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Homogeneous Kähler manifolds of non-degenerate Ricci curvature

Dedicated to Professor N. Tanaka on his 60th birthday

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Introduction.

Let M be a connected homogeneous Kähler manifold. Denote by $\mathrm{Aut}(M)$ the group of all holomorphic isometries of M. Let G be a connected subgroup of $\mathrm{Aut}(M)$ acting transitively on M and K the isotropy subgroup of G at a point of M. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Then there correspond to the invariant complex structure and the Kähler form of M a linear endomorphism j of $\mathfrak{g}$ and a skew-symmetric bilinear form ρ on $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ becomes an effective Kähler algebra. (For the definition of a Kähler algebra, see § 1.)

According to Vinberg and Gindikin [8], the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ is called *non-degenerate* if there exists a linear form ω on $\mathfrak{g}$ such that $\rho=d\omega$ ([8]), where the operator d means the exterior differentiation under the identification of p-forms on $\mathfrak{g}$ with left invariant p-forms on the Lie group G. Note that if the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ is non-degenerate, then the system $(\mathfrak{g}, \mathfrak{k}, j)$ becomes a j-algebra. (For the definition of a j-algebra, see § 2.)

The purpose of the present paper is to investigate the structure of j-algebras and prove the following

THEOREM. *Let $M=G/K$ be a connected homogeneous Kähler manifold where G is a subgroup of* $\mathrm{Aut}(M)$. *Then the Ricci curvature of M is non-degenerate if and only if the corresponding Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ is non-degenerate.*

We explain our method. By [3] every connected homogeneous Kähler manifold M is a holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold. Recall that the Ricci tensor of M corresponds to the canonical hermitian form introduced by Koszul [4] and it is expressed in terms of the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$. Then by a simple calculation, we can see in § 1 that *if the Ricci tensor of M is non-*

degenerate, then the flat factor of M does not appear (Theorem 1.1). On one hand we already know from [**6**, Theorems A and B] and from [**7**, Proposition 3.1] that the flat part of M vanishes if and only if the system $(\mathfrak{g}, \mathfrak{k}, j)$ becomes a j-algebra. Therefore the study of homogeneous Kähler manifolds of non-degenerate Ricci curvature has great concern with the study of j-algebras.

In §§ 2 and 3, starting from the decomposition theorem of a j-algebra in [**9**] with respect to an abelian ideal and using the structure theorem of a homogeneous convex cone in [**3**], we will describe the structure of a j-algebra in more detail. From our descriptions, we can see in § 4 that *every closed 2-form ρ on an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ satisfying the conditions: $\rho(\mathfrak{k}, \mathfrak{g})=0$ and $\rho(jx, jy)=\rho(x, y)$ for $x, y \in \mathfrak{g}$ is an exact form* (Theorem 4.1). In particular, for an effective Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ the non-degeneracy is equivalent to the condition that $(\mathfrak{g}, \mathfrak{k}, j)$ is a j-algebra.

§ 5 is not needed for the proof of our theorem, but is devoted to giving an invariant meaning to the decomposition of a j-algebra obtained in §§ 2 and 3 (Theorem 5.3).

In §§ 6 and 7 we will prove that *the canonical hermitian form of every effective j-algebra is non-degenerate* (Theorem 6.1). This can be done by direct computations, using the root space decomposition due to [**9**].

Summing up our results, we have the following implications:

$$
\begin{tikzcd}
\text{the Ricci curvature is non-degenerate} \arrow[r, "\text{Theorem 1.1}"] & \text{the flat factor does not appear} \arrow[d, leftrightarrow, "{[6]}"] \\
(\mathfrak{g}, \mathfrak{k}, j, \rho) \text{ is non-degenerate} \arrow[r, shift right, "\text{trivial}"'] & (\mathfrak{g}, \mathfrak{k}, j) \text{ is a } j\text{-algebra.} \arrow[l, shift right, "\text{Theorem 4.1}"'] \arrow[ul, "\text{Theorem 6.1}"']
\end{tikzcd}
$$

Thus we get our theorem. At the same time, we also obtain that *the Ricci curvature of a connected homogeneous Kähler manifold M is non-degenerate if and only if M is a holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is a compact simply connected homogeneous Kähler manifold.* We would like to remark that the last condition is equivalent to say that M is, as a complex manifold, the product of a homogeneous bounded domain and a compact simply connected homogeneous complex manifold ([**6**], [**3**]).

Throughout this paper, we use the following notations: For a Lie algebra $\mathfrak{g}$, rad$(\mathfrak{g})$ and nil$(\mathfrak{g})$ mean the radical and the nilpotent radical of $\mathfrak{g}$ respectively. Let A be a linear endomorphism of a real vector space V. Then A is uniquely decomposed as $A=R+I+N$, where all R, I and N commute, R (resp. I) is a semi-simple endomorphism with real (resp. imaginary) eigenvalues and N is a nilpotent endomorphism. We denote by $Re(A)$ the endomorphism R.

§ 1. Kähler algebras.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\boldsymbol{R}$, $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$, j a linear endomorphism of $\mathfrak{g}$ and ρ a skew-symmetric bilinear form on $\mathfrak{g}$. We then call the quadruple $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ or simply $\mathfrak{g}$ to be *a Kähler algebra* if the following conditions are satisfied:

$$j\mathfrak{k} \subset \mathfrak{k}, \qquad j^2x \equiv -x \pmod{\mathfrak{k}}, \tag{1.1}$$

$$[x, jy] \equiv j[x, y] \pmod{\mathfrak{k}} \qquad \text{for } x \in \mathfrak{k},\ y \in \mathfrak{g}, \tag{1.2}$$

$$[jx, jy] \equiv [x, y] + j[jx, y] + j[x, jy] \pmod{\mathfrak{k}} \qquad \text{for } x, y \in \mathfrak{g}, \tag{1.3}$$

$$\rho(\mathfrak{k}, \mathfrak{g}) = 0, \qquad d\rho = 0, \tag{1.4}$$

$$\rho(jx, jy) = \rho(x, y) \qquad \text{for } x, y \in \mathfrak{g}, \tag{1.5}$$

$$\rho(jx, x) > 0 \qquad \text{if } x \notin \mathfrak{k}. \tag{1.6}$$

The subalgebra $\mathfrak{k}$ will be called *the isotropy subalgebra.*

Let $M = G/K$ be a connected homogeneous Kähler manifold of a Lie group G by a closed subgroup K, equipped with a G-invariant complex structure J and a G-invariant Kähler form Ψ. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Then there exists a linear endomorphism j of $\mathfrak{g}$ such that $\pi_*(jx)_e = J_o(\pi_* x_e)$ for $x \in \mathfrak{g}$, where e denotes the identity element of G, π denotes the projection of G onto G/K and $o = \pi(e)$. We also set $\rho = \pi^*\Psi$. Then $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ becomes a Kähler algebra.

Conversely let $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ be a Kähler algebra and let G be the simply connected Lie group with $\mathfrak{g}$ as its Lie algebra. Denote by K the connected subgroup of G corresponding to $\mathfrak{k}$. Then as is proved in [**3**, Proposition 1.1], the group K is closed in G and the homogeneous space G/K admits a G-invariant Kähler structure.

Let $\mathfrak{g}$ be a Kähler algebra with an isotropy subalgebra $\mathfrak{k}$ and an operator j. We call $\mathfrak{g}$ to be *effective* if $\mathfrak{k}$ does not contain any non-trivial ideal of $\mathfrak{g}$. Let j' be another endomorphism such that $j'x \equiv jx \pmod{\mathfrak{k}}$ for all $x \in \mathfrak{g}$. Then $\mathfrak{g}$ is also a Kähler algebra relative to j'. Changing j to such a j' will be said *an inessential change* and we will not distinguish two algebras which are related to each other by inessential changes. A subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is called *a Kähler subalgebra* if it satisfies $j\mathfrak{g}' \subset \mathfrak{g}' + \mathfrak{k}$, In this case after an inessential change of j, we can assume that $j\mathfrak{g}' \subset \mathfrak{g}'$. Then $\mathfrak{g}'$ itself is a Kähler algebra with the isotropy subalgebra $\mathfrak{g}' \cap \mathfrak{k}$. Similarly *Kähler ideals* are defined.

Let $(\mathfrak{g}, \mathfrak{k}, j)$ be a system satisfying (1.1), (1.2) and (1.3). We further assume that $\operatorname{Trace} \operatorname{ad} x = 0$ for all $x \in \mathfrak{k}$. For any $x \in \mathfrak{g}$, $\operatorname{ad} jx - j \circ \operatorname{ad} x$ leaves $\mathfrak{k}$ invariant and hence induces an endomorphism of $\mathfrak{g}/\mathfrak{k}$. According to Koszul [**4**], we define a linear form ψ, called *the Koszul form*, by

$$\psi(x) = \mathrm{Trace}(\mathrm{ad}\, jx - j\circ \mathrm{ad}\, x)|_{\mathfrak{g}/\mathfrak{k}} \qquad \text{for } x \in \mathfrak{g}\,.$$

Let us set

$$\eta(x, y) = \psi([jx, y]) \qquad \text{for } x, y \in \mathfrak{g}\,.$$

We can see that η is a symmetric bilinear form on $\mathfrak{g}$ satisfying the following properties ([**4**]):

$$\eta(\mathfrak{k}, \mathfrak{g}) = 0 \qquad \text{and} \qquad \eta(jx, jy) = \eta(x, y) \quad \text{for } x, y \in \mathfrak{g}\,.$$

By the above properties, the form η induces a hermitian symmetric bilinear form on $\mathfrak{g}/\mathfrak{k}$, which will be called *the canonical hermitian form*. It is standard that for a Kähler algebra $\mathfrak{g}$, the canonical hermitian form thus obtained can be identified with the Ricci tensor of the homogeneous Kähler manifold corresponding to $\mathfrak{g}$. Using the result of [**3**], we will calculate the canonical hermitian form and prove the following

THEOREM 1.1. *Let M be a connected homogeneous Kähler manifold. Assume that the Ricci curvature of M is non-degenerate. Then M is, as a complex manifold, the product of a homogeneous bounded domain and a compact simply connected homogeneous complex manifold.*

PROOF. By [**3**, Theorems 2.1 and 2.5], we can find a subgroup G of $\mathrm{Aut}(M)$ acting on M transitively and having the following properties: Let us denote by $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ the Kähler algebra attached to the expression $M=G/K$. Then $\mathfrak{g}$ is decomposed as $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{h}$ where

(1) $\mathfrak{a}$ is an abelian Kähler ideal of $\mathfrak{g}$;

(2) $\mathfrak{h}$ is a Kähler subalgebra containing $\mathfrak{k}$ and the homogeneous Kähler manifold associated with the Kähler algebra $(\mathfrak{h}, \mathfrak{k}, j, \rho)$ is, as a complex manifold, the product of a homogeneous bounded domain and a compact simply connected homogeneous Kähler manifold.

