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On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{ev}$ of semisimple graded Lie algebras

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Introduction.

Graded Lie algebras (abbreviated as GLA's in the sequel), even those of finite dimensions, play important roles in many fields in mathematics. In this article we shall always assume that Lie algebras are defined over $\boldsymbol{R}$ and are finite-dimensional. Let $\mathfrak{g}=\sum_{k\in Z}\mathfrak{g}_k$ be a semisimple GLA. Then there exists an integer $\nu\geqq 1$ such that $\mathfrak{g}_{\pm\nu}\neq(0)$ and $\mathfrak{g}_k=(0)$ for $|k|>\nu$. In this case we say that the GLA $\mathfrak{g}$ is *of the ν-th kind*. The family of the subspaces $(\mathfrak{g}_k)_{-\nu\leq k\leq\nu}$ is called a *gradation* on the Lie algebra $\mathfrak{g}$. Classifying GLA's is obviously equivalent to classifying gradations on each Lie algebra. We use the word "classification" in two ways. By a *weak classification* (or simply a *classification*), we mean the construction of a bijection between the set of isomorphism classes of gradations on a given Lie algebra $\mathfrak{g}$ and a certain set which is more easily accessible. By the *strong classification* we mean a weak classification plus the explicit determination of the subspaces $\mathfrak{g}_k$ of gradations. In our previous paper [11], we gave a classification of gradations on a semisimple Lie algebra $\mathfrak{g}$ in terms of its restricted fundamental root system. The same problem was treated also in Djokovič [5] from a slightly different point of view (See also Z. Hou [7]). Among semisimple GLA's, those which are most important for applications are GLA's of the first kind and of the second kind. In this direction, Kobayashi-Nagano [12] gave the strong classification of classical simple GLA's of the first kind and a classification of exceptional simple GLA's of the first kind. The strong classification of exceptional simple GLA's of the first kind was made by O. Loos [13]. J.H. Cheng [4] gave a classification of simple GLA's of the second kind under the condition that $\dim\mathfrak{g}_{-2}=1$. Afterwards, we gave in [11] a classification of classical simple GLA's of the second kind (without any assumptions) and determined the subspaces $\mathfrak{g}_{-1}$ for each case.

In the present paper, we give the strong classification of classical simple GLA's of the second kind and a classification of exceptional simple GLA's of

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the second kind (Theorems 3.2, 3.3 and Table I). For that purpose, we give a method how to read off the semisimple part of the (reductive) subalgebra $\mathfrak{g}_0$ by means of the Satake diagram of $\mathfrak{g}$ and the restricted Dynkin diagram of $\mathfrak{g}$ (Theorem 2.4). Our classification contains the afore-mentioned result of J.H. Cheng. Also Kobayashi-Nagano's one can be perfectly reproduced within our framework, although we do not perform it here. The resulting pairs $(\mathfrak{g}, \mathfrak{g}_0)$ give the infinitesimal classification of a class of homogeneous symplectic manifolds, called *simple parakähler coset spaces of the second kind* ([**10**], [**16**]). In § 3, we study how to determine the semisimple part of the subalgebra $\mathfrak{g}_{ev}=\mathfrak{g}_{-2}+\mathfrak{g}_0+\mathfrak{g}_2$ of a (effective) semisimple GLA $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ (Theorem 4.2), and give the list of the pairs $(\mathfrak{g}, \mathfrak{g}_{ev})$ for each simple GLA of the second kind (Table II). The pairs $(\mathfrak{g}, \mathfrak{g}_{ev})$ form a class of simple (affine) symmetric pairs. It seems to be interesting to study the geometric properties of the affine symmetric spaces associated with such pairs.

NOTATIONS. $\mathfrak{g}^C$ denotes the complexification of a Lie algebra (or a vector space) $\mathfrak{g}$. $\{\Gamma\}_F$ denotes the $\boldsymbol{F}$-span of a subset Γ in a vector space, where $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$. X^* denotes the transposed conjugate of a matrix X. E_n denotes the unit matrix of degree n.

§ 1. Real forms of regular semisimple subalgebras.

1.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra, τ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, where $\tau|_{\mathfrak{k}}=1$ and $\tau|_{\mathfrak{p}}=-1$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. $\mathfrak{h}$ is written as $\mathfrak{h}=\mathfrak{h}^+ +\mathfrak{a}$, where $\mathfrak{h}^+=\mathfrak{h}\cap\mathfrak{k}$. Consider the complexifications $\mathfrak{g}^C$ of $\mathfrak{g}$ and $\mathfrak{h}^C$ of $\mathfrak{h}$. $\mathfrak{h}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let $\tilde{\Delta}$ be the root system of $(\mathfrak{g}^C, \mathfrak{h}^C)$. The root space in $\mathfrak{g}^C$ for $\alpha\in\tilde{\Delta}$ is denoted by $\tilde{\mathfrak{g}}^\alpha$. The Killing form $\langle , \rangle$ of $\mathfrak{g}^C$ is positive definite on the real part $\mathfrak{h}_0:=i\mathfrak{h}^+ +\mathfrak{h}^-$ of $\mathfrak{h}^C$. We shall identify $\tilde{\Delta}$ with a subset of $\mathfrak{h}_0$ with respect to $\langle , \rangle$. τ is extended to the conjugation of $\mathfrak{g}^C$ (denoted again by τ) with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$ of $\mathfrak{g}^C$.

1.2. Let $\tilde{\Delta}'$ be a closed subsystem of $\tilde{\Delta}$ satisfying the condition that $-\tilde{\Delta}'=\tilde{\Delta}'$, that is, $-\alpha\in\tilde{\Delta}'$ for any $\alpha\in\tilde{\Delta}'$. We then have a regular complex semisimple subalgebra (cf. Dynkin [**6**]):

$$\mathfrak{g}'^C = \{\tilde{\Delta}'\}_C + \sum_{\alpha\in\tilde{\Delta}'} \tilde{\mathfrak{g}}^\alpha , \tag{1.1}$$

where $\{\tilde{\Delta}'\}_C$ is the $\boldsymbol{C}$-span of $\tilde{\Delta}'$ in $\mathfrak{h}^C$. $\{\tilde{\Delta}'\}_C$ is a Cartan subalgebra of $\mathfrak{g}'^C$ and $\tilde{\Delta}'$ can be regarded as a root system for $(\mathfrak{g}'^C, \{\tilde{\Delta}'\}_C)$.

LEMMA 1.1 *Let $\tilde{\Pi}'$ be a fundamental system of the root system $\tilde{\Delta}'$. Then*

the Dynkin diagram of $\tilde{\Pi}'$ (or of $\mathfrak{g}'^C$) can be obtained by taking the inner products between any two roots in $\tilde{\Pi}'$ with respect to $\langle , \rangle$.

PROOF. Let $\langle , \rangle'$ be the Killing form of $\mathfrak{g}'^C$. Choose the root vectors $X_\alpha \in \tilde{\mathfrak{g}}^\alpha$ in such a way that

$$\tau X_\alpha = X_{-\alpha}\,, \qquad [X_\alpha, X_{-\alpha}] = -\alpha\,. \tag{1.2}$$

Since τ leaves $\mathfrak{g}'^C$ stable, its restriction to $\mathfrak{g}'^C$ is a Cartan involution of $\mathfrak{g}'^C$ (Borel, Harish-Chandra [**2**]). Hence it follows from (1.2) that $\langle X_\alpha, X_{-\alpha}\rangle' < 0$ for $\alpha \in \tilde{\Delta}'$. Let $\alpha, \beta \in \tilde{\Delta}'$. Then we have (cf. (1.2))

$$-\langle \alpha, \beta\rangle' = \langle \beta, [X_\alpha, X_{-\alpha}]\rangle' = \langle [\beta, X_\alpha], X_{-\alpha}\rangle' = \langle \alpha, \beta\rangle \langle X_\alpha, X_{-\alpha}\rangle'. \tag{1.3}$$

Therefore $\langle \alpha, \beta\rangle = 0$ if and only if $\langle \alpha, \beta\rangle' = 0$. On the other hand, it is seen [**6**] that the two inner product $\langle , \rangle'$ and $\langle , \rangle$, restricted to each simple factor of $\mathfrak{g}'^C$ is proportional to each other. Thus we have the lemma. q.e.d.

1.3. Let σ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Assume that $\mathfrak{g}'^C$ is σ-stable. Then the intersection $\mathfrak{g}' = \mathfrak{g}'^C \cap \mathfrak{g}$ is a real form of $\mathfrak{g}'^C$. We will consider how to get the Satake diagram of $\mathfrak{g}'$. By the condition $-\tilde{\Delta}' = \tilde{\Delta}'$, $\mathfrak{g}'$ is τ-stable and has the Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$, where $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}'$ and $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{g}'$. The intersection $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$. In fact, the complexification $\mathfrak{h}'^C$ is written as $\mathfrak{h}'^C = \mathfrak{h}^C \cap \mathfrak{g}'^C$, and hence, by (1.1), $\mathfrak{h}'^C = \{\tilde{\Delta}'\}_C$ holds. Since $\mathfrak{h}'$ is τ-stable, we have

$$\mathfrak{h}' = \mathfrak{h}'^+ + \mathfrak{h}'^-\,, \tag{1.4}$$

where $\mathfrak{h}'^+ = \mathfrak{h}' \cap \mathfrak{k}'$ and $\mathfrak{h}'^- = \mathfrak{h}' \cap \mathfrak{p}'$.

