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# Reduction for Painlevé equations at the fixed singular points of the second kind

Dedicated to Professor Tosihusa Kimura on his 60th birthday

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## § 1. Introduction.

This paper gives a simple reduction theorem for Painlevé equations near the fixed singular points of the second kind in the framework of Hamiltonian mechanics.

It is known that each Painlevé equation $P_J$ ($J=\mathrm{I}, \cdots, \mathrm{VI}$) is equivalent to a Hamiltonian system: $d\lambda/dt=\partial H_J/\partial\mu$, $d\mu/dt=-\partial H_J/\partial\lambda$, where the Hamiltonian function $H_J=H_J(t, \lambda, \mu)$ is a polynomial of $\lambda$ and $\mu$ of which the coefficients are rational functions of $t$ ([**14**]). We call these Hamiltonian systems Painlevé systems. Then the fixed singular points are formally classified as follows: a fixed singular point of a Painlevé equation is *of the first kind* or *of the second kind* if Poincaré rank of the corresponding Painlevé system at the point is zero or positive respectively.

We want to construct a 2-parameter family of solutions of each Painlevé system at each fixed singular point, in other words, to obtain a local biholomorphic transformation which reduces it to a solvable system. As is well known, concerning the construction of an $n$-parameter family of solutions of an $n$-system at a fixed singular point, we have a general theorem by J. Malmquist under the so-called Poincaré's condition ([**12**], [8]). However, we can not apply the theorem to Painlevé systems because Poincaré's condition is completely violated for them.

Recently, having been stimulated by the idea of M. Iwano ([**9**]), several authors have obtained 2-parameter families of solutions of Painlevé systems at the fixed singular points of the second kind ([**16**], [**15**], [**19**], [**20**]). Their works especially those by S. Yoshida explain, from a general point of view, the reason why the formal transformations for Painlevé systems without

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Poincaré's condition are convergent.

On the other hand, for fixed singular points of the first kind, the present author gave a reduction theorem for general Hamiltonian systems containing Painlevé systems which shows the convergence of the formal canonical transformation in an unbounded domain is derived from the boundedness of the Hamiltonian function in a similar domain ([**17**]). In this paper, we give an analogous result in the case where the fixed singular point is of the second kind. We note that each $P_J$ ($J=\mathrm{I}, \cdots, \mathrm{V}$) has a fixed singular point of the second kind at the point at infinity, whereas all fixed singular points of $P_{\mathrm{VI}}$ and the origin of $P_{\mathrm{III}}$ and $P_{\mathrm{V}}$ are of the first kind.

We can verify that each Hamiltonian function $H_J$ ($J=\mathrm{I}, \cdots, \mathrm{V}$) associated with Painlevé equation $P_J$ is reduced to a Hamiltonian in a normal form as

$$H(t, q, p) = \lambda_0 qp + t^{-1} \sum_{i, j \geqq -1} h_{ij}(t) q^{i+1} p^{j+1}$$

at the fixed singular point of the second kind. Here

(i) $\lambda_0$ is a nonzero constant,

(ii) $h_{ij}(t)$'s are holomorphic for $|t|>R$,

(iii) $\sum_{i, j \geqq -1} h_{ij}(t) q^{i+1} p^{j+1}$ converges absolutely and uniformly for

$$|t| > R, \qquad |q|, |t^{-1}p|, |qp| < \rho$$

and represents there a bounded holomorphic function. Note that Poincaré rank of the system is one.

Considering the above fact on Painlevé systems and other applications, we now formulate our problem. Let $\lambda(t)$ be a polynomial of degree $\sigma-1$ as

$$\lambda(t) = \sum_{k=0}^{\sigma-1} \lambda_k t^{\sigma-k-1}, \qquad \lambda_0 \neq 0. \tag{1.1}$$

Set

$$\Lambda(t) = \int_0^t \lambda(t) dt = (\lambda_0/\sigma) t^\sigma + \cdots + \lambda_{\sigma-1} t. \tag{1.2}$$

We say that $(\theta)$ is a *singular direction* of $\Lambda(t)$ if

$$\cos(\sigma\theta + \arg \lambda_0) = 0. \tag{1.3}$$

A half line through $t=0$ with argument $\theta$ satisfying (1.3) is called a *singular line* of $\Lambda(t)$. Remark that $\mathrm{Re}\,\Lambda(t) = [(|\lambda_0|/\sigma)\cos(\sigma\theta + \arg \lambda_0)]|t|^\sigma + O(t^{\sigma-1})$, where $\theta = \arg t$, $O$ denoting Landau's symbol.

Let

$$S(\underline{\theta}, \bar{\theta}, R) := \{t \in \boldsymbol{C} \mid |t|>R, \underline{\theta} < \arg t < \bar{\theta}\},$$

$$D(\underline{\theta}, \bar{\theta}, R, \rho) := \{(t, q, p) \in \boldsymbol{C}^3 \mid t \in S(\underline{\theta}, \bar{\theta}, R), |q|, |t^{-1}p|, |qp| < \rho\}.$$

For a function $f=f(t, q, p)$ holomorphic in $D=D(\underline{\theta}, \bar{\theta}, R, \rho)$, we say that $f$ has singularity *of the first kind at $t=\infty$ with respect to $D$* if $tf$ is bounded in $D$.

Let $f$ be such a function with $|tf|\leqq M$ in $D$, $R\rho\geqq 1$ and let $tf(t, q, p)=\sum_{i,j\geqq-1}f_{ij}(t)q^{i+1}p^{j+1}$ be the Taylor expansion in $q$ and $p$. Then $f_{ij}(t)$'s are holomorphic in $S=S(\underline{\theta}, \bar{\theta}, R)$ and satisfy

$$|f_{ij}(t)|\leqq M|t|^{-(j-i)^{+}}/\rho^{\max(i,j)+1},\qquad t\in S \tag{1.4}$$

where $a^{+}=\max(a, 0)$ for $a\in\boldsymbol{R}$.

In this paper, we study Hamiltonian systems with Poincaré rank $\sigma$ where Hamiltonian functions are of the form

$$H(t, q, p)=\lambda(t)qp+H'(t, q, p). \tag{1.5}$$

Here $\lambda(t)$ is a polynomial as (1.1) and $H'$ is holomorphic in $D=D(\underline{\theta}, \bar{\theta}, R, \rho)$ having singularity of the first kind at $t=\infty$ with respect to $D$. We assume, moreover, that the coefficients $h_{ij}(t)$'s in the expansion

$$tH'(t, q, p)=\sum_{i,j\geqq-1}h_{ij}(t)q^{i+1}p^{j+1} \tag{1.6}$$

admit asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$, $t\in S(\underline{\theta}, \bar{\theta}, R)$. Remark that the Hamiltonian system with Hamiltonian $\lambda(t)qp$ is one of the simplest systems with Poincaré rank $\sigma$. We may say that our Hamiltonian (1.5) is a perturbation of the $\lambda(t)qp$ with a perturbation term having singularity of the first kind.

We suppose the following assumption:

(A) *$S(\underline{\theta}, \bar{\theta}, R)$ contains one and only one singular direction of $\Lambda(t)=\int_0^t\lambda(t)dt$ and neither $\underline{\theta}$ nor $\bar{\theta}$ is a singular direction of $\Lambda(t)$.*

Our main result consists of two parts: the formal one (Theorem 1) and the analytic one (Theorem 2).

THEOREM 1. *Let $H=H(t, q, p)$ be a function stated above expressed as*

$$H(t, q, p)=\lambda(t)qp+t^{-1}\sum_{i,j\geqq-1}h_{ij}(t)q^{i+1}p^{j+1}. \tag{1.7}$$

*If* (A) *is satisfied, then there exists a unique formal canonical transformation of the form*

$$q=\sum_{j\geqq 0}a_{-1,j}(t)P^{j}+Q\sum_{i,j\geqq 0}a_{ij}(t)Q^{i}P^{j},\qquad p=\sum_{i\geqq 0}b_{i,-1}(t)Q^{i}+P\sum_{i,j\geqq 0}b_{ij}(t)Q^{i}P^{j} \tag{1.8}$$

*which changes $(H)$: $dq/dt=\partial H/\partial p$, $dp/dt=-\partial H/\partial q$ to $(H_{\infty})$: $dQ/dt=\partial H_{\infty}/\partial P$, $dP/dt=-\partial H_{\infty}/\partial Q$ with*

$$H_{\infty}=\lambda(t)QP+t^{-1}\sum_{i\geqq 0}h_{ii}(\infty)(QP)^{i+1}. \tag{1.9}$$

*Here, $a_{ij}(t)$'s and $b_{ij}(t)$'s are holomorphic in $S=S(\underline{\theta}, \bar{\theta}, R')$ having asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$, $t\in S$ with*

$$a_{00}(\infty)=b_{00}(\infty)=1 \tag{1.10}$$

$$a_{ij},\ b_{ij}=O(t^{-(j-i)^{+}-\delta_{ij}-(1-\delta_{ij})\sigma}), \tag{1.11}$$

*$\delta_{ij}$ being Kronecker's delta, provided $R'$ ($\geqq R$) is sufficiently large.*

Set

$$\eta=h_{00}(\infty), \tag{1.12}$$

$$h(w)=\sum_{i\geqq 1}(i+1)h_{ii}(\infty)w^{i}, \tag{1.13}$$

then $h(w)$ is holomorphic for $|w|<\rho$. The general solution of $(H_{\infty})$ is given by

$$Q(t)=c_1 t^{\eta+h(c_1c_2)}\exp\Lambda(t), \qquad P(t)=c_2 t^{-\eta-h(c_1c_2)}\exp(-\Lambda(t)), \tag{1.14}$$

$c_1$ and $c_2$ being arbitrary constants.