In order to prove our theorem it is sufficient to show that $\mathfrak{a}=0$. Let Ψ denote the Koszul form of the Kähler algebra $(\mathfrak{g}, \mathfrak{k}, j, \rho)$. After an inessential change of j, we can assume that $\mathfrak{a}$ is j-invariant. Then it is clear that $\Psi(\mathfrak{a})=0$. Hence we have $\Psi([\mathfrak{a}, \mathfrak{g}])=0$. This means that $\mathfrak{a}\subset\mathfrak{k}$, because the canonical hermitian form is non-degenerate in our case. Since $\mathfrak{k}\cap\mathfrak{a}=0$ holds, we can conclude $\mathfrak{a}=0$, proving the theorem. q.e.d.

§2. The structure of j-algebras.

Let $(\mathfrak{g}, \mathfrak{k}, j)$ be a system satisfying (1.1), (1.2) and (1.3). We call $(\mathfrak{g}, \mathfrak{k}, j)$ or simply $\mathfrak{g}$ *a j-algebra* if there exists a linear form ω on $\mathfrak{g}$ such that $(\mathfrak{g}, \mathfrak{k}, j, d\omega)$

is a Kähler algebra. Such a form ω will be called *an admissible form* of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$. Clearly if $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ is a non-degenerate Kähler algebra, then $(\mathfrak{g}, \mathfrak{k}, j)$ is a j-algebra. For the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$, we can also define effectiveness, j-subalgebras, the Koszul form, etc, similarly to Kähler algebras. In this section, we first recall a result of [**9**] concerning to j-algebras.

Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra. An abelian ideal $\mathfrak{r}$ of $\mathfrak{g}$ is called *of the first kind* if there exists an element e of $\mathfrak{r}$ such that

$$[jx, e] = x \qquad \text{for all } x \in \mathfrak{r}. \tag{2.1}$$

The element e is called *the principal idempotent* of $\mathfrak{r}$.

The following fact is standard.

PROPOSITION 2.1 ([**9**]). *Let $\mathfrak{r}$ be an abelian ideal of the first kind with the principal idempotent e and let $\mathfrak{g}_\lambda$ be the eigenspace of the operator $Re(\mathrm{ad}\, je)$ with eigenvalue λ. Then $\mathfrak{g}$ is decomposed into the sum of subspaces as*

$$\mathfrak{g} = \mathfrak{r} \oplus j\mathfrak{r} \oplus \mathfrak{w} \oplus \mathfrak{s}$$

in the following way:

(1) $\mathfrak{g}_0 = j\mathfrak{r} \oplus \mathfrak{s}$, $\mathfrak{g}_{1/2} = \mathfrak{w}$ *and* $\mathfrak{g}_1 = \mathfrak{r}$.

(2) $\mathfrak{s}$ *is a j-subalgebra containing* $\mathfrak{k}$ *and given by* $\mathfrak{s} = \{x \in \mathfrak{g}_0 ; [x, e] = 0\}$.

(3) $j\mathfrak{w} \subset \mathfrak{w} \oplus \mathfrak{k}$.

Moreover let us denote by τ the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{r}$ and by G_0 the connected subgroup of $GL(\mathfrak{r})$ generated by $\tau(\mathfrak{g}_0)$. Then

(4) $\Omega = G_0 e$ *is an open convex cone in $\mathfrak{r}$ not containing any straight line. Assume further that $\mathfrak{r}$ is a maximal abelian ideal of the first kind. Then*

(5) $\mathfrak{s}$ *is reductive.*

In what follows, $\mathfrak{r}$ always denotes a maximal abelian ideal of the first kind.

Let us denote by $\mathfrak{F}$ the algebraic hull of $\tau(\mathfrak{g}_0)$. Then by [**3**, Theorem 6.2], we have

PROPOSITION 2.2 ([**3**]). *There exist elements $e_1, \cdots, e_m$ of $\mathfrak{r}$, commutative elements $f_1, \cdots, f_m \in \mathfrak{F}$, decompositions $\mathfrak{r} = \sum_{1 \leq i \leq j \leq m} \mathfrak{r}_{ij}$, $\mathfrak{F} = \sum_{1 \leq i \leq j \leq m} \mathfrak{F}_{ij} \oplus \mathfrak{F}_0$ and irreducible self dual cones $\Omega_i \subset \mathfrak{r}_{ii}$ such that $f_i \in \mathfrak{F}_{ii}$ and*

(1) $f_i = (\delta_{ij} + \delta_{ik})/2$ *on* $\mathfrak{r}_{jk}$ *and* $\mathrm{ad}\, f_i = (\delta_{ij} - \delta_{ik})/2$ *on* $\mathfrak{F}_{jk}$;

(2) $\mathfrak{F}_0 = \{f \in \mathfrak{F} ; fx = 0$ *for all* $x \in \sum_{i=1}^m \mathfrak{r}_{ii}\}$;

(3) $[\mathfrak{F}_0, \mathfrak{F}_{ii}] = 0$ *for all i and* $[\mathfrak{F}_{ii}, \mathfrak{F}_{jj}] = 0$, $\mathfrak{F}_{ii}\mathfrak{r}_{jj} = 0$ *for* $i \neq j$.

By the property (1), *each $\mathfrak{r}_{jk}$ is invariant under $\mathfrak{F}_{ii}$. Then*

(4) *the restriction of $\mathfrak{F}_{ii}$ to $\mathfrak{r}_{ii}$ gives an isomorphism between $\mathfrak{F}_{ii}$ and Lie* $\mathrm{Aut}(\Omega_i)$, *the Lie algebra of the group of all automorphisms of the cone Ω_i;*

(5) $e = \sum_{i=1}^m e_i$, $e_i \in \Omega_i$ *and* $\Omega_1 \times \cdots \times \Omega_m = \Omega \cap \sum_{i=1}^m \mathfrak{r}_{ii}$;

(6) *the isotropy subalgebra $\mathfrak{F}_e$ of $\mathfrak{F}$ at the point e is decomposed as* $\mathfrak{F}_e = \sum_{i=1}^m \mathfrak{F}_e \cap \mathfrak{F}_{ii} \oplus \mathfrak{F}_0$ *and* $\mathfrak{F}_e \cap \mathfrak{F}_{ii} = \{f \in \mathfrak{F}_{ii} ; fe_i = 0\}$.

From the above properties, we can see

$$\mathfrak{F}_e e_i = 0 \qquad \text{for all } i=1, \cdots, m. \tag{2.2}$$

Since $\tau(\mathfrak{k})\subset\tau(\mathfrak{s})\subset\mathfrak{F}_e$, we have

$$[\mathfrak{s}, e_i] = 0 \qquad \text{and} \qquad [\mathfrak{k}, e_i] = 0. \tag{2.3}$$

Moreover from (4), we also know that $\mathfrak{F}_{ii}$ is reductive and its center is the 1-dimensional subspace generated by f_i. Therefore

$$\mathfrak{F}_{ii} = \boldsymbol{R} f_i \oplus \mathfrak{H}_i, \tag{2.4}$$

where $\mathfrak{H}_i=[\mathfrak{F}_{ii}, \mathfrak{F}_{ii}]$.

After an inessential change of j, we can assume that $j\mathfrak{r}$ is a solvable subalgebra ([**9**]). Let $e_1, \cdots, e_m$ be as in Proposition 2.2. We consider the operators $R_i=Re(\mathrm{ad}\, je_i)$. We put $f'_i=\tau(je_i)$. Then $f_i e - f'_i e = e_i - e_i = 0$. Therefore $f_i - f'_i \in \mathfrak{F}_e$. Since $[f_i, \mathfrak{F}_e]=0$ and since every element of $\mathfrak{F}_e$ has only imaginary eigenvalues, we have $Re(f_i)=Re(f'_i)$. Therefore

$$R_i = \frac{1}{2}(\delta_{ij}+\delta_{ik}) \qquad \text{on } \mathfrak{r}_{jk}. \tag{2.5}$$

LEMMA 2.3. $\quad [je_i, je_j] = 0.$

PROOF. Since $\mathfrak{F}_e e_i=0$ by (2.2), we have

$$[je_i, je_j] = j[je_i, e_j]+j[e_i, je_j] = jf'_i e_j = jf'_j e_i = j(f_i e_j - f_j e_i) = 0.$$

q. e. d.

By Lemma 2.3, we can decompose $\mathfrak{g}$ into the sum of root spaces $\mathfrak{g}^\Gamma$ relative to the abelian space of linear endomorphisms generated by $R_1, \cdots, R_m$. Since all $\mathfrak{r}$, $j\mathfrak{r}$, $\mathfrak{g}_0$ and $\mathfrak{w}$ are ad $j\mathfrak{r}$-invariant, we also have $\mathfrak{r}=\sum\mathfrak{r}^\Gamma$, $j\mathfrak{r}=\sum(j\mathfrak{r})^\Gamma$, $\mathfrak{g}_0=\sum\mathfrak{g}_0^\Gamma$ and $\mathfrak{w}=\sum\mathfrak{w}^\Gamma$. If we define Δ_i for $i=1, \cdots, m$ by $\Delta_i(R_j)=\delta_{ij}$, then by (2.5) we have immediately the following

LEMMA 2.4. $\quad \mathfrak{r}=\sum_{1\leq i\leq j\leq m} \mathfrak{r}^{(\Delta_i+\Delta_j)/2}$ *and* $\mathfrak{r}^{(\Delta_i+\Delta_j)/2}=\mathfrak{r}_{ij}$.

Next we show the following

LEMMA 2.5. $\quad j\mathfrak{r}_{ij} = (j\mathfrak{r})^{(\Delta_i-\Delta_j)/2}$ *for* $i<j$ *and* $j\mathfrak{r}_{ii} \subset \mathfrak{g}_0^0$.

PROOF. From (2.1), we have $[j\mathfrak{r}_{ij}, e]\subset\mathfrak{r}_{ij}$. It is clear that the correspondence: $\mathfrak{F}\ni f\to fe\in\mathfrak{r}$ gives a linear map of $\mathfrak{F}_{ij}$ onto $\mathfrak{r}_{ij}$. This means that $\tau(j\mathfrak{r}_{ij})\subset \mathfrak{F}_{ij}+\mathfrak{F}_e$. We then have for $x\in\mathfrak{r}_{ij}$ $[jx, e_k]\in\mathfrak{F}_{ij}e_k=0$ if $j\neq k$ and $[jx, e_j]=[jx, e]=x$. Therefore $[je_k, jx]=j[e_k, jx]+j[je_k, x]=-\delta_{jk}x+j[je_k, x]$. Hence $j\mathfrak{r}_{ij}$ is invariant under R_k and $R_k=(\delta_{ik}-\delta_{jk})/2$ on $j\mathfrak{r}_{ij}$. q. e. d.