LEMMA 1.2. (1) *$\mathfrak{h}'^-$ is a maximal abelian subspace of $\mathfrak{p}'$.* (2) *The $\boldsymbol{R}$-span $\{\tilde{\Delta}'\}_R$ is σ-stable and coincides with $\mathfrak{h}_0' := i\mathfrak{h}'^+ + \mathfrak{h}'^-$.*

PROOF. (1) Let $\mathfrak{h}_1'^-$ be an abelian subspace of $\mathfrak{p}'$ containing $\mathfrak{h}'^-$. By the conjugacy of maximal abelian subspaces in $\mathfrak{p}$, one can assume $\mathfrak{h}_1'^- \subset \mathfrak{a}$. Then $\mathfrak{h}'^+ + \mathfrak{h}_1'^- \subset \mathfrak{h}^+ + \mathfrak{a} = \mathfrak{h}$, and so $\mathfrak{h}'^+ + \mathfrak{h}_1'^-$ is an abelian subalgebra of $\mathfrak{g}'$ containing $\mathfrak{h}'$. Since $\mathfrak{h}'$ is maximal abelian in $\mathfrak{g}'$, one has $\mathfrak{h}_1'^- = \mathfrak{h}'^-$. (2) We have $\{\tilde{\Delta}'\}_R = \{\tilde{\Delta}'\}_C \cap \mathfrak{h}_0 = \mathfrak{h}'^C \cap \mathfrak{h}_0 = i\mathfrak{h}'^+ + \mathfrak{h}'^- = \mathfrak{h}_0'$. $\mathfrak{h}'^C$ and $\mathfrak{h}_0$ being σ-stable, $\{\tilde{\Delta}'\}_R$ is σ-stable. q.e.d.

Since the subset $\tilde{\Delta} \subset \mathfrak{h}_0$ is σ-stable, we see from Lemma 1.2 (2) that $\tilde{\Delta}'$ is σ-stable. σ is equal to -1 on $i\mathfrak{h}'^+$ and 1 on $\mathfrak{h}'^-$. Therefore, in view of Lemma 1.2, one can introduce in $\tilde{\Delta}'$ a σ-order in the sense of Satake [**14**]. Let $\tilde{\Pi}'$ be the σ-fundamental system of $\tilde{\Delta}'$ with respect to this σ-order. Then one can construct the Satake diagram of the σ-fundamental system $\tilde{\Pi}'$ (which amounts to the Satake diagram of $\mathfrak{g}'$ (cf. Lemma 1.2(1)), by taking the inner products

between two elements of $\tilde{\Pi}'$ with respect to the Killing form $\langle , \rangle$ of $\mathfrak{g}^C$ (cf. Lemma 1.1).

From the above arguments, we have

LEMMA 1.3. *Suppose that the σ-fundamental system $\tilde{\Pi}'$ of $\tilde{\Delta}'$ is a subset of a σ-fundamental system $\tilde{\Pi}$ of $\tilde{\Delta}$. Then the Satake diagram of $\tilde{\Pi}'$ (or of $\mathfrak{g}'$) is obtained from that of $\tilde{\Pi}$ by deleting the vertices not belonging to $\tilde{\Pi}'$ and the rods and arrows emanating from those vertices.*

§2. Determination of $\mathfrak{g}_0$.

2.1. We go back to the situation in 1.1 to recall some results in [**11**]. Let $\mathfrak{g}$ be a real semisimple Lie algebra, and τ be the Cartan involution in 1.1. We extend it to the conjugation (denoted again by τ) of $\mathfrak{g}^C$ with respect to $\mathfrak{g}_u = \mathfrak{k}+i\mathfrak{p}$. Let σ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$. Then the root system $\tilde{\Delta}$ of $(\mathfrak{g}^C, \mathfrak{h}^C)$ is σ-stable. Let $\tilde{\Delta}_\bullet$ denote the set of roots $\alpha \in \tilde{\Delta}$ satisfying the condition $\sigma(\alpha) = -\alpha$. Let ϖ be the orthogonal projection of $\mathfrak{h}_0$ onto $\mathfrak{a}$ with respect to $\langle , \rangle$. If we put $\Delta = \varpi(\tilde{\Delta} - \tilde{\Delta}_\bullet)$, then Δ is the root system for $(\mathfrak{g}, \mathfrak{a})$. Choose a σ-fundamental system $\tilde{\Pi} = \{\alpha_1, \cdots, \alpha_s\}$ of $\tilde{\Delta}$. Put $\tilde{\Pi}_\bullet = \tilde{\Pi} \cap \tilde{\Delta}_\bullet$. Then $\Pi := \varpi(\tilde{\Pi} - \tilde{\Pi}_\bullet)$ is a fundamental system of Δ (=a restricted fundamental system of $\mathfrak{g}$). Consider a partition of Π

$$\Pi = \Pi_0 \amalg \Pi_1 . \tag{2.1}$$

Let $\Pi = \{\gamma_1, \cdots, \gamma_r\}$. A root $\gamma \in \Delta$ can be expressed as an integral linear combination of γ_i's: $\gamma = \sum_{i=1}^{r} m_i(\gamma)\gamma_i$. For the partition (2.1), we define an integer-valued function h_{Π_1} on Δ by

$$h_{\Pi_1}(\gamma) = \sum_{\gamma_i \in \Pi_1} m_i(\gamma) . \tag{2.2}$$

Let ϑ be the dominant root of Δ with respect to Π, and let $\nu = h_{\Pi_1}(\vartheta)$. Let

$$\Delta_k = \{\gamma \in \Delta : h_{\Pi_1}(\gamma) = k\}, \qquad -\nu \leqq k \leqq \nu . \tag{2.3}$$

Then we have a partition of Δ

$$\Delta = \coprod_{k=-\nu}^{\nu} \Delta_k . \tag{2.4}$$

Consider the following subalgebra $\mathfrak{g}_0$ and subspaces $\mathfrak{g}_k$ of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{c}(\mathfrak{a}) + \sum_{\gamma \in \Delta_0} \mathfrak{g}^\gamma , \\ \mathfrak{g}_k &= \sum_{\gamma \in \Delta_k} \mathfrak{g}^\gamma , \qquad -\nu \leqq k \leqq \nu,\ k \neq 0 , \end{aligned} \tag{2.5}$$

where $\mathfrak{c}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and $\mathfrak{g}^\gamma$ is the root space in $\mathfrak{g}$ for $\gamma \in \Delta$. Then $\mathfrak{g}$ can be expressed as a GLA of the ν-th kind ([**11**])

$$\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k \,. \tag{2.6}$$

A GLA $\mathfrak{G}=\sum_{k\in Z}\mathfrak{G}_k$ is said to be *of type* α_0, if $\sum_{k\geq 1}\mathfrak{G}_{\varepsilon k}$ are generated by $\mathfrak{G}_{\varepsilon 1}$, where $\varepsilon=+$ or $-$.

THEOREM 2.1 ([11]). *The family of the subspaces* $(\mathfrak{g}_k)_{-\nu\leq k\leq\nu}$ *in* (2.5) *is a gradation of type* α_0 *on* $\mathfrak{g}$. *Conversely, every gradation of type* α_0 *of the* ν*-th kind on* $\mathfrak{g}$ *is obtained (up to isomorphisms) in the above manner, starting from a partition* (2.1).

For the GLA (2.6), we define an element $Z\in\mathfrak{a}$ by

$$\begin{aligned} &\langle \gamma_i, Z\rangle = 0\,, \qquad \gamma_i \in \Pi_0\,, \\ &\langle \gamma_j, Z\rangle = 1\,, \qquad \gamma_j \in \Pi_1\,. \end{aligned} \tag{2.7}$$

Z is the characteristic element ([11]) of the GLA (2.6). Δ_k is characterized by Z as

$$\Delta_k = \{\gamma \in \Delta : \langle \gamma, Z\rangle = k\}, \qquad -\nu \leqq k \leqq \nu\,. \tag{2.8}$$

2.2. Let us put

$$\begin{aligned} &\tilde{\Delta}_0 = (\varpi^{-1}(\Delta_0)\cap\tilde{\Delta})\cup\tilde{\Delta}_{\bullet}\,, \\ &\tilde{\Delta}_k = \varpi^{-1}(\Delta_k)\cap\tilde{\Delta}\,, \qquad -\nu \leqq k \leqq \nu,\ k \neq 0\,. \end{aligned} \tag{2.9}$$

Then $\tilde{\Delta}_k$ $(-\nu\leqq k\leqq\nu)$ is characterized by (cf. (2.8))

$$\tilde{\Delta}_k = \{\alpha \in \tilde{\Delta} : \langle \alpha, Z\rangle = k\}, \tag{2.10}$$

and we have

$$\tilde{\Delta} = \coprod_{k=-\nu}^{\nu} \tilde{\Delta}_k\,. \tag{2.11}$$

Let

$$\begin{aligned} &\tilde{\Pi}_0 = (\varpi^{-1}(\Pi_0)\cap\tilde{\Pi})\cup\tilde{\Pi}_{\bullet'}\,, \\ &\tilde{\Pi}_1 = \varpi^{-1}(\Pi_1)\cap\tilde{\Pi}\,. \end{aligned} \tag{2.12}$$

