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On small data scattering with cubic convolution nonlinearity

Dedicated to Professor Takeyuki Hida on his sixtieth birthday

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1. Introduction.

We shall consider the Schrödinger equation

$$\frac{1}{i}\partial_t w = \Delta w + f(w), \tag{1.1}$$

the Klein-Gordon equation

$$\partial_t^2 w = \Delta w - w + f(w), \tag{1.2}$$

and the wave equation

$$\partial_t^2 w = \Delta w + f(w) \tag{1.3}$$

for $(x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}$, where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^n \partial_j^2$ $(\partial_j = \partial/\partial x_j)$ and $f(u)$ represents the cubic convolution nonlinearity:

$$f(w) = (V * |w|^2)w = \left(\int_{R^n} V(x-y)|w(y)|^2 dy\right) w(x). \tag{1.4}$$

The steady state equations corresponding to (1.1), (1.2) and (1.3) have the same form and are given by

$$-\Delta v - f(v) = \mu v \qquad (\mu \in \boldsymbol{R}). \tag{1.5}$$

This equation has been studied e.g., in Gross [**6**], Lions [**10**] and Menzala [**12**]. In case $V = |x|^{-1}$, (1.5) is known as the Hartree equation for the helium atom. The time dependent equation (1.1) has been studied by Glassey [**5**], Ginibre-Velo [**4**], Dias-Figueira [**3**], Hayashi-Tsutsumi [**7**] and Hayashi-Ozawa [**8**], and equations (1.2) and (1.3) have been studied by Menzala-Strauss [**13**]. The positivity $V(x) \geqq 0$ and the symmetry $V(-x) = V(x)$ are required there. Then the well-posedness of the Cauchy problem and the asymptotic behaviors of solutions

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(including the scattering theory) are obtained under suitable decaying conditions on V.

In the following we do not require V to be non-negative. As is proved in [13] (see also Matsumoto-Mochizuki [11] for the Schrödinger equation (1.1)), in our case, solutions may blow up in finite time unless the initial data is restricted to have small energy.

The aim of this paper is to extend the small data scattering theory and apply it to equations (1.1), (1.2) and (1.3). The theory has been studied in Strauss [17] for equations (1.1) and (1.2) with V satisfying

$$|V(x)| \leqq C|x|^{-\sigma} \quad \text{or} \quad V(x) \in L^z. \tag{1.6}$$

It was shown that if

$$2 \leqq \sigma < \min\{n, 4\} \quad \text{or} \quad \frac{n}{4} \leqq z \leqq \frac{n}{2} \quad \text{and} \quad z \geqq 1 \tag{1.7}$$

for equation (1.1), and if

$$2 \leqq \sigma \leqq \frac{4n}{n+1} \quad \text{and } \sigma < \text{n} \quad \text{or} \quad \frac{n+1}{4} \leqq z \leqq \frac{n}{2} \quad \text{and} \quad z \geqq 1 \tag{1.8}$$

for equation (1.2), then the scattering operators exist in whole neighborhoods of 0 in the energy spaces $X=H^{1,2}$ and $X=H^{1,2}\times L^2$, respectively. Note that a more general $V(x)$ is allowed if we dispense with the requirement that the data is "arbitrary" within a neighborhood of 0 in X (see [17] and also Mochizuki-Motai [14] for the wave equation (1.3)).

We shall extend the above mentioned results of Strauss. In our theory the conditions (1.7) and (1.8) can be weakened to

$$2 \leqq \sigma \leqq 4 \text{ and } \sigma < n \quad \text{or} \quad \frac{n}{4} \leqq z \leqq \frac{n}{2} \text{ and } z \geqq 1. \tag{1.9}$$

Moreover, in case of the Schrödinger equation (1.1), we can construct the scattering operator in a neighborhood of 0 in $X=L^2$ if

$$\sigma = 2 \text{ and } \sigma < n \quad \text{or} \quad z = \frac{n}{2} \text{ and } z \geqq 1. \tag{1.10}$$

As for the wave equation (1.3) we can have the following result. If

$$\sigma = 4 \text{ and } \sigma < n \quad \text{or} \quad z = \frac{n}{4} \text{ and } z \geqq 1, \tag{1.11}$$

then the scattering operator exists in a neighborhood of 0 in the energy space $X=H^{1,0,2}\times L^2$.

The key estimates for the results are the well known decay properties in t of the fundamental solutions of the unperturbed linear problems. We shall combine these properties with the Hardy-Littlewood-Sobolev inequality or the Young inequality. In this sense our proof is very close to that developed by

Pecher [**16**] for the wave and Klein-Gordon equation with power nonlinearity.

The paper is organized as follows: In §2 we give notation and several lemmas which will be used throughout this paper. In §3 we state the existence theorem of the scattering operator in an abstract form and prove it. Applications of this theorem to equations (1.1), (1.2) and (1.3) are respectively given in the following §§4, 5 and 6.

2. Notation and preliminary lemmas.

First we give notation which will be freely used in the sequel: $C_0^\infty = C_0^\infty(\boldsymbol{R}^n) = \bigcap_{k=1}^\infty C_0^k(\boldsymbol{R}^n)$, where $C_0^k(\boldsymbol{R}^n)$ is the space of all k-times continuously differentiable functions with compact support in $\boldsymbol{R}^n$. $L^p = L^p(\boldsymbol{R}^n)$, $1 \leqq p \leqq \infty$, is the usual space of all L^p-functions in $\boldsymbol{R}^n$. For a Banach space Z, $L^p(Z) = L^p(\boldsymbol{R}; Z)$ is the space of all Z-valued L^p-functions in $\boldsymbol{R}$. $\mathscr{S}' = \mathscr{S}'(\boldsymbol{R}^n)$ is the space of all tempered distributions in $\boldsymbol{R}^n$. $\hat{}$ denotes the spacial Fourier transform and $\mathcal{F}^{-1}$ is its inverse. For $r \in \boldsymbol{R}$, $\varepsilon > -n$ and $1 \leqq p \leqq \infty$, let $H^{r,p} = H^{r,p}(\boldsymbol{R}^n)$ and $H^{\varepsilon,r,p} = H^{\varepsilon,r,p}(\boldsymbol{R}^n)$ be the completions of $C_0^\infty(\boldsymbol{R}^n)$ with respect to

$$\|f\|_{H^{r,p}} = \|\mathcal{F}^{-1}\{\langle\xi\rangle^r \hat{f}(\xi)\}\|_{L^p} \quad \text{and} \tag{2.1}$$

$$\|f\|_{H^{\varepsilon,r,p}} = \|\mathcal{F}^{-1}\{|\xi|^\varepsilon \langle\xi\rangle^r \hat{f}(\xi)\}\|_{L^p}, \tag{2.2}$$

respectively. Here $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Conjugate exponents are denoted by q, q'; s, s' etc. Positive constants are denoted by c. If necessary, by $c(*, \cdots, *)$ we denote constants depending on the quantities in parentheses. They might change from line to line.