By virture of Lemmas 2.4 and 2.5, we obtain the following fact using the similar argument in [**9**].

LEMMA 2.6. $$\mathfrak{w}=\sum_{i=1}^{m}\mathfrak{w}^{\Delta_i/2}.$$

Next we show

LEMMA 2.7. $$\mathfrak{s}\subset\mathfrak{g}_0^0.$$

PROOF. By (2.3), $[e_i, \mathfrak{s}]=0$. Since $[jx, s]\equiv j[x, s] \pmod{\mathfrak{s}}$ holds for $x\in\mathfrak{r}$ and $s\in\mathfrak{s}$ (cf. [**9**]), we know that $\mathrm{ad}\,je_i$ leaves $\mathfrak{s}$ invariant. Let $\mathfrak{c}$ and $\mathfrak{h}$ denote the center and the semi-simple part of the reductive Lie algebra $\mathfrak{s}$. Note that $\mathfrak{c}\subset\mathfrak{k}$ and we can assume that $\mathfrak{h}$ is j-invariant, Both $\mathfrak{c}$ and $\mathfrak{h}$ are invariant under $\mathrm{ad}\,je_i$. Therefore there exists $k_i\in\mathfrak{h}$ such that $[je_i-k_i, \mathfrak{h}]=0$. Since $[je_i, \mathfrak{k}]\equiv j[e_i, \mathfrak{k}]\equiv 0 \pmod{\mathfrak{k}}$, we know that k_i is contained in the normalizer of $\mathfrak{h}\cap\mathfrak{k}$ in $\mathfrak{h}$. Since $(\mathfrak{h}, \mathfrak{h}\cap\mathfrak{k}, j)$ is a semi-simple j-algebra, the normalizer of $\mathfrak{h}\cap\mathfrak{k}$ in $\mathfrak{h}$ coincides with $\mathfrak{h}\cap\mathfrak{k}$ ([**5**, p. 59]). Therefore $k_i\in\mathfrak{k}$, whence $Re(\mathrm{ad}\,je_i)|_{\mathfrak{h}}=Re(\mathrm{ad}\,k_i)|_{\mathfrak{h}}=0$.

Clearly $\mathfrak{c}$ is an ideal of the subalgebra $\boldsymbol{R}je_i\oplus\mathfrak{c}$. Since $\mathrm{ad}\,\mathfrak{c}$ is completely reducible, there exists a 1-dimensional subspace $\mathfrak{v}$ invariant under $\mathrm{ad}\,\mathfrak{c}$ such that $\boldsymbol{R}je_i\oplus\mathfrak{c}=\mathfrak{v}\oplus\mathfrak{c}$. But then $[\mathfrak{v}, \mathfrak{c}]=0$. Therefore $\boldsymbol{R}je_i\oplus\mathfrak{c}$ is abelian, implying $\mathfrak{c}\subset\mathfrak{g}_0^0$.
q. e. d.

Summing up the results, we have proved

PROPOSITION 2.8. *$\mathfrak{g}$ is decomposed as $\mathfrak{g}=\sum\mathfrak{g}^{\Gamma}$, where $\Gamma\in\{\Delta_i/2, (\Delta_i\pm\Delta_j)/2; 1\leqq i\leqq j\leqq m\}$ and the following hold;*

$$\mathfrak{g}^{(\Delta_i+\Delta_j)/2}=\mathfrak{r}_{ij}\quad(i\leqq j),\qquad \mathfrak{g}^{(\Delta_i-\Delta_j)/2}=j\mathfrak{r}_{ij}\quad(i<j),$$

$$\mathfrak{g}^{\Delta_i/2}=\mathfrak{w}^{\Delta_i/2},\qquad \mathfrak{g}^0=\sum_{i=1}^{m}j\mathfrak{r}_{ii}\oplus\mathfrak{s}.$$

We put

$$\mathfrak{r}^{\#}=\sum_{i=1}^{m}\mathfrak{r}_{ii}.$$

LEMMA 2.9. $$\tau(\mathfrak{g}^0)|_{\mathfrak{r}^{\#}}=\sum_{i=1}^{m}Lie\,\mathrm{Aut}(\Omega_i).$$

PROOF. It is clear that $\tau(\mathfrak{g}^0)\subset\sum_{i=1}^{m}\mathfrak{F}_{ii}\oplus\mathfrak{F}_0$. Therefore

$$\tau(\mathfrak{g}^0)|_{\mathfrak{r}^{\#}}\subset\sum_{i=1}^{m}Lie\,\mathrm{Aut}(\Omega_i).$$

Since $\mathfrak{F}$ is the algebraic hull of $\tau(\mathfrak{g}_0)$, we know that $[\mathfrak{F}, \mathfrak{F}]=[\tau(\mathfrak{g}_0), \tau(\mathfrak{g}_0)]$. Therefore $\mathfrak{h}_i\subset\tau(\mathfrak{g}_0)$, where $\mathfrak{h}_i$ is the semi-simple part of $\mathfrak{F}_{ii}$. Moreover the eigenvalue of $\mathrm{ad}\,je_k$ has 0 on $\mathfrak{F}_{ii}$, we have $\mathfrak{h}_i\subset\tau(\mathfrak{g}^0)$. Let $g\in(\sum_{i=1}^{m}\boldsymbol{R}\tau(je_i)|_{\mathfrak{r}^{\#}})\cap(\sum_{i=1}^{m}\mathfrak{h}_i)|_{\mathfrak{r}^{\#}}$. Then from the equation $\mathrm{Trace}\,g|_{\mathfrak{r}_{ii}}=0$ for all i, we know that $g=0$. Since $\mathfrak{F}_{ii}\cong Lie\,\mathrm{Aut}(\Omega_i)$ and since $\mathfrak{F}_{ii}=\boldsymbol{R}f_i\oplus\mathfrak{h}_i$, we get the assertion. q. e. d.

By Lemma 2.9, $\tau(\mathrm{nil}(\mathfrak{g}^0))|_{\mathfrak{r}^\#}=0$. In particular, $\tau(\mathrm{nil}(\mathfrak{g}^0))e=0$, whence $\mathrm{nil}(\mathfrak{g}^0)\subset\mathfrak{s}$. Then $\mathrm{nil}(\mathfrak{g}^0)$ is a nilpotent ideal of $\mathfrak{s}$, whence it is contained in the center $\mathfrak{c}$ of $\mathfrak{s}$. Recall that $\mathfrak{c}\subset\mathfrak{k}$. Therefore $\mathrm{nil}(\mathfrak{g}^0)\subset\mathfrak{k}$ and hence $\mathrm{nil}(\mathfrak{g}^0)=0$, proving that $\mathfrak{g}^0$ is reductive.

It is clear that $\sum_{\Gamma\neq 0}\mathfrak{g}^\Gamma$ is a solvable ideal of $\mathfrak{g}$ contained in $[\mathfrak{g}, \mathfrak{g}]$. Therefore by Proposition 2.8, we have

PROPOSITION 2.10. *The subalgebra $\mathfrak{g}^0$ is reductive and $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})\oplus[\mathfrak{g}^0, \mathfrak{g}^0]$. Moreover*

$$\mathrm{rad}(\mathfrak{g}) = \sum_{\Gamma\neq 0}\mathfrak{g}^\Gamma\oplus \textit{the center of } \mathfrak{g}^0, \qquad \mathrm{nil}(\mathfrak{g}) = \sum_{\Gamma\neq 0}\mathfrak{g}^\Gamma .$$

Let us set

$$\mathfrak{g}^\# = \mathfrak{r}^\#\oplus\mathfrak{g}^0 \quad (=\mathfrak{r}^\#\oplus j\mathfrak{r}^\#\oplus\mathfrak{s}). \tag{2.6}$$

Then $\mathfrak{g}^\#$ is a j-invariant subalgebra containing $\mathfrak{k}$. Since $\mathfrak{g}^0$ is reductive, we have

$$\mathfrak{r}^\# = \mathrm{nil}(\mathfrak{g}^\#). \tag{2.7}$$

Let us put

$$\mathfrak{n} = \mathrm{nil}(\mathfrak{g})\cap(j\,\mathrm{nil}(\mathfrak{g})\oplus\mathfrak{k}). \tag{2.8}$$

By Proposition 2.10, we have

$$\mathfrak{n} = \mathfrak{w}\oplus\sum_{i<j}\mathfrak{r}_{ij}\oplus\sum_{i<j}j\mathfrak{r}_{ij}. \tag{2.9}$$

Therefore we get

PROPOSITION 2.11. $\mathfrak{g}=\mathfrak{g}^\#\oplus\mathfrak{n}$, $[\mathfrak{g}^\#, \mathfrak{n}]\subset\mathfrak{n}$ *and* $[\mathfrak{n}, \mathfrak{n}]\subset\mathfrak{n}\oplus\mathfrak{r}^\#$.

§ 3. The subalgebra $\mathfrak{g}^\#$.

In this section, we investigate the structure of the j-algebra $(\mathfrak{g}^\#, \mathfrak{k}, j)$ and give a description of $\mathrm{rad}(\mathfrak{g}^0)$.

We define

$$\mathfrak{s}_0^\# = \{x\in\mathfrak{s}\,;\,[x, \mathfrak{r}^\#]=0\}.$$

Then $\mathfrak{s}_0^\#$ is an ideal of $\mathfrak{g}^\#$.

LEMMA 3.1. *After an inessential change of j if necessary, there exists an ideal $\hat{\mathfrak{s}}^\#$ of $\mathfrak{s}$ satisfying the conditions*

(1) $\mathfrak{g}^\#=\mathfrak{r}^\#\oplus j\mathfrak{r}^\#\oplus\hat{\mathfrak{s}}^\#\oplus\mathfrak{s}_0^\#$,

(2) $\mathfrak{s}=\hat{\mathfrak{s}}^\#\oplus\mathfrak{s}_0^\#$, $\mathfrak{k}=(\mathfrak{k}\cap\hat{\mathfrak{s}}^\#)\oplus(\mathfrak{k}\cap\mathfrak{s}_0^\#)$,

(3) $\mathfrak{r}^\#\oplus j\mathfrak{r}^\#\oplus\hat{\mathfrak{s}}^\#$ *is a j-invariant ideal of* $\mathfrak{g}^\#$.