Then $\tilde{\Pi}$ admits a partition ([11])

$$\tilde{\Pi} = \tilde{\Pi}_0 \coprod \tilde{\Pi}_1\,. \tag{2.13}$$

By the definition, $\tilde{\Delta}_0$ is a closed subsystem of $\tilde{\Delta}$ satisfying $-\Delta_0=\Delta_0$. The complexification $\mathfrak{g}_0^C$ of $\mathfrak{g}_0$ can be written as [11]

$$\mathfrak{g}_0^C = \mathfrak{h}^C + \sum_{\alpha\in\tilde{\Delta}_0} \mathfrak{g}^{\alpha}\,. \tag{2.14}$$

Let us consider the regular semisimple subalgebra of $\mathfrak{g}_0^C$:

$$\mathfrak{g}^C(\tilde{\Delta}_0) = \{\tilde{\Delta}_0\}_C + \sum_{\alpha\in\tilde{\Delta}_0}\tilde{\mathfrak{g}}^\alpha . \tag{2.15}$$

LEMMA 2.2. *Let* $\{\tilde{\Delta}_0\}^\perp = \{X\in\mathfrak{h}^C : \langle X, \tilde{\Delta}_0\rangle = 0\}$. *Then the Levi decomposition of the complexification* $\mathfrak{g}_0^C$ *is given by*

$$\mathfrak{g}_0^C = \mathfrak{g}^C(\tilde{\Delta}_0)\oplus\{\tilde{\Delta}_0\}^\perp , \tag{2.16}$$

where $\mathfrak{g}^C(\tilde{\Delta}_0)$ *is the maximal semisimple ideal of* $\mathfrak{g}_0^C$ *and* $\{\tilde{\Delta}_0\}^\perp$ *is the center of* $\mathfrak{g}_0^C$.

PROOF. First note that $\mathfrak{h}^C = \{\tilde{\Delta}_0\}_C\oplus\{\tilde{\Delta}_0\}^\perp$. Therefore, by (2.14) and (2.15), we have $\mathfrak{g}_0^C = \{\tilde{\Delta}_0\}^\perp + \mathfrak{g}^C(\tilde{\Delta}_0)$ (vector space direct sum). Let $X\in\{\tilde{\Delta}_0\}^\perp$ and let $E_\alpha\in\tilde{\mathfrak{g}}^\alpha$, $\alpha\in\tilde{\Delta}_0$. Then $[X, E_\alpha] = \langle\alpha, X\rangle E_\alpha = 0$, which implies the inclusion $\{\tilde{\Delta}_0\}^\perp \subset \mathfrak{z}(\mathfrak{g}_0^C)$, the center of $\mathfrak{g}_0^C$. To prove the converse inclusion, let $X\in\mathfrak{z}(\mathfrak{g}_0^C)$ and write $X = X_1 + X_2$, $X_1\in\{\tilde{\Delta}_0\}^\perp$, $X_2\in\mathfrak{g}^C(\tilde{\Delta}_0)$. Let $\alpha\in\tilde{\Delta}_0$. Then $0 = [X, \tilde{\mathfrak{g}}^\alpha] = [X_2, \tilde{\mathfrak{g}}^\alpha]$. For an element $Y\in\{\tilde{\Delta}_0\}_C$, we have $0 = [X, Y] = [X_2, Y]$. Hence X_2 lies in the center of $\mathfrak{g}^C(\tilde{\Delta}_0)$ and so $X_2 = 0$, from which we get the inclusion $\mathfrak{z}(\mathfrak{g}_0^C)\subset\{\tilde{\Delta}_0\}^\perp$.

q.e.d.

$\tilde{\Delta}_0$ is σ-stable. In fact, let $\alpha\in\tilde{\Delta}_0$. Then, noting that $\tilde{\Delta}$ is σ-stable, we have

$$\langle\sigma(\alpha), Z\rangle = \langle\sigma(\alpha), \sigma(Z)\rangle = \langle\alpha, Z\rangle = 0 , \tag{2.17}$$

which implies $\sigma(\alpha)\in\tilde{\Delta}_0$ (cf. (2.10)). Hence $\mathfrak{g}^C(\tilde{\Delta}_0)$ is σ-stable and the intersection $\mathfrak{g}(\tilde{\Delta}_0) = \mathfrak{g}^C(\tilde{\Delta}_0)\cap\mathfrak{g}$ is a real form of $\mathfrak{g}^C(\tilde{\Delta}_0)$.

LEMMA 2.3. *The semisimple part* $\mathfrak{g}_0^s$ *and the center* $\mathfrak{z}(\mathfrak{g}_0)$ *of the reductive subalgebra* $\mathfrak{g}_0$ *are given by*

$$\mathfrak{g}_0^s = \mathfrak{g}(\tilde{\Delta}_0) , \qquad \mathfrak{z}(\mathfrak{g}_0) = \{\tilde{\Delta}_0\}^\perp\cap\mathfrak{g}_0 . \tag{2.18}$$

PROOF. Since $\mathfrak{g}_0^s$ and $\mathfrak{g}^C(\tilde{\Delta}_0)$ are σ-stable, (2.18) follows from (2.16) by taking the intersection of each member in (2.16) with $\mathfrak{g}$. q.e.d.

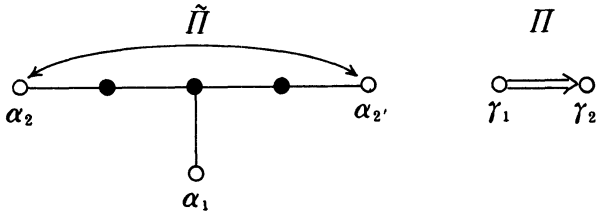
The following theorem enables us to describe the semisimple part of $\mathfrak{g}_0$ for a semisimple GLA $\mathfrak{g} = \sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ of type α_0 (cf. Theorem 2.1).

THEOREM 2.4. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra and* σ *be the conjugation of* $\mathfrak{g}^C$ *with respect to* $\mathfrak{g}$. *Let* $\tilde{\Pi}$ *be a* σ*-fundamental system of* $\mathfrak{g}$, *and* Π *be the restricted fundamental system of* $\mathfrak{g}$ *obtained from* $\tilde{\Pi}$. *Then the Satake diagram of the semisimple part* $\mathfrak{g}_0^s$ *of the subalgebra* $\mathfrak{g}_0$ *in the gradation on* $\mathfrak{g}$ *corresponding to a partition* (2.1) *of* Π *is obtained from the Satake diagram* $\tilde{\Pi}$ *of* $\mathfrak{g}$ *by deleting all the vertices whose* ϖ*-images lie in* Π_1 *and by deleting all the rods and arrows emanating from those vertices.*

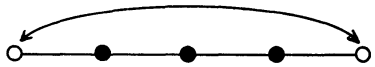
PROOF. Since $\tau|_{\mathfrak{h}_0} = -1$, τ leaves $\mathfrak{g}_0^s = \mathfrak{g}(\tilde{\Delta}_0)$ stable. Therefore the arguments

in 1.3 can be applied to the subalgebra $\mathfrak{g}(\tilde{\Delta}_0)$. We have that $\tilde{\Pi}_0$ is a σ-fundamental system of $\tilde{\Delta}_0$, viewed as a root system of $\mathfrak{g}^C(\tilde{\Delta}_0)$. From Lemma 1.3 and (2.13) it follows that the Satake diagram of $\tilde{\Pi}_0$ (=the Satake diagram of $\mathfrak{g}(\tilde{\Delta}_0)$) is obtained from that of $\tilde{\Pi}$ by deleting all vertices of $\tilde{\Pi}_1$ and by deleting rods and arrows emanating from those vertices. Let $\alpha_i \in \tilde{\Pi}$. Then, by (2.12), $\alpha_i \in \tilde{\Pi}_1$ if and only if $\varpi(\alpha_i) \in \Pi_1$. q.e.d.

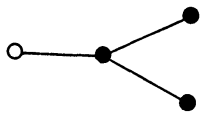
EXAMPLES 2.5. (1) Let $\mathfrak{g}=E_{6(-14)}$. Then $\tilde{\Pi}$ and Π $(=BC_2)$ are given by:



In this case, $\varpi(\alpha_1)=\gamma_1$ and $\varpi(\alpha_2)=\varpi(\alpha_{2'})=\gamma_2$. $\vartheta=2\gamma_1+2\gamma_2$. Case I. Let $\Pi_1=\{\gamma_1\}$. Then $h_{\Pi_1}(\vartheta)=m_1(\vartheta)=2$, and consequently the corresponding gradation is of type α_0 of the second kind. Theorem 2.4 implies that $\tilde{\Pi}_0$ is given by:

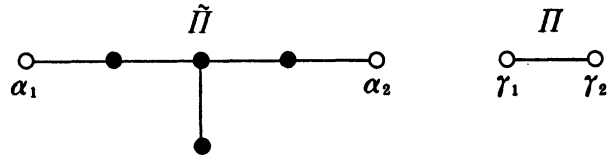


Therefore we have $\mathfrak{g}_0^s=\mathfrak{su}(1, 5)$. Case II. Let $\Pi_1=\{\gamma_2\}$. The corresponding gradation is also of type α_0 of the second kind. By Theorem 2.4 we have that $\tilde{\Pi}_0$ is given by:

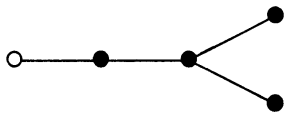


Therefore $\mathfrak{g}_0^s=\mathfrak{so}(1, 7)$.