Next we summarize several lemmas which will be used throughout this paper.

LEMMA 2.1. *If* $1 < p < q < \infty$ *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{\nu}{n}, \tag{2.3}$$

then

$$\||x|^{-n+\nu} * f\|_{L^q} \leqq c(p, \nu)\|f\|_{L^p} \quad \textit{for} \quad f \in L^p(\boldsymbol{R}^n). \tag{2.4}$$

LEMMA 2.2. *If* $1 \leqq p, q \leqq \infty$ *and* $V(x) \in L^z(\boldsymbol{R}^n)$ *with*

$$\frac{1}{q} = \frac{1}{p} - \left(1 - \frac{1}{z}\right), \tag{2.5}$$

then

$$\|V * f\|_{L^q} \leqq c(p, z)\|f\|_{L^p} \quad \textit{for} \quad f \in L^p(\boldsymbol{R}^n). \tag{2.6}$$

LEMMA 2.3. *Let* $\rho(\xi) \in \mathscr{S}'$ *satisfy*

(2.7) $$|\xi|^{|\alpha|}|\partial^{\alpha}\rho(\xi)| \leqq c(\rho, m) < \infty$$

for $|\alpha|=\alpha_1+\cdots+\alpha_n\leqq m$ *and* $\xi\in \boldsymbol{R}^n-\{0\}$, *where* $\partial^{\alpha}=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ *and* $m\geqq n/2$ *is an appropriate integer. Then* ρ *is a Fourier multiplier on* L^p, $1<p<\infty$, *i.e.*,

(2.8) $$\|\mathcal{F}^{-1}\rho * f\|_{L^p} \leqq c(\rho, p)\|f\|_{L^p} \quad \text{for} \quad f\in L^p(\boldsymbol{R}^n).$$

LEMMA 2.4. *Let* $1\leqq p, q\leqq\infty$, $r, t\in \boldsymbol{R}$ *and* $\varepsilon, \delta>-n$.

(i) *If* $1/p\geqq 1/q\geqq 1/p-(r-t)/n$, *then*

(2.9) $$H^{r,p} \hookrightarrow H^{t,q}.$$

(ii) *If* $\varepsilon\geqq 0$ *and* $1/p-\varepsilon/n\geqq 1/q\geqq 1/p-(r+\varepsilon-t)/n$, *then*

(2.10) $$H^{\varepsilon,r,p} \hookrightarrow H^{t,q}.$$

(iii) *If* $\varepsilon\geqq\delta$ *and* $r+\delta\geqq s+\varepsilon$, *then*

(2.11) $$H^{\delta,r,p} \hookrightarrow H^{\varepsilon,s,p}.$$

Here $K\hookrightarrow L$ *means that* K *is continuously embedded in* L.

LEMMA 2.5. *Let* $1\leqq p\leqq\infty$. *If* r *and* ε *are non-negative integers, then the norms* (2.1) *of* $H^{r,p}$ *and* (2.2) *of* $H^{\varepsilon,r,p}$ *are equivalent to*

(2.12) $$\sum_{|\alpha|\leqq r}\|\partial^{\alpha}f\|_{L^p} \quad \text{and} \quad \sum_{\varepsilon\leqq|\alpha|\leqq r}\|\partial^{\alpha}f\|_{L^p},$$

respectively.

Lemmas 2.1 and 2.2 are known as the Hardy-Littlewood-Sobolev and the Young inequalities, respectively. The proof can be found, e.g., in Hörmander [**9**], Chapter IV. Lemmas 2.3 and 2.4 are the so called Mihlin multiplier and the Sobolev embedding theorems, respectively. As for the proof see, e.g., Bergh-Löfström [**1**], Chapter 6. As is easily seen, Lemma 2.5 is a result of Lemma 2.3.

3. The small data scattering operator.

Let X be a Hilbert space with norm $\|\ \|_X$, and A be a selfadjoint operator in X with dense domain $\mathcal{D}(A)\subset X$. We consider the evolution equation

(3.1) $$\begin{cases} i\partial_t u = Au+F(u), & t\in\boldsymbol{R} \\ \|u(t)-U_0(t)\varphi_-\|_X \longrightarrow 0 & \text{as } t\longrightarrow -\infty, \end{cases}$$

where $\varphi_-\in X$ and

(3.2) $$U_0(t)=\exp\{-iAt\}, \qquad t\in\boldsymbol{R}.$$

It is convenient to rewrite (3.1) into the integral form:

$$u(t)=U_0(t)\varphi_-+\int_{-\infty}^{t}U_0(t-\tau)F(u(\tau))d\tau. \tag{3.3}$$

We make the following hypotheses.

(I) There exist Banach spaces Y and Z such that X, Y and Y' are continuously embedded in Z, and Z' is dense in each X, Y and Y'. Here Y' and Z' are the dual spaces, with respect to X, of Y and Z, respectively.

(II) $U_0(t)$ restricted to $X\cap Y'$ has a continuous linear extension (still denoted $U_0(t)$) which maps Y' to Y, and there exist $c>0$ and $0<d<1$ such that

$$\|U_0(t)\phi\|_Y\leqq c|t|^{-d}\|\phi\|_{Y'} \qquad \text{for} \quad t\neq 0 \text{ and } \phi\in Y'. \tag{3.4}$$

$U_0(t)$ also has a continuous extension from Y to Z such that

$$U_0(t)U_0(s)\phi=U_0(t+s)\phi \qquad \text{for} \quad \phi\in Y'. \tag{3.5}$$

(III) F maps $X\cap Y$ to Y', $F(0)=0$ and we have

$$\begin{aligned}\|F(u)-F(v)\|_{Y'}\leqq\ & c\|u-v\|_X\{\|u\|_Y^{s-1}+\|v\|_Y^{s-1}\}\\ &+c\{\|u\|_X+\|v\|_X\}\|u-v\|_Y\{\|u\|_Y^{s-2}+\|v\|_Y^{s-2}\}\end{aligned} \tag{3.6}$$

for $u, v\in X\cap Y$, where $s=2/d$.

(IV) Moreover, F maps Y into X and we have

$$\|F(u)-F(v)\|_X\leqq c\|u-v\|_Y\{\|u\|_Y^{s-1}+\|v\|_Y^{s-1}\} \qquad \text{for} \quad u, v\in Y. \tag{3.7}$$

The integral equation (3.3) will be considered in the following space of functions $u(t)$:

$$W=L^s(\boldsymbol{R};\ Y)\cap L^\infty(\boldsymbol{R};\ X). \tag{3.8}$$

THEOREM 3.1. *Under* (I)$\sim$(IV) *there exists a* $\delta>0$ *with the following properties: If* $\varphi_-\in X$ *and* $\|\varphi_-\|_X\leqq\delta$, *then there exists a unique solution* $u(t)\in W$ *of the integral equation* (3.3) *such that*

$$\|u\|_W\leqq\frac{4}{3}\|U_0(t)\varphi_-\|_W\leqq\frac{4c}{3}\|\varphi_-\|_X, \tag{3.9}$$

$$\|u(t)-U_0(t)\varphi_-\|_X\longrightarrow 0 \qquad as \quad t\longrightarrow-\infty. \tag{3.10}$$

Furthermore, there exists a unique $\varphi_+\in X$ *such that*

$$\|u(t)-U_0(t)\varphi_+\|_X\longrightarrow 0 \qquad as \quad t\longrightarrow+\infty. \tag{3.11}$$

Thus, we can define the scattering operator $S:\varphi_-\rightarrow\varphi_+$ *on a neighborhood of* 0 *in* X.