Concerning the convergence of formal canonical transformation (1.8), we have

THEOREM 2. (i) *$\sum_{j\geqq 0}a_{-1,j}(t)P^{j}$ converges absolutely and uniformly in*

$$D''(\underline{\theta},\bar{\theta},R',\rho'):=\{(t,P)\in \boldsymbol{C}^2 \mid t\in S(\underline{\theta},\bar{\theta},R'),\ |t^{-1}P|<\rho'\}$$

*and represents there a holomorphic function of order $O(t^{-1-\sigma})$.*

(ii) *$\sum_{i\geqq 0}b_{i,-1}(t)Q^{i}$ converges absolutely and uniformly in*

$$D'(\underline{\theta},\bar{\theta},R',\rho'):=\{(t,Q)\in \boldsymbol{C}^2 \mid t\in S(\underline{\theta},\bar{\theta},R'),\ |Q|<\rho'\}$$

*and represents there a holomorphic function of order $O(t^{-\sigma})$.*

(iii) *$\sum_{i,j\geqq 0}c_{ij}(t)Q^{i}P^{j}$ ($c=a$ or $b$) converges absolutely and uniformly in $D(\underline{\theta},\bar{\theta},R',\rho')$ and represents there a holomorphic function of order $O(1)$, provided $R'$ ($\geqq R$) and $1/\rho'$ ($\geqq 1/\rho$) are large.*

By using these theorems, we can obtain all known 2-parameter families of solutions for Painlevé equations ([**16**], [**20**]) and new ones. We can also see that the transformations obtained by S. Yoshida ([**20**]) are modified to canonical transformations. We explain here a simple result for Painlevé system $(H_{\mathrm{V}})$, while the others are shown in the last section. The Hamiltonian $H_{\mathrm{V}}$ is given by

$$H_{\mathrm{V}}=t^{-1}[\lambda(\lambda-1)^2\mu^2-\{\kappa_0(\lambda-1)^2+\theta\lambda(\lambda-1)-\eta t\lambda\}\mu+\kappa(\lambda-1)]$$

where $\lambda$ and $\mu$ are canonical variables, $t$ is an independent variable and the other letters stand for constants ([**14**]). Consider the following successive canonical transformations

$$\lambda=x-t^{-1}f_1(t^{-1}y), \qquad \mu=y,$$

$$x=q/[1+f_2(t^{-1}p)+t^{-1}pf_2'(t^{-1}p)], \qquad y=p[1+f_2(t^{-1}p)]$$

where $f_1(X)$ and $f_2(X)$ are functions holomorphic at $X=0$ defined by $f_1=-\kappa_0/(\eta+X)$ and $f_2'=(1+f_2)^2/\eta$ with $f_2(0)=0$ respectively. Then $H_{\mathrm{V}}$ is changed to $H$ of a normal form (1.7) where $\sigma=1$ and

$$\lambda_0=\eta, \quad h_{00}(\infty)=2\kappa_0+\theta, \quad h_{11}(\infty)=-2, \quad h_{jj}(\infty)=0, \quad j\geqq 2.$$

Let $\lambda=\Phi_1(t, Q, P)$, $\mu=\Phi_2(t, Q, P)$ be the composition of the above transformations and that given by Theorems 1 and 2. Then $\Phi_1$ and $\Phi_2$ admit analogous expansions as the right hand sides of (1.8) satisfying (1.10) and (1.11) except for $a_{-1,j}$, $j\geqq 0$, $a_{0j}$, $b_{0j}$, $j\geqq 1$, while $a_{-1,j}=O(t^{-j-1})$, $j\geqq 0$, $a_{0j}$, $b_{0j}=O(t^{-j})$, $j\geqq 1$. We can verify that the transformation $(\lambda, \mu)=\Phi(t, Q, P)=(\Phi_1(t, Q, P), \Phi_2(t, Q, P))$ is just the transformation constructed by the present author in [**16**] and $(\lambda, \mu)=\Phi(t, V_1, tV_2)$ coincides with the transformation obtained by S. Yoshida ([**20**]). The 2-parameter family of solutions $\Phi(t)=\Phi(t, Q(t), P(t))$ of $(H_V)$ thus obtained behaves as

$$\Phi(t)=((1+o(1))Q(t), (1+o(1))P(t))$$

as $t\to\infty$ along a curve $\gamma(t_0)$ which is tangent to the singular line of $\eta t$. Here $o$ is Landau's small $o$ and $\gamma(t_0)$ is a curve which is explained in Section 5 as path of integration.

In Section 2, we prove Theorem 1. The remaining sections except the last section are devoted to the proof of Theorem 2.

In the proof of Theorems 1 and 2, we take step by step procedure of successive canonical transformations. In order to avoid complicated expressions, at each step of transformation, we denote the old canonical variables and Hamiltonian function by $q$, $p$ and $H$ and the new ones by $Q$, $P$ and $K$. We remark that the canonical transformation $q=-W_p$, $P=-W_Q$ (or $p=W_q$, $P=-W_Q$) generated by $W=W(t, Q, p)$ (or $W(t, q, Q)$) changes $H$ to $K=H+W_t$. The special transformation $q=-t^{-1}p^*$, $p=tq^*$ generated by $W=tqq^*$ has the following remarkable property: if $H(t, q, p)$ has the singularity of the first kind with respect to $D(\underline{\theta}, \bar{\theta}, R, \rho)$, then the new Hamiltonian $H^*(t, q^*, p^*)$ has also the same type singularity.

We denote by $v$ a valuation of the ring of bounded holomorphic functions in $S(\underline{\theta}, \bar{\theta}, R)$ admitting asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$ or of the ring of formal power series of $t^{-1}$ defined by $v(a)=n$ for $a\sim\sum_{k\geqq n}a_k t^{-k}$, $a_n\neq 0$. Since $\underline{\theta}$ and $\bar{\theta}$ are fixed in this paper, we often denote $S(\underline{\theta}, \bar{\theta}, R)$, $D(\underline{\theta}, \bar{\theta}, R, \rho)$, $D'(\underline{\theta}, \bar{\theta}, R, \rho)$ and $D''(\underline{\theta}, \bar{\theta}, R, \rho)$ by $S(R)$, $D(R, \rho)$, $D'(R, \rho)$ and $D''(R, \rho)$ respectively.

## § 2. Proof of Theorem 1.

Let

$$H(t, q, p)=\lambda(t)qp+t^{-1}\sum_{i,j\geqq -1}h_{ij}(t)q^{i+1}p^{j+1} \tag{2.1}$$

be the expansion of $H$ where $h_{ij}(t)$'s are holomorphic in $S=S(R)$ having asymptotic expansions in powers of $t^{-1}$ in $S$ with

(2.2) $$v(h_{ij}) \geqq (j-i)^{+}.$$

**2.1. Elimination of $h_{-1,0}$ and $h_{0,-1}$.** Consider a canonical transformation

(2.3) $$q = Q+a(t), \qquad p = P+b(t)$$

generated by $W=-Qp-w_{0,-1}Q-w_{-1,0}p$ where $a=w_{-1,0}$, $b=-w_{0,-1}$. In order that $k_{-1,0}(t)=k_{0,-1}(t)=0$ for the new Hamiltonian $K=\lambda(t)QP+t^{-1}\sum k_{ij}(t)Q^{i+1}P^{j+1}$, it is necessary and sufficient that the pair $(a, b)$ is a solution of

(2.4) $$da/dt = H_p(t, a, b), \qquad db/dt = -H_q(t, a, b).$$

We see that (2.4) has a solution given by formal power series of $t^{-1}$ with

(2.5) $$v(a) \geqq 1+\sigma, \qquad v(b) \geqq \sigma.$$

Since $S(\underline{\theta}, \bar{\theta}, *)$ is a proper domain of $\pm\Lambda(t)$ with respect to 0 containing singular directions of $\pm\Lambda(t)$, there exists a unique solution of (2.4) bounded holomorphic in $S(R')$ and asymptotically developable to the formal power series solution, provided $R'$ ($\geqq R$) is large. From (2.5), it follows that

(2.6) $$v(k_{ij}) \geqq (j-i)^{+}$$

(2.7) $$k_{ii}(\infty) = h_{ii}(\infty).$$

**2.2. Elimination of $h_{-1,1}$ and $h_{1,-1}$.** We can suppose $h_{-1,0}=h_{0,-1}=0$ in (2.1). Consider a canonical transformation

(2.8) $$q = (1-4w_{-1,1}w_{1,-1})Q+2w_{-1,1}P, \qquad p = -2w_{1,-1}Q+P$$

generated by $W=-Qp-w_{1,-1}Q^2-w_{-1,1}p^2$. In order that $k_{-1,1}=k_{1,-1}=0$, it is necessary and sufficient that $(w_{-1,1}, w_{1,-1})$ is a solution of

(2.9) $$\begin{aligned} dx/dt &= t^{-1}h_{-1,1}+2(\lambda+t^{-1}h_{00})x+4t^{-1}h_{1,-1}x^2 \\ dy/dt &= t^{-1}h_{1,-1}(1-4xy)^2-2(\lambda+t^{-1}h_{00})(1-4xy)y+4t^{-1}h_{-1,1}y^2 \end{aligned}$$

where $x=w_{-1,1}$, $y=w_{1,-1}$. By the same way as in 2.1, we can verify that (2.9) has a unique solution bounded holomorphic in $S(R')$ with

(2.10) $$v(w_{-1,1}) \geqq 2+\sigma, \qquad v(w_{1,-1}) \geqq \sigma$$

admitting asymptotic expansions in powers of $t^{-1}$. If we write (2.8) as

(2.11) $$q = a_{00}(t)Q+a_{-1,1}(t)P, \qquad p = b_{1,-1}(t)Q+b_{00}(t)P$$

then we have

(2.12) $$a_{00}(t) = 1+O(t^{-1}), \qquad b_{00}(t) = 1,$$

(2.13) $$v(a_{-1,1}) \geqq 2+\sigma, \qquad v(b_{1,-1}) \geqq \sigma,$$

which imply the properties (2.6) and (2.7).

