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Asymptotic completeness for three-body Schrödinger operators with short-range interactions

By Hideo TAMURA

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Introduction.

In this work we prove the asymptotic completeness for three-body Schrödinger operators with short-range pair interactions. In the remarkable work by Sigal-Soffer [11], the problem of asymptotic completeness has been already studied for general N-body systems with short-range interactions. The proof there is based on intuitive and beautiful ideas but some of its technical details seems to be quite complicated. The present work is the outcome of the author's efforts in trying to make the original proof by [11] as transparent as possible in the case of three-body systems.

The proof here is in principle based on the same idea as in [11] but several new ingredients are added to the techniques developed there. In particular, the arguments used in the proof of Lemmas 1.1 and 1.3 may make the proof considerably transparent. The most essential step in the original proof by [11] is to construct a ceratin phase space partition of unity with the property that the boundaries of its support lie in the classically forbidden region and the construction seems to require an elaborate argument even in the case of three-body systems. The lemmas above enable us to prove the asymptotic completeness without constructing such a phase space partition of unity.

We shall formulate the obtained result precisely. The formulation requires several basic notations and definitions in scattering theory.

For brevity, we restrict ourselves to a system of three particles with identical masses $m_j=1$, $1\leqq j\leqq 3$, moving in the space R^3. For such a system, the configuration space X in the center of mass frame is given by

$$X=\{r=(r_1, r_2, r_3)\in R^{3\times 3}: \sum_{j=1}^{3} r_j=0\}$$

with the scalar product

$$\langle r, \tilde{r}\rangle=\sum_{j=1}^{3} r_j\cdot\tilde{r}_j \tag{0.1}$$

and also the energy operator (Schrödinger operator) takes the form

$$H=-\frac{1}{2}\Delta+V \qquad \text{on } L^2(X), \tag{0.2}$$

where Δ denotes the Laplacian on X and the interaction $V(r)$ is given by a sum of pair potentials

$$V(r)=\sum_{1\leq j<k\leq 3} V_{jk}(r_j-r_k).$$

The pair potential $V_{jk}(y)$, $y\in R^3$, is assumed to be real and to have the following decaying property:

$$|V_{jk}(y)|\leqq C(1+|y|)^{-\rho} \qquad \text{for some } \rho>1. \tag{V}$$

We may assume that $1<\rho<2$. Throughout the entire discussion, the constant ρ is used with the meaning ascribed above and also assumption (V) is always assumed to be satisfied. By assumption (V), the operator H formally defined by (0.2) admits a unique self-adjoint realization in $L^2(X)$. We denote by the same notation H this self-adjoint realization.

Let $P: L^2(X)\to L^2(X)$ be the eigenprojection of H associated with point spectrum. The problem of asymptotic completeness is to study the asymptotic behavior as $t\to\pm\infty$ of the scattering state $\exp(-itH)\phi$ with $\phi\in\text{Range }(Id-P)$, Id being the identity operator. Roughly speaking, the three-body system which we consider here has the two types of such behaviors: (i) states are broken up into three clusters; (ii) states are broken up into two clusters.

Let H_0 be the free Hamiltonian defined by

$$H_0=-\frac{1}{2}\Delta \qquad \text{on } L^2(X).$$

Then we define the wave operator $\Omega_0^{\pm}: L^2(X)\to L^2(X)$ by

$$\Omega_0^{\pm}=\underset{t\to\pm\infty}{s\text{-lim}}\exp(itH)\exp(-itH_0).$$

If $\phi\in\text{Range }\Omega_0^{\pm}$, then by definition it follows that there exist $\phi_0^{\pm}\in L^2(X)$ such that $\exp(-itH)\phi$ behaves like $\exp(-itH_0)\phi_0^{\pm}$ as $t\to\pm\infty$. Thus the state is broken up into three clusters.

Next we consider the second type of asymptotic behavior. Let $a=\{(j, k), l\}$ with $1\leqq j<k\leqq 3$ and $l\neq j, k$ be a two-cluster decomposition. For such a decomposition, we define the configuration space Y_a of the internal motion within the cluster (j, k) by

$$Y_a=\{r\in X: r_j+r_k=0\}$$

and the configuration space Z_a of the relative motion of the clusters (j, k) and (l) by

$$Z_a=\{r\in X: r_j=r_k\}.$$

We introduce the coordinates $y_a \in R^3$ on Y_a by

$$y_a = \sqrt{1/2}\,(r_j - r_k) \tag{0.3}$$

and the coordinates $z_a \in R^3$ on Z_a by

$$z_a = \sqrt{2/3}\{(r_j + r_k)/2 - r_l\} \tag{0.4}$$

with the normalization constants $\sqrt{1/2}$ and $\sqrt{2/3}$. As is easily seen, the spaces Y_a and Z_a are orthogonal to each other with respect to the scalar product (0.1) and the space X is represented as the orthogonal sum of spaces Y_a and Z_a; $X = Y_a \oplus Z_a$. The truncated Hamiltonian H_a is defined by

$$H_a = -\frac{1}{2}\Delta + V_{jk}(r_j - r_k) \qquad \text{on } L^2(X).$$

The operator H_a has the decomposition

$$H_a = h_a \otimes Id + Id \otimes T_a \tag{0.5}$$

on $L^2(X) = L^2(Y_a) \otimes L^2(Z_a)$, where $h_a = -(1/2)\Delta + V_{jk}$ acting on $L^2(Y_a)$ is the two-particle subsystem Hamiltonian associated with the cluster (j, k), while $T_a = -(1/2)\Delta$ acting on $L^2(Z_a)$ is the kinetic energy operator of the center of mass motion of the clusters (j, k) and (l).

Let $a = \{(j, k), l\}$ be again a two-cluster decomposition. Under assumption (V), the operator h_a has no positive bound state energies. We denote by $m(a)$, $0 \leqq m(a) \leqq \infty$, the number of (non-positive) bound state energies of h_a with repetition according to the multiplicities. For a channel $\alpha = (a, m)$, $1 \leqq m \leqq m(a)$, we further introduce the following notations: (0) $\varepsilon_\alpha (\leqq 0)$, m-th bound state energy of h_a; (i) $\psi_\alpha \in L^2(Y_a)$, normalized eigenstate associated with ε_α; (ii) $H_\alpha = \varepsilon_\alpha + T_a$, channel Hamiltonian on $L^2(Z_a)$; (iii) $J_\alpha : L^2(Z_a) \to L^2(X)$; $J_\alpha u = \psi_\alpha \otimes u$, channel identification operator; (iv) $\Omega_\alpha^\pm : L^2(Z_a) \to L^2(X)$, channel wave operator, defined by

$$\Omega_\alpha^\pm = \operatorname*{s\text{-}lim}_{t \to \pm\infty} \exp(itH) J_\alpha \exp(-itH_\alpha).$$

If $\psi \in \text{Range}\, \Omega_\alpha^\pm$, then there exist $\phi_\alpha^\pm \in L^2(Z_a)$ such that the state $\exp(-itH)\psi$ behaves like $\exp(-it\varepsilon_\alpha)\psi_\alpha \otimes \exp(-itT_a)\phi_\alpha^\pm$ as $t \to \pm\infty$ and hence it is broken up into two clusters.

As is well known (see, for example, [**10**]), under assumption (V), the wave operators $\Omega_0^\pm$ and $\Omega_\alpha^\pm$ exist and have the property that their ranges are orthogonal to each other. We are now in a position to state the main theorem.

THEOREM (asymptotic completeness). *Assume* (V). *Let the notations be as above. Then*

$$Range\,(Id-P) = \bigoplus_{a} Range\,\Omega_a^{\pm} \oplus Range\,\Omega_0^{\pm}.$$

Besides the work [11], there are many works dealing with the problem of asymptotic completeness for many-body systems. Among them, we should refer to the work by Enss [4] in which the asymptotic completeness for three-body systems has been already proved by a different method, including the case of longrange interactions. We also make a comment that the method due to Enss is recently extended by Kitada [7] to the case of N-body systems under an additional assumption that all subsystem Hamiltonians have only a finite number of bound state energies. An extensive list of related references can be found in [2] and [11]. We do not here intend to make a detailed review of these works.

