\beta_i x)$$

for $f\in\mathrm{Hom}_\Gamma(D_0, \mathcal{M})$ when $\Gamma\alpha\Gamma = \coprod_i \Gamma\beta_i$, then (7.3.12) is $R(\Gamma, \Delta)$-equivariant with respect to the operators $[\Gamma\alpha\Gamma]$.

7.4. Hecke operators (continued). For a pro-l $\mathfrak{G}$-module $\mathcal{M}$, we write $H_P^1(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}})$ for the image of $H_c^1(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}}) \to H^1(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}})$ in the following. Thus this group is canonically isomorphic to $H_P^1(\mathfrak{F}, \mathcal{M})$ ((3.2.8), (3.2.9)). Also, for notational convenience, we shall put $T(q, q) = T^*(q, q) \equiv 0$ if q is not prime to the level under consideration, in the following discussions.

LEMMA (7.4.1). *Assume that a pro-l $\mathfrak{G}$-module $\mathcal{M}$ is equipped with the structure of a Δ-module so that the action of $\Delta\cap\mathfrak{G}$ is the original one. Then for integers $n\geqq m\geqq 1$, the trace mappings:*

$$H_\dagger^1(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}}) \longrightarrow H_\dagger^1(Y_1(Nl^m)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\mathcal{M}})$$

commute with all $T^*(r)$ and $T^*(q, q)$, where $H_\dagger^1$ means either one of H^1, H_P^1, or H_c^1. If l divides N, then this also holds for $m=0$. When l does not divide N and $m=0$, the mappings above still commute with $T^*(r)$ and $T^*(q, q)$ for r and q prime to l.

PROOF. It is enough to prove our assertion for H^1 and H^1_c. Also, we may assume that $\mathcal{M}$ is finite; and hence that $\mathcal{M}$ is a $\boldsymbol{Z}/l^a\boldsymbol{Z}$-module for some $a\geqq 1$. Put $\check{\mathcal{M}}:=\mathrm{Hom}_{\boldsymbol{Z}/l^a\boldsymbol{Z}}(\mathcal{M}, \boldsymbol{Z}/l^a\boldsymbol{Z})$, and consider it as a Δ^ι-module as in 7.3. This action of Δ^ι coincides with the natural action of $\mathfrak{G}$ on $\Delta^\iota\cap\mathfrak{G}$. Thus by Poincaré duality theorem, in view of (the remark after) (7.3.11), our claim is equivalent to the assertion that the canoniacal mappings:

$$H^1_\dagger(Y_1(Nl^m)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\check{\mathcal{M}}})\longrightarrow H^1_\dagger(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\check{\mathcal{M}}})$$

commute with $T(r)$ and $T(q, q)$ for $H^1_\dagger=H^1$ or H^1_c. But this is obvious from (7.3.6) and (7.3.14), because when we decompose the double cosets appearing in the definitions of $T(r)$ and $T(q, q)$ into a sum of left cosets, we can take the same set of representatives for levels Nl^n and Nl^m; and also for N either when $l|N$, or when r and q are prime to l ([**Sh1**] 3.3). ■

By this lemma, we can consider the operators $\varprojlim_{n\geq 0}T^*(r)$ and $\varprojlim_{n\geq 0}T^*(q, q)$ acting on $\varprojlim_{n\geq 0}H^1_\dagger(Y_1(Nl^n)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, \boldsymbol{Z}_l)$, which will be denoted by $T^*(r)$ and $T^*(q, q)$, respectively. Here we assume that r and q are prime to l when N is not divisible by l.