**2.3.** Suppose $h_{ij}=0$ for $i+j\leqq 0$, $(i, j)\neq(0, 0)$. Let

(2.14) $$q = w_{00}(t)Q, \qquad p = P/w_{00}(t)$$

be a canonical transformation generated by $W=-w_{00}Qp$. Then the new Hamiltonian $K$ is $[\lambda(t)+t^{-1}h_{00}(t)-(1/w_{00})dw_{00}/dt]QP+t^{-1}\sum_{i+j\geqq 1}h_{ij}(t)w_{00}^{i-j}Q^{i+1}P^{j+1}$. If we take the solution of

$$(2.15) \qquad dw_{00}/dt=t^{-1}[h_{00}(t)-\eta]w_{00}, \qquad w_{00}=1+O(t^{-1}),$$

then the new Hamiltonian satisfies

$$(2.16) \qquad k_{ij}=0, \quad i+j\leqq 0, \ (i, j)\neq(0, 0), \qquad k_{00}(t)=\eta$$

$$(2.17) \qquad v(k_{ij})\geqq(j-i)^{+}, \qquad k_{ii}(\infty)=h_{ii}(\infty), \quad i\geqq 1.$$

**2.4.** Suppose that $h_{ij}=0$, $i+j\leqq 0$, $(i, j)\neq(0, 0)$ and $h_{00}=\eta$. We successively make canonical transformations

$$(2.18)_N \qquad q=Q+\sum_{i+j\geqq N}a_{ij}(t)Q^{i+1}P, \qquad p=P+\sum_{i+j\geqq N}b_{ij}(t)Q^{i}P^{j+1},$$

$N=1, 2, \cdots$, generated by $W=-Qp-\sum_{i+j=N}w_{ij}(t)Q^{i+1}p^{j+1}$. Let $c_{ij}(t)$'s be functions defined by

$$(2.19) \qquad QP+\sum_{i+j\geqq N}c_{ij}Q^{i+1}P^{j+1}=(Q+\sum_{i+j\geqq N}a_{ij}Q^{i+1}P^{j})(P+\sum_{i+j\geqq N}b_{ij}Q^{i}P^{j+1}).$$

We first notice

LEMMA 2.1. (i) *All $a_{ij}$'s, $b_{ij}$'s and $c_{ij}$'s are polynomials of $w_{ij}$'s of which each term is*

$$(2.20) \qquad w_{i_1j_1}\cdots w_{i_nj_n}, \qquad i=\sum_{m=1}^{n}i_m, \quad j=\sum_{m=1}^{n}j_m.$$

(ii) *If $w_{ij}=0$ for all $i, j$ with $i\neq j$, then $c_{ij}=0$ for all $i, j$.*

The main part in the proof of Theorem 1 is the following

PROPOSITION 2.2. *Let $h_{ij}(t)$'s be defined in $S=S(R)$. Then we can uniquely determine $a_{ij}(t)$'s and $b_{ij}(t)$'s in the same sector $S$ admitting asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$ through $S$ with*

$$(2.21) \qquad v(a_{ij}),\ v(b_{ij})\geqq(j-i)^{+}+\delta_{ij}+(1-\delta_{ij})\sigma$$

*so that $(2.18)_N$ changes $H$ to a new Hamiltonian $K=(\lambda(t)+\eta t^{-1})QP+t^{-1}\sum_{i+j\geqq 1}k_{ij}(t)Q^{i+1}P^{j+1}$ where*

(i) *$k_{ij}(t)$'s are bounded holomorphic in $S$ having asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$, $t\in S$ with*

$$(2.22) \qquad v(k_{ij})\geqq(j-i)^{+}$$

(ii) $\qquad k_{ij}(t)=h_{ij}(t), \qquad i+j<N$

(iii) $\qquad k_{ij}(t)=\delta_{ij}h_{ij}(\infty), \qquad i+j=N$

(iv) $\qquad k_{ii}(\infty)=h_{ii}(\infty), \qquad i\geqq 1.$

PROOF. It is easy to see that (ii) holds and that for $i+j=N$

$$k_{ij}(t)=(j-i)[t\lambda(t)+\eta]w_{ij}+h_{ij}(t)-tdw_{ij}/dt.$$

For $i\neq j$, we can uniquely determine bounded holomorphic functions $w_{ij}$ having asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$ in $S$ with

$$v(w_{ij})\geqq(j-i)^{+}+\sigma \tag{2.23}$$

so that $k_{ij}=0$. For $i=j$, there exists a unique function $w_{ii}$ having the same asymptotic properties as $w_{ij}$'s, $i\neq j$ with

$$v(w_{ii})\geqq 1 \tag{2.24}$$

such that $k_{ii}(t)=h_{ii}(\infty)$. We notice that the equations for $w_{ij}$'s are linear. For these functions $w_{ij}$'s, we see that (2.21) holds by Lemma 2.1, (i) and by the following inequalities

$$\sum_{m=1}^{n}[(j_m-i_m)^{+}+\delta_{i_m j_m}]\geqq(j-i)^{+}+\delta_{ij} \tag{2.25}$$

$$\sum_{m=1}^{n}(1-\delta_{i_m j_m})\geqq 1-\delta_{ij} \tag{2.26}$$

where $i=\sum_{m=1}^{n}i_m$, $j=\sum_{m=1}^{n}j_m$.

In order to verify the other properties, we have to investigate the coefficients of $[\lambda(t)+\eta t^{-1}]\sum_{i+j\geqq N}c_{ij}(t)Q^{i+1}P^{j+1}$ where $c_{ij}$'s are defined by (2.19). By Lemma 2.1, (i), (2.25) and (2.26), we have $v(c_{ij})\geqq(j-i)^{+}+\sigma$ for $i\neq j$, therefore

$$v((\lambda(t)+\eta t^{-1})c_{ij})\geqq(j-i)^{+}+1,\qquad i\neq j. \tag{2.27}$$

By Lemma 2.1, (ii), each term in $c_{ii}$ has a factor $w_{i'j'}$, $i'\neq j'$ and hence both factors $w_{i'j'}$ $(j'>i')$ and $w_{i''j''}$ $(j''<i'')$. Therefore we have $v(c_{ii})\geqq 1+\sigma$ and then

$$v((\lambda(t)+\eta t^{-1})c_{ii})\geqq 2. \tag{2.28}$$

By (2.27) and (2.28), we can verify the other properties in the proposition.

We can complete the proof of Theorem 1 by noting

LEMMA 2.3. *If the coefficients of* $(2.18)_M$ *and* $(2.18)_N$ *satisfy the order condition* (2.21), *then those of the composition of the two transformations also satisfy* (2.21).

## § 3. Convergence of simple power series.

**3.1. Preliminary reduction.** We see that our Hamiltonian $H$ is supposed to have an expansion of the form

$$H=qp\{[\lambda(t)+\eta t^{-1}]+t^{-1}\sum_{i+j\geqq 1}h_{ij}(t)q^{i}p^{j}\}. \tag{3.1}$$

By the results in 2.1, 2.2 and 2.3, we can first suppose that $h_{ij}=0$ for $i+j\leqq 0$, $(i,j)\neq(0,0)$ and $h_{00}(t)=\eta$. Next, by the argument as in § 6 of [17], we can verify that the successive canonical transformations of the form

(3.2) $$q=Q, \qquad p=P+\sum_{i\geqq 2} b_{i,-1}(t)Q^i$$

(3.3) $$q=Q+\sum_{j\geqq 2} a_{-1,j}(t)P^j, \qquad p=P$$

change the Hamiltonian to that of the form (3.1). Here, (i) $b_{i,-1}$'s and $a_{-1,j}$'s are bounded holomorphic functions in $S=S(R')$ admitting asymptotic expansions in powers of $t^{-1}$ as $t\to\infty$ in $S$ with $v(b_{i,-1})\geqq\sigma$, $v(a_{-1,j})\geqq j+1+\sigma$ (ii) $\sum_{i\geqq 2}b_{i,-1}(t)Q^i$ and $\sum_{j\geqq 2}a_{-1,j}(t)P^j$ converge in $D'(R',\rho')$ and $D''(R',\rho')$ and represent there bounded holomorphic functions of order $O(t^{-\sigma}Q^2)$ and $O(t^{-3-\sigma}P^2)$ respectively, where $R'$ and $1/\rho'$ are sufficiently large.