1. Reduction to main lemmas.

Before going into the detailed proof, we here explain the strategy of the proof. The proof is done by reducing it to the proof of several main lemmas below (Lemmas 1.0~1.4).

We first fix an energy E arbitrarily. Assume that E is neither a threshold energy nor a bound state energy of H. Under assumption (V), we know that the set of threshold and bound state energies is closed and countable. Hence we can take a small open interval Γ around E avoiding the threshold and bound state energies of H. Let $g_0 \in C_0^\infty(\Gamma)$ be a non-negative function such that $g_0=1$ in a small neighborhood of E. To prove the theorem, it suffices, by following the standard argument as in [11], to show that for any $\varepsilon>0$ small enough, there exists $\phi_{a,\varepsilon}^{\pm} \in L^2(X)$ such that

$$\lim_{t\to\pm\infty} \|\exp(-itH)g_0(H)\phi - \textstyle\sum_a \exp(-itH_a)\phi_{a,\varepsilon}^{\pm}\|_X < \varepsilon\,, \tag{1.1}$$

where $\|\cdot\|_X$ denotes the L^2 norm in $L^2(X)$ and the summation $\sum_a$ is taken over all two-cluster decompositions $a=\{(j,\ k),\ l\}$. We shall prove (1.1) for the case $E>0$ and $t\to\infty$. The other cases can be dealt with in a similar way. In particular, the case $E<0$ is much easier to deal with than the case $E>0$.

The strategy of proof is explained through a series of steps.

1.1. Let $a=\{(j,\ k),\ l\}$ be a two-cluster decomposition. Let S be the unit sphere in X. Define the closed subset S_a in S by

$$S_a = \{\omega=(\omega_1,\ \omega_2,\ \omega_3)\in S : \omega_j=\omega_k\}.$$

We now fix a coordinate system on X arbitrarily and write it as $x=(x_1,\ x_2)$ with $x_j \in R^3$, $1\leqq j\leqq 2$. We introduce a non-negative smooth partition of unity on X, $\{j_a(x)\}$, with the following properties: $(j.0)$ $\sum_a j_a=1$ on X; $(j.1)$ for $|x|>1$,

j_a has support in a conical neighborhood of S_a; (j.2) $|\partial_x^\alpha j_a| \leqq C_\alpha(1+|x|)^{-|\alpha|}$ for any multi-index α; (j.3) for $|x|>1$, $\nabla_x j_a$ vanishes on $Y=U_a Y_a$. It is not difficult to see that such a partition of unity really exists. By properties (j.0) and (j.1), the gradient $\nabla_x j_a$ vanishes in a small conical neighborhood of S_a for $|x|>1$.

1.2. Let $g_0 \in C_0^\infty(\Gamma)$ be as above and let $g_1 \in C_0^\infty(\Gamma)$ be such that $g_1 g_0 = g_0$. By property (j.1), $H-H_a=O(|x|^{-\rho})$ as $|x|\to\infty$ on the support of j_a and hence $j_a(g_1(H)-g_1(H_a))$: $L^2(X)\to L^2(X)$ is compact. By property (j.2), we have that the commutator $[g_1(H_a), j_a]$ is also compact. Since $\exp(-itH)g_0(H)\psi$ converges to zero weakly as $t\to\infty$, we see that

$$\lim_{t\to\infty} \|\exp(-itH)g_0(H)\psi - \textstyle\sum_a \exp(-itH_a)\psi_a(t)\|_X = 0\,,$$

where

$$\psi_a = g_1(H_a)\exp(itH_a)j_a \exp(-itH)g_0(H)\psi\,.$$

1.3. We analyze the term $\psi_a(t)$. This can be rewritten in the following integral form:

$$\psi_a(t) = g_1(H_a)j_a g_0(H)\psi + i\phi_{0a}(t) + i\phi_a(t)\,,$$

where

$$\phi_{0a} = \int_0^t g_1(H_a)\exp(isH_a)j_a(H_a-H)\exp(-isH)g_0(H)\psi ds \tag{1.2}$$

and

$$\phi_a = \int_0^t g_1(H_a)\exp(isH_a)[H_0, j_a]\exp(-isH)g_0(H)\psi ds\,.$$

LEMMA 1.0. *$\phi_{0a}(t)$ is strongly convergent as $t\to\infty$.*

1.4. To analyze the term $\phi_a(t)$, we further introduce a non-negative smooth partition of unity on R^1, $\{f_j\}_{j=1}^4$, with the following properties: (f.0) $\sum_{j=1}^4 f_j^2=1$ on R^1;

(f.1) $\operatorname{supp} f_1 \subset (-\infty, -\sqrt{2E}/3)$, $f_1=1$ on $(-\infty, -\sqrt{2E}/2]$;

(f.2) $\operatorname{supp} f_2 \subset (-\sqrt{2E}/2, \sqrt{2E}(1-\kappa))$, $f_2=1$ on $[-\sqrt{2E}/3, \sqrt{2E}(1-2\kappa)]$;

(f.3) $\operatorname{supp} f_3 \subset (\sqrt{2E}(1-2\kappa), M)$, $f_3=1$ on $[\sqrt{2E}(1-\kappa), M/2]$;

(f.4) $\operatorname{supp} f_4 \subset (M/2, \infty)$, $f_4=1$ on $[M, \infty)$,

where κ, $0<\kappa\ll 1$, is fixed small enough and $M\gg 1$ is taken large enough. How large M should be chosen will be determined in the proof of Lemma 1.4 (section 4).

As in [11], we define the operator γ as

$$\gamma = \frac{1}{2i}\{(x/\langle x\rangle)\cdot\nabla_x + \nabla_x\cdot(x/\langle x\rangle)\} \tag{1.3}$$

with

$$\langle x\rangle = (1+|x|^2)^{1/2}.$$

We know ([**11**], Theorem 3.2) that γ extends uniquely to a self-adjoint operator on $L^2(X)$. The operator γ plays an important role in our analysis.

According to $(f.0)$, we decompose $\phi_a(t)$ into

$$\phi_a(t) = \sum_{j=1}^{4} \phi_{ja}(t),$$

where

$$\phi_{ja} = \int_0^t g_1(H_a)\exp(isH_a)f_j(\gamma)^2[H_0, j_a]\exp(-isH)g_0(H)\phi ds. \tag{1.4}$$

1.5. The following four lemmas, together with Lemma 1.0, prove (1.1) for the case $E>0$ and $t\to\infty$.

LEMMA 1.1. *$\phi_{1a}(t)$ is strongly convergent as $t\to\infty$.*

LEMMA 1.2. *$\phi_{2a}(t)$ is strongly convergent as $t\to\infty$.*

LEMMA 1.3. *For any $\varepsilon>0$ small enough, there exist $\phi_{a,\varepsilon}\in L^2(X)$ such that*

$$\limsup_{t\to\infty} \|\textstyle\sum_a \exp(-itH_a)(\phi_{3a}(t)-\phi_{a,\varepsilon})\|_X<\varepsilon.$$

LEMMA 1.4. *$\phi_{4a}(t)$ is strongly convergent as $t\to\infty$.*

We will prove Lemma 1.0 in section 3, Lemmas 1.2 and 1.4 in section 4, Lemma 1.1 in section 5 and Lemma 1.3 in section 6.

2. Commutator calculus.

In the present section we collect several properties of commutator operators which are used without further references as a basic tool to prove Lemmas 1.0 $\sim$1.4. These properies can be easily proved or can be proved in almost the same way as in [3] and [**11**].

2.0. We keep the same notations as in section 1 and further introduce new notations.

Let $B: L^2(X)\to L^2(X)$ be bounded. We denote by $\|B\|$ the operator norm of B. Let $X_\beta: \phi(x)\to\langle x\rangle^\beta\phi(x)$ be the multiplication operator by $\langle x\rangle^\beta$. For given operator B (not necessarily bounded) on $L^2(X)$, we write $B\in O(\langle x\rangle^m)$ if $X_{-m+k}BX_{-k}$ extends to a bounded operator on $L^2(X)$ for any real k. We also use the notation $B_1=B_2+O(\langle x\rangle^m)$ if the difference B_1-B_2 is of class $O(\langle x\rangle^m)$.