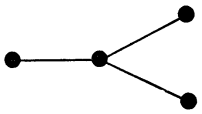
(2) Let $\mathfrak{g}=E_{6(-26)}$. Then $\tilde{\Pi}$ and Π are given by:



In this case $\varpi(\alpha_i)=\gamma_i$ $(i=1, 2)$. $\vartheta=\gamma_1+\gamma_2$. Case I. Let $\Pi_1=\{\gamma_1\}$. Then $h_{\Pi_1}(\vartheta)=m_1(\vartheta)=1$ and so the corresponding gradation is of the first kind. By Theorem 2.4, we have that $\tilde{\Pi}_0$ is given by:



Therefore $\mathfrak{g}_0^s=\mathfrak{so}(1, 9)$. Case II. Let $\Pi_1=\{\gamma_1, \gamma_2\}$. Then $h_{\Pi_1}(\vartheta)=m_1(\vartheta)+m_2(\vartheta)=2$, and so the corresponding gradation is of the second kind. By Theorem 2.4, we have that $\tilde{\Pi}_0$ is given by:



Therefore $\mathfrak{g}_0^s=\mathfrak{so}(8)$.

§ 3. Semisimple GLA's of the second kind (Classification).

3.1. Let $\mathfrak{G}=\sum\mathfrak{G}_k$ be a GLA. $\mathfrak{G}$ is called *effective*, if $\mathfrak{G}_0$ contains no non-zero ideal of $\mathfrak{G}$. Suppose that $\mathfrak{G}$ is not effective. We then choose the maximal ideal $\mathfrak{N}$ of $\mathfrak{G}$ which is contained in $\mathfrak{G}_0$. It follows that the quotient algebra $\mathfrak{G}/\mathfrak{N}$ has a natural graded Lie algebra structure (induced from that of $\mathfrak{G}$) which is effective. Therefore we can assume that the GLA $\mathfrak{G}$ is effective. The following lemma can be proved analogously as for Lemma 2.2 [8].

LEMMA 3.1. *Let* $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$ *be an effective semisimple GLA of the* ν*-th kind, and let* $\mathfrak{g}=\mathfrak{g}^1\oplus\cdots\oplus\mathfrak{g}^s$ *be the decomposition of* $\mathfrak{g}$ *into simple ideals. Then each simple factor* $\mathfrak{g}^i$ *is a graded ideal of the* μ_i*-th kind, where* $1\leqq\mu_i\leqq\nu$, *and* $\mathfrak{g}$ *is the direct sum of the graded ideals* $\mathfrak{g}^1, \cdots, \mathfrak{g}^s$.

By Lemma 3.1, the classification of (effective) semisimple GLA's of the second kind is reduced to that of simple GLA's of the first and the second kinds. The simple GLA's of the first kind were classified (up to isomorphisms) by Kobayashi-Nagano [12] (see also [11]). In [11] we have classified classical simple GLA's of the second kind and determined the subspace $\mathfrak{g}_{-1}$ for each gradation. In the sequel, we will give more perfect determination of real simple GLA's of the second kind. First of all, we use the following notations for Jordan triple systems.

$M_{n,m}(\boldsymbol{F})$ the space of $n\times m$ matrices with entries in $\boldsymbol{F}$, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ (=quaternions),

$H(n, \boldsymbol{F})$ the space of $\boldsymbol{F}$-hermitian $n\times n$ matrices, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$,

$SH(n, \boldsymbol{F})$ the space of $\boldsymbol{F}$-skew-hermitian $n\times n$ matrices, $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$,

$Alt_n(\boldsymbol{F})$ the space of $n\times n$ skew-symmetric matrices with entries in $\boldsymbol{F}$, where $\boldsymbol{F}=\boldsymbol{R}$ or $\boldsymbol{C}$,

$Sym_n(\boldsymbol{C})$ the space of $n\times n$ complex symmetric matrices.

THEOREM 3.2. *Classical real simple GLA's* $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ *of the second kind are classified (up to isomorphisms) as follows:*

(c1) $\mathfrak{g}=\mathfrak{sl}(n, \boldsymbol{F})$, $n\geqq 3$, $\boldsymbol{F}=\boldsymbol{R}$ *or* $\boldsymbol{C}$,

$\Pi=\{\gamma_1, \cdots, \gamma_{n-1}\}$ *of type* A_{n-1},

$\Pi_1=\{\gamma_p, \gamma_{p+q}\}$, $1\leqq p\leqq [n/2]$, $1\leqq q\leqq n-2p$,

$\mathfrak{g}_0=\mathfrak{sl}(p, \boldsymbol{F})+\mathfrak{sl}(q, \boldsymbol{F})+\mathfrak{sl}(n-p-q, \boldsymbol{F})+\boldsymbol{F}+\boldsymbol{F}$,

$\mathfrak{g}_{-1}=M_{p,q}(\boldsymbol{F})\times M_{q,n-p-q}(\boldsymbol{F})$,

$\mathfrak{g}_{-2}=M_{p,n-p-q}(\boldsymbol{F})$.

(c2) $\mathfrak{g}=\mathfrak{sl}(n, \boldsymbol{H})$, $n\geqq 3$,

Π, Π_1 *the same as in* (c1) *with the same conditions,*

$\mathfrak{g}_0=\mathfrak{sl}(p, \boldsymbol{H})+\mathfrak{sl}(q, \boldsymbol{H})+\mathfrak{sl}(n-p-q, \boldsymbol{H})+\boldsymbol{R}+\boldsymbol{R}$,

$\mathfrak{g}_{-1}=M_{p,q}(\boldsymbol{H})\times M_{q,n-p-q}(\boldsymbol{H})$,

$\mathfrak{g}_{-2}=M_{p,n-p-q}(\boldsymbol{H})$.

(c3) $\mathfrak{g}=\mathfrak{su}(p, q)$, $1\leqq p<q$ or $3\leqq p=q$,

$\Pi=\{\gamma_1, \cdots, \gamma_p\}$ *of type* BC_p $(p<q)$, *or type* C_p $(p=q)$,

$\Pi_1=\{\gamma_k\}$, $1\leqq k\leqq p$ *if* $p<q$, *or* $1\leqq k\leqq p-1$ *if* $p=q$,

$\mathfrak{g}_0=\mathfrak{sl}(k, \boldsymbol{C})+\mathfrak{su}(p-k, q-k)+\boldsymbol{R}+i\boldsymbol{R}$,

$\mathfrak{g}_{-1}=M_{k,p+q-2k}(\boldsymbol{C})$,

$\mathfrak{g}_{-2}=H(k, \boldsymbol{C})$.

(c4) $\mathfrak{g}=\mathfrak{so}(p, q)$, $2\leqq p<q$, *or* $4\leqq p=q$,

$\Pi=\{\gamma_1, \cdots, \gamma_p\}$ *of type* B_p $(p<q)$, *or type* D_p $(p=q)$,

$\Pi_1=\{\gamma_k\}$, $2\leqq k\leqq p$ *if* $p<q$, *or* $2\leqq k\leqq p-2$ *if* $p=q$,

$\mathfrak{g}_0=\mathfrak{sl}(k, \boldsymbol{R})+\mathfrak{so}(p-k, q-k)+\boldsymbol{R}$,

$\mathfrak{g}_{-1}=M_{k,p+q-2k}(\boldsymbol{R})$,

$\mathfrak{g}_{-2}=Alt_k(\boldsymbol{R})$.

(c5) $\mathfrak{g}=\mathfrak{sp}(n, \boldsymbol{F})$, $n\geqq 3$, $\boldsymbol{F}=\boldsymbol{R}$ *or* $\boldsymbol{C}$,

$\Pi=\{\gamma_1, \cdots, \gamma_n\}$ *of type* C_n,

$\Pi_1=\{\gamma_k\}$, $1\leqq k\leqq n-1$,

$\mathfrak{g}_0 = \mathfrak{gl}(k, \boldsymbol{F})+\mathfrak{sp}(n-k, \boldsymbol{F})+\boldsymbol{F}$,

$\mathfrak{g}_{-1} = M_{k,2n-2k}(\boldsymbol{F})$,

$\mathfrak{g}_{-2} = Sym_k(\boldsymbol{C})$ *if* $\boldsymbol{F}=\boldsymbol{C}$, *or* $H(k, \boldsymbol{R})$ *if* $\boldsymbol{F}=\boldsymbol{R}$.