Hereafter, we suppose that $H$ is of the form (3.1). Note that the corresponding formal canonical transformation (1.8) which changes the $H$ to $H_\infty$ is of the form

(3.4) $$q=Q\sum_{i+j\geqq 0} a_{ij}Q^iP^j, \qquad p=P\sum_{i+j\geqq 0} b_{ij}Q^iP^j, \qquad a_{00}=b_{00}=1.$$

In order to show Theorem 2, it is sufficient to verify

PROPOSITION 3.1. (i) $\sum_{n\geqq 1}c_{m+n,m}(t)Q^n$ $(c=a$ *or* $b)$, $m\geqq 0$, *converge absolutely and uniformly for* $(t, Q)\in D'(R',\rho')$ *and represent there bounded holomorphic functions* $c'_m(t, Q)$*'s of order* $O(Q)$,

(ii) $\sum_{n\geqq 1}c_{m,m+n}(t)P^n$ $(c=a$ *or* $b)$, $m\geqq 0$, *converge absolutely and uniformly for* $(t, P)\in D''(R',\rho')$ *and represent there bounded holomorphic functions* $c''_m(t, P)$*'s of order* $O(t^{-1}P)$,

(iii) $\sum_{m\geqq 0}[c_{mm}(t)+c'_m(t, Q)+c''_m(t, P)](QP)^m$ $(c=a$ *or* $b)$ *converges absolutely and uniformly for* $(t, Q, P)\in D(R',\rho')$ *and represents there a bounded holomorphic function, provided* $R'$ *and* $1/\rho'$ *are large.*

**3.2. Proof of (i) in Proposition 3.1.** Set

$$A_j(t, Q)=\sum_{i\geqq 0}a_{ij}(t)Q^i, \qquad B_j(t, Q)=\sum_{i\geqq 0}b_{ij}(t)Q^i, \quad j\geqq 0.$$

Let $Q(t)$ be a general solution of $dQ/dt=[\lambda(t)+\eta t^{-1}]Q$, namely, $Q(t)=ct^\eta\exp\Lambda(t)$, $c$ being an arbitrary constant. Then $A_j=A_j(t, Q(t))$, $B_j=B_j(t, Q(t))$, $j\geqq 0$, satisfy

$$dA_0/dt=t^{-1}f(t, Q(t), A_0), \qquad dB_0/dt=-t^{-1}g(t, Q(t), A_0)B_0,$$
$$dA_j/dt=\{j[\lambda(t)+\eta t^{-1}]+t^{-1}g(t, Q(t), A_0)\}A_j+c_j,$$
$$dB_j/dt=\{j[\lambda(t)+\eta t^{-1}]-t^{-1}g(t, Q(t), A_0)\}B_j+d_j, \quad j\geqq 1,$$

where $f(t, x, y)=\sum_{m\geqq 1}h_{m0}(t)x^my^{m+1}$, $g(t, x, y)=\partial f(t, x, y)/\partial y$, $c_j=c_j(t, Q, A_0, B_0, \cdots, A_{j-1}, B_{j-1}, \partial A_0/\partial Q, \partial B_0/\partial Q, \cdots, \partial A_{j-1}/\partial Q, \partial B_{j-1}/\partial Q)$, $d_j=d_j(t, Q, A_0, B_0, \cdots, A_{j-1}, B_{j-1}, A_j, \partial A_0/\partial Q, \partial B_0/\partial Q, \cdots, \partial A_{j-1}/\partial Q, \partial B_{j-1}/\partial Q)$ are polynomials of the variables other than $t$ and $Q$ of which the coefficients are bounded holomorphic functions of $t$ and $Q$ of order $O(t^{-1})$. Therefore, by virtue of a well known theorem ([8]), $\sum_{i\geqq 0}a_{ij}(t)Q^i$, $\sum_{i\geqq 0}b_{ij}(t)Q^i$, $j\geqq 0$, converge in $D'(R',\rho')$ and represent there bounded holomorphic functions, provided $R'$ and $\rho'$ are large.

**3.3. Proof of (ii) in Proposition 3.1.** Let $H^*(t, q^*, p^*)$ be the Hamiltonian obtained from $H$ by the canonical transformation $q=-t^{-1}p^*$, $p=tq^*$ generated by $W=tqq^*$. Then

$$H^* = q^*p^*\{[-\lambda(t)-(1+\eta)t^{-1}]+t^{-1}\sum_{i+j\geqq 1} h^*_{ij}(t)(q^*)^i(p^*)^j\}$$

where $h^*_{ij}(t)=(-1)^{j+1}t^{-(j-i)}h_{ji}(t)$. Let

$$q^* = Q^*\sum_{i+j\geqq 0} a^*_{ij}Q^{*i}P^{*j}, \qquad p^* = P^*\sum_{i+j\geqq 0} b^*_{ij}Q^{*i}P^{*j}$$

with $a^*_{00}(t)=b^*_{00}(t)=1$ be the formal canonical transformation which changes $H^*$ to $H^*_\infty=Q^*P^*\{[-\lambda(t)-(1+\eta)t^{-1}]+t^{-1}\sum_{i\geqq 1}h^*_{ii}(\infty)(Q^*P^*)^i\}$. Since $Q=-t^{-1}P^*$, $P=tQ^*$ changes $H_\infty$ to $H^*_\infty$, we have

$$a_{ij} = (-1)^i t^{-(j-i)}b^*_{ji}, \qquad b_{ij} = (-1)^i t^{-(j-i)}a^*_{ji},$$

by the uniqueness in Theorem 1. The convergence of $\sum_{n\geqq 1}c_{m,m+n}(t)P^n$, $m\geqq 0$, in $D''(R', \rho')$ follows from that of $\sum_{n\geqq 1}c^*_{m+n,m}(t)Q^{*n}$, $m\geqq 0$, in $D'(R', \rho')$ proved in 3.2, $c$ representing $a$ or $b$.

## § 4. Fundamental lemma.

The rest of this paper will be devoted to the proof of (iii) in Proposition 3.1. Let $N$ be an arbitrary positive integer. Put

$$q = Q[a_N(t, Q, P)+\phi], \qquad p = P[b_N(t, Q, P)+\psi] \tag{4.1}$$

where

$$c_N = \sum_{m=0}^{N-1}(QP)^m[c_{mm}(t)+c'_m(t, Q)+c''_m(t, P)] \tag{4.2}$$

$c$ representing $a$ or $b$. Then in order that (4.1) changes $H$ of the form (3.1) to $H_\infty$, it is necessary and sufficient that $(\phi, \psi)=(\phi(t, Q, P), \psi(t, Q, P))$ satisfies

$$D\phi = t^{-1}F_N(t, Q, P, \phi, \psi), \qquad D\psi = t^{-1}G_N(t, Q, P, \phi, \psi) \tag{4.3$_N$}$$

where $D$ denotes an operator $\partial/\partial t+[\lambda(t)+\eta t^{-1}+t^{-1}h(QP)]Q\partial/\partial Q-[\lambda(t)+\eta t^{-1}+t^{-1}h(QP)]P\partial/\partial P$ and

$$\begin{aligned} F_N &= \Big[\sum_{i+j\geqq 1}(j+1)h_{ij}Q^iP^j(a_N+\phi)^i(b_N+\psi)^j-h(QP)\Big](a_N+\phi)-tDa_N, \\ G_N &= -\Big[\sum_{i+j\geqq 1}(i+1)h_{ij}Q^iP^j(a_N+\phi)^i(b_N+\psi)^j-h(QP)\Big](b_N+\psi)-tDb_N. \end{aligned} \tag{4.4}$$

We can verify

PROPOSITION 4.1. *The following inequalities hold*

$$(4.5)\quad |F_N(t, Q, P, \phi, \psi)|,\ |G_N(t, Q, P, \phi, \psi)| \\ \leqq c_N(|t|^{-1}+|Q|+|t^{-1}P|)|QP|^N+M(|QP|+|Q|+|t^{-1}P|)(|\phi|+|\psi|),$$

$$(4.6)\quad |F_N(t, Q, P, \phi_1, \psi_1)-F_N(t, Q, P, \phi_2, \psi_2)|, \\ |G_N(t, Q, P, \phi_1, \psi_1)-G_N(t, Q, P, \phi_2, \psi_2)| \\ \leqq M(|QP|+|Q|+|t^{-1}P|)(|\phi_1-\phi_2|+|\psi_1-\psi_2|)$$

*for* $(t, Q, P)\in D(R_N, \rho_N)$, $|\phi|$, $|\phi_j|$, $|\psi|$, $|\psi_j|<\Delta_N$, $j=1, 2$, *where* $M$ *is a constant independent of* $N$, *while* $c_N$ *is one depending on* $N$, *provided* $R_N$, $1/\rho_N$ *and* $1/\Delta_N$ *are large.*

In the proof of (iii) in Proposition 3.1, the following lemma plays an essential role.

