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The set of vector fields with transverse foliations

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Introduction.

There exist precise criterions to decide whether a given C^1 flow ϕ on an m dimensional closed manifold M admits a cross-section. For example, one has the asymptotic cycles [**Sc**] as well as homology directions [**Fr**]. Both of these make use of the first real homology group of M. On the other hand, there does not exist a general criterion to decide whether a flow admits a transverse foliation. However, in the case of a three manifold this problem is solved for certain types of flows, like flows whose orbits are compact [**Mi**], [**Wo**], [**E-H-N**], Morse-Smale flows [**Go1**] and Smale flows [**Go2**]. In this paper we treat the problem of extending the result of Goodman's criterion to a general vector field on a three manifold. We found that the natural extension should be in terms of what we call "homotopy direction" [**An2**]. Using this notion we define the set $\mathcal{L}(M)$ of vector fields whose flows are homotopically linked (§2). Although we were not completely successful, we obtained unexpected properties which are described in the theorems below.

Let M be a smooth three dimensional closed manifold. We assume that M is oriented and for convenience we shall fix a Riemannian metric. Every flow ϕ appearing henceforth is generated by a vector field $\dot{\phi}$ in $NSX(M)$, the space of C^1 non-singular vector fields on M endowed with the C^0 topology and every foliation $\mathcal{F}$ is a codimension one transversely oriented foliation on M given by a C^1 coordinate systems. We denote by $\pitchfork(M)$ the topological subspace of $NSX(M)$ of vector fields whose flows admit a transverse foliation and by $\pitchfork(\overline{M})$ its closure.

0.1 THEOREM. *The sets $\pitchfork(M)$ and $\mathcal{L}(M)$ are open and not dense in $NSX(M)$ and satisfy the inclusions*

$$\pitchfork(M) \subset \mathcal{L}(M) \subset \pitchfork(\overline{M}).$$

We construct a flow to show the following

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0.2 THEOREM. $\pitchfork(M) \subseteqq \mathcal{L}(M)$.

As a by-product we have

0.3 COROLLARY. $\pitchfork(M) \subsetneqq \mathrm{Int} \pitchfork(\overline{M})$.

The above example of flow has a foliation which is "non negatively transverse" to it. Thus we are lead to the problem about when the flow under this circumstance has a transverse foliation. To do that we study the frontier of $\pitchfork(\mathcal{F})$, the set of vector fields positively $\mathcal{F}$-transverse.

0.4 DEFINITION. Let $\dot{\phi} \in \pitchfork(\overline{\mathcal{F}})$. A point $p \in M$ escapes from $\mathcal{F}$ by ϕ provided there exists a point q on the positive ϕ-orbit of p such that $\dot{\phi}(q)$ is positively $\mathcal{F}$-transverse.

0.5 THEOREM. *Let $\dot{\phi} \in \pitchfork(\overline{\mathcal{F}})$. If each point in the Birkhoff center of ϕ escapes from $\mathcal{F}$ by ϕ, then $\dot{\phi} \in \pitchfork(M)$.*

Finally we also obtained the invariance of the set $\pitchfork(M)$ under topological conjugacy (§3 Theorem 3.3).

0.6 REMARK. We don't know if $\mathcal{L}(M) = \mathrm{Int} \pitchfork(\overline{M})$. However S. Matsumoto and A. Sato [**M-S**] working with the C^1 topology in $NSX(M)$ have shown that $\mathcal{L}(M) \subseteqq \mathrm{Int} \pitchfork(\overline{M})$. They proved that a vector field tangent to the Hopf fibration on the three sphere S^3 lies in the $\mathrm{Int} \pitchfork(S^3)$. On the other hand, the corresponding flow is not homotopically linked.

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§1. Background.

a) The chain recurrent set.

Let ϕ be a flow on M. A point $p \in M$ is called chain recurrent for ϕ if for any $\varepsilon > 0$ there exists a sequence $\Gamma = \{p = p_0, p_1, \cdots, p_n = p, t_0, t_1, \cdots, t_{n-1}, p_i \in M$ and $t_i > 1\}$ such that $d(\phi_{t_i}(p_i), p_{i+1}) < \varepsilon$, for $0 \leqq i \leqq n-1$. The set of chain recurrent points is called the chain recurrent set and will be denoted by $\mathcal{R}_\phi$. It is a compact ϕ invariant set [**Co**] and cannot be exploded in the following sense:

1.1 THEOREM. [**An1**] *Let $\mathcal{R}_\phi$ be the chain recurrent set of the flow ϕ. Then given a neighbourhood U of $\mathcal{R}_\phi$ in M there exists a C^0 neighbourhood $\mathcal{V}$ of $\dot{\phi}$ in $NSX(M)$ such that $\mathcal{R}_\psi \subset U$ for every $\dot{\psi} \in \mathcal{V}$.*

Now, the condition that a flow has a hyperbolic chain recurrent set is equivalent to Axiom A and the no cycle property [**F-S**]. Therefore $\mathcal{R}_\phi$ is the union of a finite number of disjoint, compact, invariant pieces called basic sets, each of which contains a dense orbit. We say that ϕ is a Smale flow if the set $\mathcal{R}_\phi$ is one dimensional, has a hyperbolic structure and the flow satisfies the transversality condition. In particular, a Smale flow whose chain recurrent set consists of finitely many closed orbits is called a Morse-Smale flow.

b) Lyapunov functions and filtrations.

Denote by $\sum f$ the set of critical points of the C^∞ map $f: M \to R$ and by $\dot{\phi}(f)_p$ its $\dot{\phi}$ directional derivative at $p \in M$. We say that f is a Lyapunov function for ϕ provided

a) $\dot{\phi}(f)_p < 0$ for any $p \notin \mathcal{R}_\phi$

b) $\mathcal{R}_\phi = \sum f$

c) If $r \in f(\mathcal{R}_\phi)$, then $f^{-1}(r) \cap \mathcal{R}_\phi$ is a connected component of $\mathcal{R}_\phi$.

By using a combination of results from [**Co**] and [**N-S**], we show that there always exists a Lyapunov function for ϕ. So, taking regular values of f, say $-\infty = r_0 < r_1 < \cdots < r_k = \infty$ and $r_i \in f(M)$ $(1 < i < k)$, the collection of submanifolds $\{M_i; M_i = f^{-1}(-\infty, r_i]\}_{i=0}^{k}$ is a filtration for ϕ, i.e.

a) $\{\} = M_0 \subset M_1 \subset \cdots \subset M_k = M$

b) $\dim M_i = \dim M \quad \forall i$

c) $\phi_t[M_i] \subset \operatorname{Int} M_i \quad \forall t > 0$ and $\forall i$

d) $\dot{\phi}$ is transverse to the boundary ∂M_i, $1 < i < k$.