(c6) $\mathfrak{g} = \mathfrak{sp}(p, q)$, $1 \leqq p < q$, *or* $2 \leqq p = q$,

$\Pi = \{\gamma_1, \cdots, \gamma_p\}$ *of type* BC_p $(p<q)$, *or type* C_p $(p=q)$,

$\Pi_1 = \{\gamma_k\}$, $1 \leqq k \leqq p$ *if* $p<q$, *or* $1 \leqq k \leqq p-1$ *if* $p=q$,

$\mathfrak{g}_0 = \mathfrak{gl}(k, \boldsymbol{H})+\mathfrak{sp}(p-k, q-k)+\boldsymbol{R}$,

$\mathfrak{g}_{-1} = M_{k,p+q-2k}(\boldsymbol{H})$,

$\mathfrak{g}_{-2} = SH(k, \boldsymbol{H})$.

(c7) $\mathfrak{g} = \mathfrak{so}^*(2n)$, n *even* $\geqq 6$, *or* n *odd* $\geqq 5$,

$\Pi = \{\gamma_1, \cdots, \gamma_{[n/2]}\}$ *of type* $C_{[n/2]}$ $(n$ *even*$)$, *or of type* $BC_{[n/2]}$ $(n$ *odd*$)$,

$\Pi_1 = \{\gamma_k\}$, $1 \leqq k \leqq [n/2]-1$ *if* n *even*, *or* $1 \leqq k \leqq [n/2]$ *if* n *odd*,

$\mathfrak{g}_0 = \mathfrak{gl}(k, \boldsymbol{H})+\mathfrak{so}^*(2n-4k)+\boldsymbol{R}$,

$\mathfrak{g}_{-1} = M_{k,n-2k}(\boldsymbol{H})$,

$\mathfrak{g}_{-2} = H(k, \boldsymbol{H})$.

(c8) $\mathfrak{g} = \mathfrak{so}(n, n; \boldsymbol{F})$, $\boldsymbol{F}=\boldsymbol{R}$ *or* $\boldsymbol{C}$,

$\Pi = \{\gamma_1, \cdots, \gamma_n\}$ *of type* D_n,

a) $\Pi_1 = \{\gamma_{n-1}, \gamma_n\}$ $(n \geqq 4)$,

$\mathfrak{g}_0 = \mathfrak{gl}(n-1, \boldsymbol{F})+\boldsymbol{F}+\boldsymbol{F}$,

$\mathfrak{g}_{-1} = M_{n-1,2}(\boldsymbol{F})$,

$\mathfrak{g}_{-2} = Alt_{n-1}(\boldsymbol{F})$.

b) $\Pi_1 = \{\gamma_1, \gamma_n\}$ $(n \geqq 5)$,

$\mathfrak{g}_0 = \mathfrak{gl}(n-1, \boldsymbol{F})+\boldsymbol{F}+\boldsymbol{F}$,

$\mathfrak{g}_{-1} = M_{1,n-1}(\boldsymbol{F}) \times Alt_{n-1}(\boldsymbol{F})$,

$\mathfrak{g}_{-2} = \boldsymbol{F}^{n-1}$.

(c9) $\mathfrak{g} = \mathfrak{so}(n, \boldsymbol{C})$, n *odd* $\geqq 5$, *or* n *even* $\geqq 8$,

$\Pi = \{\gamma_1, \cdots, \gamma_{[n/2]}\}$ *of type* $B_{[n/2]}$ $(n$ *odd*$)$, *or type* $D_{n/2}$ $(n$ *even*$)$,

$\Pi_1 = \{\gamma_k\}$, $2 \leqq k \leqq [n/2]$ *if* n *odd*, *or* $2 \leqq k \leqq (n/2)-2$ *if* n *even*,

$\mathfrak{g}_0 = \mathfrak{gl}(k, \boldsymbol{C})+\mathfrak{so}(n-2k, \boldsymbol{C})+\boldsymbol{C}$,

$\mathfrak{g}_{-1} = M_{k,n-2k}(\boldsymbol{C})$,

$\mathfrak{g}_{-2} = Alt_k(\boldsymbol{C})$.

PROOF. Since a simple GLA of the second kind is of type α_0 (Tanaka [**15**]), one can apply Theorem 2.4 to determine $\mathfrak{g}_0$. In order to determine $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$, we have to choose a nice realization adapted to the gradation for each simple Lie algebra. We give the proof only for the case $\mathfrak{g}=\mathfrak{su}(p, q)$, $p<q$. Other cases can be done analogously. We use the following realization of $\mathfrak{g}=\mathfrak{su}(p, q)$:

$$\mathfrak{su}(p, q)=\{X\in\mathfrak{gl}(p+q, \boldsymbol{C}): X^*A_{p,q}+A_{p,q}X=0\}, \tag{3.1}$$

where

$$A_{p,q}=\begin{pmatrix} 0 & 0 & iE_p \\ 0 & E_{q-p} & 0 \\ -iE_p & 0 & 0 \end{pmatrix}.$$

Then $X\in\mathfrak{su}(p, q)$ if and only if

$$X=\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & -iX_{12}^* \\ X_{31} & -iX_{21}^* & -X_{11}^* \end{pmatrix},\quad X_{13}, X_{31}\in H(p, \boldsymbol{C}),\ X_{22}\in\mathfrak{su}(q-p). \tag{3.2}$$

If we put $\tau(X)=-X^*$, then τ is a Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{a}=\{X=\operatorname{diag}(x_1, \cdots, x_p, 0, \cdots, 0, -x_1, \cdots, -x_p): x_i\in\boldsymbol{R}\}. \tag{3.3}$$

Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The root system Δ of $(\mathfrak{g}, \mathfrak{a})$ is given by

$$\Delta=\{\pm(x_i\pm x_j)\ (1\leqq i<j\leqq p),\ \pm x_i,\ \pm 2x_i\ (1\leqq i\leqq p)\}, \tag{3.4}$$

which is of type BC_p. A fundamental system $\Pi=\{\gamma_1, \cdots, \gamma_p\}$ of Δ is chosen to be

$$\gamma_i=x_{i+1}-x_i\ \ (1\leqq i\leqq p-1),\qquad \gamma_p=-x_p. \tag{3.5}$$

The restricted dominant root ϑ is given by $2\gamma_1+\cdots+2\gamma_p$, and hence $h_{\Pi_1}(\vartheta)=2$ if and only if $\Pi_1=\{\gamma_k\}$, $1\leqq k\leqq p$. Therefore, by 2.1 and 2.2, every gradation of the second kind in $\mathfrak{g}$ arises from $\Pi_1=\{\gamma_k\}$ for some k $(1\leqq k\leqq p)$. The corresponding partition $\Delta=\coprod_{j=-2}^{2}\Delta_j$ in (2.4) is given by ([**11**])

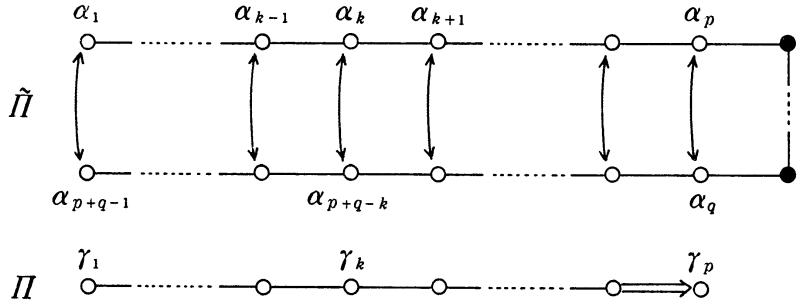
$$\begin{aligned}
\Delta_0 &= \{\pm(x_i-x_j)\ (1\leqq i<j\leqq k),\ \pm(x_i\pm x_j)\ (k+1\leqq i<j\leqq p),\\
&\qquad \pm x_i,\ \pm 2x_i\ (k+1\leqq i\leqq p)\},\\
\Delta_{-1} &= -\Delta_1=\{x_i\pm x_j\ (1\leqq i\leqq k,\ k+1\leqq j\leqq p),\ x_i\ (1\leqq i\leqq k)\},\\
\Delta_{-2} &= -\Delta_2=\{x_i+x_j\ (1\leqq i<j\leqq k),\ 2x_i\ (1\leqq i\leqq k)\}.
\end{aligned} \tag{3.6}$$

By using (3.2), (3.3) and (3.4) one can determine the root space $\mathfrak{g}^\gamma\subset\mathfrak{g}$ for each

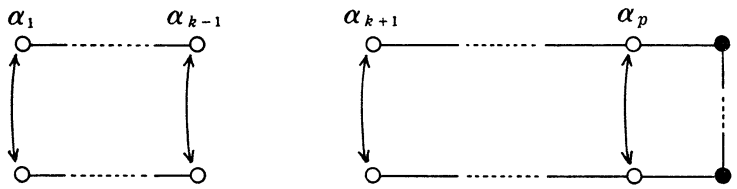
root $\gamma \in \Delta$. Thus it follows from (2.5) and (3.6) that

$$\mathfrak{g}_{-1} = M_{k,\,p+q-2k}(\boldsymbol{C}), \qquad \mathfrak{g}_{-2} = H(k,\,\boldsymbol{C}). \tag{3.7}$$