LEMMA 4.2 (Fundamental lemma). *System* $(4.3)_N$ *has a solution* $(\phi(t, Q, P), \psi(t, Q, P))$ *with the properties:*

(i) $\phi(t, Q, P)$ *and* $\psi(t, Q, P)$ *are holomorphic in* $D=D(R_N, \rho_N)$,

(ii) $\phi, \psi=O(|QP|^N(|t|^{-1}+|Q|+|t^{-1}P|))$ *in* $D$, *provided* $R_N$ *and* $1/\rho_N$ *are sufficiently large.*

*Furthermore a solution with these properties is unique.*

## § 5. Path of integration and stable domain.

The fundamental lemma will be proved by solving a system of integral equations which is equivalent to system $(4.3)_N$. In this section, we give a path of integration and a domain $\mathcal{D}$ which is a deformation of $D$ and is usually called *a stable domain.*

For the sake of simplicity, we assume hereafter that

$$(5.1)\qquad \lambda_0 = 1.$$

In case where (5.1) does not hold, we make a scale transformation of $t$ so that the new Hamiltonian satisfies it.

For a set $E$ in the $t$-plane, we denote by $\Lambda(E)$ the set in the $z$-plane defined by $\{z\in \boldsymbol{C} \mid z=\Lambda(t), t\in E\}$ where $\Lambda(t)$ is the function (1.2).

**5.1. Sectorial domain.** We define a sectorial domain $\mathcal{S}=\mathcal{S}(R)=\mathcal{S}(\underline{\theta}, \bar{\theta}, R)$ which is a deformation of $S=S(R)=S(\underline{\theta}, \bar{\theta}, R)$.

By assumption (A), we see that $\Lambda(S(R))$ with $R\gg 1$ contains a half line $\{\arg z=\pi/2, |z|\gg 1\}$ or $\{\arg z=-\pi/2, |z|\gg 1\}$. We consider for example the case where $\Lambda(S)\supset\{\arg z=\pi/2, |z|\gg 1\}$, because the other case can be treated by the same way. In our case, we can choose a small number $\varepsilon>0$ so that

$$(5.2)\qquad \Lambda(S(R))\subset\{z\in \boldsymbol{C} \mid |z|>(1-\varepsilon)R^\sigma/\sigma,\ |\arg z-\pi/2|<\pi-\varepsilon\}$$

for every large $R$.

Let $\underline{l}$ and $\bar{l}$ be half lines in $t$-plane defined by $\underline{l}=\{\arg t=\underline{\theta},\ |t|\gg 1\}$ and $\bar{l}=\{\arg t=\bar{\theta},\ |t|\gg 1\}$ and let

$$\underline{c}=\Lambda(\underline{l}),\qquad \bar{c}=\Lambda(\bar{l}). \tag{5.3}$$

Define a curve $c(R)$ in $z$-plane by

$$\begin{aligned} |z| &= R^{\sigma}/\sigma, && \text{for } |\Theta-\pi/2|\leqq\pi/2-\omega \\ |z| &= (R^{\sigma}/\sigma)|\cos\omega/\cos\Theta|, && \text{for } \pi/2-\omega\leqq|\Theta-\pi/2|\leqq\pi-\varepsilon, \end{aligned} \tag{5.4}$$

$\Theta=\arg z$. Here $\omega$ is a constant with $\pi/4<\omega<\pi/2$ which will be determined later (see (5.15)). The domain in $z$-plane bounded by $\underline{c}$, $\bar{c}$ and $c(R)$ is denoted by $\mathfrak{S}=\mathfrak{S}(R)=\mathfrak{S}(\underline{\theta},\bar{\theta},R)$. Then we define a domain $\mathcal{S}=\mathcal{S}(R)=\mathcal{S}(\underline{\theta},\bar{\theta},R)$ by

$$\Lambda(\mathcal{S}(\underline{\theta},\bar{\theta},R))=\mathfrak{S}(\underline{\theta},\bar{\theta},R). \tag{5.5}$$

We see that $\mathcal{S}(\underline{\theta},\bar{\theta},R)$ is a sectorial domain containing every direction $(\theta)$ with $\underline{\theta}<\theta<\bar{\theta}$.

Divide $\mathfrak{S}=\mathfrak{S}(R)$ into three parts as follows: $\mathfrak{S}_1=\mathfrak{S}\cap\{|\Theta-\pi/2|\leqq\pi/2-\omega\}$, $\mathfrak{S}_2=\mathfrak{S}\cap\{-\pi/2+\varepsilon\leqq\Theta\leqq\omega\}$ and $\mathfrak{S}_3=\mathfrak{S}\cap\{\pi-\omega\leqq\Theta\leqq 3\pi/2-\varepsilon\}$, where $\Theta=\arg z$. Then we define a decomposition $\mathcal{S}=\bigcup_{j=1}^{3}\mathcal{S}_j$ by

$$\Lambda(\mathcal{S}_j)=\mathfrak{S}_j,\qquad 1\leqq j\leqq 3. \tag{5.6}$$

**5.2. Path of integration.** In order to define a curve $\gamma(t_0)$ in $t$-plane joining $\infty$ and $t_0\in\mathcal{S}$, we define $\Gamma(z_0)$ in $z$-plane related with $\gamma(t_0)$ by

$$\Lambda(\gamma(t_0))=\Gamma(z_0),\qquad \Lambda(t_0)=z_0. \tag{5.7}$$

Set

$$\eta_1=-\eta,\qquad \eta_2=1+\eta. \tag{5.8}$$

From $\mathrm{Re}(\eta_1+\eta_2)=1$, it follows that $\mathrm{Re}\,\eta_1>0$ or $\mathrm{Re}\,\eta_2>0$.

**5.2.1. The case** $\mathrm{Re}\,\eta_1>0$. Take constants $\kappa>0$ and $0<\delta\ll 1$ so that

$$\mathrm{Re}\,\eta_1-\delta>\sigma/\kappa>-\mathrm{Re}\,\eta_2+\delta. \tag{5.9}$$

Then we define a curve $\Gamma(z_0)$ which generally consists of two parts $\Gamma_1(z_0)$ and $\Gamma_2(z_0)$.

In the case where $z_0\in\mathfrak{S}_1$. $\Gamma(z_0)$ consists of a part $\Gamma_1(z_0)$ only and the variable point $z=z(\tau)$ on $\Gamma_1(z_0)$ is given by

$$z(\tau)=\tau+x_0+iy_0e^{\kappa\tau},\qquad \tau\geqq 0 \tag{5.10}$$

where $z_0=x_0+iy_0$, $i=\sqrt{-1}$. In the case where $z_0\in\mathfrak{S}_2$ (or $\mathfrak{S}_3$). $\Gamma(z_0)$ consists of two parts $\Gamma_j(z_0)$, $j=1, 2$. The variable point $z=z(\Theta)$ on $\Gamma_2(z_0)$ is given by

$$z(\Theta)=|z_0|e^{i\Theta}\cos\Theta_0/\cos\Theta \tag{5.11}$$

for $\Theta_0\leqq\Theta\leqq\omega$ (or $\pi-\omega\leqq\Theta\leqq\Theta_0$) with $\Theta=\arg z$, $\Theta_0=\arg z_0$ and $\Gamma_1(z_0)$ is the curve joining $\infty$ and $z(\omega)$ (or $z(\pi-\omega)$) defined by (5.10).

We denote by $\gamma_j(t_0)$, $j=1, 2$, the parts of $\gamma(t_0)$ defined by

$$\Lambda(\gamma_j(t_0)) = \Gamma_j(z_0), \qquad j=1, 2. \tag{5.12}$$

**5.2.2. The case** $\mathrm{Re}\,\eta_1 \leqq 0$**.** In this case, $\mathrm{Re}\,\eta_2 > 0$. Let $\kappa > 0$ and $0 < \delta \ll 1$ be constants such that

$$\mathrm{Re}\,\eta_2 - \delta > \sigma/\kappa > -\mathrm{Re}\,\eta_1 + \delta. \tag{5.13}$$

The curve $\Gamma_2(z_0)$ is defined by the same way as 3.2.1, while $\Gamma_1(z_0)$ is defined by

$$z(\tau) = -\tau + x_0 + iy_0 e^{\kappa\tau}, \qquad \tau \geqq 0 \tag{5.14}$$

where $z_0 = x_0 + iy_0$.

**5.2.3.** In this paper, we only explain the case $\mathrm{Re}\,\eta_1 > 0$. Define a constant $\pi/4 < \omega < \pi/2$ by

$$\tan\omega = [1 + (3\kappa+4)\sigma^{-1}\max(|\eta_j| + \delta) + \max \nu_j]/\min \nu_j \tag{5.15}$$

where

$$\nu_j = (-1)^j + (\kappa/\sigma)\mathrm{Re}(\eta_j - \delta) > 0, \qquad j=1, 2, \tag{5.16}$$

then we have

PROPOSITION 5.1. *If* $R \gg 1$, *then* $t_0 \in \mathcal{S} = \mathcal{S}(R)$ *implies* $\gamma(t_0) \subset \mathcal{S}$, *in particular,* $t_0 \in \mathcal{S}_1 = \mathcal{S}_1(R)$ *implies* $\gamma(t_0) \subset \mathcal{S}_1$.