Conversely, any filtration is obtained from a Lyapunov function by the above method.

1.2 Definition. A block system for ϕ is a family $\mathcal{B} = \{N_i\}_{i=0}^{k}$ of compact connected submanifolds of M, called blocks, satisfying

a) $N_i \cap \mathcal{R}_\phi \neq \{\}$ $\forall i$

b) $N_i \cap N_j = \partial N_i \cap \partial N_j$ if $i \neq j$

c) $\{\} = N_0 \subset N_1 \subset N_1 \cup N_2 \subset \cdots \subset N_1 \cup \cdots \cup N_k = M$ is a filtration for ϕ.

A Smale flow ϕ admits a block system $\mathcal{B} = \{N_i\}_{i=0}^{k}$ where each block N_i contains only a basic set $\wedge_i$ and those blocks containing an attracting or repelling periodic orbit are diffeomorphic to the solid torus $D^2 \times S^1$. The topological structure of a block containing a basic set which is neither an attractor nor a repellor is described in [**B-W**]. The periodic orbits of a basic set $\wedge_i$ are in one to one correspondence with the periodic orbits of a semi-flow on the knot-holder (see too [**Go2**]).

c) Homotopy directions.

Given a block N for ϕ, let $[S^1, N]$ be the set of homotopy classes of continuous maps $\gamma: S^1 \to N$. For $[\gamma] \in [S^1, N]$ and $k \in Z^+$ we denote by $[\gamma]^k$ the homotopy class of the map $\alpha(z) = \gamma(z^k)$.

Take a point $p \in \mathcal{R}_\phi \cap N$ and $\varepsilon > 0$. An εp-sequence is a sequence

$$\Gamma_{\varepsilon p} = \{p = p_0, \cdots, p_n = p, t_0, \cdots, t_{n-1}; p_i \in \mathcal{R}_\phi \cap N \text{ and } t_i > 1\}$$

such that $d(\phi_{t_i}(p_i), p_{i+1}) < \varepsilon$ for $i = 0, \cdots, n-1$. Recall that there always exists an εp-sequence since $\mathcal{R}_\phi = \mathcal{R}_{\phi/\mathcal{R}}$ **[Co]**. If ε is small enough then each εp-sequence gives rise to an εp-closed orbit of ϕ, i.e., a loop $\gamma_{\varepsilon p}: [0, T] \to N$ such that

$$\gamma_{\varepsilon p}([0, T]) = \cup_{i=0}^{n-1} \{(\text{the trajectory from } p_i \text{ to } \phi_{t_i}(p_i))$$
$$+ (\text{a minimal geodesic from } \phi_{t_i}(p_i) \text{ to } p_{i+1})\}.$$

To define the closed path $\gamma_{\varepsilon p}$ we arrange by turns the parametrization of the ϕ orbit segments $[p_i, \phi_{t_i}(p_{i+1})]$ with the geodesic parametrizations and construct the continuous function $\gamma_{\varepsilon p}: [0, T] \to N$. By using the exponential map $E_T: [0, T] \to S^1$, $E_T(t) = e^{2\pi i t/T}$, we define a class in $[S^1, N]$. A class $[\gamma] \in [S^1, N]$ is said to be an ε-homotopy direction for ϕ provided that $[\gamma]^m = [\gamma_{\varepsilon p}]^n$ for some εp-closed orbit $\gamma_{\varepsilon p}$ and some $m, n \in Z^+$. Denote by $H^\varepsilon_{\phi/N}$ the set of all ε-homotopy directions for ϕ in the block N. One can easily see that $H^\varepsilon_{\phi/N}$ is a non-empty set and that $[\gamma]^k \in H^\varepsilon_{\phi/N}$ for every $[\gamma] \in H^\varepsilon_{\phi/N}$ and $k \in Z^+$. Observe that the set of ε-homotopy directions detects all closed orbits of ϕ in the block N. Let $P_{\phi/N}$ consist of those ε-homotopy directions defined by closed orbits of ϕ. Again we have $[\gamma]^k \in P_{\phi/N}$ for every $[\gamma] \in P_{\phi/N}$ and $k \in Z^+$. On the other hand, it follows from the definition that if $0 < \delta < \varepsilon$ then a δp-closed orbit is an εp-closed orbit. Therefore we have the following inclusions

$$P_{\phi/N} \subset H^\delta_{\phi/N} \subset H^\varepsilon_{\phi/N}.$$

1.3 DEFINITION. Set $H_{\phi/N} = \cap_{\varepsilon > 0} H^\varepsilon_{\phi/N}$. An element $[\gamma] \in H_{\phi/N}$ is said to be a homotopy direction for ϕ in the block N.

Recall that a block N for ϕ is also a block for any flow ψ C^0 close to ϕ because there do not exist $\mathcal{R}$ explosions. So, by a straightforward adaptation of the method used in **[An2]** one can prove the next two theorems.

1.4 THEOREM. *Let $\mathcal{B} = \{N_i\}_{i=0}^k$ be a block system for a flow ϕ on M. Then $\forall \varepsilon > 0$ there exist a $\delta > 0$ and a C^0 neighbourhood $\mathcal{V}$ of $\dot{\phi}$ in $NSX(M)$ such that for every $\dot{\psi} \in \mathcal{V}$ the following hold*

i) *$\mathcal{B}$ is a block system for ψ*

ii) *$H^\delta_{\psi/N} \subset H^\varepsilon_{\phi/N}$ for each $N \in \mathcal{B}$.*

1.5 THEOREM. *Let $\mathcal{B} = \{N_i\}_{i=0}^k$ be a block system for a flow ϕ with a hyper-*

bolic chain recurrent set. Then the following hold

i) $H_{\phi/N}=P_{\phi/N}$ *for each* $N\in\mathcal{B}$

ii) *There exists a neighbourhood* $\mathcal{V}$ *of* $\dot{\phi}$ *in* $NSX(M)$ *such that for every* $\dot{\psi}\in\mathcal{V}$, $\mathcal{B}$ *is a block system for* ψ *and* $H_{\psi/N}=H_{\phi/N}$ *for each* $N\in\mathcal{B}$.

§ 2. **Homotopically linked flow.**

In this section we prove Theorem 0.1 and Theorem 0.2.