The Satake diagram of $\tilde{\Pi}$ and the restricted Dynkin diagram Π of $\mathfrak{su}(p,\,q)$ are given by:



By Theorem 2.4 the Satake diagram of $\mathfrak{g}_0^s$ is given by:



Therefore one has

$$\mathfrak{g}_0^s = \mathfrak{sl}(k,\,\boldsymbol{C}) + \mathfrak{su}(p-k,\,q-k). \tag{3.8}$$

Let $\mathfrak{z}(\mathfrak{g}_0)$ denote the center of $\mathfrak{g}_0$. Then it follows from (3.7) and (3.8) that

$$\dim \mathfrak{z}(\mathfrak{g}_0) = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_0^s = \dim \mathfrak{g} - 2(\dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-2}) - \dim \mathfrak{g}_0^s = 2\,,$$

$$\dim (\text{vector part of } \mathfrak{z}(\mathfrak{g}_0)) = \mathrm{rk}_R \mathfrak{g}_0 - \mathrm{rk}_R \mathfrak{g}_0^s = \mathrm{rk}_R \mathfrak{g} - \mathrm{rk}_R \mathfrak{g}_0^s = 1\,,$$

where rk_R denotes the real rank. Therefore we have that $\mathfrak{g}_0 = \mathfrak{g}_0^s + \mathfrak{z}(\mathfrak{g}_0) = \mathfrak{sl}(k,\,\boldsymbol{C}) + \mathfrak{su}(p-k,\,q-k) + \boldsymbol{R} + i\boldsymbol{R}$. q.e.d.

THEOREM 3.3. *Exceptional real simple GLA's* $\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k$ *of the second kind are classified (up to isomorphisms) as listed in Table I.*

PROOF. We give only the sketch of the proof. By looking into the coefficients $m_i(\vartheta)$ of the restricted dominant root ϑ, we can find all possible choices of partitions $\Pi = \Pi_0 \amalg \Pi_1$ which yield gradations of the second kind in $\mathfrak{g}$. Then we get the partitions (2.4) of Δ with $\nu = 2$. The dimension of $\mathfrak{g}_{-k}$ $(k=1,\,2)$ can be computed by the formula $\dim_R \mathfrak{g}_{-k} = \sum_{\gamma \in \Delta_k} \dim \mathfrak{g}^\gamma$ (cf. (2.5)); $\dim \mathfrak{g}^\gamma = m(\gamma)$ (=the multiplicity of the restricted root γ) can be determined by finding the ϖ-image $\varpi(\alpha)$ for each $\alpha \in \tilde{\Delta}$. $\mathfrak{g}_0^s$ is determined by the method given

in Theorem 2.4. Then we can go along the same line as in the proof of Theorem 3.2 to determine $\mathfrak{g}_0$ itself. We will show the details only for $\mathfrak{g}=E_{6(-14)}$ treated in Examples 2.5(1). In this case $\Pi=\{\gamma_1, \gamma_2\}$ is of type BC_2. We then have $m(\gamma_1)=6$, $m(\gamma_2)=8$ and $m(2\gamma_2)=1$ (cf. Araki [1]). Let $\Pi_1=\{\gamma_2\}$. Then, by (3.6), we get $\Delta_{-1}=\{x_1, x_2\}$ and $\Delta_{-2}=\{x_1+x_2, 2x_1, 2x_2\}$. Then, considering the action of the Weyl group on Δ, we have from (3.5) and (2.5) that $\dim \mathfrak{g}_{-1}=m(x_1)+m(x_2)=16$ and $\dim \mathfrak{g}_{-2}=m(x_1+x_2)+m(2x_1)+m(2x_2)=8$. Hence we get $\dim \mathfrak{g}_0=30$. But we know by Examples 2.5 that $\mathfrak{g}_0^s=\mathfrak{so}(1, 7)$. Therefore $\dim \mathfrak{z}(\mathfrak{g}_0)=2$. Furthermore $\mathrm{rk}_R\mathfrak{g}_0=\mathrm{rk}_R\mathfrak{g}=2$ and $\mathrm{rk}_R\mathfrak{g}_0^s=1$, and hence $\mathrm{rk}_R\mathfrak{z}(\mathfrak{g}_0)=1$. Thus we conclude that $\mathfrak{g}_0=\mathfrak{so}(1, 7)+\boldsymbol{R}+i\boldsymbol{R}$. q.e.d.

In Table I we are adopting the numbering of simple roots as in Bourbaki [3] for exceptional simple Lie algebras.

REMARK 3.4. The GLA's (c3) $k=p<q$, (c7) n odd and $k=[n/2]$ and (e7) are the GLA's of infinitesimal holomorphic automorphisms of irreducible symmetric Siegel domains of the second kind.

Table I.

	$\mathfrak{g}$	Π	Π_1	$\mathfrak{g}_0$	$\dim_R\mathfrak{g}_{-1}$	$\dim_R\mathfrak{g}_{-2}$
(e1)	$E_{6(6)}$	E_6	$\{\gamma_3\}$	$\mathfrak{sl}(5, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R}$	20	5
(e2)	$E_{6(6)}$	E_6	$\{\gamma_2\}$	$\mathfrak{sl}(6, \boldsymbol{R})+\boldsymbol{R}$	20	1
(e3)	$E_{6(6)}$	E_6	$\{\gamma_1, \gamma_6\}$	$\mathfrak{so}(4, 4)+\boldsymbol{R}+\boldsymbol{R}$	16	8
(e4)	$E_{6(2)}$	F_4	$\{\gamma_1\}$	$\mathfrak{su}(3, 3)+\boldsymbol{R}$	20	1
(e5)	$E_{6(2)}$	F_4	$\{\gamma_4\}$	$\mathfrak{so}(3, 5)+\boldsymbol{R}+i\boldsymbol{R}$	16	8
(e6)	$E_{6(-14)}$	BC_2	$\{\gamma_1\}$	$\mathfrak{su}(1, 5)+\boldsymbol{R}$	20	1
(e7)	$E_{6(-14)}$	BC_2	$\{\gamma_2\}$	$\mathfrak{so}(1, 7)+\boldsymbol{R}+i\boldsymbol{R}$	16	8
(e8)	$E_{6(-26)}$	A_2	$\{\gamma_1, \gamma_2\}$	$\mathfrak{so}(8)+\boldsymbol{R}+\boldsymbol{R}$	16	8
(e9)	$E_{7(7)}$	E_7	$\{\gamma_6\}$	$\mathfrak{so}(5, 5)+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R}$	32	10
(e10)	$E_{7(7)}$	E_7	$\{\gamma_1\}$	$\mathfrak{so}(6, 6)+\boldsymbol{R}$	32	1
(e11)	$E_{7(7)}$	E_7	$\{\gamma_2\}$	$\mathfrak{sl}(7, \boldsymbol{R})+\boldsymbol{R}$	35	7
(e12)	$E_{7(-5)}$	F_4	$\{\gamma_1\}$	$\mathfrak{so}^*(12)+\boldsymbol{R}$	32	1
(e13)	$E_{7(-5)}$	F_4	$\{\gamma_4\}$	$\mathfrak{so}(3, 7)+\mathfrak{su}(2)+\boldsymbol{R}$	32	10
(e14)	$E_{7(-25)}$	C_3	$\{\gamma_1\}$	$\mathfrak{so}(2, 10)+\boldsymbol{R}$	32	1
(e15)	$E_{7(-25)}$	C_3	$\{\gamma_2\}$	$\mathfrak{so}(1, 9)+\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R}$	32	10
(e16)	$E_{8(8)}$	E_8	$\{\gamma_8\}$	$E_{7(7)}+\boldsymbol{R}$	56	1
(e17)	$E_{8(8)}$	E_8	$\{\gamma_1\}$	$\mathfrak{so}(7, 7)+\boldsymbol{R}$	64	14