The proof of this theorem will be done based on a contraction mapping principle. For this aim we prepare three propositions.

PROPOSITION 3.2. *$f(t)\to\int_{-\infty}^{\infty}U_0(-t)f(t)dt$ is a continuous map of $L^{s'}(\boldsymbol{R};Y')$, where $1/s'=1-1/s$, to X. Namely, there exists a $C_1>0$ such that*

$$\left\|\int_{-\infty}^{\infty}U_0(-t)f(t)dt\right\|_X\leqq C_1\|f\|_{L^{s'}(Y')}\qquad \textit{for}\quad f(t)\in L^{s'}(\boldsymbol{R};Y'). \tag{3.14}$$

PROOF. We have only to prove (3.14) for $f(t)$ in a dense set of $L^{s'}(\boldsymbol{R};Y')$. Let $f(t)\in C_0^\infty(\boldsymbol{R};Z')$. In this case we can change the order of integrations to obtain

$$\left\|\int_{-\infty}^{\infty}U_0(-t)f(t)dt\right\|_X^2=\left|\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}U_0(t-\tau)f(\tau)d\tau,\ f(t)\right)_X dt\right|.$$

Here $(\ ,\ )_X$ denotes the innerproduct in X, or more generally, the duality between Z and Z'. Using (I), (II) and the Hölder inequality, we then have

$$\begin{aligned}&\leqq\int_{-\infty}^{\infty}\left\|\int_{-\infty}^{\infty}U_0(t-\tau)f(\tau)d\tau\right\|_Y\|f(t)\|_{Y'}dt\\&\leqq\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}c|t-\tau|^{-d}\|f(\tau)\|_{Y'}d\tau\right]\|f(t)\|_{Y'}dt\\&\leqq c\left\|\int_{-\infty}^{\infty}|t-\tau|^{-d}\|f(\tau)\|_{Y'}d\tau\right\|_{L^s}\|f\|_{L^{s'}(Y')}.\end{aligned}$$

The requirement $1/s=d/2$ implies that $1/s=1/s'-(1-d)$. Thus, we can apply Lemma 2.1 with $n=1$ and $\nu=1-d$ to obtain

$$\leqq cc(s',1-d)\|f\|_{L^{s'}(Y')}^2.$$

This proves (3.14) if we put $C_1=\sqrt{cc(s',1-d)}$. $\square$

PROPOSITION 3.3. *Let $\varphi\in X$. Then $U_0(t)\varphi\in L^s(\boldsymbol{R};Y)$ and we have*

$$\|U_0(t)\varphi\|_{L^s(Y)}\leqq C_1\|\varphi\|_X\qquad \textit{for}\quad \varphi\in X, \tag{3.15}$$

where C_1 is the constant given in (3.14).

PROOF. Let $\varphi\in X$ and $f(t)\in C_0^\infty(\boldsymbol{R};Z')$. Then we have from the above proposition

$$\left|\int_{-\infty}^{\infty}(f(t),U_0(t)\varphi)_X dt\right|=\left|\left(\int_{-\infty}^{\infty}U_0(-t)f(t)dt,\ \varphi\right)_X\right|\leqq C_1\|f\|_{L^{s'}(Y')}\|\varphi\|_X.$$

This proves (3.15) since $C_0^\infty(\boldsymbol{R};Z')$ is dense in $L^{s'}(\boldsymbol{R};Y')$. $\square$

PROPOSITION 3.4. *There exists a $C_2>0$ such that*

$$\left\|\int_{-\infty}^{t}U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_W\leqq C_2\|u-v\|_W\{\|u\|_W^{\frac{s}{2}-1}+\|v\|_W^{\frac{s}{2}-1}\} \tag{3.16}$$

for $u(t),v(t)\in W$.

PROOF. By (II) and (III)

$$\left\|\int_{-\infty}^{t} U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_{L^s(Y)}$$

$$\leqq \left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\|F(u(\tau))-F(v(\tau))\|_{Y'}d\tau\right\|_{L^s}$$

$$\leqq \left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\|u(\tau)-v(\tau)\|_X\{\|u(\tau)\|_Y^{s-1}+\|v(\tau)\|_Y^{s-1}\}d\tau\right\|_{L^s}$$

$$+\left\|\int_{-\infty}^{\infty} c|t-\tau|^{-d}\{\|u(\tau)\|_X+\|v(\tau)\|_X\}\|u(\tau)-v(\tau)\|_Y$$

$$\times\{\|u(\tau)\|_Y^{s-2}+\|v(\tau)\|_Y^{s-2}\}d\tau\right\|_{L^s}.$$

Noting $1/s=1/s'-(1-d)$, we can apply Lemma 2.1 to obtain

$$\leqq C\|u-v\|_{L^\infty(X)}\{\|u\|_{L^s(Y)}^{s-1}+\|v\|_{L^s(Y)}^{s-1}\}$$

$$+C\{\|u\|_{L^\infty(X)}+\|v\|_{L^\infty(X)}\}\|u-v\|_{L^s(Y)}\{\|u\|_{L^s(Y)}^{s-2}+\|v\|_{L^s(Y)}^{s-2}\}.$$

On the other hand, by (IV)

$$\left\|\int_{-\infty}^{t} U_0(t-\tau)\{F(u(\tau))-F(v(\tau))\}d\tau\right\|_{L^\infty(X)} \leqq \int_{-\infty}^{\infty}\|F(u(\tau))-F(v(\tau))\|_X d\tau$$

$$\leqq \tilde{C}\int_{-\infty}^{\infty}\|u(\tau)-v(\tau)\|_Y\{\|u(\tau)\|_Y^{s-1}+\|v(\tau)\|_Y^{s-1}\}d\tau$$

$$\leqq \tilde{C}\|u-v\|_{L^s(Y)}\{\|u\|_{L^s(Y)}^{s-1}+\|v\|_{L^s(Y)}^{s-1}\}.$$

Summarizing these inequalities, we obtain (3.16) with $C_2\leqq 2C+\tilde{C}$. □

PROOF OF THEOREM 3.1. We put

$$(\Phi u)(t)=U_0(t)\varphi_-+\int_{-\infty}^{t}U_0(t-\tau)F(u(\tau))d\tau, \tag{3.17}$$

and consider it in the ball $\mathscr{B}(\delta_1)=\{u\in W;\ \|u\|_W\leqq\delta_1\}$, where the constants $\delta_1>0$ and $\delta>0$ in the theorem are chosen to satisfy

$$2C_2\delta_1^{s-1}\leqq\frac{1}{2} \quad\text{and}\quad (1+C_1)\delta\leqq\frac{3}{4}\delta_1. \tag{3.18}$$