We introduce the following function classes:

$$C_1^\infty(R^1)=\{f\in C^\infty(R^1): f'\in C_0^\infty(R^1)\};$$

$$C_2^\infty(R^1)=\{f\in C^\infty(R^1): f''\in C_0^\infty(R^1)\}.$$

For $f\in C_1^\infty(R^1)$ or $C_2^\infty(R^1)$, we denote its (distributional) Fourier transform by

$$\hat{f}(\sigma)=(2\pi)^{-1/2}\int e^{-i\sigma s}f(s)ds,$$

where the integration with no domain attached is taken over the whole space. This abbreviation is used throughout the argument in the sequel. By definition, it follows that $\sigma\hat{f}(\sigma)\in S(R^1)$ (the Schwartz space over R^1) for $f\in C_1^\infty(R^1)$ and $\sigma^2\hat{f}(\sigma)\in S(R^1)$ for $f\in C_2^\infty(R^1)$.

2.1. The Hamiltonian H under consideration is semi-bounded. We can choose $c>1$ so large that $(H+c)^{-1}: L^2(X)\to L^2(X)$ is bounded. The following lemma is easy to prove.

LEMMA 2.1. *One has the following statements.*

(i) *The operators below are all of class* $O(\langle x\rangle^0)$:

a) $(H+c)^{-1}$;

b) $(H+c)^{-1}\partial_x^\alpha$ *and* $\partial_x^\alpha(H+c)^1$ *with* $|\alpha|=1$ *or* 2;

c) $\partial_x^\alpha(H+c)^{-1}\partial_x^\beta$ *with* $|\alpha|=|\beta|=1$.

(ii) *The commutator* $[X_\beta, (H+c)^{-1}]\in O(\langle x\rangle^{\beta-1})$.

REMARK. The Hamiltonians H_a and H_0 have the same properties as in the lemma. The future statements are formulated for H only but these remain true for H_a and H_0.

Let γ be defined by (1.3) and let A be defined by

$$A=\frac{1}{2i}(x\cdot\nabla_x+\nabla_x\cdot x)\,. \tag{2.1}$$

The operator A also extends uniquely to a self-adjoint operator on $L^2(X)$.

LEMMA 2.2. *Let* γ *and* A *be as above. Let* ρ, $1<\rho<2$, *be as in* (V). *Then one has the following statements:*

a) $[\gamma, (H+c)^{-1}]\in O(\langle x\rangle^{-1})$;

b) $[A, (H+c)^{-1}]\in O(\langle x\rangle^0)$;

c) $[X_\beta, [\gamma, (H+c)^{-1}]]\in O(\langle x\rangle^{\beta-2})$;

d) $[X_\beta, [A, (H+c)^{-1}]]\in O(\langle x\rangle^{\beta-1})$;

e) $[\gamma, [\gamma, (H+c)^{-1}]]\in O(\langle x\rangle^{-\rho})$;

f) $[\gamma, [A, (H+c)^{-1}]]\in O(\langle x\rangle^{-\rho+1})$.

PROOF. We give only a sketch for the proof. a) We write

$$[\gamma, (H+c)^{-1}] = (H+c)^{-1}[H, \gamma](H+c)^{-1}.$$

By assumption (V), the commutator $[H, \gamma]$ takes the form

$$[H, \gamma] \sim \sum_{|\alpha|+|\beta|\leq 1} \partial_x^\alpha O(\langle x\rangle^{-1})\partial_x^\beta.$$

Hence, Lemma 2.1 proves a).

The same argument as above applies to b). And also c) and d) are verified in a similar way.

e) By assumption (V) again, the double commutator $[\gamma, [\gamma, H]]$ takes the form

$$[\gamma, [\gamma, H]] \sim \sum_{|\alpha|+|\beta|\leq 2} \partial_x^\alpha O(\langle x\rangle^{-\rho})\partial_x^\beta.$$

This, together with Lemma 2.1, implies e). The statement f) can be similarly verified. □

2.2. The following three results can be found in [**3**] and [**11**].

PROPOSITION 2.3 ([**11**], Lemma A.1). *Let B and C be self-adjoint operators on $L^2(X)$. Let f be of class $C_1^\infty(R^1)$. Assume that B and $[C, B]$ are bounded. Then $[B, f(C)]$ is also bounded and one has*

$$[B, f(C)] = (2\pi)^{-1/2}\int ds\, \hat{f}(s)\int_0^s d\tau \exp(i(s-\tau)C)i[B, C]\exp(i\tau C).$$

LEMMA 2.4 ([**3**], Lemma 5.4). *The operator* $\exp(it\gamma)$, $t\in R^1$, *is of class* $O(\langle x\rangle^0)$ *and one has*

$$\|X_k \exp(it\gamma)X_{-k}\| \leq c_k(1+|t|)^{|k|}.$$

PROPOSITION 2.5 ([**11**], Lemma A.7). *Let B and C be selfadjoint operators on $L^2(X)$. Let f be of class $C_2^\infty(R^1)$. Assume that B, $[C, B]$ and $[C, [C, B]]$ are bounded. Then $[B, f(C)]$ is also bounded and one has*

$$[B, f(C)] = f'(C)[B, C]+R$$

with

$$R = -(2\pi)^{-1/2}\int ds\, \hat{f}(s)\int_0^s d\tau\,(s-\tau)\exp(i(s-\tau)C)[C, [C, B]]\exp(i\tau C).$$

2.3. By making use of these results, we can obtain several useful properties of commutator operators which plays a basic role in our analysis.

LEMMA 2.6. *Let $g\in C_0^\infty(R^1)$. Then $g(H)\in O(\langle x\rangle^0)$ and*

$$[X_\beta, g(H)]\in O(\langle x\rangle^{\beta-1}). \tag{2.2}$$

PROOF. The first assertion follows from the second one. Since

$$[X_\beta, g(H)] = X_\beta[g(H), X_{-\beta}]X_\beta,$$

it suffices to prove (2.2) for the case $\beta<0$. We can represent $g(H)$ as $g(H)=\tilde{g}((H+c)^{-1})$ with $\tilde{g}\in C_0^\infty(R^1)$. As is easily seen from Lemma 5.4 of [**3**],

$$\|X_k \exp(it(H+c)^{-1})X_{-k}\| \leqq c_k(1+|t|)^{|k|}.$$

Therefore, Proposition 2.3 with $B=X_\beta$, $\beta<0$, and $C=(H+c)^{-1}$ proves the lemma. □

The same reasoning as above applies to prove the following

LEMMA 2.7. *Let f be of class $C_1^\infty(R^1)$. Then $f(\gamma)$ is of class $O(\langle x\rangle^0)$ and*

$$[X_\beta, f(\gamma)] \in O(\langle x\rangle^{\beta-1}).$$

LEMMA 2.8. *Let f be of class $C_2^\infty(R^1)$. Then one has*

$$[f(\gamma), (H+c)^{-1}] \in O(\langle x\rangle^{-1}).$$

and

$$[f(\gamma), (H+c)^{-1}] = f'(\gamma)[\gamma, (H+c)^{-1}]+O(\langle x\rangle^{-\rho}).$$

PROOF. We use Proposition 2.5 with $B=(H+c)^{-1}$ and $C=\gamma$. By Lemma 2.2, $[\gamma, [\gamma, (H+c)^{-1}]]$ is of class $O(\langle x\rangle^{-\rho})$. This proves the lemma. □

LEMMA 2.9. *Let $f\in C_1^\infty(R^1)$ and let $g\in C_0^\infty(R^1)$. Then the commutator $[f(\gamma), g(H)]$ is of class $O(\langle x\rangle^{-1})$.*

PROOF. We again write $g(H)=\tilde{g}((H+c)^{-1})$ with $\tilde{g}\in C_0^\infty(R^1)$ and use Proposition 2.3 with $B=f(\gamma)$ and $C=(H+c)^{-1}$. Then the lemma follows from Lemma 2.8 at once. □

LEMMA 2.10. *Let $f\in C_1^\infty(R^1)$. Then one has;*

a) $[f(\gamma), [\gamma, (H+c)^{-1}]]\in O(\langle x\rangle^{-\rho})$,

b) $[f(\gamma), [A, (H+c)^{-1}]]\in O(\langle x\rangle^{-\rho+1})$.