(e18)	$E_{8(-24)}$	F_4	$\{\gamma_1\}$	$E_{7(-25)}+\boldsymbol{R}$	56	1
(e19)	$E_{8(-24)}$	F_4	$\{\gamma_4\}$	$\mathfrak{so}(3, 11)+\boldsymbol{R}$	64	14
(e20)	$F_{4(4)}$	F_4	$\{\gamma_1\}$	$\mathfrak{sp}(3, \boldsymbol{R})+\boldsymbol{R}$	14	1
(e21)	$F_{4(4)}$	F_4	$\{\gamma_4\}$	$\mathfrak{so}(3, 4)+\boldsymbol{R}$	8	7
(e22)	$F_{4(-20)}$	BC_1	$\{\gamma_1\}$	$\mathfrak{so}(7)+\boldsymbol{R}$	8	7
(e23)	$G_{2(2)}$	G_2	$\{\gamma_2\}$	$\mathfrak{sl}(2, \boldsymbol{R})+\boldsymbol{R}$	4	1
					$\dim_C \mathfrak{g}_{-1}$	$\dim_C \mathfrak{g}_{-2}$
(e24)	E_6^C	E_6	$\{\gamma_3\}$	$\mathfrak{sl}(5, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C}$	20	5
(e25)	E_6^C	E_6	$\{\gamma_2\}$	$\mathfrak{sl}(6, \boldsymbol{C})+\boldsymbol{C}$	20	1
(e26)	E_6^C	E_6	$\{\gamma_1, \gamma_6\}$	$\mathfrak{so}(8, \boldsymbol{C})+\boldsymbol{C}+\boldsymbol{C}$	16	8
(e27)	E_7^C	E_7	$\{\gamma_6\}$	$\mathfrak{so}(10, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C}$	32	10
(e28)	E_7^C	E_7	$\{\gamma_2\}$	$\mathfrak{so}(12, \boldsymbol{C})+\boldsymbol{C}$	32	1
(e29)	E_7^C	E_7	$\{\gamma_2\}$	$\mathfrak{sl}(7, \boldsymbol{C})+\boldsymbol{C}$	35	7
(e30)	E_8^C	E_8	$\{\gamma_8\}$	$E_7^C+\boldsymbol{C}$	56	1
(e31)	E_8^C	E_8	$\{\gamma_1\}$	$\mathfrak{so}(14, \boldsymbol{C})+\boldsymbol{C}$	64	14
(e32)	F_4^C	F_4	$\{\gamma_1\}$	$\mathfrak{sp}(3, \boldsymbol{C})+\boldsymbol{C}$	14	1
(e33)	F_4^C	F_4	$\{\gamma_4\}$	$\mathfrak{so}(7, \boldsymbol{C})+\boldsymbol{C}$	8	7
(e34)	G_2^C	G_2	$\{\gamma_2\}$	$\mathfrak{sl}(2, \boldsymbol{C})+\boldsymbol{C}$	4	1

§ 4. The subalgebra $\mathfrak{g}_{ev}$.

Let $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ be an effective semisimple GLA of the second kind, and consider the graded subalgebra

$$\mathfrak{g}_{ev}=\mathfrak{g}_{-2}+\mathfrak{g}_0+\mathfrak{g}_2\,. \tag{4.1}$$

By Lemma 3.1, the GLA $\mathfrak{g}$ is of type α_0, since each simple factor of $\mathfrak{g}$ is a graded ideal of the first or the second kind. Consequently we may go back to the situation in § 2 (cf. Theorem 2.1). Let τ be the Cartan involution in 2.1. τ extends to the conjugation of $\mathfrak{g}^C$ (denoted again by τ) with respect to $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$, as in 2.1. Thus the gradation $(\mathfrak{g}_k)_{-2\leq k\leq 2}$ is viewed as the one corresponding to a partition (2.1) of Π. From (2.11), we have

$$\tilde{\Delta}=\perp\!\!\!\perp_{k=-2}^{2}\tilde{\Delta}_k\,. \tag{4.2}$$

Let

$$\tilde{\Delta}_{ev}=\tilde{\Delta}_{-2}\cup\tilde{\Delta}_0\cup\tilde{\Delta}_2\,. \tag{4.3}$$

One sees easily that $\tilde{\Delta}_{ev}$ is a closed subsystem of $\tilde{\Delta}$ with $-\tilde{\Delta}_{ev}=\tilde{\Delta}_{ev}$. Therefore we have a regular complex semisimple subalgebra of $\mathfrak{g}^C$:

$$\mathfrak{g}^C(\tilde{\Delta}_{ev}) = \{\tilde{\Delta}_{ev}\}c+\sum_{\alpha\in\tilde{\Delta}_{ev}}\tilde{\mathfrak{g}}^\alpha . \tag{4.4}$$

To the partition (4.3) there corresponds the complexified graded subalgebra of $\mathfrak{g}^C$

$$\mathfrak{g}^C_{ev} = \mathfrak{g}^C_{-2}+\mathfrak{g}^C_0+\mathfrak{g}^C_2 = \mathfrak{h}^C+\sum_{\alpha\in\tilde{\Delta}_{ev}}\tilde{\mathfrak{g}}^\alpha . \tag{4.5}$$

$\mathfrak{g}^C(\tilde{\Delta}_{ev})$ is a subalgebra of $\mathfrak{g}^C_{ev}$. Let $\{\tilde{\Delta}_{ev}\}^\perp$ be the totality of elements $X\in\mathfrak{h}^C$ such that $\langle X, \tilde{\Delta}_{ev}\rangle=0$. Then, as for Lemma 2.2, we have the Levi decomposition

$$\mathfrak{g}^C_{ev} = \mathfrak{g}^C(\tilde{\Delta}_{ev})\oplus\{\tilde{\Delta}_{ev}\}^\perp , \tag{4.6}$$

where $\mathfrak{g}^C(\tilde{\Delta}_{ev})$ is a semisimple ideal and $\{\tilde{\Delta}_{ev}\}^\perp$ is the center of $\mathfrak{g}^C_{ev}$. $\tilde{\Delta}_{ev}$ is σ-stable. In fact, as was proved in (2.17), $\tilde{\Delta}_0$ is σ-stable, and analogously we have that $\tilde{\Delta}_{\pm 2}$ are σ-stable (cf. (2.10)). Hence $\mathfrak{g}^C(\tilde{\Delta}_{ev})$ is σ-stable and the intersection $\mathfrak{g}(\tilde{\Delta}_{ev})=\mathfrak{g}^C(\tilde{\Delta}_{ev})\cap\mathfrak{g}$ is a real form of $\mathfrak{g}^C(\tilde{\Delta}_{ev})$. As in Lemma 2.3, we have from (4.6) that the semisimple part $\mathfrak{g}^s_{ev}$ and the center $\mathfrak{z}(\mathfrak{g}_{ev})$ of the reductive subalgebra $\mathfrak{g}_{ev}$ are given by

$$\mathfrak{g}^s_{ev} = \mathfrak{g}(\tilde{\Delta}_{ev}) , \qquad \mathfrak{z}(\mathfrak{g}_{ev}) = \{\tilde{\Delta}_{ev}\}^\perp\cap\mathfrak{g} . \tag{4.7}$$

Let $\tilde{\Pi}=\{\alpha_1, \cdots, \alpha_s\}$ be a σ-fundamental system for $\tilde{\Delta}$ chosen in 2.1. $\tilde{\Pi}$ admits the partition (2.13). Let $\alpha_0=-\tilde{\vartheta}$, where $\tilde{\vartheta}$ is the dominant root in $\tilde{\Delta}$ with respect to $\tilde{\Pi}$. Let

$$\tilde{\Pi}_{ev} = \{\alpha_0\}\cup\tilde{\Pi}_0 . \tag{4.8}$$

LEMMA 4.1. *$\tilde{\Pi}_{ev}$ is a σ-fundamental system of $\tilde{\Delta}_{ev}$ (viewed as the root system for $\mathfrak{g}^C(\tilde{\Delta}_{ev})$).*

PROOF. Since $-\tilde{\Delta}_{ev}=\tilde{\Delta}_{ev}$, $\mathfrak{g}^C(\tilde{\Delta}_{ev})$ is τ-stable and so is $\mathfrak{g}(\tilde{\Delta}_{ev})$. Therefore we can apply the argument for $\mathfrak{g}'$ in 1.3 to the real form $\mathfrak{g}(\tilde{\Delta}_{ev})$, and hence the same assertion as in Lemma 1.2 is valid for $\{\tilde{\Delta}_{ev}\}_R$. $\tilde{\Pi}_0$ contains $\tilde{\Pi}_\bullet$ and is a σ-fundamental system for $\tilde{\Delta}_0$. Let $\alpha\in\tilde{\Delta}_{-2}$. We claim that it can be written as

$$\alpha = \alpha_0+\sum_{\alpha_i\in\tilde{\Pi}_0}n_i\alpha_i , \qquad n_i\geqq 0 . \tag{4.9}$$

In fact, α can be written as $\alpha=\alpha_0+\sum_{\alpha_i\in\tilde{\Pi}}n_i\alpha_i$ with $n_i\geqq 0$. Hence one has

$$-2 = \langle\alpha, Z\rangle = \langle\alpha_0, Z\rangle+\sum_{\alpha_i\in\tilde{\Pi}}n_i\langle\alpha_i, Z\rangle = -2+\sum_{\alpha_i\in\tilde{\Pi}_1}n_i ,$$

which implies (4.9). $\tilde{\Pi}_{ev}$ is thus a fundamental system for $\tilde{\Delta}_{ev}$, and $\tilde{\Delta}_{-2}$ consists of positive roots with respect to $\tilde{\Pi}_{ev}$. As was noted before, $\tilde{\Delta}_{-2}$ is σ-stable, and so we conclude that the linear order in $\tilde{\Delta}_{ev}$ defined by $\tilde{\Pi}_{ev}$ is a σ-order, or equivalently, $\tilde{\Pi}_{ev}$ is a σ-fundamental system for $\tilde{\Delta}_{ev}$. q.e.d.