Let $u\in\mathscr{B}(\delta_1)$. Then by Proposition 3.3 with $\varphi=\varphi_-$ and Proposition 3.4 with $v=0$,

$$\|\Phi u\|_W\leqq(1+C_1)\|\varphi_-\|_X+C_2\|u\|_W^s\leqq\delta_1. \tag{3.19}$$

On the other hand, it follows from Proposition 3.4 that

$$\|\Phi u-\Phi v\|_W\leqq C_2\{\|u\|_W^{s-1}+\|v\|_W^{s-1}\}\|u-v\|_W\leqq\frac{1}{2}\|u-v\|_W \tag{3.20}$$

for any $u, v\in\mathscr{B}(\delta_1)$. (3.19) and (3.20) show that Φ defines a contraction map of $\mathscr{B}(\delta_1)$ into itself. Thus, there exists a unique fixed point $u\in\mathscr{B}(\delta_1)$ which solves (3.3). (3.19) and the first inequality of (3.18) imply that this u satisfies (3.9) also. Moreover, we have

$$\|u(t)-U_0(t)\varphi_-\|_X \leqq \int_{-\infty}^{t} c\|u(\tau)\|_Y^{3} d\tau \longrightarrow 0 \qquad \text{as} \quad t \longrightarrow -\infty,$$

and (3.10) follows.

Next, put

$$\varphi_+ = \varphi_- + \int_{-\infty}^{\infty} U_0(-\tau)F(u(\tau))d\tau. \tag{3.21}$$

Then $\varphi_+ \in X$ by Proposition 3.2, and we have noting (II),

$$U_0(t)\varphi_+ = u(t) + \int_t^{\infty} U_0(t-\tau)F(u(\tau))d\tau.$$

Thus, letting $t \to +\infty$, we obtain (3.11). $\square$

4. The Schrödinger equation with a cubic convolution.

In this section we consider equation (1.1) with the cubic convolution nonlinearity (1.4) requiring

$$|V(x)| \leqq c|x|^{-\sigma} \qquad \text{with} \quad 2 \leqq \sigma < n, \quad \text{or} \tag{4.1}$$

$$V(x) \in L^z(\boldsymbol{R}^n) \qquad \text{with} \quad 1 \leqq z \leqq \frac{n}{2}. \tag{4.2}$$

We put $X = H^{k,2}$ for $k \in \boldsymbol{Z}$, and define

$$A = -\Delta \qquad \text{with domain } \mathcal{D}(A) = H^{k+2,2}, \tag{4.3}$$

$$F(u) = -f(u) = -(V * |u|^2)u, \tag{4.4}$$

where $u = w$.

THEOREM 4.1. *Let $k \in \boldsymbol{Z}$ be chosen to satisfy*

$$\sigma \leqq 2(k+1) \quad \text{or} \quad \frac{n}{2(k+1)} \leqq z. \tag{4.5}$$

Put $X = H^{k,2}$, $Y = H^{k,q}$ and $Z = H^{k-m,q}$, where

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{3n} \tag{4.6}$$

and $m \geqq 1$ (integer) is sufficiently large. Then (1.1) can be written in the form (3.3), and all the assertions of Theorem 3.1 hold.

The proof will be done in a series of lemmas.

LEMMA 4.2. *We have $Y' = H^{k,q'}$ and $Z' = H^{k+m,q'}$.*

PROOF. Obvious from definition of Y and Z. $\square$

LEMMA 4.3. *A is a positive selfadjoint operator in X, and $U_0(t)=\exp\{-iAt\}$ satisfies the following estimate:*

$$\|U_0(t)\phi\|_Y \leqq c(q)|t|^{-n/q'+n/2}\|\phi\|_{Y'}, \quad t\neq 0 \text{ and } \phi\in C_0^\infty(\boldsymbol{R}^n). \tag{4.7}$$

PROOF. The first assertion is well known. Since we have $\|f\|_Y=\|(1+A)^{k/2}f\|_{L^q}$, we have only to show (4.7) in case $Y=L^q$. Note the formula

$$U_0(t)\phi=(4\pi t)^{-n/2}\int_{R^n}\exp\Big\{i\Big(\frac{|x-y|^2}{4t}-\frac{n\pi}{4}\Big)\Big\}\phi(y)dy, \qquad t\neq 0.$$

Then we have $\|U_0(t)\phi\|_{L^\infty}\leqq c|t|^{-n/2}\|\phi\|_{L^1}$. Interpolating this and the unitarity $\|U_0(t)\phi\|_{L^2}=\|\phi\|_{L^2}$, we obtain the desired estimate. □

LEMMA 4.4. *$U_0(t)$ maps Z' continuously to Y':*

$$\|U_0(t)\phi\|_{Y'} \leqq c(t, m)\|\phi\|_{Z'}\,. \tag{4.8}$$

PROOF. We put $\rho(\xi\,;t)=\langle\xi\rangle^{-m}\exp\{-i|\xi|^2t\}$, where m is chosen so large that this ρ satisfies conditions of Lemma 2.3. Then since we have $\mathcal{F}^{-1}\rho*\phi=(1+A)^{-m/2}U_0(t)\phi$, it follows that

$$\begin{aligned}\|U_0(t)\phi\|_{Y'} &= \|U_0(t)(1+A)^{k/2}\phi\|_{L^{q'}}\\ &= \|\mathcal{F}^{-1}\rho*(1+A)^{m/2+k/2}\phi\|_{L^{q'}} \leqq c(t, m)\|\phi\|_{Z'}\,. \quad \square\end{aligned}$$

LEMMA 4.5. *For $1\leqq a, b, h\leqq\infty$ satisfying*

$$\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{h}-1\Big)\frac{1}{q'}=\begin{cases}1-\dfrac{\sigma}{n} & \textit{if } V \textit{ satisfies } (4.1),\\[2ex] 1-\dfrac{1}{z} & \textit{if } V \textit{ satisfies } (4.2),\end{cases} \tag{4.9}$$

we have

$$\begin{aligned}\|F(u)-F(v)\|_{Y'} \leqq c &\sum_{|\alpha+\beta+\gamma|\leqq k}\|\partial^\gamma u-\partial^\gamma v\|_{L^{hq'}}\\ &\times\{\|\partial^\alpha u\|_{L^{aq'}}+\|\partial^\alpha v\|_{L^{aq'}}\}\{\|\partial^\beta u\|_{L^{bq'}}+\|\partial^\beta v\|_{L^{bq'}}\}\,.\end{aligned} \tag{4.10}$$