PROOF. We prove a) only. We again use Proposition 2.3 with $B=[\gamma, (H+c)^{-1}]$ and $C=\gamma$. Then the lemma follows from Lemma 2.2 at once. □

3. Limiting absorption principle

In this section we prove Lemma 1.0. The proof is done by making use of the limiting absorption principle proved by [8] and [9] and of the smooth perturbation theory developed by [**6**].

3.1. We start by making a brief review of the principle of limiting absorption for the three-body Hamiltonian H under consideration. Recall the notations in section 1. Let $E>0$ be a fixed energy. Assume that E is neither a threshold energy nor a bound state energy of H. Let Γ be a small open interval around E avoiding the threshold and bound state energies of H.

PROPOSITION 3.1. *Let $E>0$ be as above. Let A be defined by* (2.1). *Let κ, $0<\kappa\ll 1$, be as in* (f.2). *Then one can take an interval Γ around E so small that*

$$E_\Gamma(H)i[H, A]E_\Gamma(H) \geqq 2E(1-\kappa/2)^2 E_\Gamma(H), \tag{3.1}$$

where $E_\Gamma(H)$ denotes the spectral projection of H onto Γ and the relation $\geqq$ should be understood in the form sense.

This proposition follows as a special case of the result obtained by [**5**]. The inequality (3.1) is called the Mourre estimate and plays a central role in proving the principle of limiting absorption. This inequality is also used to prove Lemmas 1.1 and 1.2 in the sequel.

PROPOSITION 3.2 (limiting absorption principle). *Let E and Γ be as in Proposition* 3.1. *Then one has*

$$\|X_{-\beta}(H-(\lambda\pm i\kappa))^{-1}X_{-\beta}\| < C_{\beta\Gamma}$$

for $\beta>1/2$ uniformly in $\lambda\in\Gamma$ and κ, $0<\kappa\leqq 1$, and also there exist boundary values of $X_{-\beta}(H-(\lambda\pm i\kappa))^{-1}X_{-\beta}$ as $\kappa\to 0$ in the uniform operator topology.

We should here make a comment on the above proposition. The principle of limiting absorption has been first proved by [**8**] in the case of three-body systems and then has been extended by [**9**] to the case of N-body systems, including the case of long-range interactions, under the additional assumptions that:

$$\langle y\rangle^\theta \nabla_y V_{jk}(-\Delta_y+1)^{-1} \text{ is bounded on } L^2(R_y^3) \quad \text{for } \theta>1; \tag{3.2}$$

$$(y\cdot\nabla_y)^2 V_{jk}(-\Delta_y+1)^{-1} \text{ is bounded on } L^2(R_y^3), \tag{3.3}$$

where Δ_y denotes the Laplacian on R_y^3. Recently, this result has been improved by [**1**] and [**12**] to remain true without such restrictions. In particular, in the case of short-range interactions, the principle of limiting absorption holds true under assumption (V) only.

PROPOSITION 3.3. *Let E and Γ be again as in Proposition* 3.1. *Then one has*

$$\int \|X_{-\beta}\exp(-itH)E_\Gamma(H)\phi\|_X^2 dt \leqq C\|\phi\|_X^2$$

for $\beta > 1/2$.

This proposition follows from Proposition 3.2 as an immediate consequence of the smoothness theorem by Kato [**6**].

REMARKS. (i) If E (not necessarily $E>0$) is neither a threshold energy nor a bound state energy of H, then we have the Mourre inequality $E_\Gamma(H)i[H, A]E_\Gamma(H) \geqq kE_\Gamma(H)$ with $k>0$ for a small interval Γ around E avoiding threshold and bound state energies of H and also the statements of Propositions 3.2 and 3.3 hold true for such an interval Γ. (ii) The statements of Propositions 3.1~3.3 remain true for the Hamiltonians H_a and H_0.

3.2. In the discussion below, we occasionally have to study the strong convergence as $t\to\infty$ in $L^2(X)$ of the integral

$$\int_0^t \exp(isK_2)B_2^*B_0B_1\exp(-isK_1)\psi ds$$

with self-adjoint operators K_j, $1\leqq j\leqq 2$, and bounded operators B_j, $0\leqq j\leqq 2$. The following convenient criterion is used to prove the existence of such a limit: If

$$\int_0^\infty \|B_j\exp(-isK_j)\psi\|_X^2 ds \leqq C\|\psi\|_X^2, \qquad 1\leqq j\leqq 2, \tag{3.4}$$

then the above integral has a strong limit as $t\to\infty$.

3.3. We now prove Lemma 1.0.

PROOF OF LEMMA 1.0. Recall the representation (1.2) for $\phi_{0a}(t)$. By the construction of the partition of unity $\{j_a(x)\}$ on X, it follows that $j_a(H-H_a)\in O(\langle x\rangle^{-\rho})$, $\rho>1$. Hence, by Proposition 3.3, the basic criterion (3.4) proves the lemma. □

4. Non-propagation estimates.

In this section we shall prove Lemmas 1.2 and 1.4. These lemmas follow as a special case of the non-propagation estimates proved by [**3**] and [**11**], including the case of N-body systems with long-range interactions. However, for completeness, we here prove these lemmas, restricting ourselves to the case of three-body systems. In the works [**3**] and [**11**], these estimates have been proved under the additional assumptions (3.2) and (3.3), but such restrictions are essentially used to prove the principle of limiting absorption principle only (and hence Proposition 3.3). Thus we first note that the non-propagation estimates hold true under assumption (V) only in the case of short-range interactions.

4.1. In the present and following two subsections, we make use of commutator calculus in section 2 to derive a certain basic formula which plays an important role in proving Lemmas 1.2 and 1.4.

Recall the notations $C_1^\infty(R^1)$ and $C_2^\infty(R^1)$. Let $F\in C_2^\infty(R^1)$ be a real function such that $F'=f^2$ with $f\in C_1^\infty(R^1)$. Assume that energy $E>0$ and interval Γ around E are as in section 1. Let $g\in C_0^\infty(\Gamma)$. Then we have the relation

$$\langle F(\gamma)\exp(-isH)g(H)\phi, \exp(-isH)g(H)\phi\rangle_X \,|_{s=0}^t \tag{4.1}$$
$$=\int_0^t \langle i[H, F(\gamma)]\exp(-isH)g(H)\phi, \exp(-isH)g(H)\phi\rangle_X ds,$$

where $\langle\, ,\rangle_X$ denotes the scalar product in $L^2(X)$. Since $F(\gamma)(H+c)^{-1}$ is a bounded operator on $L^2(X)$, the term on the left side is dominated by $C\|\phi\|_X^2$ uniformly in $t\gg 1$.

4.2. We study the commutator $[H, F(\gamma)]$ on the right side of (4.1). We first write

$$g(H)i[H, F(\gamma)]g(H) = g(H)(H+c)i[F(\gamma), (H+c)^{-1}](H+c)g(H)$$

and use Lemma 2.8 to obtain that this is equal to

$$g(H)(H+c)f(\gamma)^2 i[\gamma, (H+c)^{-1}](H+c)g(H)+O(\langle x\rangle^{-\rho}).$$

Let A be defined by (2.1). Then, by definition, the two operators γ and A are related through $\gamma=AX_{-1}+Q(x)$, where $Q(x)$ satisfies the estimate $|\partial_x^\alpha Q|\leqq C_\alpha\langle x\rangle^{-1-|\alpha|}$. This yields

$$i[\gamma, (H+c)^{-1}] = i[A, (H+c)^{-1}]X_{-1}+\gamma X_1 i[X_{-1}, (H+c)^{-1}]+O(\langle x\rangle^{-2}).$$

By a simple calculation,

$$i[H_0, X_{-1}] = -\gamma X_{-2}+O(\langle x\rangle^{-3})$$

and hence the second term on the right side is equal to

$$-\gamma X_1(H+c)^{-1}\gamma X_{-2}(H+c)^{-1}+O(\langle x\rangle^{-2}).$$

Thus we have

$$i[\gamma, (H+c)^{-1}] = X_{-1/2}i[A, (H+c)^{-1}]X_{-1/2}$$
$$-X_{-1/2}(H+c)^{-1}\gamma^2(H+c)^{-1}X_{-1/2}+O(\langle x\rangle^{-2}).$$