PROOF. Since $\mathfrak{s}_0^\#\subset\mathfrak{s}$, $\mathfrak{s}_0^\#$ is reductive. The center $\mathfrak{c}_0$ of $\mathfrak{s}_0^\#$ is contained in $\mathfrak{k}$ and $[\mathfrak{s}_0^\#, \mathfrak{s}_0^\#]$ is a semi-simple ideal of $\mathfrak{g}^\#$. Therefore $\mathfrak{g}^\#=[\mathfrak{s}_0^\#, \mathfrak{s}_0^\#]\oplus\mathfrak{g}'$, where $\mathfrak{g}'$

is the centralizer of $[\mathfrak{s}_0^\#, \mathfrak{s}_0^\#]$ in $\mathfrak{g}^\#$. Clearly $\mathfrak{g}'$ is an ideal of $\mathfrak{g}^\#$. By [**3**, Proposition 5.13], we can assume that both $\mathfrak{g}'$ and $[\mathfrak{s}_0^\#, \mathfrak{s}_0^\#]$ is j-invariant. But then $\mathfrak{g}' \supset \mathfrak{r}^\# \oplus j\mathfrak{r}^\#$. Therefore $\mathfrak{g}' = \mathfrak{r}^\# \oplus j\mathfrak{r}^\# \oplus \mathfrak{s}'$, where $\mathfrak{s}' = \mathfrak{s} \cap \mathfrak{g}'$. Clearly $\mathfrak{s}' \cap \mathfrak{s}_0^\# = \mathfrak{c}_0$. Then $\mathfrak{c}_0$ coincides with the largest ideal of $\mathfrak{g}'$ contained in $\mathfrak{k} \cap \mathfrak{g}'$. Therefore there exists an ideal $\mathfrak{g}''$ of $\mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{g}'' \oplus \mathfrak{c}_0$ and $\mathfrak{g}'' \supset \mathfrak{r}^\#$ ([**3**, Proof of Lemma 1.4]). After an inessential change of j, we can assume that $\mathfrak{g}''$ is j-invariant. Then $\mathfrak{g}'' = \mathfrak{r}^\# \oplus j\mathfrak{r}^\# \oplus (\mathfrak{s} \cap \mathfrak{g}'')$. If we put $\hat{\mathfrak{s}}^\# = \mathfrak{s} \cap \mathfrak{g}''$, then $\mathfrak{s} = \hat{\mathfrak{s}}^\# \oplus \mathfrak{s}_0^\#$. Since $\hat{\mathfrak{s}}^\#$ is an ideal of $\mathfrak{s}$, we know $\mathfrak{k} = (\mathfrak{k} \cap \hat{\mathfrak{s}}^\#) \oplus (\mathfrak{k} \cap \mathfrak{s}_0^\#)$ by [**1**] and [**5**]. It is now clear that $\hat{\mathfrak{s}}^\#$ has the desired properties. q. e. d.

Let $\hat{\mathfrak{s}}^\#$ be as in Lemma 3.1. Then by Lemma 2.6 we have

LEMMA 3.2. *There exists ideals $\mathfrak{f}_i$ $(i=1, \cdots, m)$ of $j\mathfrak{r}^\# \oplus \hat{\mathfrak{s}}^\#$ such that*

$$j\mathfrak{r}^\# \oplus \hat{\mathfrak{s}}^\# = \sum_{i=1}^{m} \mathfrak{f}_i, \qquad [\mathfrak{f}_i, \mathfrak{r}_{jj}] = 0 \ (i \neq j), \qquad [\mathfrak{f}_i, \mathfrak{r}_{ii}] \subset \mathfrak{r}_{ii},$$

and the adjoint representation of $\mathfrak{f}_i$ on $\mathfrak{r}_{ii}$ gives an isomorphism of $\mathfrak{f}_i$ onto Lie $\mathrm{Aut}(\Omega_i)$.

Since $\hat{\mathfrak{s}}^\#$ is identified with the isotropy subalgebra of $\sum_{i=1}^{m} \mathit{Lie}\,\mathrm{Aut}(\Omega_i)$, we have

$$\hat{\mathfrak{s}}^\# = \sum_{i=1}^{m} \mathfrak{s}_i, \qquad \mathfrak{s}_i \subset \mathfrak{f}_i.$$

Since $\mathfrak{s}_i$ is an ideal of $\hat{\mathfrak{s}}^\#$, we have from [**1**] and [**5**]

$$\hat{\mathfrak{s}}^\# \cap \mathfrak{k} = \sum_{i=1}^{m} \mathfrak{s}_i \cap \mathfrak{k}.$$

LEMMA 3.3. *By a suitable change of j, $j\mathfrak{r}_{ii}$ is contained in $\mathfrak{f}_i$.*

PROOF. Let $x \in \mathfrak{r}_{ii}$. We decompose as $jx = y + \sum_{j \neq i} x_j$, where $y \in \mathfrak{f}_i$ and $x_j \in \mathfrak{f}_j$. Consider the equation $[jx, e] = [y, e] + \sum_{j \neq i} [x_j, e]$. Since $[jx, e] = x \in \mathfrak{r}_{ii}$, we have $[x_j, e] = 0$, whence $x_j \in \mathfrak{s}_j$ for all $j \neq i$. Let $k_j \in \mathfrak{s}_j \cap \mathfrak{k}$. Then $[x_j, k_j] = [jx, k_j] \equiv j[x, k_j] \ (\mathrm{mod}\,\mathfrak{k})$ and $[x, k_j] = 0$. Therefore $[x_j, k_j] \in \mathfrak{f}_j \cap \mathfrak{k} = \mathfrak{s}_j \cap \mathfrak{k}$. Note that $\mathfrak{s}_j$ is an ideal of the reductive Lie algebra $\mathfrak{s}$. Then using the result of [**1**], we can assume that $\mathfrak{s}_j$ is j-invariant and $(\mathfrak{s}_j, \mathfrak{s}_j \cap \mathfrak{k}, j)$ is a j-algebra. Then the center of $\mathfrak{s}_j$ is contained in $\mathfrak{s}_j \cap \mathfrak{k}$. We then derive from [**5**] that the normalizer of $\mathfrak{s}_j \cap \mathfrak{k}$ in $\mathfrak{s}_i$ coincides with $\mathfrak{s}_j \cap \mathfrak{k}$. Therefore $x_j \in \mathfrak{s}_j \cap \mathfrak{k}$. We change j on $\mathfrak{r}_{ii}$ to j' as $j'x = y$. It is easy to see that $j'\mathfrak{r}_{ii}$ and $j'\mathfrak{r}$ are still subalgebra of $\mathfrak{g}$. q. e. d.

By Lemma 3.3, we can assume

$$\mathfrak{f}_i = j\mathfrak{r}_{ii} \oplus \mathfrak{s}_i.$$

Since $\mathfrak{f}_i$ is reductive, we have

$$\mathfrak{f}_i = \mathfrak{c}_i \oplus \mathfrak{h}_i ,$$

where $\mathfrak{c}_i$ is the center of $\mathfrak{f}_i$ and $\mathfrak{h}_i$ is the semi-simple part of $\mathfrak{f}_i$. Note that $\mathfrak{c}_i$ is generated by the element g_i such that $\operatorname{ad} g_i=1$ on $\mathfrak{r}_{ii}$ and $\mathfrak{h}_i$ consists of all $f\in\mathfrak{f}_i$ satisfying Trace $\operatorname{ad} f|_{\mathfrak{r}_{ii}}=0$. In particular, $\mathfrak{s}_i\subset\mathfrak{h}_i$ and $j\mathfrak{r}_{ii}=\boldsymbol{R}je_i\oplus(\mathfrak{h}_i\cap j\mathfrak{r}_{ii})$ holds. Moreover from $[je_i-c_i, e_i]=0$, we know that $je_i-c_i\in\mathfrak{s}_i$. Clearly $[je_i-c_i, \mathfrak{k}\cap\mathfrak{s}_i]\subset\mathfrak{k}\cap\mathfrak{s}_i$, whence we know $je_i-c_i\in\mathfrak{k}\cap\mathfrak{s}_i$ by the same reason as before.

We can now change j to j' on $\mathfrak{r}_{ii}$ as follows:

$$j'e_i = c_i \qquad \text{and} \qquad j'x = jx \quad \text{for } x\in\{\mathfrak{r}_{ii};\ jx\in\mathfrak{h}_i\}.$$

It is clear that $j'\mathfrak{r}_{ii}$ is still a subalgebra because $[j\mathfrak{r}_{ii}, j\mathfrak{r}_{ii}]\subset\mathfrak{h}_i$. This inessential change can be extended on whole $\mathfrak{g}$ keeping the property that $j'\mathfrak{r}$ is a solvable subalgebra. Thus we have proved

PROPOSITION 3.4. *By a suitable change of j, we have*

$$\mathfrak{g}^{\#} = \sum_{i=1}^{m} \mathfrak{g}_i\oplus\mathfrak{s}_0^{\#} \qquad (\textit{direct sum of ideals}),$$

$$\mathfrak{s} = \sum_{i=1}^{m} \mathfrak{s}_i\oplus\mathfrak{s}_0^{\#} \qquad (\textit{direct sum of ideals}),$$

$$\mathfrak{k} = \sum_{i=1}^{m} \mathfrak{k}_i\oplus\mathfrak{k}_0 \qquad (\textit{direct sum of ideals}),$$

where $\mathfrak{g}_i=\mathfrak{r}_{ii}\oplus j\mathfrak{r}_{ii}\oplus\mathfrak{s}_i$, $\mathfrak{k}_i=\mathfrak{g}_i\cap\mathfrak{k}$ and $\mathfrak{k}_0=\mathfrak{s}_0^{\#}\cap\mathfrak{k}$. Moreover all $\mathfrak{g}_i$, $\mathfrak{s}_i$ and $\mathfrak{s}_0^{\#}$ are j-invariant and the following equations hold:

$$j\mathfrak{r}_{ii}\oplus\mathfrak{s}_i \cong \textit{Lie}\ \mathrm{Aut}(\Omega_i)\,, \qquad \boldsymbol{R}je_i = \textit{the center of } j\mathfrak{r}_{ii}\oplus\mathfrak{s}_i\,.$$

COROLLARY 3.5. *The center of $\mathfrak{g}^0=\sum_{i=1}^{m}\boldsymbol{R}je_i\oplus$the center of $\mathfrak{s}_0^{\#}$.*

Recall that the center of $\mathfrak{s}_0^{\#}$ is contained in $\mathfrak{k}$. Then the above results combined with Proposition 2.10 yield

PROPOSITION 3.6. $\quad \mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k} = \mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{w}\oplus\mathfrak{k}\,.$

REMARK 1. Let us denote by $D(\Omega_i)$ the Siegel domain of the first kind associated with the convex cone Ω_i. Then $D(\Omega_i)$ is an irreducible symmetric domain and the Lie algebra $\mathfrak{g}_i$ in Proposition 3.4 coincides with the Lie algebra of the group of all affine transformations of $D(\Omega_i)$.