Next recall the isomorphisms (7.1.2), and that $\mathcal{A}/\mathcal{A}I_{\mathfrak{U}}\cong\varprojlim_{n\in N}\boldsymbol{Z}_l[\mathfrak{G}_n/\mathfrak{U}_n]\cong\varprojlim_{n\in N}\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]$. When N is divisible by l, a simple calculation shows that

$$\text{(7.4.2)}\qquad \begin{cases} \Gamma_1(N)\Delta_1(Nl^n)=\Delta_1(N) \\ \Gamma_1(N)\cap\Delta_1(Nl^n)\Delta_1(Nl^n)^{-1}=\Gamma_1(Nl^n)\,. \end{cases}$$

Let $\Gamma_1(N)=\coprod_i a_i\Gamma_1(Nl^n)$ be the disjoint decomposition. Then, for $\delta\in\Delta_1(N)$ and i, we have: $\delta a_i=a_{i'}\delta'$ with unique index i' and $\delta'\in\Delta_1(Nl^n)$ by (7.4.2). We hereafter consider $\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]$ as a $\Delta_1(N)$-module by letting $\delta\in\Delta_1(N)$ act on this group by: $a_i\Gamma_1(Nl^n)\mapsto a_{i'}\Gamma_1(Nl^n)$ with the notation as above. Thus the groups $H^1_\dagger(\Gamma_1(N), \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])\cong H^1_\dagger(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])\cong H^1_\dagger(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]})$ ($\dagger=\emptyset$ or P) and $H^1_c(Y_1(N)\otimes_{\boldsymbol{Q}}\bar{\boldsymbol{Q}}, F_{\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]})\cong\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])\cong\mathrm{Hom}_{\Gamma_1(N)}(D_0, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])$ become modules over the Hecke ring $R(\Gamma_1(N), \Delta_1(N))$. Since the natural mappings $\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]\to\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^m)]$ are homomorphisms of $\Delta_1(N)$-modules for $n\geqq m\geqq 0$, we can take projective limits to define the operators $T^*(r)$ and $T^*(q, q)$ on $H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$, $H^1_P(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ and $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ for all r and q. When N is not divisible by l, replacing $\Delta_1(Nl^n)$ by:

$$\{\alpha\in\Delta_1(Nl^n)\mid(\det(\alpha), l)=1\}$$

for all $n\geqq 0$ in the argument above, we can define $T^*(r)$ and $T^*(q, q)$ as above for r and q prime to l, exactly in the same manner.

LEMMA (7.4.3). *The isomorphisms* (7.1.2) *commute with the operators* $T^*(r)$ *and* $T^*(q, q)$ *defined above.*

PROOF. We give the proof under the assumption that N is divisible by l; the other case can be treated similarly. It is enough to prove our assertion for the first and the third isomorphisms in (7.1.2). For this, we claim that the isomorphisms (2.5.1) and (5.5.3):

$$\begin{cases} H^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \longrightarrow H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l) \\ \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \longrightarrow \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l) \end{cases}$$

commute with all $T^*(r)$ and $T^*(q, q)$. In fact, as is perhaps well-known, the inverse of these isomorphisms are given by the composites of:

$$\begin{cases} H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l) \xrightarrow{i_*} H^1(\mathfrak{F}_1^n, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \longrightarrow H^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \\ \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l) \xrightarrow{i_*} \mathrm{Hom}_{c,\mathfrak{F}_1^n}(\mathcal{D}_0, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \\ \qquad\qquad \longrightarrow \mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]), \end{cases}$$

where the first arrows are induced from the obvious injection: $i\colon \boldsymbol{Z}_l \hookrightarrow \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]$ of $\mathfrak{F}_1^n$-modules; and the second arrows in the first (resp. the second) sequence is the corestriction (resp. the left vertical arrow appearing in (5.5.2)). Now since i above is a homomorphism of $\Delta_1(Nl^n)$-modules, the two mappings labelled i_* above commute with $T^*(r)$ and $T^*(q, q)$ of level Nl^n. That the second arrows commute with $T^*(r)$ and $T^*(q, q)$ follows from (7.4.1), in view of the commutativity of (2.5.4) and (5.5.2). We obtain our assertion by taking projective limits. ■

PROPOSITION (7.4.4). *The notation and the conventions being as above, the specialization mappings* (7.2.1) *commute with* $T^*(r)$ *and* $T^*(q, q)$ *for all* $d \geqq 0$.