4.3. We first analyze the operator

$$T_1 = g(H)(H+c)f(\gamma)^2 X_{-1/2}i[A, (H+c)^{-1}]X_{-1/2}(H+c)g(H).$$

This is equal to

$$T_1 = X_{-1/2} f(\gamma) g(H) i[H, A] g(H) f(\gamma) X_{-1/2} + O(\langle x\rangle^{-\rho}).$$

Next we analyze the operator

$$T_2 = g(H)(H+c) f(\gamma)^2 X_{-1/2}(H+c)^{-1}\gamma^2 (H+c)^{-1} X_{-1/2}(H+c) g(H).$$

This is equal to

$$T_2 = X_{-1/2} g(H)\gamma^2 f(\gamma)^2 g(H) X_{-1/2} + O(\langle x\rangle^{-\rho}).$$

Thus we have the following basic formula:

$$\begin{aligned} g(H) i[H, F(\gamma)] g(H) = {} & X_{-1/2} f(\gamma) g(H) i[H, A] g(H) f(\gamma) X_{-1/2} \\ & - X_{-1/2} g(H)\gamma^2 f(\gamma)^2 g(H) X_{-1/2} + O(\langle x\rangle^{-\rho}). \end{aligned} \tag{4.2}$$

4.4. We now prove Lemma 1.2.

PROOF OF LEMMA 1.2. Let $f_2 \in C_0^\infty(R^1)$ be as in the lemma. We use the formula (4.2) with

$$F(s) = \int_{-\infty}^{s} f_2(s)^2 ds \in C_1^\infty(R^1).$$

By property $(f.2)$, f_2 has support in $(-\sqrt{2E}/2, \sqrt{2E}(1-\kappa))$ and hence it follows that

$$\gamma^2 f_2(\gamma)^2 \leqq 2E(1-\kappa)^2 f_2(\gamma)^2$$

in the form sense. Thus, by Proposition 3.1, we can take an interval Γ around E so small that

$$\int_0^\infty \| X_{-1/2} f_2(\gamma) \exp(-itH) g(H)\psi\|_X^2 dt \leqq C\|\psi\|_X^2 \tag{4.3}$$

for $g \in C_0^\infty(\Gamma)$. A similar estimate holds true for the truncated Hamiltonian H_a. Hence the criterion (3.4) proves the lemma. □

4.5. Next we shall prove Lemma 1.4.

PROOF OF LEMMA 1.4. The lemma is verified in a way similar to Lemma 1.2. Let $f_4 \in C_1^\infty(R^1)$ be as in the lemma. We again use the formula (4.2) with

$$F(s) = \int_{-\infty}^{s} f_4(s)^2 ds \in C_2^\infty(R^1).$$

We note that $E_\Gamma(H) i[H, A] E_\Gamma(H)\colon L^2(X) \to L^2(X)$ is bounded;

$$\|E_\Gamma(H) i[H, A] E_\Gamma(H)\| \leqq C_\Gamma.$$

By property $(f.4)$, f_4 has support in $(M/2, \infty)$ for $M \gg 1$ large enough and hence

$$\gamma^2 f_4(\gamma)^2 \geqq (M^2/4) f_4(\gamma)^2$$

in the form sense. Thus we can take M so large that

$$\int_0^\infty \|X_{-1/2}f_4(\gamma)\exp(-itH)g(H)\phi\|_X^2 dt \leqq C\|\phi\|_X^2 \tag{4.4}$$

for $g\in C_0^\infty(\Gamma)$. This completes the proof of the lemma. □

The estimates (4.3) and (4.4) are called the non-propagation estimates, because, roughly speaking, the phase space supports of f_2 and f_4 lie in the classically forbidden region.

5. Analysis for incoming states.

In the present section we prove Lemma 1.1. The proof of this lemma is based on the following

PROPOSITION 5.1. *Let energy $E>0$ and interval Γ around E be as in section 1. Let f be of class $C_1^\infty(R^1)$ with support in $(-\infty, 0)$. Then the incoming state $f(\gamma)\exp(-itH)E_\Gamma(H)\phi$ is strongly convergent to zero as $t\to\infty$.*

5.1. We first complete the proof of Lemma 1.1, accepting the above proposition as proved.

PROOF OF LEMMA 1.1. Let $f_1\in C_1^\infty(R^1)$ be as in the lemma. Set $F=f_1^2$. Then, by property $(f.1)$, $F\in C_1^\infty(R^1)$ and F has support in $(-\infty, -\sqrt{2E}/3)$.

We start with the following relation:

$$\begin{aligned}&\frac{d}{dt}\{\exp(itH_a)F(\gamma)j_a\exp(-itH)\}\\&\qquad=\exp(itH_a)i\{H_aF(\gamma)j_a-F(\gamma)j_aH\}\exp(-itH).\end{aligned}$$

We further calculate

$$H_aF(\gamma)j_a-F(\gamma)j_aH=[H_a,\ F(\gamma)]j_a+F(\gamma)[H_0,\ j_a]+F(\gamma)j_a(H_a-H).$$

Recall the representation (1.4) for $\phi_{1a}(t)$. Then $\phi_{1a}(t)$ can be rewritten as

$$\phi_{1a}(t)=\sum_{j=1}^3\theta_{ja}(t),$$

where

$$\theta_{1a}=-i\int_0^t\frac{d}{ds}\{g_1(H_a)\exp(isH_a)F(\gamma)j_a\exp(-isH)g_0(H)\phi\}ds,$$

$$\theta_{2a}=-\int_0^t g_1(H_a)\exp(isH_a)F(\gamma)j_a(H_a-H)\exp(-isH)g_0(H)\phi ds,$$

$$\theta_{3a}=-\int_0^t E_\Gamma(H_a)\exp(isH_a)g_1(H_a)[H_a,\ F(\gamma)]j_ag_0(H)\exp(-isH)E_\Gamma(H)\phi ds.$$

By Proposition 5.1, the term $\theta_{1a}(t)$ strongly converges as $t\to\infty$. Since $F(\gamma)j_a(H_a-H)\in O(\langle x\rangle^{-\rho})$, the term $\theta_{2a}(t)$ also strongly converges. Thus it remains to prove the strong convergence as $t\to\infty$ of $\theta_{3a}(t)$ only.

We calculate the commutator $[H_a, F(\gamma)]$. By Lemma 2.8, it follows that

$$\begin{aligned} &g_1(H_a)[H_a, F(\gamma)]j_a g_0(H) \\ &\quad = g_1(H_a)(H_a+c)F'(\gamma)[\gamma, (H_a+c)^{-1}](H_a+c)g_0(H)j_a+O(\langle x\rangle^{-\rho}). \end{aligned}$$

By property $(f.1)$, F' has support in $(-\sqrt{2E}/2, -\sqrt{2E}/3)$. We now choose $f\in C_0^\infty(R^1)$ with support in $(-\sqrt{2E}/2, -\sqrt{2E}/3)$ such that $F'(\gamma)=f(\gamma)F'(\gamma)f(\gamma)$. Then the non-propagation estimate (4.3) obtained in the proof of Lemma 1.2 implies that

$$\int_0^\infty \|X_{-1/2}f(\gamma)\exp(-itH)E_\Gamma(H)\psi\|_X^2 dt \leqq C\|\psi\|_X^2$$

for f as above. Hence, this, together with criterion (3.4), proves the desired convergence of $\theta_{3a}(t)$ and completes the proof of the lemma. □

5.2. The proof of Proposition 5.1 is done through a series of lemmas.

We first take c_1 and c_2, $c_2>c_1>0$, arbitrarily. Assume that $(-c_2, -c_1)\subset(\sqrt{2E}, 0)$. Next we introduce a real function $h\in C_1^\infty(R^1)$ with the following properties: (i) $h=h_0^2$ with $h_0\in C_1^\infty(R^1)$ and $h'=-h_1^2$ with $h_1\in C_0^\infty(R^1)$; (ii) $h_0=1$ on $(-\infty, -c_2]$ and $h_0=0$ on $[-c_1, \infty)$; (iii) h_1 has support in $(-c_2, -c_1)$. It is not difficult to see that such a function really exists.