Let ϕ be a C^1 flow on a three manifold M. We say that ϕ satisfies the weak linking property if there is a periodic orbit σ of ϕ which bounds an imbedded 2-disk D in M then the interior of D must intersect a periodic orbit. Observe that if ϕ admits a transverse foliation $\mathcal{F}$, then ϕ satisfies this property since the Novikov's result [**No**] asserts that a 2-disk whose boundary is transverse to $\mathcal{F}$ must intersect some Reeb component N which contains a ϕ closed orbit [**Go2**]. On the other hand, the weak linking property is not, in general, sufficient to insure that $\dot{\phi}\in\pitchfork(M)$. For example, a flow tangent to a Hopf fibration $S^1\to S^3\to S^2$ satisfies the weak linking property but is not transverse to any foliation, since any foliation on S^3 has a Reeb component N and the boundary ∂N cannot meet any closed transversal. However, for a Morse-Smale flow ϕ the weak linking property is also a sufficient condition for $\dot{\phi}\in\pitchfork(M)$ [**Go1**]. Notice that K. Yano proved a similar result in an equivalent terminology [**Ya**]. Now, to extend the above result to Smale flows ϕ requires a stronger property. We say that ϕ satisfies the linking property if there is a periodic orbit σ of ϕ which bounds an imbedded 2-disk D in M then the interior of D must intersect an attracting or repelling periodic orbit. A Smale flow ϕ satisfies the linking property if and only if $\dot{\phi}\in\pitchfork(M)$ [**Go2**]. To enlarge the concept above, we introduce the following

2.1 DEFINITION. A flow ϕ on M is said to be homotopically linked provided there exists $\mathcal{B}=\{N_i\}_{i=1}^k$ a block system for ϕ such that $[*]\notin H_{\phi/N}$, for each $N\in\mathcal{B}$.

Recall that $[*]$ denotes the null homotopic class of $[S^1, N]$, where N is a submanifold of M.

Let $\mathcal{L}(M)$ be the set of $NSX(M)$ consisting of vector fields whose flows are homotopically linked. Proposition 2.2 below shows that these two notions are the same for Smale flows.

2.2 PROPOSITION. *Let* ϕ *be a Smale flow on* M. *The following are equivalent*

(i) ϕ *satisfies the linking property*

(ii) ϕ *admits a transverse foliation*

(iii) ϕ *is homotopically linked.*

PROOF. (i)↔(ii) This is S. Goodman's result [**Go2**].

(ii)→(iii) Suppose that ϕ is positively $\mathcal{F}$-transverse. Take a family of submanifolds $\mathcal{B}=\{\{\ \}=N_0, N_1^+, \cdots, N_{k-1}^+, N_k, N_{k+1}^-, \cdots, N_l^-\}$ where N_i^+ (resp. N_j^-) denotes the Reeb components for which the flow is exiting (resp. entering), N_k is the closure of the exterior of the union of all the Reeb components and $M=N_1^+\cup\cdots\cup N_k\cup\cdots\cup N_l^-$. Since the boundary of each $N\in\mathcal{B}$ does not meet the chain recurrent set of the Smale flow ϕ, it is easy to show that $\mathcal{B}$ is a block system for ϕ. Now, let $H_{\phi/N}$ be the homotopy direction for ϕ in a given block N. From Theorem 1.5 $H_{\phi/N}$ consists of those homotopy directions defined by closed orbits. If $N=N_i^+$ (resp. N_j^-) then each $[\sigma]\in H_{\phi/N}$ clearly represents a nontrivial element of the fundamental group $\pi_1(N)\approx Z$. Therefore $[\sigma]\neq[*]$. If $N=N_k$, then no $[\sigma]\in H_{\phi/N}$ can be the null homotopic class in N, otherwise there could be a positive power of a closed orbit of ϕ null homotopic in N and by Novikov's theorem there exists a block N_i^+ or N_j^- inside N which is a contradiction. Therefore $[\sigma]\neq[*]$ and we have proved that ϕ is homotopically linked.

(iii)→(i) First of all we recall Corollary 3.3 in [**Go2**]. Let N be a block for a Smale flow ϕ containing a unique component $\wedge_0$ of the basic set. If no closed orbit in $\wedge_0$ is null homotopic in N then there is a ϕ transverse foliation $\mathcal{F}$ tangent to the boundary ∂N. Therefore if $[*]\notin H_{\phi/N}$ the same conclusion holds because $H_{\phi/N}=P_{\phi/N}$ (Theorem 1.5). Now suppose that ϕ is a homotopically linked Smale flow. Let $\mathcal{B}=\{N_i\}_{i=0}^k$ be a block system for ϕ such that the null homotopic class is not a homotopy direction for ϕ in any block $N\in\mathcal{B}$. If there exist blocks containing more than one component of the basic set of ϕ, one can use a Lyapunov function $f: M\to R$ to define the filtration $\{\ \}=N_0\subset N_1\subset N_3\cup N_2\subset\cdots\subset M$ and break these blocks and construct a new block system $\mathcal{B}'=\{N_i'\}_{i=0}^l$ where each N_i' contains only one component of the basic set. It is easy to see that $[*]$ is not a homotopy direction for ϕ in any N_i'. So there exists a ϕ transverse foliation.

2.3 LEMMA. *Let N be a block for ϕ. If $[\gamma]\notin H_{\phi/N}$ then there exists a C^0 neighbourhood $\mathcal{V}$ of $\dot\phi$ in $NSX(M)$ such that N is a block for ψ and $[\gamma]\notin H_{\psi/N}$ for every $\dot\psi\in\mathcal{V}$.*

PROOF. Recall that $H_{\phi/N}=\bigcap_{\varepsilon>0}H^{\varepsilon}_{\phi/N}$. So if $[\gamma]\notin H_{\phi/N}$ then there is an $\varepsilon_0>0$ such that $[\gamma]\notin H^{\varepsilon}_{\phi/N}$. From Theorem 1.4 one can find a $\delta_0>0$ and a neighbourhood $\mathcal{V}$ of $\dot\phi$ such that $H^{\delta_0}_{\psi/N}\subset H^{\varepsilon_0}_{\phi/N}$ for every $\dot\psi\in\mathcal{V}$. Thus $[\gamma]\notin H_{\psi/N}$ for every $\dot\psi\in\mathcal{V}$.

PROOF OF THEOREM 0.1. Take a homotopically linked flow ϕ. Let $\mathcal{B}=$

$\{N_i\}_{i=0}^{k}$ be a block system for ϕ such that $[*]\notin H_{\phi/N}$ for each $N\in\mathcal{B}$. From Lemma 2.3 it follows that there exists a neighbourhood $\mathcal{V}$ of $\dot{\phi}$ such that $[*]\notin H_{\psi/N}$ for each $N\in\mathcal{B}$ and for every $\dot{\psi}\in\mathcal{V}$. This shows that $\mathcal{L}(M)$ is open.