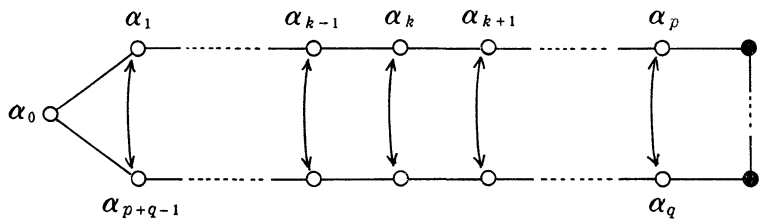
We obtain the *extended Satake diagram* of $\tilde{\Pi}$ by adding the white vertex α_0 to the Satake diagram of $\tilde{\Pi}$ and by joining it with appropriate vertices of $\tilde{\Pi}$ in the usual way.

THEOREM 4.2. *Let* $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ *be an effective real semisimple GLA of the second kind corresponding to a partition* $\Pi=\Pi_0 \amalg \Pi_1$. *Then, under the same notations as in Theorem* 2.4, *the Satake diagram of the semisimple part* $\mathfrak{g}_{ev}^s$ *of* $\mathfrak{g}_{ev}$ *is obtained from the extended Satake diagram* $\tilde{\Pi}\cup\{\alpha_0\}$ *of* $\mathfrak{g}$ *by deleting all the vertices whose* ϖ*-images lie in* Π_1 *and by deleting all rods and arrows emanating from those vertices.*

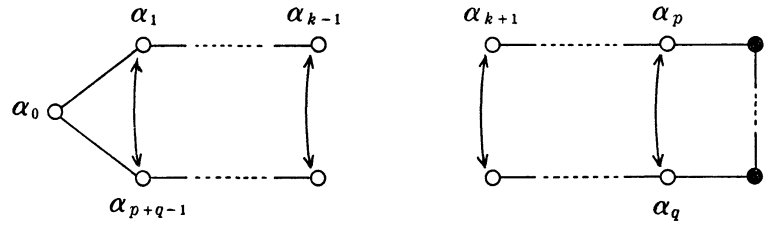
PROOF. As a subset of $\tilde{\Delta}_{ev}$, $\tilde{\Pi}_{ev}$ is a subset of $\tilde{\Pi}\cup\{\alpha_0\}$. Therefore, by Lemmas 4.1, 1.1 and (4.7), the Satake diagram of $\mathfrak{g}_{ev}^s$ is the diagram of the σ-fundamental system $\tilde{\Pi}_{ev}$ which is viewed as a subdiagram of the extended Satake diagram of $\mathfrak{g}$. By (2.13) we have $\tilde{\Pi}_{ev}=(\tilde{\Pi}\cup\{\alpha_0\})-\tilde{\Pi}_1$. Hence, by the same reason as in Theorem 2.4, we conclude the assertion of the theorem.

q.e.d.

EXAMPLE 4.3. Let $\mathfrak{g}=\mathfrak{su}(p, q)$, $p<q$. Then the extended Satake diagram $\tilde{\Pi}\cup\{\alpha_0\}$ is given by:



The restricted fundamental system Π was given in the proof of Theorem 3.2. Let $\Pi_1=\{\gamma_k\}$. Then $\tilde{\Pi}_{ev}$ is given by:



Therefore we conclude that $\mathfrak{g}_{ev}^s=\mathfrak{su}(k, k)+\mathfrak{su}(p-k, q-k)$.

Let $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ be a semisimple GLA, and Z be its characteristic element. Put $\varepsilon=\mathrm{Ad}\exp \pi i Z$. Then the triple $(\mathfrak{g}, \mathfrak{g}_{ev}, \varepsilon)$ is a symmetric triple (cf. [9]). The pair $(\mathfrak{g}, \mathfrak{g}_{ev})$ is called a *symmetric pair associated with the GLA* $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$. By using Theorem 4.2, we get

THEOREM 4.3. *The symmetric pairs* $(\mathfrak{g}, \mathfrak{g}_{ev})$ *associated with simple GLA's of the second kind are all given, as listed in Table II. (The symbols for simple GLA's (e.g.* (c1), $\cdots$, *etc.) are the same as in Theorem* 3.2 *and Table I.)*

Table II.

		$(\mathfrak{g}, \mathfrak{g}_{ev})$
(c1)		$(\mathfrak{sl}(n, \boldsymbol{F}), \mathfrak{sl}(n-q, \boldsymbol{F})+\mathfrak{sl}(q, \boldsymbol{F})+\boldsymbol{F})$, $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$
(c2)		$(\mathfrak{sl}(n, \boldsymbol{H}), \mathfrak{sl}(n-q, \boldsymbol{H})+\mathfrak{sl}(q, \boldsymbol{H})+\boldsymbol{R})$
(c3)		$(\mathfrak{su}(p, q), \mathfrak{su}(k, k)+\mathfrak{su}(p-k, q-k)+i\boldsymbol{R})$
(c4)		$(\mathfrak{so}(p, q), \mathfrak{so}(k, k)+\mathfrak{so}(p-k, q-k))$
(c5)		$(\mathfrak{sp}(n, \boldsymbol{F}), \mathfrak{sp}(k, \boldsymbol{F})+\mathfrak{sp}(n-k, \boldsymbol{F}))$, $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}$
(c6)		$(\mathfrak{sp}(p, q), \mathfrak{sp}(k, k)+\mathfrak{sp}(p-k, q-k))$
(c7)		$(\mathfrak{so}^*(2n), \mathfrak{so}^*(4k)+\mathfrak{so}^*(2n-4k))$
(c8)	a.	$(\mathfrak{so}(n, n), \mathfrak{so}(1, 1)+\mathfrak{so}(n-1, n-1))$
	b.	$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \boldsymbol{R})+\boldsymbol{R})$
	a^c.	$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{so}(2, \boldsymbol{C})+\mathfrak{so}(2n-2, \boldsymbol{C}))$
	b^c.	$(\mathfrak{so}(2n, \boldsymbol{C}), \mathfrak{sl}(n, \boldsymbol{C})+\boldsymbol{C})$
(c9)		$(\mathfrak{so}(n, \boldsymbol{C}), \mathfrak{so}(2k, \boldsymbol{C})+\mathfrak{so}(n-2k, \boldsymbol{C}))$
(e1)		$(E_{6(6)}, \mathfrak{sl}(6, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$
(e2)		$(E_{6(6)}, \mathfrak{sl}(6, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$
(e3)		$(E_{6(6)}, \mathfrak{so}(5, 5)+\boldsymbol{R})$
(e4)		$(E_{6(2)}, \mathfrak{su}(3, 3)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e5)		$(E_{6(2)}, \mathfrak{so}(4, 6)+i\boldsymbol{R})$
(e6)		$(E_{6(-14)}, \mathfrak{su}(1, 5)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e7)		$(E_{6(-14)}, \mathfrak{so}(2, 8)+i\boldsymbol{R})$
(e8)		$(E_{6(-26)}, \mathfrak{so}(1, 9)+\boldsymbol{R})$
(e9)		$(E_{7(7)}, \mathfrak{so}(6, 6)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e10)		$(E_{7(7)}, \mathfrak{so}(6, 6)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e11)		$(E_{7(7)}, \mathfrak{sl}(8, \boldsymbol{R}))$
(e12)		$(E_{7(-5)}, \mathfrak{so}^*(12)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e13)		$(E_{7(-5)}, \mathfrak{so}(4, 8)+\mathfrak{su}(2))$
(e14)		$(E_{7(-25)}, \mathfrak{so}(2, 10)+\mathfrak{sl}(2, \boldsymbol{R}))$
(e15)		$(E_{7(-25)}, \mathfrak{so}(2, 10)+\mathfrak{sl}(2, \boldsymbol{R}))$

(e16) $(E_{8(8)}, E_{7(7)}+\mathfrak{sl}(2, \boldsymbol{R}))$

(e17) $(E_{8(8)}, \mathfrak{so}(8, 8))$

(e18) $(E_{8(-24)}, E_{7(-25)}+\mathfrak{sl}(2, \boldsymbol{R}))$

(e19) $(E_{8(-24)}, \mathfrak{so}(4, 12))$

(e20) $(F_{4(4)}, \mathfrak{sp}(3, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$

(e21) $(F_{4(4)}, \mathfrak{so}(4, 5))$

(e22) $(F_{4(-20)}, \mathfrak{so}(1, 8))$

(e23) $(G_{2(2)}, \mathfrak{sl}(2, \boldsymbol{R})+\mathfrak{sl}(2, \boldsymbol{R}))$

(e24) $(E_6^c, \mathfrak{sl}(6, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

(e25) $(E_6^c, \mathfrak{sl}(6, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

(e26) $(E_6^c, \mathfrak{so}(10, \boldsymbol{C})+\boldsymbol{C})$

(e27) $(E_7^c, \mathfrak{so}(12, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

(e28) $(E_7^c, \mathfrak{so}(12, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

(e29) $(E_7^c, \mathfrak{sl}(8, \boldsymbol{C}))$

(e30) $(E_8^c, E_7^c+\mathfrak{sl}(2, \boldsymbol{C}))$

(e31) $(E_8^c, \mathfrak{so}(16, \boldsymbol{C}))$

(e32) $(F_4^c, \mathfrak{sp}(3, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

(e33) $(F_4^c, \mathfrak{so}(9, \boldsymbol{C}))$

(e34) $(G_2^c, \mathfrak{sl}(2, \boldsymbol{C})+\mathfrak{sl}(2, \boldsymbol{C}))$

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