In order to verify Proposition 5.1, it suffices to prove the following

LEMMA 5.2. *Let $h\in C_1^\infty(R^1)$ be as above. Then*

$$\|(H+\lambda)^{-1}h(\gamma)\exp(-itH)E_\Gamma(H)\psi\|_X \longrightarrow 0$$

as $t\to\infty$ for some $\lambda\gg 1$ large enough.

We further continue the reduction. As is easily seen, the above lemma follows from the following

LEMMA 5.3. *Let $g\in C_0^\infty(\Gamma)$. Define*

$$u(t)=(H+\lambda)^{-1}h(\gamma)\exp(-itH)g(H)\psi, \qquad t\geqq 0,$$

for $\psi\in\mathcal{S}(X)$ (Schwartz space on X). Then, for some δ, $0<\delta\ll 1$, small enough, one has

$$\|X_\delta u(t)\|_X \leqq C_\delta$$

uniformly in $t\geqq 0$.

Let $u=u(t)$ be as in the lemma. Then u obeys the equation $\partial_t u=-iHu+w$, where

$$w(t)=i[h(\gamma),(H+\lambda)^{-1}]\exp(-itH)(H+\lambda)g(H)\phi. \tag{5.1}$$

LEMMA 5.4.

$$\operatorname{Re}\int_0^t \langle -iHu, X_{2\delta}u\rangle_X dt \leqq C_\delta.$$

LEMMA 5.5.

$$\operatorname{Re}\int_0^t \langle w, X_{2\delta}u\rangle_X dt \leqq C_\delta.$$

If the two lemmas above are proved, then Lemma 5.3 and hence Proposition 5.1 follows immediately.

5.3. PROOF OF LEMMA 5.4. We calculate

$$\operatorname{Re}\langle -iHu, X_{2\delta}u\rangle_X=\frac{1}{2}\langle i[H_0, X_{2\delta}]u, u\rangle_X.$$

Furthermore, the term on the right side is written as

$$\delta\langle X_{\delta-1/2}\gamma X_{\delta-1/2}u, u\rangle_X+\langle O(\langle x\rangle^{2\delta-2})u, u\rangle_X.$$

Take δ so small that $2\delta-2<-1$. Then, by Proposition 3.3, the integral over $(0, t)$ of the second term is bounded uniformly in t.

We study the first term. Set

$$v(t)=\exp(-itH)g(H)\phi,$$

so that $u=(H+\lambda)^{-1}h(\gamma)v$ and hence the first term under consideration is represented as

$$\langle h(\gamma)(H+\lambda)^{-1}X_{\delta-1/2}\gamma X_{\delta-1/2}(H+\lambda)^{-1}h(\gamma)v, v\rangle_X.$$

We make repeated use of commutator calculus in section 2 to obtain that the above operator is equal to

$$X_{\delta-1/2}(H+\lambda)^{-1}\gamma h(\gamma)^2(H+\lambda)^{-1}X_{\delta-1/2}+O(\langle x\rangle^{2\delta-2}).$$

Since h is supported in $(-\infty, 0)$, the operator $\gamma h(\gamma)^2$ is nonpositive. Thus, by Proposition 3.3 again, the lemma follows immediately. □

5.4. Before going into the proof of Lemma 5.5, we here introduce the new notation $\operatorname{Re}B=(B+B^*)/2$ for given bounded operator B on $L^2(X)$.

PROOF OF LEMMA 5.5. Set

$$v_1(t)=\exp(-itH)(H+\lambda)E_\Gamma(H)\phi.$$

Recall the representation (5.1) for w. Then we have

$$w = i[h(\gamma), (H+\lambda)^{-1}]g(H)v_1$$

and also

$$u = (H+\lambda)^{-1}h(\gamma)(H+\lambda)^{-1}g(H)v_1 .$$

Thus the term in question is written as

$$\mathrm{Re}\langle w, X_{2\delta}u\rangle_X = \langle(\mathrm{Re}\, T)v_1, v_1\rangle_X ,$$

where

$$T = g(H)(H+\lambda)^{-1}h(\gamma)(H+\lambda)^{-1}X_{2\delta}i[h(\gamma), (H+\lambda)^{-1}]g(H) .$$

By Lemma 2.8, we have

$$i[h(\gamma), (H+\lambda)^{-1}] = h'(\gamma)i[\gamma, (H+\lambda)^{-1}]+O(\langle x\rangle^{-\rho}) .$$

Recall the forms of h and h'; $h=h_0^2$ with $h_0\in C_1^\infty(R^1)$ and $h'=-h_1^2$ with $h_1\in C_0^\infty(R^1)$. Set $h_2=h_0h_1$. Then we can decompose T into

$$T = T_1+T_2+O(\langle x\rangle^{-\rho+2\delta}) ,$$

where

$$T_1 = -X_\delta h_2(\gamma)^2 g(H)\lambda^{-2}i[\gamma, (H+\lambda)^{-1}]g(H)X_\delta ,$$

$$T_2 = -X_\delta h_2(\gamma)^2 g(H)((H+\lambda)^{-2}-\lambda^{-2})i[\gamma, (H+\lambda)^{-1}]g(H)X_\delta .$$

Define the operator S as

$$S = X_{\delta-1/2}h_2(\gamma)g(H)(H+\lambda)^{-1} .$$

Since h_2 has support in $(-\sqrt{2E}, 0)$, the same reasoning as in section 4 proves that

$$\mathrm{Re}\, T_1 \leqq -d\lambda^{-2}S^*S+O(\langle x\rangle^{-\rho+2\delta})$$

for some $d>0$ independent of $\lambda\gg 1$. On the other hand, we have

$$\mathrm{Re}\, T_2 \leqq O(\lambda^{-3})S^*S+O(\langle x\rangle^{-\rho+2\delta}) ,$$

because

$$\|E_\Gamma(H)((H+\lambda)^{-2}-\lambda^{-2})E_\Gamma(H)\| = O(\lambda^{-3}) , \qquad \lambda\to\infty .$$

We now take δ, $0<\delta<(\rho-1)/2$, small enough and $\lambda\gg 1$ large enough. Then, Proposition 3.3 proves the lemma. □

6. Analysis for outgoing states.

The last section is devoted to proving Lemma 1.3. The proof of Lemmas 1.0, 1.1, 1.2 and 1.4 does not have made an essential use of the restriction that the Hamiltonian H under consideration is a three-body system and, in fact, the arguments used in the proof of these lemmas extend to the case of general N-body systems with a slight modification. The restriction is used to prove Lemma 1.3 only. The analysis for outgoing states will become the hardest part of the future study towards the proof of asymptotic completeness for N-body systems.

The proof of Lemma 1.3 is long and is divided into several steps.

6.1. We start by introducing new notations. For $u(t)$ and $v(t)$, $t\geqq 0$, with values in $L^2(X)$, we write $u(t)\sim v(t)$ if the difference $u(t)-v(t)$ is strongly convergent in $L^2(X)$ as $t\to\infty$.

Let $\chi_R\in C^\infty(X)$ be a non-negative function such that: (i) $\chi_R=1$ for $|x|>2R$ and $\chi_R=0$ for $|x|<R$; (ii) $|\partial_x^\alpha\chi_R|\leqq C_\alpha\langle x\rangle^{-|\alpha|}$ uniformly in $R\gg 1$ for any multi-index α.