Of course $\pitchfork(M)$ is an open subset of $NSX(M)$. Let us show that $\pitchfork(M)$ is dense in $\mathcal{L}(M)$. Take $\phi\in\pitchfork(M)$ and suppose that ϕ is positively $\mathcal{F}$-transverse. Consider the family of submanifolds $\mathcal{B}=\{\{\ \}=N_0, N_1^+, \cdots, N_{k-1}^+, N_k, N_{k+1}^-, \cdots, N_l^-\}$ as described in Proposition 2.2. Observe that the chain recurrent set does not intersect the boundary of ∂N_k. Otherwise if $p\in\mathcal{R}_\phi\cap\partial N_k$, choose an εp-sequence $\Gamma_{\varepsilon p}=\{p=p_0, \cdots, p_n=p, t_0, \cdots, t_{n-1};\ p_i\in\mathcal{R}_\phi$ and $t_i>1\}$, consider the corresponding εp-closed orbit $\gamma_{\varepsilon p}:[0, T]\to M$. For $\varepsilon>0$ small enough we can guarantee that p_i and p_{i+1} belong to the same coordinate system of the foliation which allows us to construct a closed $\mathcal{F}$-transverse curve meeting a boundary component of ∂N_k. This is a contradiction, since each boundary component of ∂N_k is the boundary of a Reeb component. Now it is easy to show that $\mathcal{B}$ is a block system for ϕ and by using the argument of Proposition 2.2 one can show that $[*]\notin H_{\phi/N}$ for each $N\in\mathcal{B}$. Thus we have the inclusion $\pitchfork(M)\subset\mathcal{L}(M)$. The density follows from C^0 density of Smale flows [**Ol**] and from Proposition 2.2. This also shows that $\mathcal{L}(M)\subset\overline{\pitchfork(M)}$.

In order to prove that $\pitchfork(M)$ is not dense we use a well known result which says that the exterior of the union of all the Reeb components is an irreducible manifold, i.e., any embedded two sphere bounds a three ball.

Given a C^∞ Smale flow ϕ on M, let N be a block for ϕ diffeomorphic to the solid torus $S^1\times D^2$ containing only an attracting basic set in its core. We modify the flow ϕ to obtain a new Smale flow ψ which agrees with ϕ outside N and has two basic sets contained in the block. These new basic sets are an attracting closed orbit σ_1 and a saddle orbit σ_2, each of which bounds in N an embedded two disk without intersection [**As**]. Of course ψ does not satisfy the linking property, therefore $\dot{\psi}\notin\pitchfork(M)$. Let us show that a vector field $\dot{\xi}$ C^0 close to $\dot{\psi}$ does not admit a transverse foliation. First of all choose disjoint compact tubular neighbourhoods V_1, V_2 of σ_1, σ_2 respectively, of small radius to insure that the following conditions hold: (a) $V_1\cup V_2\subset N$; (b) $\dot{\psi}$ is transverse to the disks of the tubular neighbourhoods; (c) $\dot{\psi}$ is also transverse to the torus ∂V_1 toward σ_1. Now consider a C^0 neighbourhood $\mathcal{V}$ of $\dot{\psi}$ in $NSX(M)$ such that for every $\dot{\xi}\in\mathcal{V}$, the flow ξ satisfies the conditions (b) and (c). Moreover we may assume that N is a block for ξ and that the chain recurrent set $\mathcal{R}_{\xi/N}$ is contained in $V_1\cup V_2$ [Theorem 1.1]. The next step is to show that $\mathcal{V}\cap\pitchfork(M)=\{\ \}$. By contradiction, suppose that $\dot{\xi}\in\mathcal{V}$ is positively $\mathcal{F}_0$-transverse. From the density property [**Ol**], we may assume that ξ is a Smale flow. Since $\dot{\xi}$ satisfies conditions (b) and (c), given a disk D^2 of the tubular neighbourhood V_1, one can define the first return map for ξ, $r:D^2\to D^2$. Observe that r must have a fixed point and that each fixed point corresponds to a closed orbit of ξ contained in V_1. On the other hand, the fixed points of r are isolated because ξ has a hyperbolic chain recurrent set, so one can show by using an index argument that some fixed point is a source or sink, i.e., there exists an attracting or repelling, closed orbit ξ contained in V_1. Now we can "turbularize" the foliation $\mathcal{F}_0$ near those closed orbits to construct a new ξ-transverse foliation $\mathcal{F}_1$ with Reeb components inside the tubular neighbourhood V_1. From the fact that there is an embedded two sphere which bounds a three ball D^3 contained in the interior of the block N and such that $V_1 \subset \operatorname{Int} D^3$, it follows that the exterior of the union of all Reeb components of $\mathcal{F}_1$ is not an irreducible manifold. That is the contradiction. Thus $\pitchfork(M)$ is not dense in $NSX(M)$. Since $\pitchfork(M)$ is dense in $\mathcal{L}(M)$, we conclude that $\mathcal{L}(M)$ is not dense.

PROOF OF THEOREM 0.2. First of all we construct a vector field on the manifold $V=R\times S^1\times S^1$ whose flow is homotopically linked but does not admit a transverse foliation. Notice that $S^1=R/Z$ and that $\{\partial/\partial t, \partial/\partial x, \partial/\partial y\}$ is the canonical global frame on V. Let $\dot{\mu}$ be the C^∞ unit vector field on V tangent to the tori $T_t=\{t\}\times S^1\times S^1$ defined by

$$\dot{\mu}(t)=\cos t\frac{\partial}{\partial x}+\sin t\frac{\partial}{\partial y}.$$

By standard methods, construct a C^∞ function $\lambda: R\to[0, 1]$ such that $\lambda^{-1}(0)=[0, \pi]$ and that $\lambda^{-1}(1)=(-\infty, -\pi]\cup[2\pi, \infty)$. Now, consider the C^∞ non singular vector field $\dot{\xi}$ on V defined by

$$\dot{\xi}(t, x, y)=\lambda(t)\frac{\partial}{\partial t}+(1-\lambda(t))\dot{\mu}(t).$$