REMARK 2. Let D be a homogeneous bounded domain and G a group of holomorphic transformations of D acting transitively on D. We then have $D=G/K$. Let $(\mathfrak{g}, \mathfrak{k}, j)$ be the corresponding j-algebra. Assume that $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$ holds. Then by Proposition 3.6 and [**9**, § 1, Theorem 2], D is realized as a Siegel domain of the second kind in such a way that G acts as an affine transformation group. (J. Dorfmeister also obtained this fact by different method.) Conversely, we can easily see that every j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ corresponding to a transitive affine automorphism group of a Siegel domain satisfies the equation $\mathfrak{g}=\mathrm{rad}(\mathfrak{g})+j\,\mathrm{rad}(\mathfrak{g})+\mathfrak{k}$.

§ 4. Closed forms on j-algebras.

The purpose of this section is to prove the following

THEOREM 4.1. *Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra and let ρ be a skew-symmetric bilinear form on $\mathfrak{g}$ satisfying*

$$d\rho=0, \quad \rho(\mathfrak{k}, \mathfrak{g})=0 \quad \text{and} \quad \rho(jx, jy)=\rho(x, y) \qquad \text{for all } x, y\in\mathfrak{g}\,.$$

Then there exists a linear form ω on $\mathfrak{g}$ such that $\rho=d\omega$.

In the special case where $\mathfrak{k}=0$ and $\mathfrak{g}$ is solvable, this fact is obtained by Dorfmeister [2].

Let $(\mathfrak{g}, \mathfrak{k}, j)$ and ρ be as in Theorem 4.1. We keep the notations used in the previous sections. The following fact is well known. But we put a proof because we use the similar technique in later.

LEMMA 4.2.
(1) $\rho(\mathfrak{w}, \mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{s}) = 0$.
(2) $\rho(\mathfrak{r}, \mathfrak{r}) = 0$.

PROOF. Consider the function $A(t)=\rho(e^{t\,\mathrm{ad}\,je}x, e^{t\,\mathrm{ad}\,je}y)$ for $x\in\mathfrak{w}$ and $y\in\mathfrak{r}$. Roughly speaking, $A(t)$ grows like $e^{3t/2}$ if $A(t)\neq 0$, because $x\in\mathfrak{g}_{1/2}$ and $y\in\mathfrak{g}_1$. On the other hand, since $[\mathfrak{w}, \mathfrak{r}]=0$, we have $dA(t)/dt=\rho(je, e^{\mathrm{ad}\,je}[x, y])=0$. Therefore $A(t)\equiv 0$, proving $\rho(\mathfrak{w}, \mathfrak{r})=0$. Similarly, we have $\rho(\mathfrak{r}, \mathfrak{r})=0$. Since $j\mathfrak{w}\subset\mathfrak{w}+\mathfrak{k}$, we also have $\rho(\mathfrak{w}, j\mathfrak{r})=\rho(\mathfrak{w}, \mathfrak{r})=0$.

Finally we consider the function $A(t)$ for $x\in\mathfrak{w}$, $y\in\mathfrak{s}$. Then $A(t)$ grows like $e^{t/2}$. But $dA(t)/dt=\rho(je, e^{t\,\mathrm{ad}\,je}[x, y])=0$, because $[x, y]\in\mathfrak{w}$ and $\rho(je, \mathfrak{w})=0$. Thus we also have $A(t)\equiv 0$, proving $\rho(\mathfrak{w}, \mathfrak{s})=0$. q. e. d.

Next we show

LEMMA 4.3. $\qquad \rho(\mathfrak{r}_{ij}\oplus j\mathfrak{r}_{ij}, \mathfrak{g}^{\#}) = 0 \qquad$ *for* $i<j$.

PROOF. Since $\mathfrak{g}^{\#}$ is j-invariant, it is sufficient to show $\rho(\mathfrak{r}_{ij}, \mathfrak{g}^{\#})=0$. Moreover since $\rho(\mathfrak{r}, \mathfrak{r})=0$ by Lemma 4.2, we only have to show $\rho(\mathfrak{r}_{ij}, \mathfrak{g}^0)=0$. Consider the function $A(t)=\rho(e^{t\,\mathrm{ad}\,je_i}x, e^{t\,\mathrm{ad}\,je_i}y)$ for $x\in\mathfrak{r}_{ij}$, $y\in\mathfrak{g}^0$. Then $A(t)$ grows like $e^{t/2}$. We have $dA(t)/dt=\rho(je_i, e^{t\,\mathrm{ad}\,je_i}[x, y])\subset\rho(je_i, \mathfrak{r}_{ij})=\rho(e_i, j\mathfrak{r}_{ij})$. We consider the function $B(t)=\rho(e^{t\,\mathrm{ad}\,je_j}e_i, e^{t\,\mathrm{ad}\,je_j}z)$ for $z\in j\mathfrak{r}_{ij}$. Noting that $[e_i, j\mathfrak{r}_{ij}]=0$, we have $dB(t)/dt=0$. Since $B(t)$ grows like $e^{-t/2}$, we have $B(t)\equiv 0$. From this, we also have $A(t)\equiv 0$. q. e. d.

By Lemmas 4.2 and 4.3 we have

PROPOSITION 4.4. $\rho(\mathfrak{n}, \mathfrak{g}^{\#}) = 0$.

Recall that $\mathfrak{g}^0$ is reductive. Therefore

$$\mathfrak{g}^0 = \mathfrak{c}^{\#} \oplus \mathfrak{h}^{\#},$$

where $\mathfrak{c}^{\#}$ is the center and $\mathfrak{h}^{\#}$ is the semi-simple part of $\mathfrak{g}^0$. Since the center of $\mathfrak{s}_0^{\#}$ is contained in $\mathfrak{k}$, we have from Corollary 3.5 the following

LEMMA 4.5. $\rho(\mathfrak{c}^{\#}, \mathfrak{c}^{\#}) = 0$.

We now define a linear form ω on $\mathfrak{g}$. We have obtained the decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{r}^{\#} \oplus \mathfrak{c}^{\#} \oplus \mathfrak{h}^{\#}.$$

Since $\mathfrak{h}^{\#}$ is semi-simple, there exists a linear form ω on $\mathfrak{h}^{\#}$ such that $\rho = d\omega$ on $\mathfrak{h}^{\#}$. We extend ω to a linear form on $\mathfrak{g}$ by setting

$$\omega(\mathfrak{n} \oplus \mathfrak{c}^{\#}) = 0 \quad \text{and} \quad \omega(x) = -\omega(je, x) \quad \text{for } x \in \mathfrak{r}^{\#}.$$

Then from Proposition 4.4

$$\omega(x) = -\rho(je, x) \qquad \text{for all } x \in \mathfrak{n} \oplus \mathfrak{r}^{\#}. \tag{4.1}$$

We have to show that $d\omega = \rho$. We can assume that $j\mathfrak{r}$ is a solvable subalgebra. Then $(\mathfrak{r} \oplus j\mathfrak{r} \oplus \mathfrak{w}, 0, j)$ is a solvable j-algebra corresponding to a homogeneous Siegel domain ([**9**]). Then the following lemma is essentially proved in Dorfmeister [**2**, § 3].

LEMMA 4.6 ([**2**]). *Let* $I = \mathrm{ad}\, je - Re(\mathrm{ad}\, je)$. *Then the restriction of* I *on* $\mathfrak{r} \oplus j\mathfrak{r} \oplus \mathfrak{w}$ *is skew-symmetric relative to* ρ *and commutes with* j.

Using the above lemma, we show

LEMMA 4.7.
(1) $\rho(x, y) = d\omega(x, y)$ *for* $x, y \in \mathfrak{w}$.
(2) $\rho(x, y) = d\omega(x, y) = 0$ *for* $x \in \mathfrak{w}$, $y \in \mathfrak{r} \oplus j\mathfrak{r} \oplus \mathfrak{s}$.

PROOF. (1) Since $[x, y] \in \mathfrak{r}$, using (4.1) and Lemma 4.6 we have $d\omega(x, y) = -\omega([x, y]) = \rho(je, [x, y]) = \rho([je, x], y) + \rho(x, [je, y]) = \rho(x/2, y) + \rho(x, y/2) + \rho(Ix, y) + \rho(x, Iy) = \rho(x, y)$.

(2) In this case, $[x, y] \in \mathfrak{w}$. Therefore $\omega([x, y]) = 0$. On the other hand $\rho(x, y) = 0$, by Lemma 4.2. q.e.d.

Recall that $\mathfrak{r} \oplus j\mathfrak{r} \oplus \mathfrak{s} = \sum_{i<j} \mathfrak{r}_{ij} \oplus j \sum_{i<j} \mathfrak{r}_{ij} \oplus \mathfrak{g}^{\#}$.

LEMMA 4.8.
(1) $\rho(x, y) = d\omega(x, y)$ *for* $x, y \in \sum_{i<j} \mathfrak{r}_{ij} \oplus j \sum_{i<j} \mathfrak{r}_{ij}$.
(2) $\rho(x, y) = d\omega(x, y) = 0$ *for* $x \in \sum_{i<j} \mathfrak{r}_{ij} \oplus j \sum_{i<j} \mathfrak{r}_{ij}$, $y \in \mathfrak{g}^{\#}$.

PROOF. (1) If $x, y\in\sum_{i<j}\mathfrak{r}_{ij}$, then $[x, y]=0$, whence $d\omega(x, y)=0$. On the other hand by Lemma 4.2, $\rho(x, y)=0$. If $x, y\in\sum_{i<j}j\mathfrak{r}_{ij}$, then $[x, y]\in\sum_{i<j}j\mathfrak{r}_{ij}$ and hence $d\omega(x, y)=0$. We also have $\rho(x, y)=\rho(jx, jy)=0$. Finally, if $x\in\sum_{i<j}\mathfrak{r}_{ij}$ and $y\in j\sum_{i<j}\mathfrak{r}_{ij}$, then $[x, y]\in\mathfrak{r}$. Therefore $d\omega(x, y)=\rho(je, [x, y])$ by (4.1). Moreover using Lemma 4.6, $\rho(je, [x, y])=\rho([je, x], y)+\rho(x, [je, y])=\rho(x, y)+\rho(Ix, y)+\rho(x, Iy)=\rho(x, y)$.

(2) In this case $[x, y]\in\mathfrak{n}$, whence $d\omega(x, y)=-\omega([x, y])=0$. On the other hand $\rho(x, y)=0$ by Lemma 4.3. q. e. d.

By virture of Lemmas 4.7 and 4.8, for the proof of Theorem 4.1, it is enough to show $\rho(x, y)=d\omega(x, y)$ for $x, y\in\mathfrak{g}^{\#}$. Recall that $\mathfrak{g}^{\#}=\mathfrak{r}^{\#}\oplus\mathfrak{c}^{\#}\oplus\mathfrak{h}^{\#}$. In the case $x, y\in\mathfrak{r}^{\#}$, we already know $\rho(x, y)=d\omega(x, y)=0$. Assume that $x\in\mathfrak{r}^{\#}$ and $y\in\mathfrak{c}^{\#}\oplus\mathfrak{h}^{\#}$. Then $d\omega(x, y)=\rho(je, [x, y])=\rho([je, x], y)=\rho(x, y)$. Here we use the fact that $je\in\mathfrak{c}^{\#}$ and $\mathrm{ad}\,je=1$ on $\mathfrak{r}^{\#}$.