PROOF. We again give the proof under the assumption that N is divisible by l, for sp and sp_c. We first note that $S^d(\boldsymbol{Z}/l^n\boldsymbol{Z}) := S^d(\boldsymbol{Z}_l)/l^nS^d(\boldsymbol{Z}_l)$ is a $\Delta_1(N)$-module by (the same formula as) ρ_d; and that the mapping (6.1.1) (with $m_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^d$) is obtained by taking projective limit from:

$$\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)] \longrightarrow S^d(\boldsymbol{Z}/l^n\boldsymbol{Z})$$

which sends $\gamma\Gamma_1(Nl^n) \in \Gamma_1(N)/\Gamma_1(Nl^n)$ to $\rho_d(\gamma)m_0 \bmod l^nS^d(\boldsymbol{Z}_l)$. But from the definition of the action of $\Delta_1(N)$ on $\boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]$, it is easy to see that the mapping above is a homomorphism of $\Delta_1(N)$-modules; and hence our conclusion follows. ■

7.5. The action of the Iwasawa algebra. We keep the notation of previous subsections, and put

(7.5.1) $$\mathfrak{K} := \left\{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathfrak{G}\right\}.$$

Since $\mathfrak{K}$ normalizes $\mathfrak{U}$, we can consider $H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$, $H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ and $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ as $\mathfrak{K}$- or $\boldsymbol{Z}_l[[\mathfrak{K}]]$-modules (2.6, 3.4 and 5.5). We can identify $\mathfrak{K}$ with $\boldsymbol{Z}_l^\times$ (resp. $1+N\boldsymbol{Z}_l$) via the correspondence: $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \leftrightarrow a$ when $l \nmid N$ (resp. $l|N$); and hence consider the groups above as modules over the Iwasawa algebra $\boldsymbol{Z}_l[[\boldsymbol{Z}_l^\times]]$ (resp. $\boldsymbol{Z}_l[[1+N\boldsymbol{Z}_l]]$).

PROPOSITION (7.5.2). *Let q be a positive integer which is prime to l and congruent to* $1 \bmod N$. *Then the action of* $\begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \in \mathfrak{K}$ *above coincides with* $T^*(q, q)$.

PROOF. By our assumption, $T^*(q, q) = \left[\Gamma_1(N)\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}\Gamma_1(N)\right]^*$, and hence $T^*(q, q)$ sends $cl(u) \in H^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])$ to $cl\left(\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \cdot u\right)$. If $\Gamma_1(N) = \coprod_i a_i \Gamma_1(Nl^n)$, and σ_q is an element of $SL_2(\boldsymbol{Z})$ satisfying $q \cdot \sigma_q \equiv \begin{bmatrix} 1 & * \\ 0 & q^2 \end{bmatrix} \bmod Nl^n \cdot M_2(\boldsymbol{Z})$, we have:

$$\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} a_i = (a_i \sigma_q^{-1})\left(\sigma_q \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}\right)$$

with $a_i \sigma_q^{-1} \in \Gamma_1(N)$ and $\sigma_q \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \in \Delta_1(Nl^n)$. This means that the operator $T^*(q, q)$ on $H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}}) \cong \varprojlim_{n \in N} H^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)])$ is induced from the action (2.6) of $\begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \in \mathfrak{K}$. This proves our assertion for $H^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ and $H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$. The assertion for $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, \mathcal{A}/\mathcal{A}I_{\mathfrak{U}})$ is also clear from the argument above. ■

Since $T^*(q, q)$ is multiplication by q^d on $H^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l))$, $H_P^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l))$ and $\mathrm{Hom}_{c,\mathfrak{F}}(\mathcal{D}_0, S^d(\boldsymbol{Z}_l))$ for q as in the proposition, we see from (7.4.4) that the specialization mappings (7.2.1) factor through the maximal quotient of the left hand side of (7.2.1) on which any $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \mathfrak{K}$ acts as multiplication by a^d.