The main properties of $\dot{\xi}$ are the following (a) $\dot{\xi}\equiv\partial/\partial t$ on $V_1=\lambda^{-1}(1)\times S^1\times S^1$; (b) $\dot{\xi}$ is transverse to the tori T_t on $V_2=\lambda^{-1}((0, 1])\times S^1\times S^1$; (c) $\dot{\xi}\equiv\dot{\mu}$ on $V_3=\lambda^{-1}(0)\times S^1\times S^1$. So, the chain recurrent set $\mathcal{R}_\xi$ is the manifold with boundary V_3. Note that the vector field $\dot{\xi}=\dot{\mu}$ rotates in the positive direction from $\partial/\partial y$ to $-\partial/\partial y$ when t goes from 0 to π and that the $\partial/\partial x$-coefficient of $\dot{\xi}$ is always non negative on V_3. This behavior forbids the existence of a transverse foliation. Indeed, by contradiction suppose that $\mathcal{F}$ is transverse to $\dot{\xi}$ on V_3. Let $\mathcal{G}_t$ be the C^1 oriented one dimensional foliation on the torus $T_t=\{t\}\times S^1\times S^1$ defined by the intersection of leaves of $\mathcal{F}$ with T_t. We can construct a trivialization for $\mathcal{F}$ in the following sense: there exists a C^1 diffeomorphism $F: [0, \pi]\times S^1\times S^1\to[0, \pi]\times S^1\times S^1$ such that $F^*(\mathcal{F})=[0, \pi]\times\mathcal{G}_0$ and that $F^*(\mathcal{G}_t)=\{t\}\times\mathcal{G}_0$. For this consider the projection $\pi_1: R\times S^1\times S^1\to R$, $\pi(t, x, y)=t$ and take the vector field $\dot{\tau}$ tangent to $\mathcal{F}$ on a neighbourhood of V_3 such that at each point (t, x, y) is projected under the derivative $d\pi_1$ onto $\partial/\partial t$. If τ_t is the semi-flow of $\dot{\tau}$ define $F(t, x, y)=(t_1\tau_t(0, x, y))$. Now, let $\dot{\gamma}_0$ be a vector field on T_0 tangent to $\mathcal{G}_0$. So, from this trivialization we conclude that $\{\dot{\gamma}_0, \partial/\partial y\}$ and $\{\dot{\gamma}_0, -\partial/\partial y\}$ define the same orientation on T_0 which is a contradiction. Thus ξ does not admit a transverse foliation on V_3.

Now we shall show that ξ is a homotopically linked flow. Let $\mathscr{B}=\{V\}$ be the trivial block system for ξ. Given $\varepsilon>0$ and $p\in\mathscr{R}_\xi=[0, \pi]\times S^1\times S^1$, consider an εp-sequence, say $\Gamma_{\varepsilon p}=\{p_0=p, p_1, \cdots, p_n=p, t_0, \cdots, t_{n-1}, p_i\in\mathscr{R}_\xi \text{ and } t_i>1\}$ and the corresponding εp-closed orbit $\gamma_{\varepsilon p}: [0, T]\to[0, \pi]\times S^1\times S^1$. Recall that

$$\gamma_{\varepsilon p}([0, T])=\bigcup_{i=0}^{n-1}\{(\text{the trajectory from } p_i \text{ to } \phi_{t_i}(p_{i+1}))$$
$$+(\text{a minimal geodesic from } \phi_{t_i}(p_i) \text{ to } p_{i+1})\}.$$

To prove that $\gamma_{\varepsilon p}$ cannot be null homotopic, we show that $\pi_0\gamma_{\varepsilon p}$ is not null homotopic, where $\pi: [0, \pi]\times S^1\times S^1\to\{0\}\times S^1\times S^1$ is the natural projection along the t-axis, or equivalently that the lifted curve $\widetilde{\pi_0\gamma_{\varepsilon p}}$ to R^2, the universal covering of $\{0\}\times S^1\times S^1$, is not a closed curve. We may assume that the lifted curve $\widetilde{\pi_0\gamma_{\varepsilon p}}$ starts at (0, 0). Observe that the trajectory from p_i to $\phi_{t_i}(p_i)$ gives rise to a line segment from $\widetilde{\pi(p_i)}$ to $\widetilde{\pi(\phi_{t_i}(p_i))}$ of length equal to $t_i>1$ and that the x-coordinate of $\widetilde{\pi(\phi_{t_i}(p_i))}$ is not less than the x-coordinate of $\widetilde{\pi(p_i)}$ because the $\partial/\partial x$-coefficient of $\dot{\xi}=\dot{\mu}$ is non negative on $V_3=\mathscr{R}_\xi$. On the other hand, the distance between the points $\widetilde{\pi\phi_{t_i}(p_i)}$ and $\widetilde{\pi(p_{i+1})}$ is less than ε. So, for $\varepsilon=1/3$ it is not hard to show that the end point of $\widetilde{\pi_0\gamma_{\varepsilon p}}$ is a point with integral coordinates (m, n) satisfying $m^2+n^2>1/9$. So $\widetilde{\pi_0\gamma_{\varepsilon p}}$ is not a closed curve on R^2 therefore $\gamma_{\varepsilon p}$ is no null homotopic on V. Since $H_\varepsilon=\bigcap_{\varepsilon>0}H_\xi^\varepsilon$ and $[*]\notin H_\xi^{1/3}$ we have shown that ξ is a homotopically linked flow.

Let us show that $\pitchfork(M)\subsetneqq\mathcal{L}(M)$. Given a C^∞ Smale flow ϕ which is homotopically linked we may construct a block system for ϕ, $\mathscr{B}=\{N_i\}_{i=0}^k$ such that each block contains one and only one basic set of ϕ. Moreover, $[*]\notin H_\phi$ for every $N\in\mathscr{B}$. Since ϕ is a Smale flow there exists an attracting closed orbit σ. Let us assume that σ is contained in the block N_k which is diffeomorphic to the solid torus $S^1\times D^2$. Note that the boundary of N_k is diffeomorphic to the torus $S^1\times S^1$ and the flow is transverse to ∂N_k, by using ϕ we can define a C^∞ diffeomorphism $F: R\times S^1\times S^1\to M$ such that $F(-1, x, y)\in\partial N_k$ and that $F_*(\partial/\partial t)=\dot{\phi}$. Now we modify $\dot{\phi}$ by $F_*(\dot{\xi})$ inside the submanifold $F(V_1)$ and construct a C^∞ vector field $\dot{\psi}$ on M. We claim that the flow ψ is homotopically linked. Indeed, consider the block system for ψ, $\mathscr{B}'=\{\{\ \}=N_0, N_1, \cdots, N_{k-1}, N_k', N_{k+1}'\}$ where $N_k'=F_*(V_1)$ and N_{k+1}' is the closure of N_k/N_k'. Since ξ is homotopically linked on V_1 and N_k/N_k' contains in its core the closed orbit σ it follows that $\dot{\phi}\in\mathcal{L}(M)$. Since ξ does not admit a transverse foliation on V_1 then ψ does not admit a transverse foliation on $F_*(V_1)$. Thus $\dot{\psi}\notin\pitchfork(M)$.

§ 3. Conjugations.