PROOF. Let $w=u-v$. Then it follows from (4.4) that

$$F(u)-F(v)=(V*|u|^2)w+(V*[u\bar{w}])v+(V*[w\bar{v}])v\,.$$

We shall estimate $(V*|u|^2)w$. The other terms of the right can be similarly estimated. By means of Lemma 2.5 we see that the norm $\|(V*|u|^2)w\|_{Y'}$ is equivalent to $\sum_{|\eta|\leqq k}\|\partial^\eta\{(V*|u|^2)w\}\|_{L^{q'}}$. Repeated differentiations of $(V*|u|^2)w$ give

$$\partial^\eta\{(V*|u|^2)w\}=\sum_{\alpha+\beta+\gamma=\eta}c(\alpha, \beta, \gamma)(V*[\partial^\alpha u\partial^\beta\bar{u}])\partial^\gamma w\,. \tag{4.11}$$

By the Hölder inequality

$$\|(V*[\partial^\alpha u\partial^\beta \bar{u}])\partial^\gamma w\|_{L^{q'}} \leqq \|\partial^\gamma w\|_{L^{hq'}}\|V*[\partial^\alpha u\partial^\beta \bar{u}]\|_{L^{h'q'}}.$$

We can use Lemma 2.1 or Lemma 2.2 to obtain

$$\leqq \|\partial^\gamma w\|_{L^{hq'}}\, c\|\partial^\alpha u\partial^\beta \bar{u}\|_{L^{pq'}},$$

where p solves $1/h'q'=1/pq'-(1-\sigma/n)$, or $1/h'q'=1/pq'-(1-1/z)$. Dividing $1/p$ into $1/a+1/b$ and using again the Hölder inequality, we conclude

$$\leqq c\|\partial^\gamma w\|_{L^{hq'}}\|\partial^\alpha u\|_{L^{aq'}}\|\partial^\beta u\|_{L^{bq'}}.$$

This and (4.11) imply the desired estimate for $(V*\partial^\alpha u\partial^\beta \bar{u})\partial^\gamma w$. □

LEMMA 4.6. *For $1\leqq a, b, h\leqq\infty$ satisfying*

$$\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{h}-1\right)=\begin{cases}1-\dfrac{\sigma}{n} & \textit{if } V \textit{ satisfies } (4.1)\\[2mm] 1-\dfrac{1}{z} & \textit{if } V \textit{ satisfies } (4.2),\end{cases} \tag{4.12}$$

we have

$$\|F(u)-F(v)\|_X \leqq c\sum_{|\alpha+\beta+\gamma|\leqq k}\|\partial^\gamma u-\partial^\gamma v\|_{L^{2h}} \times\{\|\partial^\alpha u\|_{L^{2a}}+\|\partial^\alpha v\|_{L^{2a}}\}\{\|\partial^\beta u\|_{L^{2b}}+\|\partial^\beta v\|_{L^{2b}}\}. \tag{4.13}$$

PROOF. To obtain (4.13) we can follow the same argument as in the above proof. □

PROOF OF THEOREM 4.1. We shall show Hypotheses (I)~(IV).

First note that how we have chosen q as in (4.6). By (4.7) we should choose $d=n/q'-n/2$. On the other hand, as we see from (4.10) or (4.13), $s=3$ in our cubic case. Since we require $s=2/d$ in Hypotheses (III) and (IV), it follows that $1/q=1/2-2/(3n)$ and $d=2/3$.

(I) It follows from (4.6) that

$$\text{any of } \left\{\frac{1}{2}, \frac{1}{q}, \frac{1}{q'}\right\} \geqq \frac{1}{q} \geqq \text{any of } \left\{\frac{1}{2}, \frac{1}{q}, \frac{1}{q'}\right\}-\frac{1}{n},$$

$$\frac{1}{q'} \geqq \text{any of } \left\{\frac{1}{2}, \frac{1}{q}, \frac{1}{q'}\right\} \geqq \frac{1}{q'}-\frac{1}{n}.$$

Then applying Lemma 2.4 (i) and noting Lemma 4.2, we see that $X=H^{k,2}$, $Y=H^{k,q}$ and $Z=H^{k-m,q}$ with $m\geqq 1$ satisfy Hypothesis (I).

(II) With $d=n/q'-n/2=2/3$ is already known in Lemmas 4.3 and 4.4 except relation (3.5), which is verified as follows: If $\varphi\in Z'$, then

$$(U_0(t)U_0(s)f, \varphi)_X=(U_0(s)f, U_0(-t)\varphi)_X=(f, U_0(-t-s)\varphi)_X=(U_0(t+s)f, \varphi)_X.$$

(III) That $F(0)=0$ is obvious. Further, in the present case, $s=2/d=3$.

Thus, noting Lemmas 2.4 (i), 2.5 and 4.5, we conclude (3.6) if there exist $1 \leqq a, b, h \leqq \infty$ satisfying (4.9) and also

$$\frac{1}{2} \geqq \frac{1}{hq'} \geqq \frac{1}{2} - \frac{k-|\gamma|}{n}, \quad \frac{1}{q} \geqq \frac{1}{aq'} \geqq \frac{1}{q} - \frac{k-|\alpha|}{n}$$

$$\text{and} \quad \frac{1}{q} \geqq \frac{1}{bq'} \geqq \frac{1}{q} - \frac{k-|\beta|}{n}.$$

Add up each term of these inequalities and use (4.9). Then we have

$$\frac{1}{2} + \frac{2}{q} \geqq 1 - \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} + \frac{1}{q'} \geqq \frac{1}{2} + \frac{2}{q} - \frac{2k}{n}. \tag{4.14}$$

Conversely, it is easy to show the existence of a desirable triplet a, b, h from this condition. Now, substitute (4.6) for the above q. Then (4.14) is reduced to what we have required in (4.1), (4.2) and (4.5).

(IV) Note Lemmas 2.4 (i), 2.5 and 4.6. Then we can use the same argument as above to establish (3.7). Namely, we are enough to show that

$$\frac{3}{q} \geqq 1 - \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} + \frac{1}{2} \geqq \frac{3}{q} - \frac{2k}{n}.$$

As is easily seen, this is also equivalent to conditions (4.1), (4.2) and (4.5). □

5. The Klein-Gordon equation with cubic convolution.

In this section we consider equation (1.2) with the cubic convolution nonlinearity (1.4). We require again that $V(x)$ satisfies (4.1) or (4.2). We put $X = H^{k,2} \times H^{k-1,2}$ for $k \in \boldsymbol{Z}$, and define

$$A = i\begin{pmatrix} 0 & 1 \\ \Delta - 1 & 0 \end{pmatrix} \quad \text{with domain } \mathscr{D}(A) = H^{k+1,2} \times H^{k,2}, \tag{5.1}$$

$$F(u) = \{0, if(u_1)\} = \{0, i(V * |u_1|^2)u_1\}, \tag{5.2}$$

where $u = \{u_1, u_2\} = \{w, w_t\}$.