PROOF. It is easy to see that $t_0 \in \mathcal{S}_2$ (or $\mathcal{S}_3$) implies $\gamma_2(t_0) \subset \mathcal{S}_2$ (or $\mathcal{S}_3$). Hence we have only to see the latter assertion. From $\tan\omega > 1$, it follows

$$d|z(\tau)|^2/d\tau > 0, \qquad \tau \geqq 0. \tag{5.17}$$

In order that $|\arg z(\tau) - \pi/2| \leqq \pi/2 - \omega$, it is sufficient $g(\tau) := y_0^2 e^{2\kappa\tau} - (\tau + x_0)^2 \tan^2\omega \geqq 0$, which is verified if $R > 0$ is large such that

$$\kappa \cdot y_0 \geqq \tan\omega, \qquad x_0 + iy_0 \in \mathfrak{S}_1(R). \tag{5.18}$$

**5.3. Stable domain.** Set

$$h_1(w) = -h(w), \qquad h_2(w) = h(w), \tag{5.19}$$

$$T(t) = t\Lambda(t)^{-1/\sigma} = (1/\sigma)^{1/\sigma} + O(t^{-1}), \tag{5.20}$$

where $h(w)$ is the function (1.13), and

$$C_j(t, w) = \begin{cases} 1, & t \in \mathcal{S}_1 \\ |\cos(\arg \Lambda(t))/\cos\omega|^{\mathrm{Re}(\eta_j + h_j(w))/\sigma}, & t \in \mathcal{S}_2 \cup \mathcal{S}_3, \end{cases} \tag{5.21}$$

$$E_j(t, w) = |T(t)|^{-\mathrm{Re}(\eta_j + h_j(w))} \exp[\arg t \cdot \mathrm{Im}(\eta_j + h_j(w))], \tag{5.22}$$

$j=1, 2$. Then we define a domain $\mathcal{D} = \mathcal{D}(R, \rho)$ by

$$\mathcal{D} = \{(t, Q, P) \in \boldsymbol{C}^3 \mid t \in \mathcal{S}(R),\ |Q| < \rho C_1(t, QP) E_1(t, QP), \\ |t^{-1}P| < \rho C_2(t, QP) E_2(t, QP)\}. \tag{5.23}$$

Note that there exists a constant $\alpha>1$ such that

(5.24) $$\mathcal{D}(R\alpha, \rho/\alpha) \subset D(R, \rho) \subset \mathcal{D}(R/\alpha, \rho\alpha)$$

for $R\gg 1$, $0<\rho\ll 1$.

For the functions $Q(t)$ and $P(t)$ given by (1.14), put $X_1(t)=Q(t)$, $X_2(t)=t^{-1}P(t)$, then

(5.25) $$X_j(t) = c_j t^{-(\eta_j+h_j(c_1c_2))}\exp[(-1)^{j-1}\Lambda(t)], \qquad j=1, 2.$$

Let $u_j(t)$ be a principal factor of $X_j(t)$ defined by

(5.26) $$u_j(t) = c_j\Lambda(t)^{-\mathrm{Re}(\eta_j+h_j(c_1c_2))/\sigma}\exp[(-1)^{j-1}\Lambda(t)], \qquad j=1, 2.$$

Then we have the following proposition.

PROPOSITION 5.2. *Let* $t_0\in\mathcal{S}_1=\mathcal{S}_1(R)$, *then*

(5.27) $$d\log|u_j(t(\tau))|/d\tau \leqq -(3/4)\nu_j, \qquad \tau\geqq 0,\ j=1, 2$$

*provided* $R>0$ *is large.*

PROOF. (5.27) is equivalent to

(5.28) $$\begin{aligned} I_j(\tau) := {}& [(-1)^j+\kappa\,\mathrm{Re}(\eta_j+h_j)]y_0^2e^{2\kappa\tau}-[(-1)^{j-1}+3\kappa\,\mathrm{Re}(\eta_j+h_j)](\tau+x_0)^2 \\ & +[4\,\mathrm{Re}(\eta_j+h_j)/\sigma](\tau+x_0) \geqq 0, \qquad j=1, 2 \end{aligned}$$

for every constants $h_j$'s with $|h_j|<\delta$, which follows from

(5.29) $$I_j(0)\geqq 0, \qquad I_j'(0)\geqq 0, \qquad I_j''(\tau)\geqq 0, \quad \text{for } \tau\geqq 0.$$

We can verify (5.29) by the same way as in [**19**] under (5.18) and $y_0\geqq\tan\omega>1$, so we omit its proof.

PROPOSITION 5.3 (Stability of $\mathcal{D}$). *For every point* $(t_0, Q_0, P_0)\in\mathcal{D}=\mathcal{D}(R, \rho)$, *we have* $(t, Q(t), P(t))\in\mathcal{D}$ *for all* $t\in\gamma(t_0)$, *where* $(Q(t), P(t))$ *is the solution of* $(H_\infty)$: $dQ/dt=\partial H_\infty/\partial P$, $dP/dt=-\partial H_\infty/\partial Q$ *with* $Q(t_0)=Q_0$, $P(t)=P_0$.

PROOF. *Let* $(t_0, Q_0, P_0)\in\mathcal{D}$. If $t_0\in\mathcal{S}_2$ (or $\mathcal{S}_3$), then it is readily verified that $(t, Q(t), P(t))\in\mathcal{D}$ for $t\in\gamma_2(t_0)$. In case $t_0\in\mathcal{S}_1$, it follows from Proposition 5.2 that $|u_j(t(\tau))|$ is monotone decreasing in $\tau\geqq 0$, which proves $(t, Q(t), P(t))\in\mathcal{D}$ for $t\in\gamma_1(t_0)$.

## § 6. Proof of the fundamental lemma.

System $(4.3)_N$ is equivalent to a system of integral equations

$(6.1)_N$ $$\begin{aligned} \phi(t_0, Q_0, P_0) &= \int_{\gamma(t_0)} t^{-1}F_N(t, Q(t), P(t), \phi(t, Q(t), P(t)), \psi(t, Q(t), P(t)))dt, \\ \psi(t_0, Q_0, P_0) &= \int_{\gamma(t_0)} t^{-1}G_N(t, Q(t), P(t), \phi(t, Q(t), P(t)), \psi(t, Q(t), P(t)))dt \end{aligned}$$

where $(Q(t), P(t))$ is the solution of $(H_\infty)$ with $Q(t_0)=Q_0$, $P(t_0)=P_0$. Therefore, in order to prove the fundamental lemma, we show that system $(6.1)_N$ has a unique solution $(\phi(t, Q, P), \psi(t, Q, P))$ holomorphic in $\mathcal{D}(R_N/\alpha, \rho_N\alpha)$ satisfying the order condition (ii) in Lemma 4.2, provided $R_N$ and $1/\rho_N$ are large. Here $\alpha>1$ is a constant for which (5.24) holds. Define a family $\mathcal{F}$ as a set of all $(\phi(t, Q, P), \psi(t, Q, P))$ such that $\phi$ and $\psi$ are holomorphic in $\mathcal{D}(R_N/\alpha, \rho_N\alpha)$ satisfying there

$$(6.2)\qquad |\phi(t, Q, P)|,\ |\psi(t, Q, P)| \leqq K_N(|t|^{-1}+|Q|+|t^{-1}P|)|QP|^N.$$

Then we define, for $(\phi, \psi)\in\mathcal{F}$, the functions $\Phi(t_0, Q_0, P_0)$ and $\Psi(t_0, Q_0, P_0)$ by the integrals on the right hand sides of $(6.1)_N$. Note that we must assume

$$(6.3)\qquad K_N\alpha^N(1/R_N+2\rho_N)\rho_N^N < \Delta_N,$$

for the integrals to be defined.