Recall the representation (1.4) for $\phi_{3a}(t)$. Set $F_3=f_3^2\in C_0^\infty(R^1)$. Since $(g_0(H)-g(H_0))\nabla_x j_a\in O(\langle x\rangle^{-\rho-1})$, we obtain $\phi_{3a}(t)\sim\eta_{aR}(t)$, where

$$\eta_{aR}=\int_0^t g_1(H_a)\exp(isH_a)F_3(\gamma)g_0(H_0)[H_0,\ j_a]\chi_R\exp(-isH)E_\Gamma(H)ds\,.$$

6.2. The next task is to approximate $F_3(\gamma)g_0(H_0)$ by a pseudodifferential operator.

For given $f\in C_0^\infty(R^1)$, we define the pseudodifferential operator $f^\gamma(x,\ D_x)$ by

$$f^\gamma(x,\ D_x)u=(2\pi)^{-6/2}\int\exp(ix\cdot\xi)f(x\cdot\xi/\langle x\rangle)\hat{u}(\xi)d\xi\,,$$

where

$$\hat{u}(\xi)=(2\pi)^{-6/2}\int\exp(-ix\cdot\xi)u(x)dx$$

is the Fourier transform of u. Let $g\in C_0^\infty(R^1)$. Then the operator $f^\gamma(x,\ D_x)g(H_0)$ is defined as the pseudodifferential operator with symbol $f(x\cdot\xi/\langle x\rangle)g(|\xi|^2/2)$.

LEMMA 6.1. *Let both f and g belong to $C_0^\infty(R^1)$. Then one has*

$$f(\gamma)g(H_0)-f^\gamma(x,\ D_x)g(H_0)\in O(\langle x\rangle^{-1})\,.$$

We proceed with the argument, accepting the above lemma as proved. The proof of the lemma is given in the last subsection.

6.3. For a two-cluster decomposition $a=\{(j,\ k),\ l\}$, let the coordinates y_a and z_a be defined by (0.3) and (0.4), respectively. We denote by p_a and q_a the coordinates dual to y_a and z_a, respectively. Then $\{y_a,\ z_a,\ p_a,\ q_a\}$ forms the coordinate system over the phase space $X\times X'$ associated with the two-cluster decomposition a.

By property $(f.3)$, $F_3=f_3^2\in C_0^\infty(R^1)$ is supported in $(\sqrt{2E}(1-2\kappa),\ M)$ for κ, $0<\kappa\ll 1$, small enough. Hence, by property $(j.3)$ of the partition $\{j_a\}$, we obtain the following inclusion relation:

$$\begin{aligned}(6.1)\qquad &\bigcup_a \{(x, \xi)\in X\times X' : x\in \operatorname{supp}\chi_R,\ x\in \operatorname{supp}\nabla_x j_a\\ &\qquad x\cdot\xi/\langle x\rangle \in \operatorname{supp} F_3,\ |\xi|^2/2\in \Gamma\}\\ &\subset \bigcap_a \{(y_a, z_a, p_a, q_a)\in X\times X' : d^{-1}<\langle z_a\rangle/\langle y_a\rangle<d,\\ &\qquad d^{-1}<|p_a|<d,\ d^{-1}<|q_a|<d,\ d^{-1}<y_a\cdot p_a/\langle y_a\rangle<d\}\end{aligned}$$

for some $d>1$, where a ranges over all two-cluster decompositions.

Let d be as above. Let $\omega_0\in C_0^\infty(R^3)$ be a non-negative symbol such that ω_0 has support in $\{q\in R^3 : (2d)^{-1}<|q|<2d\}$ and $\omega_0=1$ on $\{q\in R^3 : d^{-1}<|q|<d\}$. With this symbol, we associate the pseudodifferential operator Q_a defined by

$$(Q_a u)(z_a) = (2\pi)^{-3/2}\int \exp(iz_a\cdot q_a)\omega_0(q_a)\hat{u}(q_a)dq_a\,. \tag{6.2}$$

The operator Q_a is considered as an operator acting on $L^2(X)$ as well as on $L^2(Z_a)$. We further define $Q : L^2(X)\to L^2(X)$ by

$$Q = \Pi_a Q_a\,. \tag{6.3}$$

The operators Q_a commute with each other and hence Q is well-defined. By the standard calculus of pseudodifferential operators, it follows from Lemma 6.1 and relation (6.1) that

$$F_3(\gamma)g_0(H_0)[H_0, j_a]\chi_R = QF_3^\gamma(x, D_x)g_0(H_0)[H_0, j_a]\chi_R+O(\langle x\rangle^{-2})$$

and hence we obtain $\eta_{aR}(t)\sim\eta_{aR}^0(t)$, where

$$\eta_{aR}^0 = \int_0^t g_1(H_a)\exp(isH_a)QF_3^\gamma(x, D_x)g_0(H_0)[H_0, j_a]\chi_R\exp(-isH)E_\Gamma(H)\phi ds\,.$$

6.4. By the construction of $\{j_a(x)\}$, it follows that

$$\Sigma_a[H_0, j_a] = 0$$

and hence we have

$$\Sigma_a \exp(-itH_a)\eta_{aR}^0(t) = \Sigma_a \exp(-itH_a)\zeta_{aR}(t)\,,$$

where $\zeta_{aR}=\zeta_{aR}(t)$ is defined by

$$\begin{aligned}\zeta_{aR} = \int_0^t &\{g_1(H_a)\exp(isH_a)-\exp(itH_a)g_1(H_0)\exp(-i(t-s)H_0)\}\\ &\times L_{aR}\exp(-isH)E_\Gamma(H)\phi ds\end{aligned}$$

with

$$L_{aR} = QF_3^\gamma(x, D_x)g_0(H_0)[H_0, j_a]\chi_R\,. \tag{6.4}$$

We now take μ close enough to $1/2$ so that

$$1/2<\mu<\min(1, \rho-1/2)\,. \tag{6.5}$$

Then, by Proposition 3.3,

$$\int_0^\infty \|\chi_R X_{-\mu} \exp(-itH) E_\Gamma(H)\psi\|_X^2 dt = o(1)\|\psi\|_X^2, \qquad R\to\infty. \tag{6.6}$$

Let L_{aR} be as above. Define $\theta_{aR}(t, s)$, $0\leqq s\leqq t$, by

$$\theta_{aR} = X_\mu L_{aR}^*\{g_1(H_a)\exp(-isH_a)-g_1(H_0)\exp(i(t-s)H_0)\exp(-itH_a)\}\psi.$$

In order to complete the proof of Lemma 1.3, it suffices by (6.6) to show that

$$\int_0^t \|\theta_{aR}(t, s)\|_X^2 ds \leqq C\|\psi\|_X^2 \tag{6.7}$$

uniformly in $t\geqq 0$ and $R\gg 1$.

6.5. We continue the reduction. The argument here is based on the following

LEMMA 6.2. *Let Q_a be defined by* (6.2). *Then, for $\nu>1/2$, one has the estimates:*

a) $$\int_0^t \|X_{-\nu} Q_a \exp(-isH_a)\psi\|_X^2 ds \leqq C\|\psi\|_X^2,$$

b) $$\int_0^t \|X_{-\nu} Q_a \exp(i(t-s)H_0)\exp(-itH_a)\psi\|_X^2 ds \leqq C\|\psi\|_X^2$$

uniformly in $t\geqq 0$.

PROOF. By (0.5),

$$H_a = h_a\otimes Id + Id\otimes T_a$$

on $L^2(X)=L^2(Y_a)\otimes L^2(Z_a)$ and also, by definition, Q_a commutes with H_a. Hence we have

$$Q_a\exp(-isH_a) = \exp(-ish_a)\exp(-isT_a)Q_a.$$

The multiplication operator by $\langle z_a\rangle^{-\nu}$, $\nu>1/2$, is T_a-smooth in Kato's sense and hence

$$\int \|\langle z_a\rangle^{-\nu}\exp(-itT_a)Q_a\psi\|_X^2 dt \leqq C\|\psi\|_X^2.$$

This proves a).

b) Denote by h_0 the free Hamiltonian on $L^2(Y_a)$; $h_0=-1/2\Delta$. Then

$$Q_a\exp(i(t-s)H_0)\exp(-itH_a) = \exp(i(t-s)h_0)\exp(-ith_a)\exp(-isT_a)Q_a.$$

Thus, b) follows immediately by the same argument as used to prove a). $\square$

Let L_{aR} be defined by (6.4). Define $\theta_{aR}^0(t, s)$, $0\leqq s\leqq t$, by

$$\theta_{aR}^0 = X_\mu L_{aR}^*\{\exp(-isH_a)-\exp(i(t-s)H_0)\exp(-itH_a)\}\psi.$$

Since $(g_1(H_a)-g_1(H_0))\nabla_x j_a \in O(\langle x\rangle^{-\rho-1})$ and since $g_1(H_0)g_0(H_0)=g_0(H_0)$, we see by Lemma 6.2 that we have only to prove (6.7) for $\theta^0_{aR}(t, s)$ defined above.