7.6. Congruence relations. Having checked various compatibilities for Hecke operators in 7.3 and 7.4, it is now an easy matter to formulate the Eichler-Shimura congruence relation on our parabolic cohomology groups:

THEOREM (7.6.1). *The representation of* $G_{\boldsymbol{Q}}$ *on* $H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathrm{u}})$ *is unramified outside* Nl. *If we denote by* $F_p \in G_{\boldsymbol{Q}}$ *a Frobenius element at* p *for a prime* p *not dividing* Nl, *and also its image in* $\mathrm{Aut}(H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathrm{u}}))$, *we have:*

$$T^*(p) = F_p^{-1} + pT^*(p, p) \cdot F_p$$

in $\mathrm{End}(H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathrm{u}}))$.

PROOF. Let $J_1(Nl^n)$ be the Jacobian variety of $X_1(Nl^n)$ defined over $\boldsymbol{Q}$. Then we know that $H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathrm{u}}) \cong \varprojlim_{n \in N} T_l(J_1(Nl^n))(-1)$ as $G_{\boldsymbol{Q}}$-modules (3.3.4). It is therefore unramified outside Nl by Igusa's theorem.

Let ξ be the endomorphism of $J_1(Nl^n)$ corresponding to $\Gamma_1(Nl^n)\begin{bmatrix}1 & 0\\ 0 & p\end{bmatrix}\Gamma_1(Nl^n)$ in the sense of [**Sh1**] 7.2; i.e., ξ is the covariant action of the algebraic correspondence of $X_1(Nl^n)$ attached to the above double coset. Then via the isomorphism: $H_P^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]) \cong H^1(X_1(Nl^n) \otimes_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}, \boldsymbol{Z}_l) \cong T_l(J_1(Nl^n))(-1)$, it is easy to see that the operators $T^*(p)$ on the left two groups correspond with $T_l(\xi) \otimes \mathrm{id}$ on the right hand side. Therefore Shimura's congruence relation [**Sh1**] Theorem 7.9 reads as:

$$T^*(p) = F_p^{-1} + pT^*(p, p) \cdot F_p$$

in $\mathrm{End}(H_P^1(\mathfrak{F}, \boldsymbol{Z}_l[\Gamma_1(N)/\Gamma_1(Nl^n)]))$ for all $n \geqq 0$. Our assertion follows from this by taking projective limit. ∎

COROLLARY (7.6.2) (Deligne; cf. [**D**]). *For any non-negative integer* d, *the representation of* $G_{\boldsymbol{Q}}$ *on* $H_P^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l)) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ *is unramified outside* Nl; *and we have:*

$$T^*(p) = F_p^{-1} + pT^*(p, p) \cdot F_p$$

in $\mathrm{End}(H_P^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l)) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l)$ *for all primes* p *not dividing* Nl.

PROOF. Since the cokernel of the specialization mapping $sp_P: H_P^1(\mathfrak{F}, \mathcal{A}/\mathcal{A}I_{\mathrm{u}}) \to H_P^1(\mathfrak{F}, S^d(\boldsymbol{Z}_l))$ is finite (7.2.2), and sp_P commutes with $T^*(p)$, $T^*(p, p)$ and the action of $G_{\boldsymbol{Q}}$, the assertion follows from the theorem above immediately. ∎

REMARK (7.6.3). By the same method as [**O1**], using the Poincaré duality theorem for $H_P^1(Y_1(N) \otimes_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}, \mathcal{F}_{S^d(\boldsymbol{Z}_l)}) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$, one can derive from (7.6.2) the equality:

$$\begin{aligned}&\det(1 - F_p^{-1}X \mid H_P^1(Y_1(N) \otimes_{\boldsymbol{Q}} \bar{\boldsymbol{Q}}, \mathcal{F}_{S^d(\boldsymbol{Z}_l)}) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l)\\ &\quad = \det(1 - T^*(p)X + pT^*(p, p)X^2 \mid S_{d+2}(\Gamma_1(N))),\end{aligned}$$

for all primes p not dividing Nl, which is due to Deligne.

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