Let us estimate $\Phi$ and $\Psi$. From (4.5) and (6.2), it follows

$$(6.4)\qquad |\Phi(t_0, Q_0, P_0)|,\ |\Psi(t_0, Q_0, P_0)|$$
$$\leqq (c_N+6M\alpha\rho_N K_N)|Q_0P_0|^N\int_{\gamma(t_0)}[|t|^{-2}+|t^{-1}Q(t)|+|t^{-2}P(t)|]|dt|.$$

PROPOSITION 6.1. *We have the following inequalities*

$$(6.5)\qquad \int_{\gamma(t_0)}|t|^{-2}|dt| \leqq L_0|t_0|^{-1},$$

$$(6.6)\qquad \int_{\gamma(t_0)}|t^{-1}Q(t)||dt| \leqq L_1|Q_0|, \qquad \int_{\gamma(t_0)}|t^{-2}P(t)||dt| \leqq L_2|t^{-1}P_0|$$

*where $L_j$, $j=0, 1, 2$, are constants independent of $R\gg 1$ and $0<\rho\ll 1$.*

PROOF. It follows from $z=\Lambda(t)=t^\sigma[1/\sigma+O(t^{-1})]$ that

$$(6.7)\qquad |t|^{-1}|dt| \leqq (1+\delta')\sigma^{-1}|z|^{-1}|dz|,$$

$$(6.8)\qquad (1+\delta')^{-1}\sigma^{-1/\sigma}|z|^{1/\sigma} \leqq |t| \leqq (1+\delta')\sigma^{-1/\sigma}|z|^{1/\sigma},$$

for a constant $0<\delta'\ll 1$. Remark that $\Gamma_1(z_0)$ is defined by (5.10) and $|z|^{-1}|dz| \leqq\beta|d\Theta|$ on $\Gamma_2(z_0)$, with $\beta=1+\tan\varepsilon+\tan\omega$. We see that

$$\int_{\Gamma(z_0)}|z|^{-1-1/\sigma}|dz| \leqq [2\sigma\cdot\tan\omega/(\tan\omega-1)+\beta\pi](\sin\varepsilon)^{-1/\sigma}|z_0|^{-1/\sigma}.$$

Hence by (6.7) and (6.8), inequality (6.5) holds for $L_0=(1+\delta')^3\cdot[2\sigma\cdot\tan\omega/(\tan\omega-1)+\beta\pi]\sigma^{-1}(\sin\varepsilon)^{-1/\sigma}$. Let us next prove (6.6). Set

$$(6.9)\qquad v_j(t) = z^{-i\,\mathrm{Im}(\eta_j+h_j(Q_0P_0))/\sigma}T(t)^{-(\eta_j+h_j(Q_0P_0))}, \qquad j=1, 2$$

with $z=\Lambda(t)$, then we have

$$(6.10)\qquad X_j(t) = u_j(t)v_j(t), \qquad j=1, 2$$

where $X_j$ and $u_j$ are the functions defined by (5.25) and (5.26) respectively. It is easy to see that

$$V_j^{-1} \leqq |v_j(t)| \leqq V_j \tag{6.11}$$

where $V_j$, $j=1, 2$, are constants defined by $V_j=\sigma^{|\eta_j|+\delta}(1+\delta')\exp[(|\eta_j|+\delta)\cdot(3\pi/2\sigma-\varepsilon/\sigma+\delta')]$. If $z_0=\Lambda(t_0)\in\mathfrak{S}_1$, then it follows from Proposition 5.2 that $d|u_j|/|dz|\geqq[(3\nu_j)/(4\sqrt{2}\kappa)]|u_j|/|z|$, which yields

$$\int_{\Gamma_1(z_0)}(|u_j|/|z|)|dz| \leqq [(4\sqrt{2}\kappa)/(3\nu_j)]|u_j(z_0)|\,.$$

In the case where $z_0\in\mathfrak{S}_2$ or $\mathfrak{S}_3$, we have $|u_j(z)|\leqq(\sin\varepsilon)^{-(|\eta_j|+\delta)/\sigma}|u_j(z_0)|$ on $\Gamma_2(z_0)$ and hence

$$\int_{\Gamma_2(z_0)}(|u_j|/|z|)|dz| \leqq \beta\pi(\sin\varepsilon)^{-(|\eta_j|+\delta)/\sigma}|u_j(z_0)|\,.$$

Therefore we obtain

$$\int_{\Gamma(z_0)}(|u_j|/|z|)|dz| \leqq [(4\sqrt{2}\kappa)/(3\nu_j)+\beta\pi](\sin\varepsilon)^{-(|\eta_j|+\delta)/\sigma}|u_j(z_0)|\,. \tag{6.12}$$

By (6.7), (6.11) and (6.12), we see that inequalities (6.6) hold for constants $L_j=V_j^2(1+\delta')\sigma^{-1}[(4\sqrt{2}\kappa)/(3\nu_j)+\beta\pi](\sin\varepsilon)^{-(|\eta_j|+\delta)/\sigma}$, $j=1, 2$.

By virtue of (6.4), (6.5) and (6.6), we obtain

$$\begin{aligned}&|\Phi(t_0, Q_0, P_0)|,\ |\Psi(t_0, Q_0, P_0)|\\ &\leqq (c_N+6M\alpha\rho_N K_N)L(|t_0|^{-1}+|Q_0|+|t_0^{-1}P_0|)|Q_0P_0|^N\end{aligned} \tag{6.13}$$

where

$$L = \max\{L_0, L_1, L_2\}\,. \tag{6.14}$$

Now determine $K_N$ by

$$K_N = c_N/(1/L-6M\alpha\rho_N) \tag{6.15}$$

where $\rho_N>0$ is sufficiently small such that

$$6M\alpha\rho_N \leqq 1/L \tag{6.16}$$

and (6.3) hold. Then, for constants $R_N\gg 1$, $0<\rho_N\ll 1$ and $K_N>0$ chosen above, $\Phi$ and $\Psi$ also satisfy inequalities (6.2), namely $(\Phi, \Psi)\in\mathcal{F}$. It is easy to see that the operator $\mathcal{T}$ defined by $\mathcal{T}(\phi, \psi)=(\Phi, \Psi)$ has a fixed point by the use of Schauder-Tihonov fixed point theorem. The fixed point $(\phi, \psi)\in\mathcal{F}$ is a required solution of system $(6.1)_N$.

The uniqueness of solution with order condition (ii) in Lemma 4.2 is verified by Lipschitz inequality (4.6) and the inequalities obtained in this section.

## §7. Completion of the proof of Theorem 2.

For every positive integer $N$, put

$$\phi^N(t, Q, P) = a_N(t, Q, P)+\phi_N(t, Q, P), \quad \psi^N(t, Q, P) = b_N(t, Q, P)+\psi_N(t, Q, P),$$

where $(\phi_N, \psi_N)$ is the unique solution of system $(4.3)_N$. It can be supposed without loss of generality that $R_N$ and $1/\rho_N$ are monotone increasing in $N$. In order to prove (iii) of Proposition 3.1, it is sufficient to show that $(\phi^N, \psi^N)$ is independent of $N$. Let $N(1)<N(2)$ be arbitrary positive integers. Since $(a_{N(2)}-a_{N(1)}+\phi_{N(2)}, b_{N(2)}-b_{N(1)}+\psi_{N(2)})$ is a solution of $(4.3)_{N(1)}$ with order condition $O(|QP|^{N(1)}(|t|^{-1}+|Q|+|t^{-1}P|))$, it must coincide with $(\phi^{N(1)}, \psi^{N(1)})$ in $\mathcal{D}(R_{N(2)}/\alpha, \rho_{N(2)}\alpha)$, namely $(\phi^{N(1)}, \psi^{N(1)})=(\phi^{N(2)}, \psi^{N(2)})$, which shows that $(\phi^N, \psi^N)$ is independent of $N$.

It is easy to verify that the composition of the canonical transformations (2.3), (2.11), (2.14), (3.2), (3.3) and (3.4) has the convergence property stated in Theorem 2. Thus we have completed the proof of Theorem 2.

## § 8. General solutions of Painlevé systems.

Hamiltonian functions associated with Painlevé equations $P_J$, $J=\mathrm{I}, \cdots, \mathrm{IV}$, are given as follows:

$$H_{\mathrm{I}} = (1/2)\mu^2-2\lambda^3-t\lambda,$$
$$H_{\mathrm{II}} = (1/2)\mu^2-(\lambda^2+t/2)\mu-(\alpha+1/2)\lambda,$$
$$H_{\mathrm{III}} = t^{-1}[2\lambda^2\mu^2-\{2\eta_\infty t\lambda^2+(2\theta_0+1)\lambda-2\eta_0 t\}\mu+\eta_\infty(\theta_0+\theta_\infty)t\lambda],$$
$$H_{\mathrm{IV}} = 2\lambda\mu^2-\{\lambda^2+2t\lambda+2\kappa_0\}\mu+\theta_\infty\lambda$$

where $\lambda$, $\mu$ and $t$ are variables and other letters stand for constants ([**14**]). In this section, we will give canonical transformations which change $H_J$, $J=\mathrm{I}, \cdots, \mathrm{IV}$, to Hamiltonian functions of the normal form (1.7) and then compare our results with those obtained by S. Yoshida ([**20**]). The canonical transformation which reduces $H_J$ to a solvable Hamiltonian of the form (1.9) will be written as $(\lambda, \mu)=\Phi(\cdot, Q, P)$. The functions $f_k(X)$'s will stand for ones holomorphic at $X=0$.

### 8.1. Painlevé system ($H_{\mathrm{I}}$). Set

$$t = s^{4/5}$$

and consider the successive canonical transformations

$$\lambda = s^{2/5}[\kappa^2/12+x-f_2(s^{-1}y)], \qquad \mu = s^{-2/5}[y+sf_1(x-f_2(s^{-1}y))],$$
$$x = z-s^{-1}f_4(s^{-1}w), \qquad y = w+f_3(z-s^{-1}f_4(s^{-1}w)),$$
$$z = u[1+f_5(u)], \qquad w = v/[1+f_5(u)+uf_5'(u)],$$
$$u = q/[1+f_6(s^{-1}p)+s^{-1}pf_6'(s^{-1}p)], \qquad v = p[1+f_6(s^{-1}p)]$$

where $\kappa=(-24)^{1/4}$, $f_1=-\kappa X+O(X^2)$, $f_2=\kappa^3/48+O(X)$, $f_3=\kappa^2/24+O(X)$, $f_4=$

$\kappa/48+O(X)$, $f_5=-(\kappa^2/12)X+O(X^2)$ and $f_6=-(\kappa/12)X+O(X^2)$. Then $H_{\rm I}$ is normalized as (1.7) where $\sigma=1$ and

$$\lambda_0=-4\kappa/5, \quad h_{00}(\infty)=-1/2, \quad h_{11}(\infty)=-1/2, \quad h_{jj}(\infty)=0, \quad j\geqq 2.$$

Hence we have a general solution $\Phi(s)=\Phi(s, Q(s), P(s))$ which behaves as

$$\Phi(s)=(s^{2/5}(\kappa^2/2+o(1)),\ s^{3/5}o(1))$$

as $s\to\infty$ along the singular line of $-(4\kappa/5)s$. $(\lambda, \mu)=\Phi(s, V_1, 2\kappa s V_2)$ coincides with the transformation obtained by S. Yoshida.