It remains to show $\rho(x, y)=d\omega(x, y)$ for $x, y\in\mathfrak{c}^{\#}\oplus\mathfrak{h}^{\#}$. By Lemma 4.5 and the definition of ω on $\mathfrak{h}^{\#}$, it is enough to consider the case $x\in\mathfrak{c}^{\#}$ and $y\in\mathfrak{h}^{\#}$. But in this case $\omega(x, y)=-\omega([x, y])=0$ and $\rho(x, y)\in\rho(x, [\mathfrak{h}^{\#}, \mathfrak{h}^{\#}])=\rho([x, \mathfrak{h}^{\#}], \mathfrak{h}^{\#})=0$. This completes the proof of Theorem 4.1.

§5. The invariance of $\mathfrak{g}^{\#}$.

By *an automorphism* of a j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ we mean an automorphism f of the Lie algebra $\mathfrak{g}$ satisfying the following conditions:

$$f\mathfrak{k}=\mathfrak{k}, \qquad fjx\equiv jfx \pmod{\mathfrak{k}} \quad \text{for } x\in\mathfrak{g}.$$

We will show that the subalgebra $\mathfrak{g}^{\#}$ constructed in §2 is invariant under all automorphisms of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$.

We first show the following

PROPOSITION 5.1. *Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra. Then there exists an admissible form ω such that $\omega(fx)=\omega(x)$ for all automorphism f and $x\in\mathfrak{g}$.*

PROOF. Let G be the simply connected Lie group with $\mathfrak{g}$ as its Lie algebra and K the connected subgroup of G corresponding to the subalgebra $\mathfrak{k}$. Then K is closed and the homogeneous space G/K, endowed with a natural G-invariant complex structure corresponding to j, is biholomorphic to the product of a homogeneous bounded domain M_1 and a compact simply connected homogeneous complex manifold M_2 ([**6**, Theorem A]). Then every holomorphic transformation Ψ of G/K induces a holomorphic transformation ϕ of M_1 such that $\pi\circ\Psi=\phi\circ\pi$, where π denotes the projection: $G/K\to M_1$. In particular, G acts transitively on M_1. Let U denote the isotropy subgroup of G at the point $\pi(o)$, where o is the origin of the homogeneous space G/K. Every automorphism f of the j-

algebra $\mathfrak{g}$ induces an automorphism of the group G, which will be denoted by the same letter f. We want to show that f leaves U invariant. Since $fK=K$, f induces a holomorphic transformation of G/K in a natural manner, whence it also induces a holomorphic transformation $\hat{f}$ of G/U. Clearly, $\hat{f}$ fixes the origin $\pi(o)$ of G/U. Since $U/K=\pi^{-1}\pi(o)$, the above fact implies $fU=U$, proving our assertion.

Let us denote by $\mathfrak{u}$ the Lie algebra of the group U. Clearly $\mathfrak{u}$ is a j-subalgebra containing $\mathfrak{k}$. Moreover $[u, jx]\equiv j[u, x]$ (mod $\mathfrak{u}$) holds for all $u\in\mathfrak{u}$ and $x\in\mathfrak{g}$. Therefore for any $x\in\mathfrak{g}$, $\mathrm{ad}\,jx-j\circ\mathrm{ad}\,x$ leaves $\mathfrak{u}$ invariant. We now put for x in $\mathfrak{g}$,

$$\omega_1(x)=\mathrm{Trace}(\mathrm{ad}\,jx-j\circ\mathrm{ad}\,x)|_{\mathfrak{g}/\mathfrak{u}}\,,\qquad \omega_2(x)=\mathrm{Trace}(\mathrm{ad}\,jx-j\circ\mathrm{ad}\,x)|_{\mathfrak{u}/\mathfrak{k}}\,.$$

It is easy to see that $\omega_1(fx)=\omega_1(x)$ and $\omega_2(fx)=\omega_2(x)$ for all automorphism f. Note that ω_1 is the Koszul form of the j-algebra $(\mathfrak{g}, \mathfrak{u}, j)$ corresponding to the homogeneous bounded domain M_1 and the restriction of ω_2 on $\mathfrak{u}$ is the Koszul form of the j-algebra $(\mathfrak{u}, \mathfrak{k}, j)$ corresponding the compact simply connected homogeneous space M_2. Therefore as is proved in [6, §7], $\omega=\omega_1-a\omega_2$ becomes an admissible form for large enough positive number a. Then the form ω has the desired properties. q.e.d.

Let ω be an admissible form as in Proposition 5.1. Then $(\mathfrak{g}, \mathfrak{k}, j, -d\omega)$ is a Kähler algebra and the homogeneous space G/K admits a G-invariant Kähler structure with the Kähler form corresponding to $-d\omega$. Then every automorphism of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ acts on G/K as a holomorphic isometry and it fixes the origin of G/K. Therefore we have

COROLLARY 5.2. *The group of all automorphisms of an effective j-algebra is compact.*

Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra and $\mathfrak{g}^{\#}$ the subalgebra as before. Let ω be an admissible form. Then by Proposition 4.4, we have $\mathfrak{g}^{\#}=\{x\in\mathfrak{g}\,;\,d\omega(x, \mathfrak{n})=0\}$, where $\mathfrak{n}$ is the subspace given by (2.8). Assume further that ω satisfies the properties in Proposition 5.1. Then for any automorphism f, we have $d\omega(f\mathfrak{g}^{\#}, \mathfrak{n})=-\omega([f\mathfrak{g}^{\#}, \mathfrak{n}])=-\omega([\mathfrak{g}^{\#}, f^{-1}\mathfrak{n}])=d\omega([\mathfrak{g}^{\#}, \mathfrak{n}])=0$. Here we use the fact that $\mathfrak{n}$ is invariant under f^{-1}. Therefore we know $f\mathfrak{g}^{\#}=\mathfrak{g}^{\#}$, proving that $\mathfrak{g}^{\#}$ is invariant under all automorphisms of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$. Noting that $\mathfrak{r}^{\#}$ coincides with $\mathrm{nil}(\mathfrak{g}^{\#})$ (cf. (2.7)), we have from Propositions 2.11 and 3.4 the following

THEOREM 5.3. *Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra and let ω be an admissible form. We set $\mathfrak{g}^{\#}=\{x\in\mathfrak{g}\,;\,d\omega(x, \mathfrak{n})=0\}$, where $\mathfrak{n}=\mathrm{nil}(\mathfrak{g})\cap(j\,\mathrm{nil}(\mathfrak{g})+\mathfrak{k})$. Then $\mathfrak{g}^{\#}$ is a j-invariant subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$ and the following hold:*

(1) *$\mathfrak{g}^{\#}$ is independent to the choice of ω and invariant under all automor-*

phisms of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$.

(2) $\mathrm{nil}(\mathfrak{g}^{\#})$ *is abelian.*

(3) $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{g}^{\#}$, $[\mathfrak{n}, \mathfrak{g}^{\#}]\subset\mathfrak{n}$ *and* $[\mathfrak{n}, \mathfrak{n}]\subset\mathfrak{n}\oplus\mathrm{nil}(\mathfrak{g}^{\#})$.

Moreover after a suitable change of j, $\mathfrak{g}^{\#}$ *is decomposed as* $\mathfrak{g}^{\#}=\mathrm{nil}(\mathfrak{g}^{\#})\oplus j\,\mathrm{nil}(\mathfrak{g}^{\#})\oplus\hat{\mathfrak{s}}^{\#}\oplus\mathfrak{s}_0^{\#}$ *in the following way:*

(4) *Both* $\mathrm{nil}(\mathfrak{g}^{\#})\oplus j\,\mathrm{nil}(\mathfrak{g}^{\#})\oplus\hat{\mathfrak{s}}^{\#}$ *and* $\mathfrak{s}_0^{\#}$ *are ideals of* $\mathfrak{g}^{\#}$.

(5) $\mathfrak{s}_0^{\#}$ *is a reductive j-subalgebra.*

(6) $j\,\mathrm{nil}(\mathfrak{g}^{\#})\oplus\hat{\mathfrak{s}}^{\#}$ *is isomorphic to Lie* $\mathrm{Aut}(\Omega^{\#})$, *where* $\Omega^{\#}$ *is a self dual homogeneous convex cone in* $\mathrm{nil}(\mathfrak{g}^{\#})$ *and* $\hat{\mathfrak{s}}^{\#}$ *is a maximal compact subalgebra of* $j\,\mathrm{nil}(\mathfrak{g}^{\#})\oplus\hat{\mathfrak{s}}^{\#}$.

(7) $\mathfrak{k}=\mathfrak{k}\cap\hat{\mathfrak{s}}^{\#}\oplus\mathfrak{k}\cap\mathfrak{s}_0^{\#}$.

We also have the following fact which is mentioned in [9] without proof under an additional assumption.

THEOREM 5.4. *A maximal abelian ideal of the first kind of an effective j-algebra is unique.*

PROOF. Let $\mathfrak{r}$ and $\mathfrak{r}'$ be two maximal abelian ideal of the first kind of an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$. Denote by e and e' the principal idempotents of $\mathfrak{r}$ and $\mathfrak{r}'$ respectively. By Proposition 2.1, it is enough to show that $e=e'$. Let $\mathfrak{g}^{\#}$ be as in Theorem 5.3. Then both e and e' are contained in $\mathrm{nil}(\mathfrak{g}^{\#})$. Let ω be an admissible form. Then $\omega([jx, y])$ for $x, y\in\mathrm{nil}(\mathfrak{g}^{\#})$ is a positive definite symmetric bilinear form on $\mathrm{nil}(\mathfrak{g}^{\#})$. Note that $\mathrm{nil}(\mathfrak{g}^{\#})\subset\mathfrak{r}\cap\mathfrak{r}'$. Then using (2.1), we have for all $x\in\mathrm{nil}(\mathfrak{g}^{\#})$, $\omega([jx, e-e'])=\omega(x)-\omega(x)=0$. Therefore we get $e=e'$.

q. e. d.

§6. The canonical hermitian forms of j-algebras.