THEOREM 5.1. *Let $k \in \boldsymbol{Z}$ be chosen to satisfy $k \geqq 1$ and* (4.5). *Put $X = H^{k,2} \times H^{k-1,2}$, $Y = H^{k-1+e,q} \times H^{k-2+e,q}$ and $Z = H^{k-m,q} \times H^{k-m-1,q}$, where*

$$\frac{1}{q} = \frac{1}{2} - \frac{2}{3(n-1+\theta)}, \tag{5.3}$$

$$e = \frac{n+1+\theta}{2q} - \frac{n-3+\theta}{4} = \frac{2}{3}\left(1 - \frac{1}{n-1+\theta}\right) \tag{5.4}$$

with some $0 \leqq \theta \leqq 1$ which depends on σ or z, and $m \geqq 1$ is a large integer. Then (1.2) *can be written in the form* (3.3), *and all the assertions of Theorem* 3.1 *hold.*

An example of the choice of θ will be given later in (5.19).

The following lemma is obvious from definition of Y and Z.

LEMMA 5.2. *We have* $Y'=H^{k+1-e,q'}\times H^{k-e,q'}$ *and* $Z'=H^{k+m,q'}\times H^{k+m-1,q'}$.

LEMMA 5.3. (i) *A is selfadjoint in X.*

(ii) *Let B be a positive selfadjoint operator in L^2 defined by*

$$B=\sqrt{1-\Delta} \qquad \text{with domain } \mathcal{D}(B)=H^{1,2}. \tag{5.5}$$

Then

$$U_0(t)=\exp\{-iAt\}=\begin{pmatrix}\cos Bt & B^{-1}\sin Bt\\ -B\sin Bt & \cos Bt\end{pmatrix}. \tag{5.6}$$

(iii) *For any $0\leqq\theta\leqq 1$ we have*

$$\|U_0(t)\phi\|_Y\leqq c(q)|t|^{-d}\|\phi\|_{Y'}, \qquad \phi\in C_0^\infty(\boldsymbol{R}^n)\times C_0^\infty(\boldsymbol{R}^n), \tag{5.7}$$

where

$$d=(n-1+\theta)\Big(\frac{1}{2}-\frac{1}{q}\Big). \tag{5.8}$$

PROOF. (i) and (ii) are well known. Note that $\|f\|_{H^{r,p}}=\|B^r f\|_{L^p}$. Then since we have the estimate (see e.g., Brenner [2])

$$\|B^{-1}\sin(Bt)\varphi\|_{L^p}\leqq c(q)|t|^{-d}\|B^{1-2e}\varphi\|_{L^{q'}}, \qquad \varphi\in C_0^\infty(\boldsymbol{R}^n),$$

it follows that

$$\begin{aligned}\|U_0(t)\phi\|_Y&=\|B^{-1}\cos(Bt)B^{k+e}\phi_1+B^{-1}\sin(Bt)B^{k-1+e}\phi_2\|_{L^q}\\&\quad+\|-B^{-1}\cos(Bt)B^{k+e}\phi_1+B^{-1}\cos(Bt)B^{k-1+e}\phi_2\|_{L^q}\\&\leqq c|t|^{-d}\{\|B^{k+1-e}\phi_1\|_{L^{q'}}+\|B^{k-e}\phi_2\|_{L^{q'}}\},\end{aligned}$$

which proves (5.7). □

LEMMA 5.4. *$U_0(t)$ maps Z' continuously to Y':*

$$\|U_0(t)\phi\|_{Y'}\leqq c(t)\|\phi\|_{Z'}. \tag{5.9}$$

PROOF. We put

$$\rho(\xi;t)=\langle\xi\rangle^{-m+1-e}\begin{pmatrix}\cos\langle\xi\rangle t & \sin\langle\xi\rangle t\\ -\sin\langle\xi\rangle t & \cos\langle\xi\rangle t\end{pmatrix}.$$

Then ρ satisfies the conditions of Lemma 2.3 and becomes a Fourier multiplier on $L^p\times L^p$, $1<p<\infty$. Moreover, we have

$$\begin{aligned}\|U_0(t)\phi\|_{Y'}&=\|\mathcal{F}^{-1}\rho*\{B^{k+m}\phi_1, B^{k+m-1}\phi_2\}\|_{L^{q'}\times L^{q'}}\\&\leqq c(t)\|\phi\|_{H^{k+m,q'}\times H^{k+m-1,q'}}=c(t)\|\phi\|_{Z'}\end{aligned}$$

proving (5.9). □

LEMMA 5.5. *Let q'' and $1\leqq a, b, h\leqq\infty$ satisfy*

$$\frac{1}{q''}\geqq\frac{1}{q'}\geqq\frac{1}{q''}-\frac{e}{n}, \tag{5.10}$$

$$(5.11) \qquad \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{h}-1\right)\frac{1}{q''}=\begin{cases} 1-\dfrac{\sigma}{n} & \textit{if } V \textit{ satisfies } (4.1) \\ 1-\dfrac{1}{z} & \textit{if } V \textit{ satisfies } (4.2). \end{cases}$$

Then we have

$$(5.12) \qquad \begin{aligned} \|F(u)-F(v)\|_{Y'} &\leqq c\|(V*|u_1|^2)u_1-(V*|v_1|^2)v_1\|_{H^{k,q'}} \\ &\leqq c \sum_{|\alpha+\beta+\gamma|\leqq k} \|\partial^\gamma u_1-\partial^\gamma v_1\|_{L^{hq'}} \\ &\quad \times\{\|\partial^\alpha u_1\|_{L^{aq'}}+\|\partial^\alpha v_1\|_{L^{aq'}}\}\{\|\partial^\beta u_1\|_{L^{bq'}}+\|\partial^\beta v_1\|_{L^{bq'}}\}. \end{aligned}$$

PROOF. Noting (5.10), we can apply Lemma 2.4 (i) to obtain the first inequality of (5.12). The second inequality can be proved by the same argument as in the proof of Lemma 4.5. □

LEMMA 5.6. *Let a, b, h be as given in Lemma* 4.6. *Then we have*

$$(5.13) \qquad \begin{aligned} \|F(u)-F(v)\|_X &= \|(V*|u_1|^2)u_1-(V*|v_1|^2)v_1\|_{H^{k-1,2}} \\ &\leqq c \sum_{|\alpha+\beta+\gamma|\leqq k-1} \|\partial^\gamma u_1-\partial^\gamma v_1\|_{L^{2h}} \\ &\quad \times\{\|\partial^\alpha u_1\|_{L^{2a}}+\|\partial^\alpha v_1\|_{L^{2a}}\}\{\|\partial^\beta u_1\|_{L^{2b}}+\|\partial^\beta v_1\|_{L^{2b}}\}. \end{aligned}$$

PROOF. We can also follow the argument of Lemma 4.5. □

PROOF OF THEOREM 5.1. First note that (5.3) comes from (5.8) and the relation $s=3=2/d$.

(I) The same proof of Theorem 4.1 is applicable to this case if we deal with the problem in componentwise.

(II) As in the proof of Theorem 4.1, (3.5) is easily verified. The rest of (II) with $d=2/3$ is already known in Lemmas 5.3 and 5.4.