**8.2. Painlevé system ($H_{\rm II}$).** Set

$$t=s^{2/3}.$$

**8.2.1.** By the successive canonical transformations

$$\lambda=s^{-2/3}[-\alpha+y+sf_1(x-f_2(s^{-1}y))], \quad \mu=-s^{2/3}[-1/2+x-f_2(s^{-1}y)],$$
$$x=z-s^{-1}f_3(s^{-1}w+s^{-1}f_4(z)), \quad y=w+f_4(z),$$
$$z=u[1+f_5(u)], \quad w=v/[1+f_5(u)+uf_5'(u)],$$
$$u=q/[1+f_6(s^{-1}p)+s^{-1}pf_6'(s^{-1}p)], \quad v=p[1+f_6(s^{-1}p)]$$

with $f_1=X+O(X^2)$, $f_2=X/2+O(X^2)$, $f_3=-(\alpha/8+1/16)X+O(X^2)$, $f_4=-(\alpha+1/4)X+O(X^2)$, $f_5=-X/2+O(X^2)$, $f_6=O(X)$, $H_{\rm II}$ is changed to (1.7) where $\sigma=1$ and

$$\lambda_0=-2/3, \quad h_{00}(\infty)=-1/2, \quad h_{11}(\infty)=-1/2, \quad h_{jj}(\infty)=0, \quad j\geqq 2.$$

Then we have a general solution with the asymptotic property

$$\Phi(s)=(s^{-2/3}(-\alpha+o(1)s),\ s^{2/3}(1/2+o(1)))$$

as $s\to\infty$ along the singular line of $-(2/3)s$. The transformation $(\lambda, \mu)=\Phi(s, V_1, 2sV_2)$ is that obtained by S. Yoshida.

**8.2.2.** By

$$\lambda=s^{1/3}[x-f_1(s^{-1}y)-s^{-1}f_2(s^{-1}y)], \quad \mu=s^{-1/3}y, \quad x=z, \quad y=w+f_3(z),$$
$$z=u[1+f_4(u)], \quad w=v/[1+f_4(u)+uf_4'(u)],$$
$$u=q/[1+f_5(s^{-1}p)+s^{-1}pf_5'(s^{-1}p)], \quad v=p[1+f_5(s^{-1}p)]$$

with $f_1=i/\sqrt{2}+O(X)$, $f_2=-(2\alpha-1)/8+O(X)$, $f_3=-(2\alpha+1)\sqrt{2}i/4+O(X)$, $f_4=(i/\sqrt{2})X+O(X^2)$, $f_5=-X/2+O(X^2)$, $H_{\rm II}$ is changed to (1.7) where $\sigma=1$ and

$$\lambda_0=2\sqrt{2}i/3, \quad h_{00}(\infty)=-(\alpha+1/2), \quad h_{11}(\infty)=-1, \quad f_{jj}(\infty)=0, \quad j\geqq 2$$

with $i=\sqrt{-1}$. Therefore we have a general solution $\Phi(s)$ with

$$\Phi(s)=(s^{1/3}(-i/\sqrt{2}+o(1)),\ s^{-1/3}(-(2\alpha+1)\sqrt{2}i/4+o(1)s))$$

as $s\to\infty$ along a curve which is tangent to the singular line of $(2\sqrt{2}/3)is$.

This is a new general solution.

**8.3. Painlevé system** $(H_{\mathrm{III}})$.

**8.3.1.** By

$$\lambda=\sqrt{\eta_0/\eta_\infty}i+x-f_1(t^{-1}y)-t^{-1}f_2(t^{-1}y),\ \mu=\eta_\infty t+y,\quad x=z,\ y=w+f_3(z),$$
$$z=u[1+f_4(u)],\qquad w=v/[1+f_4(u)+uf_4'(u)],$$
$$u=q/[1+f_5(t^{-1}p)+t^{-1}pf_5'(t^{-1}p)],\qquad v=p[1+f_5(t^{-1}p)],$$

with $f_1=(\sqrt{\eta_0/\eta_\infty}/2\eta_\infty)iX+O(X^2)$, $f_2=-(3\theta_0+\theta_\infty+2)/(8\eta_\infty)+O(X)$, $f_3=-(\theta_0-\theta_\infty)\sqrt{\eta_\infty/\eta_0}i/4+O(X)$, $f_4=-(\sqrt{\eta_\infty/\eta_0}i/2)X+O(X^2)$, $f_5=(1/2\eta_\infty)X+O(X^2)$, $H_{\mathrm{III}}$ is changed to (1.7) where $\sigma=1$ and

$$\lambda_0=4\sqrt{\eta_0\eta_\infty}i,\qquad h_{00}(\infty)=(\theta_0-\theta_\infty)/2,\qquad h_{11}(\infty)=-1,\qquad h_{jj}(\infty)=0,\ j\geqq 2.$$

Hence we have a general solution $\Phi(t)$ with

$$\Phi(t)=(\sqrt{\eta_0/\eta_\infty}i+o(1),\ \eta_\infty t(1+o(1)))$$

as $t\to\infty$ along a curve which is tangent to the singular line of $4\sqrt{\eta_0\eta_\infty}it$. The transformation $(\lambda,\mu)=\Phi(t, V_1, 2\sqrt{\eta_0\eta_\infty}\eta_\infty itV_2)$ is that of S. Yoshida.

**8.3.2.** By

$$\lambda=\sqrt{\eta_0/\eta_\infty}+x-f_1(t^{-1}y)-t^{-1}f_2(t^{-1}y),\ \mu=y,\qquad x=z,\quad y=w+f_3(z),$$
$$z=u[1+f_4(u)],\qquad w=v/[1+f_4(u)+uf_4'(u)],$$

with $f_1=-(\sqrt{\eta_0/\eta_\infty}/2\eta_\infty)X+O(X^2)$, $f_2=(3\theta_0-\theta_\infty+2)/(8\eta_\infty)+O(X)$, $f_3=(\theta_0+\theta_\infty)\sqrt{\eta_\infty/\eta_0}/4+O(X)$, $f_4=(\sqrt{\eta_\infty/\eta_0}/2)X+O(X^2)$, $f_5=-(1/2\eta_\infty)X+O(X^2)$, $H_{\mathrm{III}}$ is changed to (1.7) where $\sigma=1$ and

$$\lambda_0=-4\sqrt{\eta_0\eta_\infty},\qquad h_{00}(\infty)=(\theta_0+\theta_\infty)/2,\qquad h_{11}(\infty)=-1,\qquad h_{jj}(\infty)=0,\ j\geqq 2.$$

Then we have a new general solution $\Phi(t)$ with

$$\Phi(t)=(\sqrt{\eta_0/\eta_\infty}+o(1),\ (\theta_0+\theta_\infty)\sqrt{\eta_\infty/\eta_0}/4+o(1)t)$$

as $t\to\infty$ along a curve which is tangent to the singular line of $-4\sqrt{\eta_0\eta_\infty}t$.

**8.4. Painlevé system** $(H_{\mathrm{IV}})$. Set

$$t=s^{1/2}.$$

By

$$\lambda=s^{1/2}x,\qquad \mu=s^{-1/2}(y+f_1(x)),$$
$$x=z-s^{-1}f_2(s^{-1}w),\qquad y=w,$$
$$z=u[1+f_3(u)],\qquad w=v/[1+f_3(u)+uf_3'(u)],$$
$$u=q/[1+f_4(s^{-1}p)+s^{-1}pf_4'(s^{-1}p)],\qquad v=p[1+f_4(s^{-1}p)],$$

with $f_1=\theta_\infty/2+O(X)$, $f_2=\kappa_0+O(X)$, $f_3=X/2+O(X^2)$, $f_4=-X+O(X^2)$, $H_{\mathrm{IV}}$ is changed to (1.7) where $\sigma=1$ and

$$\lambda_0=-1, \qquad h_{00}(\infty)=\kappa_0+\theta_\infty-1/2, \qquad h_{11}(\infty)=-3/2, \qquad h_{jj}(\infty)=0, \quad j\geqq 2.$$

Hence we obtain a general solution $\Phi(s)$ with the property

$$\Phi(s)=(o(1)s,\ s^{-1/2}(\theta_\infty/2+o(1)))$$

as $s\to\infty$ along a curve which is tangent to the singular line of $-s$. The transformation $(\lambda, \mu)=\Phi(s, V_1, sV_2)$ coincides with that of S. Yoshida.

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