Let $(\mathfrak{g}, \mathfrak{k}, j)$ be an effective j-algebra. In this and the next sections, we calculate the Koszul form of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ and prove the following

THEOREM 6.1. *The canonical hermitian form of an effective j-algebra is non-degenerate.*

Let $\mathfrak{r}$ be the maximal abelian ideal of the first kind with the principal idempotent e and let $\mathfrak{g}=\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{s}\oplus\mathfrak{w}$ be the decomposition as in Proposition 2.1. Consider the subalgebra $\mathfrak{g}_0\oplus\mathfrak{g}_1$ $(=\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{s})$. Let us put

$$\mathfrak{s}_0=\{x\in\mathfrak{g}_0\,;\,[x, \mathfrak{r}]=0\}. \tag{6.1}$$

It is easy to see that $\mathfrak{s}_0$ is an ideal of $\mathfrak{g}_0\oplus\mathfrak{g}_1$ contained in $\mathfrak{s}$. The following lemma can be proved by the similar way as Lemma 3.1.

LEMMA 6.2. *After an inessential change of j if necessary, there exists an* *ideal* $\hat{\mathfrak{s}}$ *of* $\mathfrak{s}$ *satisfying the following conditions:*

(1) $\mathfrak{g}_0\oplus\mathfrak{g}_1=\mathfrak{r}\oplus j\mathfrak{r}\oplus\hat{\mathfrak{s}}\oplus\mathfrak{s}_0$.

(2) $\mathfrak{s}=\hat{\mathfrak{s}}\oplus\mathfrak{s}_0, \qquad \mathfrak{k}=(\mathfrak{k}\cap\hat{\mathfrak{s}})\oplus(\mathfrak{k}\cap\mathfrak{s}_0)$.

(3) $\mathfrak{r}\oplus j\mathfrak{r}\oplus\hat{\mathfrak{s}}$ *is a j-invariant ideal of* $\mathfrak{g}_0\oplus\mathfrak{s}_1$.

We put

$$\hat{\mathfrak{g}}=\mathfrak{r}\oplus j\mathfrak{r}\oplus\hat{\mathfrak{s}}\oplus\mathfrak{w}\,.$$

Clearly $\hat{\mathfrak{g}}$ is a j-ideal of $\mathfrak{g}$ and

$$\mathfrak{g}=\hat{\mathfrak{g}}\oplus\mathfrak{s}_0\,.$$

We can assume that $\hat{\mathfrak{g}}$ is j-invariant. Let ψ denote the Koszul form of $(\mathfrak{g}, \mathfrak{k}, j)$. Let $\mathfrak{n}$ be the subspace given by (2.8). Since $\mathfrak{w}\subset\mathfrak{n}$ and $\mathfrak{w}=[je, \mathfrak{w}]$ holds, we have $\mathfrak{w}\subset[\mathfrak{n}, \mathfrak{g}^{\#}]$. Therefore applying Proposition 4.4 to the skew-symmetric bilinear form $d\psi$, we have

LEMMA 6.3. $\qquad \psi(\mathfrak{w})=0\,.$

Since $[\hat{\mathfrak{g}}, \mathfrak{s}_0]\subset\mathfrak{w}$ holds, as an immediate consequence of Lemma 6.3 we get

COROLLARY 6.4. $\qquad \psi([\hat{\mathfrak{g}}, \mathfrak{s}_0])=0\,.$

We now consider the adjoint representation of $\mathfrak{s}$ on $\mathfrak{w}$. We have chosen j so that $j\mathfrak{w}=\mathfrak{w}$. Let ψ' denote the Koszul form of the j-algebra $(\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{k}, \mathfrak{k}, j)$. By [**6**, Lemma 10], the vector space $\mathfrak{w}$, equipped with the complex structure j and the skew-symmetric bilinear form $\psi'([w, w'])$ $(w, w'\in\mathfrak{w})$, is a symplectic space in the sense of [**9**] and $\mathrm{ad}\,s|_{\mathfrak{w}}$ is a symplectic endomorphism for all $s\in\mathfrak{s}$. Furthermore for each $s\in\mathfrak{s}$, the equation $\mathrm{ad}\,js|_{\mathfrak{w}}\circ j-j\circ\mathrm{ad}\,js|_{\mathfrak{w}}-\mathrm{ad}\,s|_{\mathfrak{w}}-j\circ\mathrm{ad}\,x|_{\mathfrak{w}}\circ j=0$ holds. Therefore by [**7**, Lemma 1.1]

(6.2) $\qquad \mathrm{Trace}\,j\circ\mathrm{ad}[js, s]|_{\mathfrak{w}}\leqq 0$ for all $s\in\mathfrak{s}$ and the equality holds if and only if both $\mathrm{ad}\,js|_{\mathfrak{w}}$ and $\mathrm{ad}\,s|_{\mathfrak{w}}$ commute with j.

LEMMA 6.5. *The restriction of the canonical hermitian form of the j-algebra* $(\mathfrak{g}, \mathfrak{k}, j)$ *to the subspace* $\mathfrak{s}_0/\mathfrak{k}\cap\mathfrak{s}_0\subset\mathfrak{g}/\mathfrak{k}$ *is non-degenerate.*

PROOF. Let ψ_0 denote the Koszul form of the j-algebra $(\mathfrak{s}_0, \mathfrak{k}\cap\mathfrak{s}_0, j)$. We then have

$$\psi(x)=\mathrm{Trace}(\mathrm{ad}\,jx-j\circ\mathrm{ad}\,x)|_{\mathfrak{w}}+\psi_0(x) \qquad \text{for } x\in\mathfrak{s}_0\,. \tag{6.3}$$

Consider a Cartan decomposition of the reductive j-algebra $\mathfrak{s}_0=\mathfrak{u}\oplus\mathfrak{m}$, where $\mathfrak{u}$ denotes the sum of the center of $\mathfrak{s}_0$ and a maximal compact subalgebra of $[\mathfrak{s}_0, \mathfrak{s}_0]$ (=the semi-simple part of $\mathfrak{s}_0$) and $\mathfrak{m}$ denotes the orthogonal complement of $\mathfrak{u}$ in $[\mathfrak{s}_0, \mathfrak{s}_0]$ with respect to the killing form of $\mathfrak{s}_0$. Here we can assume that $\mathfrak{u}$ contains $\mathfrak{k}\cap\mathfrak{s}_0$. By [**4**], we can adjust j so that both $\mathfrak{u}$ and $\mathfrak{m}$ are invariant under j. We then have from [**4**] that $\psi_0([jm, m])>0$ for every non-zero element m in $\mathfrak{m}$ and that $\psi([ju, u])<0$ for every element u of $\mathfrak{u}$ which is not contained in $\mathfrak{u}\cap\mathfrak{k}$. Since $\mathrm{ad}\,x|_{\mathfrak{w}}$ is a symplectic endomorphism, we know $\mathrm{Trace}\,\mathrm{ad}\,x|_{\mathfrak{w}}=0$ for all $x\in\mathfrak{s}_0$. Moreover since the semi-simple part of $\mathfrak{u}$ is compact, we know from [**3**, Lemma 1.6] that $\mathrm{ad}\,u|_{\mathfrak{w}}$ commutes with j for all $u\in\mathfrak{u}$. Therefore from (6.2) and (6.3), we have $\psi([jm, m])>0$ for every non-zero element m in $\mathfrak{m}$ and $\psi([ju, u])<0$ for every element $u\in\mathfrak{u}$ such that $u\notin\mathfrak{k}\cap\mathfrak{u}$, proving the lemma. q. e. d.

Let $\hat{\psi}$ denote the Koszul form of the j-algebra $(\hat{\mathfrak{g}}, \mathfrak{k}\cap\hat{\mathfrak{g}}, j)$. From the fact that $\hat{\mathfrak{g}}$ is a j-ideal of $\mathfrak{g}$, it follows that $\psi(x)=\hat{\psi}(x)$ holds for all $x\in\hat{\mathfrak{g}}$. Therefore the restriction of the canonical hermitian form of the j-algebra $(\mathfrak{g}, \mathfrak{k}, j)$ to the subspace $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}\cap\mathfrak{k}$ coincides with the canonical hermitian form of the j-algebra $(\hat{\mathfrak{g}}, \mathfrak{k}\cap\hat{\mathfrak{g}}, j)$. Hence from Corollary 6.4 and from Lemma 6.5 we obtain

PROPOSITION 6.6. *Assume that the canonical hermitian form of the j-algebra $(\hat{\mathfrak{g}}, \mathfrak{k}\cap\hat{\mathfrak{g}}, j)$ is non-degenerate. Then the canonical hermitian form of $(\mathfrak{g}, \mathfrak{k}, j)$ is also non-degenerate.*

§7. Proof of Theorem 6.1.

We continue the arguments of the previous section. By Proposition 6.6, we only have to prove Theorem 6.1 for the special case where $\mathfrak{s}_0=0$. Therefore in this section we assume that the adjoint representation of $\mathfrak{s}$ on $\mathfrak{r}$ is faithful. But then $\mathfrak{s}$ is regarded as the isotropy subalgebra of the Lie algebra $j\mathfrak{r}\oplus\mathfrak{s}$ which generate a linear group acting on the cone Ω transitively and effectively. In particular the semi-simple part of $\mathfrak{s}$ is compact. Therefore by the same reason as in the previous section, we have

$$\mathrm{ad}\,s|_{\mathfrak{w}}\circ j=j\circ\mathrm{ad}\,s|_{\mathfrak{w}} \qquad \text{for all } s\in\mathfrak{s}\,. \tag{7.1}$$

It is easy to see that $[s, jx]\equiv j[s, x]$ (mod $\mathfrak{s}$) holds for all $s\in\mathfrak{s}$ and $x\in\mathfrak{r}$. From this and from (7.1), we can see that the system $(\mathfrak{g}, \mathfrak{s}, j)$ satisfies (1.1), (1.2) and (1.3). Clearly $\mathrm{Trace}\,\mathrm{ad}\,s|_{\mathfrak{g}/\mathfrak{s}}=0$ holds for all $s\in\mathfrak{s}$. Therefore we can consider the Koszul form $\check{\psi}$ of the system $(\mathfrak{g}, \mathfrak{s}, j)$. (We can prove that the system $(\mathfrak{g}, \mathfrak{s}, j)$ is a j-algebra corresponding to the homogeneous Siegel domain of the second kind. But this fact is not needed.) Let us denote by $\psi_{\mathfrak{s}}$ the Koszul form of the j-algebra $(\mathfrak{s}, \mathfrak{k}, j)$. We then have for $s, s'\in\mathfrak{s}$, $\psi([s, s'])=\check{\psi}([s, s'])+\psi_{\mathfrak{s}}([s, s'])=\psi_{\mathfrak{s}}([s, s'])$. Therefore the restriction of the canonical hermitian form of $(\mathfrak{g}, \mathfrak{k}, j)$ to the subspace $\mathfrak{s}/\mathfrak{k}$ coincides with the canonical hermitian form of $(\mathfrak{s}, \mathfrak{k}, j)$ which is negative definite because the semi-simple part of $\mathfrak{s}$ is compact. Therefore for the proof of Theorem 6.1, it is enough to show the following

PROPOSITION 7.1. $\psi([jx, x])>0$ *for all non-zero element* $x\in\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{w}$.