(III) Noting Lemmas 2.4 (i), 2.5 and 5.5, we obtain the following condition which corresponds to (4.14):

$$(5.14) \qquad \frac{1}{2}+\frac{2}{q} \geqq 1-\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}+\frac{1}{q''} \geqq \frac{1}{2}+\frac{2}{q}-\frac{k-|\alpha+\beta+\gamma|}{n}-\frac{2(k-1+e)}{n}.$$

For each $|\alpha+\beta+\gamma|\leqq k$, this defines a strip region of $(\theta, 1/q'')$, where θ moves in the interval $[0, 1]$. We denote by I_1 the narrowest region which occurs when $|\alpha+\beta+\gamma|=k$. Note that condition (5.10) also defines a strip region of $(\theta, 1/q'')$. We denote this by I_2. Then hypothesis (III) is finally obtained by showing $I_1\cap I_2\neq\emptyset$.

Use (5.3) and (5.4) to eliminate q and e. Then I_1 is represented by

$$(5.15) \qquad \frac{1}{q''} \geqq \frac{1}{2}-\frac{4}{3n}-\frac{2(k-1)}{n}-\frac{4(n-1)}{3n(n-1+\theta)}+\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\},$$

$$(5.16)\qquad \frac{1}{q''} \leqq \frac{1}{2} - \frac{4}{3(n-1+\theta)} + \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}$$

and I_2 is represented by

$$(5.17)\qquad \frac{1}{q''} \geqq \frac{1}{2} + \frac{2}{3(n-1+\theta)},$$

$$(5.18)\qquad \frac{1}{q''} \leqq \frac{1}{2} + \frac{2}{3n} + \frac{2(n-1)}{3n(n-1+\theta)}.$$

From these inequalities we see that the condition for $I_1 \cap I_2 \neq \emptyset$ is given by (4.1), (4.2) and (4.5). In fact (5.16), (5.17) with $\theta=1$ show $\{\sigma/n \text{ or } 1/z\} \geqq 2/n$, and (5.15), (5.18) with $\theta=0$ show $\{\sigma/n \text{ or } 1/z\} \leqq 2(k+1)/n$.

To determine it simply, we choose the following two straight lines included in the regions restricted by (5.15), (5.16) and (5.17), (5.18), respectively:

$$\frac{1}{q''} = \left\{\frac{4}{3n} + \frac{2(k-1)}{n}\right\}\theta + \frac{1}{2} - \frac{8}{3n} + \frac{2(k-1)}{n} + \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} \quad \text{and}$$

$$\frac{1}{q''} = \frac{-2}{3n}\theta + \frac{1}{2} + \frac{4}{3n},$$

where $0 \leqq \theta \leqq 1$. Then as the point of intersection we obtain

$$(5.19)\qquad \frac{1}{q''} = \frac{1}{2} + \frac{2(k-1)}{2nk} + \frac{1}{3k}\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}, \qquad \theta = \frac{k+1}{k} - \frac{n}{2k}\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}.$$

(IV) Noting Lemmas 2.4 (i), 2.5 and 5.6, we can reduce our problem to showing

$$(5.20)\qquad \frac{3}{q} \geqq 1 - \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} + \frac{1}{2} \geqq \frac{3}{q} - \frac{3(k-1+e) - |\alpha+\beta+\gamma|}{n}.$$

Use (5.3), (5.4) and the fact $|\alpha+\beta+\gamma| \leqq k-1$. Then (5.20) is rewritten as

$$\frac{2k}{n} + \frac{2(n-1)}{n(n-1+\theta)} \geqq \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} \geqq \frac{2}{n-1+\theta}.$$

This follows from (4.1), (4.2) and (4.5) since we have

$$\frac{2(k+1)}{n} \geqq \frac{2k}{n} + \frac{2(n-1)}{n(n-1+\theta)} \geqq \frac{2}{n-1+\theta} \geqq \frac{2}{n}$$

for any $0 \leqq \theta \leqq 1$. □

6. The wave equation with a cubic convolution.

In this section we consider equation (1.3) with the cubic convolution nonlinearity (1.4) requiring

$$(6.1)\qquad |V(x)| \leqq c|x|^{-\sigma} \qquad \text{with } 4 \leqq \sigma < n, \text{ or}$$

$$(6.2)\qquad V(x) \in L^z(\boldsymbol{R}^n) \qquad \text{with } 1 \leqq z \leqq \frac{n}{4}.$$

We put $X=H^{1,k-1,2}\times H^{k-1,2}$ for $k\in\boldsymbol{Z}$, and define

(6.3) $$A=i\begin{pmatrix}0 & 1\\ \Delta & 0\end{pmatrix} \quad \text{with domain } \mathcal{D}(A)=H^{1,k,2}\times H^{k,2},$$

(6.4) $$F(u)=\{0,\, if(w)\}=\{0,\, i(V*|u_1|^2)u_1\},$$

where $u=\{u_1, u_2\}=\{w, w_t\}$.

THEOREM 6.1. *Let $k\in\boldsymbol{Z}$ be chosen to satisfy $k\geqq 1$ and* (4.5). *Put $X=H^{1,k-1,2}\times H^{k-1,2}$, $Y=H^{e,k-1,q}\times H^{-1+e,k-1,q}$ and $Z=H^{2,k-m-2,q}\times H^{k-m-1}$, where*

(6.5) $$\frac{1}{q}=\frac{1}{2}-\frac{2}{3(n-1)},$$

(6.6) $$e=\frac{n+1}{2q}-\frac{n-3}{4}=\frac{2}{3}\Big(1-\frac{1}{n-1}\Big)$$

and $m\geqq 1$ is a large integer. Then (1.3) *can be written in the form* (3.3), *and all the assertions of Theorem* 3.1 *hold.*

The following lemma is obvious from definition of Y and Z.

LEMMA 6.2. *We have $Y'=H^{2-e,k-1,q'}\times H^{1-e,k-1,q'}$ and $Z'=H^{k+m,q'}\times H^{k+m-1,q'}$.*

LEMMA 6.3. (i) *A is selfadjoint in X.*

(ii) *Let B be a positive selfadjoint operator in L^2 defined by*

(6.7) $$B=\sqrt{-\Delta} \quad \textit{with domain } \mathcal{D}(B)=H^{1,2}.$$

Then B can be continuously extended in $H^{1,0,2}$ (still denoted B), and

(6.8) $$U_0(t)=\exp\{-iAt\}=\begin{pmatrix}\cos Bt & B^{-1}\sin Bt\\ -B\sin Bt & \cos Bt\end{pmatrix}.$$