Denote by θ_ϕ the one dimensional oriented foliation given by the orbits of the flow ϕ, for $\dot{\phi}\in NSX(M)$. A foliation θ_ψ is conjugate to θ_ϕ if there is a homeomorphism $h: M\to M$ which carries leaves of θ_ψ homeomorphically onto leaves of θ_ϕ and preserves the orientation. The homeomorphism $h: M\to M$ is said to have a C^0 positive ψ derivative if for each $p\in M$, the path $\alpha_p(t)=h(\psi_t(p))$ is C^1 and $\alpha'_p(0)=\lambda(h(p))\dot{\phi}(h(p))$ for a continuous function $\lambda: M\to R^+$. Set $\alpha'_p(0)=\dot{\psi}(h)_p$.

3.1 PROPOSITION. *If θ_ψ is conjugate to θ_ϕ by a homeomorphism $h: M\to M$, then θ_ψ is conjugate to θ_ϕ by a homeomorphism $k: M\to M$ having a C^0 positive ψ derivative.*

Before the proof of Proposition 3.1 we fix the following notation. For δ, $0\leqq\delta<1/2$, denote by I_δ the cube in $R^2\times R$ defined by

$$I_\delta =: \{(x, y)\in R^2\times R\,;\, \delta<x_i<1-\delta \text{ and } \delta<y<1-\delta\}.$$

Consider the two foliations induced from $R^2\times R$ into I_δ, namely: the two dimensional horizontal foliation $\mathcal{H}$ transverselly oriented by the canonical vertical vector field $\partial/\partial y$ and the one dimensional foliation $\theta_{\partial/\partial y}$ oriented by $\partial/\partial y$. A flow box for a flow ϕ is a C^1 diffeomorphism $f: U\subset M\to I_\delta$ such that $f_*(\dot{\phi})=\partial/\partial y$. Since $\dot{\phi}$ is a C^1 vector field, there are flow boxes around any point $p\in M$. On the other hand, a C^1 coordinate system for θ_ϕ is a C^1 diffeomorphism $f: U\subset M\to I_\delta$ such that $f^*(\theta_{\partial/\partial y})=\theta_\phi$.

3.2 LEMMA. *Let $h: I_\varepsilon\to R^2\times R$ be a homeomorphism onto the image which preserves the vertical foliation $\theta_{\partial/\partial y}$. Then given $\varepsilon<\delta<1/2$ and any C^1 function $\beta: I_\varepsilon\to[0, \infty)$ such that $\beta^{-1}(0)=I_\varepsilon\setminus I_\delta$, there exists $k: I_\varepsilon\to R^2\times R$ a homeomorphism onto the image which preserves the vertical foliation $\theta_{\partial/\partial y}$ and satisfying*

a) *$k\equiv h$ on $I_\varepsilon\setminus I_\delta$*

b) *k has a C^0 positive $\dot{\xi}=\beta\partial/\partial y$ derivative on I_δ.*

PROOF. From the hypothesis, $h(x, y)=(h_1(x), h_2(x, y))$ and for a fixed x the function $h_2(x, \cdot)$ is an increasing real function. Now, given $\varepsilon<\delta<1/2$ and any C^1 function $\beta: I_\varepsilon\to[0, \infty)$ such that $\beta^{-1}(0)=I_\varepsilon\setminus I_\delta$, consider the flow ξ on I_ε generated by the C^1 vector field $\dot{\xi}=\beta\partial/\partial y$. Define a function $k: I_\varepsilon\to R^2\times R$, $k(x, y)=(k_1(x, y), k_2(x, y))$, by setting $k_1\equiv h_1$ and $k_2(x, y)=\int_0^1 h_2(\xi_t(x, y))dt$. Of course k preserves the vertical foliation $\theta_{\partial/\partial y}$. Since $\dot{\xi}\equiv 0$ on $I_\varepsilon\setminus I_\delta$, it follows easily from the definition of k_2 that $k_2\equiv h_2$ on $I_\varepsilon\setminus I_\delta$. This proves (a). In order to prove (b) we observe that

$$\frac{k_2(\xi_s(x, y))-k_2(x, y)}{s}=\frac{1}{s}\left[\int_s^{1+s} h_2(\xi_t(x, y))dt-\int_0^s h_2(\xi_t(x, y))dt\right].$$

By taking $s\to 0$, one obtains $\dot{\xi}(h_2)=h_2\circ\xi_1-h_2$ which must be positive on I_δ because ξ is a non constant vertical flow there and $h_2(x, \cdot)$ is an increasing function. Since $\dot{\xi}(k)=\dot{\xi}(k_2)\partial/\partial y$, part (b) is proved. It remains to show that k is a homeomorphism onto its image. Recall that k preserves the vertical foliation, so from (a) we need only show that $k_2(x, \cdot)$ is a homeomorphism from the vertical segment $\sigma_x=[(x, \delta), (x, 1-\delta)]$ onto the vertical segment $h(\sigma_x)$. Indeed, the images by $h_2(x, \cdot)$ and $k_2(x, \cdot)$ of each end point of σ have the same values and k_2 is an increasing function in the interior of the segment σ. So, k_2 is a homeomorphism from σ to $k(\sigma)$.

Notice that we have adapted Kakutani's proof which approximates continuous real functions by functions having flow derivative [**Sc**, p. 272].

PROOF OF PROPOSITION 3.1. Let $\dot{\phi}$, $\dot{\psi}$ be two C^1 non singular vector fields on M. Suppose that there exists a homeomorphism $h: M\to M$ conjugating θ_ψ to θ_ϕ. Given $\{g_j: W_j\to I_{\rho_j}\}_{j=1}^m$ a family of flow boxes for ϕ whose domains form an open cover of M, consider $\{f_i: V_i\to I_{\delta_i}\}_{i=1}^n$ a family of flow boxes for ψ so that $\{V_i\}_{i=1}^n$ is an open cover of M and $h(V_i)$ is contained in some W_j. Now, choose ε_i, $0<\delta_i<\varepsilon_i$, such that $\{f_i^{-1}(I_{\varepsilon_i})\}_{i=1}^n$ is another open cover. Recall $I_{\varepsilon_i}\subset I_{\delta_i}$. Let $1=\sum_{i=1}^n\beta_i$ be a C^∞ partition of unity subordinate to $\{V_i\}_{i=1}^n$ satisfying $\beta_i^{-1}>0$ in $f_i^{-1}(I_{\varepsilon_i})$ $i=1, \cdots, n$. Let $\dot{\psi}=\sum_{i=1}^m\beta_i\dot{\psi}$. Observe that $f_{*_i}(\beta_i\dot{\psi})=\beta_i\circ f_i^{-1}(\partial/\partial y)$ is a C^1 vector field on I_{δ_i}. So from Lemma 3.2, starting at $k_0=h$ and working successively inside each V_i, we may obtain a homeomorphism $k_i: M\to M$ from the homeomorphism $k_{i-1}: M\to M$ conjugating θ_ψ to θ_ϕ and having a C^0 positive $\sum_{l=1}^i\beta_l\dot{\psi}$ derivative on $\bigcup_{l=1}^i f_l^{-1}(I_{\varepsilon_l})$. Of course $k_m: M\to M$ has a C^0 positive $\dot{\psi}$ derivative since $\{f_i^{-1}(I_{\varepsilon_i})\}_{i=1}^n$ is an open cover of M.