Let $d>1$ be as in (6.1). We introduce a non-negative symbol $\omega_1 \in C^\infty(R^9)$ with the following properties:

(i) ω_1 has support in

$$\{(y, z, p)\in R^{3\times 3}: (2d)^{-1}<\langle y\rangle/\langle z\rangle<2d,\ (2d)^{-1}<|p|<2d,$$
$$(2d)^{-1}<y\cdot p/\langle y\rangle<2d\};$$

ii) $\omega_1=1$ on

$$\{(y, z, p)\in R^{3\times 3}: d^{-1}<\langle y\rangle/\langle z\rangle<d,\ d^{-1}<|p|<d,\ d^{-1}<y\cdot p/\langle y\rangle<d\};$$

iii) $$|\partial_y^\alpha \partial_z^\beta \partial_p^\gamma \omega_1| \leqq C_{\alpha\beta\gamma}(1+|y|+|z|)^{-(|\alpha|+|\beta|)}\langle p\rangle^{-|\gamma|}.$$

With the symbol ω_1, we associate the pseudodifferential operator P_a defined by

$$P_a u = (2\pi)^{-3/2}\int \exp(iy_a\cdot p_a)\omega_1(y_a, z_a, p_a)\hat{u}(p_a, z_a)dp_a, \tag{6.8}$$

where $\hat{u}(p_a, z_a)$ is the Fourier transform of $u(y_a, z_a)$ with respect to the variables y_a. The operator P_a is considered as an operator acting on $L^2(X)$.

Let Q be defined by (6.3). Write Q as $Q=Q_aQ_{1a}$ with $Q_{1a}=\Pi_{b\neq a}Q_b$, where b ranges over two-cluster decompositions except for a. Then, by (6.1), we have

$$X_\mu L^*_{aR} = X_\mu \chi_R[j_a, H_0]g(H_0)F_3^\gamma(x, D_x)^* Q_{1a}P_aQ_a + O(\langle x\rangle^{\mu-2})Q_a.$$

Thus, by Lemma 6.2 again, the proof of Lemma 1.3 is reduced to proving that

$$\int_0^t \|\sigma_a(t, s)\|_X^2 ds \leqq C\|\psi\|_X^2 \tag{6.9}$$

uniformly in $t\geqq 0$, where $\sigma_a(t, s)$, $0\leqq s\leqq t$, is defined by

$$\sigma_a = P_a\langle z_a\rangle^{\mu-1}Q_a\{\exp(-isH_a)-\exp(i(t-s)H_0)\exp(-itH_a)\}\psi. \tag{6.10}$$

6.6. We again denote by h_0 the free Hamiltonian on $L^2(Y_a)$. Set $V_a=h_a-h_0$. Then the potential V_a is represented in terms of the coordinates y_a only and satisfies

$$|V_a| \leqq C(1+|y_a|)^{-\rho}.$$

We now calculate;

$$\langle z_a\rangle^{\mu-1}Q_a\{\exp(-isH_a)-\exp(i(t-s)H_0)\exp(-itH_a)\}$$
$$= \{\exp(-ish_a)-\exp(i(t-s)h_0)\exp(-ith_a)\}\langle z_a\rangle^{\mu-1}\exp(-isT_a)Q_a$$
$$= i\int_s^t \exp(i(\tau-s)h_0)V_a\exp(-i\tau h_a)d\tau\langle z_a\rangle^{\mu-1}\exp(-isT_a)Q_a.$$

The next lemma proves (6.9) and hence completes the proof of the lemma.

LEMMA 6.3. *Let P_a be defined by* (6.8). *Then, for $k\geqq 0$, one has*

$$\|\langle y_a\rangle^{-k}\exp(-i\tau h_0)P_a^*\| \leqq C_k(1+|\tau|+|z_a|)^{-k}$$

uniformly in $\tau\geqq 0$, *where* $\|\cdot\|$ *denotes the uniform operator norm when considered as an operator on* $L^2(Y_a)$.

We here complete the proof of Lemma 1.3 accepting this as proved.

COMPLETION OF PROOF OF LEMMA 1.3. Recall the definition (6.10) for $\sigma_a(t, s)$. We use Lemma 6.3 with $k=\rho$ to obtain that

$$\|\sigma_a(t, s)\|_X \leqq C\|\langle z_a\rangle^{-\nu}\exp(-isT_a)Q_a\psi\|_X$$

with $\nu=\rho-\mu-\varepsilon>1/2$, ε, $0<\varepsilon\ll 1$ being fixed small enough. This proves (6.9) and completes the proof. □

6.7. Lemma 6.3 is easy to prove. We give only a sketch.

PROOF OF LEMMA 6.3. For notational convenience, we write y, z and p for y_a, z_a and p_a, respectively. By interpolation, it suffices to prove the lemma for integer $k\geqq 0$.

We can write $\langle y\rangle^{-k}\exp(-i\tau h_0)P_a^*u$ with $u\in\mathcal{S}(R_y^3)$ as

$$(2\pi)^{-3}\iint\exp(i(y-y')\cdot p)\exp(-i\tau|p|^2/2)\langle y\rangle^{-k}\omega_1(y', z, p)u(y')dy'dp.$$

By the construction of the symbol ω_1,

$$|\nabla_p(\tau|p|^2/2+y'\cdot p)| \geqq C(1+|\tau|+|z|)$$

for $\tau\geqq 0$ on the support of ω_1. Hence, integrating by parts proves the lemma. □

6.8. It remains to prove Lemma 6.1.

PROOF OF LEMMA 6.1. For $u\in\mathcal{S}(X)$, we can write

$$f^\gamma(x, D_x)u = (2\pi)^{-7/2}\iint\exp(ix\cdot\xi)\exp(itx\cdot\xi/\langle x\rangle)\hat{f}(t)\hat{u}(\xi)dtd\xi$$

and hence

$$f(\gamma)u-f^\gamma(x, D_x)u = (2\pi)^{-1/2}\int\hat{f}(t)v(t, x)dt\,,$$

where

$$v = \exp(it\gamma)u-(2\pi)^{-6/2}\int\exp(ix\cdot\xi)\exp(itx\cdot\xi/\langle x\rangle)\hat{u}(\xi)d\xi\,.$$

The function v obeys the equation

$$\partial_t v = i\gamma v+iw\,, \qquad v(0, x) = 0\,,$$

where

$$w(t, x) = (2\pi)^{-6/2}\left\{\gamma\int \exp(ix\cdot\xi)\exp(itx\cdot\xi/\langle x\rangle)\hat{u}(\xi)d\xi\right.$$
$$\left. -\int \exp(ix\cdot\xi)\exp(itx\cdot\xi/\langle x\rangle)(x\cdot\xi/\langle x\rangle)\hat{u}(\xi)d\xi\right\}.$$

We can represent w in the form

$$w = (2\pi)^{-6/2}\int \exp(ix\cdot\xi)a_t(x, \xi)\exp(itx\cdot\xi/\langle x\rangle)\hat{u}(\xi)d\xi$$

with the symbol $a_t(x, \xi)$, $t\in R^1$, satisfying

$$|\partial_x^\alpha\partial_\xi^\beta a_t(x, \xi)| \leqq C_{\alpha\beta}(1+|t|)\langle x\rangle^{-1-|\alpha|}\langle \xi\rangle^{1-|\beta|}.$$

If u takes the form $u=g(H_0)X_k\phi$ with $\phi\in S(X)$, then it follows that

$$\|X_{1-k}w(t, \cdot)\|_X \leqq C_k(1+|t|)^m\|\phi\|_X$$

for some $m=m(k)>0$ and hence, by Lemma 2.4,

$$\|X_{1-k}v(t, \cdot)\|_X \leqq C_k(1+|t|)^m\|\phi\|_X$$

with another m. This completes the proof. □

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Hideo TAMURA
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310, Japan