In order to show the above proposition, we use another root system decomposition due to [9].

LEMMA 7.2 ([9]). *There exists* $r_\alpha \in \mathfrak{r}$ ($\alpha=1, \cdots, q$) *and a decomposition* $\mathfrak{r}=\sum_{\alpha\leqq\beta}\mathfrak{r}_{\alpha\beta}$ *satisfying the following:*

(1) $\mathfrak{r}_{\alpha\alpha}=\boldsymbol{R}r_\alpha$.

(2) $[jr_\alpha, jr_\beta]=0, \quad [jr_\alpha, r_\beta]=\delta_{\alpha\beta}r_\beta \quad$ *and* $\quad e=\sum r_\alpha$.

(3) $\mathfrak{r}_{\alpha\beta}$ *and* $j\mathfrak{r}_{\alpha\beta}$ *are invariant under* $\mathrm{ad}\, jr_\gamma$ *and* $Re(\mathrm{ad} jr_\gamma)=(\delta_{\alpha\gamma}+\delta_{\beta\gamma})/2$ *on* $\mathfrak{r}_{\alpha\beta}$ *and* $Re(\mathrm{ad} jr_\gamma)=(\delta_{\alpha\gamma}-\delta_{\beta\gamma})/2$ *on* $j\mathfrak{r}_{\alpha\beta}$.

We remark that this lemma can be obtained also by applying the results in §2 to the effective j-algebra ($\mathfrak{r}\oplus j\mathfrak{r}$, 0, j).

By (2) of the above lemma, the Lie algebra $\mathfrak{g}$ is decomposed into the sum of root spaces as $\mathfrak{g}=\sum\mathfrak{g}^\Gamma$ relative to the abelian space of endomorphisms generated by $\{Re(\mathrm{ad} jr_\alpha);\ \alpha=1, \cdots, q\}$. Since $\mathfrak{w}$ and $j\mathfrak{r}\oplus\mathfrak{s}$ are invariant under $\mathrm{ad} j\mathfrak{r}$, we also have the decompositions $\mathfrak{w}=\sum\mathfrak{w}^\Gamma$ and $j\mathfrak{r}\oplus\mathfrak{s}=\sum(j\mathfrak{r}\oplus\mathfrak{s})^\Gamma$. Let us denote by Δ_α the root defined by

$$\Delta_\alpha(Re(\mathrm{ad} jr_\beta))=\delta_{\alpha\beta}\,.$$

Then we know from [9]

$$\mathfrak{w}=\sum_{\alpha=1}^{m}\mathfrak{w}^{\Delta_\alpha/2}\,, \qquad j\mathfrak{w}^{\Delta_\alpha/2}=\mathfrak{w}^{\Delta_\alpha/2}\,, \tag{7.2}$$

$$j\mathfrak{r}\oplus\mathfrak{s}=\sum_{\alpha,\beta}(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}\,, \tag{7.3}$$

$$(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}=j\mathfrak{r}_{\alpha\beta}\oplus\mathfrak{s}\cap\mathfrak{g}^{(\Delta_\alpha-\Delta_\beta)/2} \qquad \text{for } \alpha\leqq\beta\,. \tag{7.4}$$

Therefore we know that

$$\text{if } \mathfrak{g}^\Gamma\neq 0, \text{ then } \Gamma\in\left\{\frac{1}{2}\Delta_\alpha,\ \frac{1}{2}(\Delta_\alpha\pm\Delta_\beta);\ \alpha, \beta=1, \cdots, q\right\}. \tag{7.5}$$

We want to improve (7.4). Let $x\in\mathfrak{s}\cap\mathfrak{g}^\Gamma$. Then $\mathrm{ad}\, x$ is a nilpotent endomorphism if $\Gamma\neq 0$. On the other hand we already know that $\mathrm{ad}\, x|_\mathfrak{r}$ is a semi-simple endomorphism with imaginary eigenvalues. Therefore $\mathrm{ad}\, x|_\mathfrak{r}=0$. This implies that $x=0$, because the representation of $\mathfrak{s}$ on $\mathfrak{r}$ is faithful. Therefore by (7.4)

$$(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}=j\mathfrak{r}_{\alpha\beta} \qquad \text{for } \alpha<\beta\,. \tag{7.6}$$

We remark also that

$$\dim(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}\leqq\dim\mathfrak{r}_{\beta\alpha} \qquad \text{for } \alpha>\beta\,. \tag{7.7}$$

In fact, let $x\in(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}$ for $\alpha>\beta$. Then $[x, e]\in\mathfrak{r}_{\beta\alpha}$. If $[x, e]=0$, then $x\in\mathfrak{s}$, whence $x=0$ follows from the fact $\mathfrak{s}\cap\mathfrak{g}^\Gamma=0$ for $\Gamma\neq 0$. This implies (7.7).

Consider the subalgebra $\mathfrak{r}\oplus j\mathfrak{r}$. It is easy to see that

$$\mathrm{nil}(\mathfrak{r}\oplus j\mathfrak{r})\cap j\,\mathrm{nil}(\mathfrak{r}\oplus j\mathfrak{r})=\sum_{\alpha<\beta}(\mathfrak{r}_{\alpha\beta}\oplus j\mathfrak{r}_{\alpha\beta})\,.$$

Then applying Proposition 4.4 to the j-algebra $\mathfrak{r}\oplus j\mathfrak{r}$ and the skew-symmetric bilinear form $d\psi|_{\mathfrak{r}\oplus j\mathfrak{r}}$, we have

LEMMA 7.3. $\qquad \psi(\mathfrak{r}_{\alpha\beta}\oplus j\mathfrak{r}_{\alpha\beta})=0 \qquad$ *for* $\alpha<\beta$.

Next we prove

LEMMA 7.4. $\qquad \psi(r_\gamma)>0 \qquad$ *for all* γ.

PROOF. Since $j\mathfrak{r}\oplus\mathfrak{k}$ is a subalgebra, we also have the decomposition $j\mathfrak{r}\oplus\mathfrak{k}=\sum(j\mathfrak{r}\oplus\mathfrak{k})^\Gamma$. Let us set $f_\gamma=\mathrm{ad}\,jr_\gamma-j\circ\mathrm{ad}\,r_\gamma$. Then

$$\begin{aligned}\psi(r_\gamma)&=\mathrm{Trace}\,f_\gamma|_{\mathfrak{r}\oplus j\mathfrak{r}}+\mathrm{Trace}\,f_\gamma|_{\mathfrak{w}}+\mathrm{Trace}\,f_\gamma|_{j\mathfrak{r}\oplus\mathfrak{s}}-\mathrm{Trace}\,f_\gamma|_{j\mathfrak{r}\oplus\mathfrak{k}}\\&=2\,\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{\mathfrak{r}}+\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{\mathfrak{w}}+\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{j\mathfrak{r}\oplus\mathfrak{s}}-\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{j\mathfrak{r}\oplus\mathfrak{k}}\,.\end{aligned}$$

By simple computations, we have from Lemma 7.2 and (7.2)

$$\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{\mathfrak{r}}=1+\frac{1}{2}\sum_{\alpha<\gamma}\dim\mathfrak{r}_{\alpha\gamma}+\frac{1}{2}\sum_{\gamma<\beta}\dim\mathfrak{r}_{\gamma\beta}$$

$$\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{\mathfrak{w}}=\frac{1}{2}\dim\mathfrak{w}^{\Delta_\gamma/2}$$

and using (7.3) and (7.6) we have

$$\begin{aligned}&\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{j\mathfrak{r}\oplus\mathfrak{s}}-\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{j\mathfrak{r}\oplus\mathfrak{k}}\\&=\sum_{\alpha>\beta}\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\beta)/2}}-\sum_{\alpha>\beta}\mathrm{Trace}\,\mathrm{ad}\,jr_\gamma|_{(j\mathfrak{r}\oplus\mathfrak{k})^{(\Delta_\alpha-\Delta_\beta)/2}}\\&=\frac{1}{2}\sum_{\gamma>\beta}(\dim(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\gamma-\Delta_\beta)/2}-\dim(j\mathfrak{r}\oplus\mathfrak{k})^{(\Delta_\gamma-\Delta_\beta)/2})\\&\quad-\frac{1}{2}\sum_{\gamma<\alpha}(\dim(j\mathfrak{r}\oplus\mathfrak{s})^{(\Delta_\alpha-\Delta_\gamma)/2}-\dim(j\mathfrak{r}\oplus\mathfrak{k})^{(\Delta_\alpha-\Delta_\gamma)/2})\,.\end{aligned}$$

Now the lemma follows from (7.7). q.e.d.

We are now in a position to prove Proposition 7.1. Let $x\in\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{w}$. We decompose as $x=\sum_{\alpha\leq\beta}r_{\alpha\beta}+\sum_{\alpha\leq\beta}jz_{\alpha\beta}+\sum_\alpha w_\alpha$, where $r_{\alpha\beta}$, $z_{\alpha\beta}\in\mathfrak{r}_{\alpha\beta}$ and $w_\alpha\in\mathfrak{w}^{\Delta_\alpha/2}$. We then have

$$[jx,x]\equiv\sum_{\alpha\leq\beta}[jr_{\alpha\beta},r_{\alpha\beta}]+[jz_{\alpha\beta},z_{\alpha\beta}]+\sum_\alpha[jw_\alpha,w_\alpha]\quad(\mathrm{mod}\,\mathfrak{w}\oplus\sum_{\alpha<\beta}(\mathfrak{r}_{\alpha\beta}\oplus j\mathfrak{r}_{\alpha\beta})).$$

Therefore by Lemmas 6.3 and 7.3, it is enough to show $\psi([jr,r])>0$ for every non-zero element $r\in\mathfrak{r}_{\alpha\beta}$ and $\psi([jw,w])>0$ for every non-zero element $w\in\mathfrak{w}^{\Delta_\alpha/2}$. But both $[jr,r]$ and $[jw,w]$ are in $\mathfrak{r}_{\alpha\alpha}$ and hence constant multiples of r_α. Let ψ' denote the Koszul form of the j-algebra $(\mathfrak{r}\oplus j\mathfrak{r}\oplus\mathfrak{w},0,j)$. This j-algebra corresponds to a homogeneous Siegel domain ([**9**]). Therefore $\psi'([jr,r])>0$ and $\psi'([jw,w])>0$ hold. Moreover since $\psi'(r_\alpha)=\psi'([jr_\alpha,r_\alpha])>0$, the above constants must be positive numbers. Therefore by Lemma 7.4, we have $\psi([jr,r])$

>0 and $\phi([jw, w])>0$, completing the proof of Proposition 7.1. This finishes the proof of Theorem 6.1.

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