Recall that a transversely oriented Lyapunov foliation on M is a pair $(\mathcal{F}, \dot{\phi})$ satisfying the following condition

i) $\dot{\phi}$ is a C^0 vector field which is uniquely integrable

ii) There exists a collection of C^0 real value function $\{f_i: W_i\subset M\to R\}_{i=1}^k$ such that (a) $\bigcup_{i=1}^k W_i=M$; (b) f_i has a C^0 positive ϕ derivative, $i=1, \cdots, k$; (c) The level set of f_i describe the foliation $\mathcal{F}$ on W_i.

If there exists a transversely oriented Lyapunov foliation $(\mathcal{F}, \dot{\phi})$ then there exists a C^1 foliation $\mathcal{G}$ transverse to $\dot{\phi}$ [**En**].

3.3 THEOREM. *The set $\pitchfork(M)$ is invariant under topological conjugacy.*

PROOF. Suppose that θ_ψ is conjugate to θ_ϕ, where $\dot{\psi}$, $\dot{\phi}\in NSX(M)$. From Theorem 3.1, we may assume that the foliations are conjugate by a homeomorphism $k: M\to M$ having a C^0 positive ψ derivative. If there is a foliation $\mathcal{F}$ transverse to $\dot{\phi}$ we consider the C^0 foliation $k^*(\mathcal{F})$. Now, it is easy to show that $(k^*(\mathcal{F}), \dot{\psi})$ is a transversely oriented Lyapunov foliation. From [**En**], there is a C^1 foliation transverse to $\dot{\psi}$. Thus $\dot{\psi}\in\pitchfork(M)$.

Next we prove Theorem 0.5.

Let $\pitchfork_\delta(\mathcal{H})$ be the set of all C^1 vector fields on the unit cube I_0 of $R^2\times R$ positively transverse to the horizontal foliation $\mathcal{H}$ and such that the one dimensional foliation θ_ϕ agrees with the vertical foliation $\theta_{\partial/\partial y}$ on $I_0\setminus I_\theta$, $0<\delta<1/2$. Notice that θ_ϕ can be extended to the closure $\bar{I}_0$ by using the orbits of $\partial/\partial y$. We use the same notation for the extended foliation.

3.4 LEMMA. *Given a vector field* $\dot{\phi}\in\pitchfork_\delta(\mathcal{H})$, *there exists a vector field* $\dot{\psi}\in\pitchfork_{\delta/3}(\mathcal{H})$ *satisfying the following properties*

(a) *The leaf of* θ_ψ *on* $\bar{I}_0$ *starting at* $(x, 0)$ *ends at* $(x, 1)$

(b) θ_ϕ *agrees with* θ_ϕ *except on a parallelepiped* R_δ, *namely*, $R_\phi=\{(x, y)\in I;\ \delta<x_i<1-\delta,\ 1-(2/3)\delta<y<1-(1/3)\delta,\ i=1, 2\}$.

PROOF. Given $\dot{\phi}\in\pitchfork_\delta(\mathcal{H})$. Let $\bar{I}_0\cup_f\bar{I}_0$ be the double compact unitary cube whose glue map is the identity of the top face of $\bar{I}_0$. Of course $\bar{I}_0\cup_f\bar{I}_0$ is canonically diffeomorphic to $K=\{(x, y)\in R^2\times R;\ 0\leqq x_i\leqq 1,\ 0\leqq y\leqq 2,\ i=1, 2\}$. Now we consider the double foliation θ_ϕ on K oriented by a C^1 vector field $\dot{\xi}$ positively transverse to $\mathcal{H}$. Since the leaf of the double θ_ϕ starting at $(x, 0)$ ends at $(x, 2)$, we may complete the proof by constructing a C^∞ diffeomorphism $k: K\to\bar{I}$, $k(x, y)=(x, k_2(y))$ where $k_2: [0, 2]\to[0, 1]$ is a C^∞ diffeomorphism such that $k_2(y)=y$ for $\delta<y\leqq 1-\delta$, and which conveniently maps $[1-\delta, 2]$ onto $[1-\delta, 1]$. Now we consider $\dot{\phi}=k_*(\dot{\xi})$ which satisfies the properties required.

Given $\mathcal{G}$ a codimension one transversely oriented foliation on the unit open cube I_0, let $\dot{\nu}$ be a normal vector field positively transverse to $\mathcal{G}$. We say that $\partial/\partial y$ is non negatively transverse to $\mathcal{G}$ provided $\langle\partial/\partial y(x, y), \dot{\nu}(x, y)\rangle\geqq 0$ for every $(x, y)\in I_0$, where $\langle\ ,\ \rangle$ denotes the canonical inner product on $R^2\times R$. Let J_δ be the parallelepiped in I_0 defined by $J_\delta=\{(x, y)\in I_0;\ \delta<y<1-\delta\}$, for $0\leqq\delta\leqq 1/2$.

3.5 LEMMA. *Let* $\mathcal{G}$ *be a two dimensional transversely oriented* C^1 *foliation on the cube* I_0 *which agrees with the horizontal foliation* $\mathcal{H}$ *on* $I_0\setminus J_\delta$. *If* $\partial/\partial y$ *is non negatively transverse to* $\mathcal{G}$, *then there exists a* C^∞ *diffeomorphism* $k: I_0\to I_0$ *satisfying*

(a) $k(x, y)=(x, y)$ *for every* $(x, y)\in I_0\setminus I_{\delta/3}$

(b) $k^*(\mathcal{G})$ *is transverse to* $\theta_{\partial/\partial y}$ *on* I_δ

(c) $k^*(\mathcal{G})=\mathcal{G}$ *on* $I_0\setminus I_\delta$.

PROOF. Let $\dot{\eta}$ be a C^∞ unitary vector field on I_0 positively transverse to $\mathcal{G}$ and $\beta : I_0 \to [0, 1]$ be a C^∞ function so that $\beta^{-1}(0) = I_0 \setminus I_\delta$. Since the vector field $\dot{\phi} = \partial/\partial y + \beta\dot{\eta}$ belongs to $\bigcap_\delta(\mathcal{H})$, from Lemma 3.4 there exists $\dot{\psi} \in \bigcap_{\delta/3}(\mathcal{H})$ such that the leaf of θ_ψ starting at $(x, 0)$ ends at $(x, 1)$ and that θ_ψ agrees with θ_ϕ except on the parallelepiped $R_\delta = \{(x, y) \in I_0 ; \delta < x_i < 1-\delta, 1-(1/3)\delta < y < 1-(2/3)\delta\}$. By construction $\dot{\psi}$ is a C^∞ vector field transverse to horizontal foliation $\mathcal{H}$, so the ψ holonomy map h_y from the bottom face of $\bar{I}_0$ to the horizontal leaf $l_{(x,y)}$ is a C^∞ diffeomorphism, namely $h_y : [0, 1]^2 \times \{0\} \to [0, 1]^2 \times \{y\}$, $h_y(x) = \{[0, 1]^2 \times \{y\}\} \cap \{\psi$ orbit starting at $(x, 0)\}$. Hence $k : I_0 \to I_0$, $k(x, y) = (h_y(x, 0), y)$, is clearly a C^∞ diffeomorphism which maps the foliation $\theta_{\partial/\partial y}$ onto the foliation θ_ψ. Property (a) follows from the facts that $\dot{\psi} = \partial/\partial y$ on $I_0/I_{\delta/3}$ and that the ψ orbit starting at $(x, 0)$ ends at $(x, 1)$. Since $k^*(\theta_\psi) = \theta_{\partial/\partial y}$ and ψ is transverse to $\mathcal{G}$ on $I_\delta(=k(I_\delta))$, then statement (b) holds. Now, for $k^*(\mathcal{H}) = \mathcal{H}$ and from the fact that $\mathcal{G} = \mathcal{H}$ on $I_0 \setminus J_\delta$ one can easily show property (c).

We denote by $\omega_\phi(p)$ (resp. $\alpha_\phi(p)$) the ω-limit set (resp. α-limit set) of a point $p \in M$ under the flow ϕ. Recall that the Birkhoff center $C(\phi)$ is the closure of the set $\{p \in M;\ p \in \omega_\phi(p) \cap \alpha_\phi(p)\}$ which is a compact ϕ-invariant nonempty set. Any compact ϕ invariant set contains points of $C(\phi)$.

PROOF OF THEOREM 0.5. Let $\dot{\phi}$ be a vector field on the frontier of $\pitchfork(\mathcal{F}_1)$ such that every point $p \in C(\phi)$ escapes from $\mathcal{F}_0$ by ϕ. First of all we observe that every point $p \in M$ escapes from $\mathcal{F}_0$. Indeed, since $C(\phi) \cap \omega_\phi(p)$ is non empty there exists a point $q \in \omega_\phi(p)$ such that $\dot{\phi}(q)$ is positively transverse to $\mathcal{F}_0$, so for some $t_p > 0$ we can insure that $\phi_{t_p}(p)$ is so close to q that $\dot{\phi}(\phi_{t_p}(p))$ is positively transverse to $\mathcal{F}_0$. By using a similar argument we prove that there is some $s_p < 0$ such that $\dot{\phi}(\dot{\phi}_{s_p}(p))$ is positively transverse to $\mathcal{F}_0$.

Consider the compact set $K = \{p \in M;\ \dot{\phi}(p)$ is tangent to $\mathcal{F}\}$. From the paragraph above, given $p \in K$ there exist real numbers s_p, t_p, $s_p < 0 < t_p$, such that the vector field $\dot{\phi}$ is positively transverse to $\mathcal{F}_0$ on a neighbourhood of the end points of the orbit segment $\sigma_p = [\phi_{s_p}(p), \phi_{t_p}(p)]$. Since ϕ is a flow without fixed points we may assume that σ_p is an embedded segment. Otherwise, σ_p is a closed orbit which is transverse to $\mathcal{F}_0$ at some point q. So, we may choose s'_p, t'_p, with $s_p < s'_p < 0 < t'_p < t_p$ such that the orbit segment $\sigma'_p = [\phi_{s'_p}(p), \phi_{t'_p}(p)]$ is embedded. Now we construct a tubular neighbourhood V_p of σ_p and a C^1 system of coordinates of θ_ϕ, $f_p : V_p \to I_0$ such that the induced foliation on the cube I_0, $\mathcal{G}_p = f_p^{-1}(\mathcal{F})$ agrees with the horizontal foliation $\mathcal{H}$ on a neighbourhood of the top and the bottom of $\bar{I}_0$. Note that $\partial/\partial y$ is non negatively transverse to $\mathcal{G}_p$. From the compactness of K, there exists a finite family of those coordinate systems say $\{f_i : V_1 \to I_0\}_{i=1}^m$ whose domains form an open cover of K. Choose $\delta > 0$ small enough to insure that $\{f_i^{-1}(I_\delta)\}_{i=1}^m$ is another open cover of

K and that the induced foliation $\mathcal{G}_i=f_i^{-1*}(\mathcal{F})$ is the horizontal foliation on $I_0\setminus J_\delta$. For $i=1$, let $k_1: I_0\to I_0$ be the C^∞ diffeomorphism constructed in Lemma 3.5 i.e., $k_1^*(\mathcal{G}_i)$ is transverse to $\theta_{\partial/\partial y}$ on I_δ. From property (a), the C^1 diffeomorphism $h_1: V_1\to V_1$, $h_1=f_1^{-1}\circ k_1\circ f_1$ can be extended as the identity outside V_1. From property (b), the foliation $\mathcal{F}_1=h_1^*(\mathcal{F})$ is transverse to θ_ϕ on $f_1^*(I_\delta)$ and from property (c) $\mathcal{F}_1$ agrees with $\mathcal{F}_0$ outside $f_1^{-1}(I_\delta)$. Applying Lemma 3.5 repeatedly one can construct a C^1 diffeomorphism $h_i: M\to M$ such that $\mathcal{F}_i=h_i^*(\mathcal{F}_{i-1})$ is transverse to θ_ϕ on $f_1^{-1}(I_\delta)\cup\cdots\cup f_i^{-1}(I_\delta)$ and that $\mathcal{F}_i$ agrees with $\mathcal{F}_{i-1}$ outside this open set. Since $\{f_i^{-1}(I_\delta)\}_{i=1}^m$ is an open cover of K, we construct a foliation $\mathcal{F}_n$ transverse to ϕ. Consequently $\dot{\phi}\in\pitchfork(M)$.

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