(iii) *We have*

(6.9) $$\|U_0(t)\phi\|_Y\leqq c(q)|t|^{-d}\|\phi\|_{Y'}, \qquad \phi\in C_0^\infty(\boldsymbol{R}^n)\times C_0^\infty(\boldsymbol{R}^n),$$

where

(6.10) $$d=(n-1)\Big(\frac{1}{2}-\frac{1}{q}\Big).$$

PROOF. (i) and (ii) are well known. On the other hand, if we use the estimate (see e.g., Pecher [15])

$$\|B^{-1}\sin(Bt)\varphi\|_{L^p}\leqq c(q)|t|^{-d}\|B^{1-2e}\varphi\|_{L^{q'}}, \qquad \varphi\in C_0^\infty(\boldsymbol{R}^n),$$

then we have

$$\begin{aligned}\|U_0(t)\phi\|_Y &= \|B^{-1}\cos(Bt)B^{1+e}\phi_1+B^{-1}\sin(Bt)B^e\phi_2\|_{H^{k-1,q}}\\ &\quad+\|-B^{-1}\sin(Bt)B^{1+e}\phi_1+B^{-1}\cos(Bt)B^e\phi_2\|_{H^{k-1,q}}\\ &\leqq c|t|^{-d}\{\|B^{2-e}\phi_1\|_{H^{k-1,q'}}+\|B^{1-e}\phi_2\|_{H^{k-1,q'}}\}\end{aligned}$$

proving (6.9). □

LEMMA 6.4. *$U_0(t)$ maps Z' continuously to Y':*

$$\|U_0(t)\phi\|_{Y'} \leqq c(t)\|\phi\|_{Z'}. \tag{6.11}$$

PROOF. We put

$$\rho(\xi\,;\,t)=|\xi|^{1-e}\langle\xi\rangle^{-m}\begin{pmatrix} |\xi|\langle\xi\rangle^{-1}\cos|\xi|t & \sin|\xi|t \\ -|\xi|\langle\xi\rangle^{-1}\sin|\xi|t & \cos|\xi|t\end{pmatrix}.$$

Then this ρ satisfies the condition of Lemma 2.3. Thus,

$$\|U_0(t)\phi\|_{Y'}=\|\mathcal{F}^{-1}\rho*\{\langle B\rangle^{k+m}\phi_1,\ \langle B\rangle^{k+m-1}\phi_2\}\|_{L^{q'}\times L^{q'}} \leqq c(t)\|\phi\|_{Z'}. \quad \square$$

LEMMA 6.5. *Let q'' be defined by*

$$\frac{1}{q'}=\frac{1}{q''}-\frac{e}{n}, \tag{6.12}$$

and let a, b, h be as given in Lemma 5.5 with this q''. Then we have

$$\begin{aligned}\|F(u)-F(v)\|_{Y'} &\leqq c\|(V*|u_1|^2)u_1-(V*|v_1|^2)v_1\|_{H^{1,k-1,q''}} \\ &\leqq c\sum_{1\leqq|\alpha+\beta+\gamma|\leqq k}\|\partial^\gamma u_1-\partial^\gamma v_1\|_{L^{hq''}} \\ &\quad\times\{\|\partial^\alpha u_1\|_{L^{aq'}}+\|\partial^\alpha v_1\|_{L^{aq'}}\}\{\|\partial^\beta u_1\|_{L^{bq'}}+\|\partial^\beta v_1\|_{L^{bq'}}\}.\end{aligned} \tag{6.13}$$

PROOF. The first inequality follows from the Sobolev embedding Lemma 2.4 (ii) if we note (6.12). The second inequality is proved by the same argument of Lemma 4.5. □

As is easily seen, Lemma 5.6 holds also in the present case without any modification.

PROOF OF THEOREM 6.1. (I) The argument of Theorem 4.1 is also applicable to this case if we deal with the problems in componentwise and if we use all the assertions of Lemma 2.4.

(II) Since (3.5) is easy, Lemmas 6.3 and 6.4 establish (II) with $d=2/3$.

(III) We shall estimate each term of the right side of (6.13). Since $|\alpha+\beta+\gamma|\geqq 1$, there exists at least one non-zero vector in each triplet α, β, γ. Suppose that $\gamma\neq 0$ (the other cases can similarly be treated). Our problem is then to obtain, e. g., the following type of inequalities:

$$\|\partial^\gamma u_1-\partial^\gamma v_1\|_{L^{hq''}}\|\partial^\alpha u_1\|_{L^{aq''}}\|\partial^\beta u_1\|_{L^{bq''}} \leqq c\|u_1-v_1\|_{H^{1,k-1,2}}\|u_1\|^2_{H^{e,k-1,q}}.$$

For this aim it is enough to show the existence of $1\leqq a$, b, $h\leqq\infty$ which satisfies (5.11) and

$$\frac{1}{2}\geqq\frac{1}{hq''}\geqq\frac{1}{2}-\frac{(k-1)-(|\gamma|-1)}{n},\qquad \frac{1}{q}\geqq\frac{1}{aq''}+\frac{e}{n}\geqq\frac{1}{q}-\frac{(k-1)-|\alpha|}{n}$$

and $\dfrac{1}{q} \geqq \dfrac{1}{aq''} + \dfrac{e}{n} \geqq \dfrac{1}{q} - \dfrac{(k-1) - |\beta|}{n}.$

Noting $|\alpha+\beta+\gamma| \leqq k$, we see that these conditions are rewritten to

$$\frac{1}{2}+\frac{2}{q} \geqq 1-\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}+\frac{1}{q''}+\frac{2e}{n} \geqq \frac{1}{2}+\frac{2}{q}-\frac{2(k-1)}{n},$$

or equivalently, to

$$\frac{4}{n} \leqq \left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\} \leqq \frac{4}{n}+\frac{2(k-1)}{n} \tag{6.14}$$

if we use (6.5), (6.6) and (6.12). This is what we have required in (6.1), (6.2) and (4.5).

(IV) We see from (5.13) that (IV) follows from the inequality

$$\frac{3}{q} \geqq 1-\left\{\frac{\sigma}{n} \text{ or } \frac{1}{z}\right\}+\frac{1}{2}+\frac{3e}{n} \geqq \frac{3}{n}-\frac{3(k-1)}{n},$$

which is also equivalent to (6.14). □

References

[1] J. Bergh and J. Löfström, Interpolation Spaces, Springer, 1976.

[2] Ph. Brenner, On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equations, J. Differential Equations, **56** (1985), 310-344.

[3] J.P. Dias and M. Figueira, Conservation laws and time decay of some nonlinear Schrödinger-Hartree equations and systems, J. Math. Anal. Appl., **84** (1981), 486-508.

[4] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations with nonlocal interaction, Math. Z., **170** (1980), 109-136.

[5] R.T. Glassey, Asymptotic behavior of solutions to certain nonlinear Schrödinger-Hartree equations, Comm. Math. Phys., **53** (1977), 9-18.

[6] E. Gross, Quantum theory of interacting bosons, Ann. Physics, **9** (1960), 292-324.

[7] N. Hayashi and Y. Tsutsumi, Scattering theory for Hartree type equations, Ann. Inst. H. Poincaré, Phys. Théor., **46** (1987), 187-213.

[8] N. Hayashi and T. Ozawa, Scattering theory in the weighted $L^2(\boldsymbol{R}^n)$ spaces for some Schrödinger equations, preprint, 1987.

[9] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Springer, 1983.

[10] P.L. Lions, The Choquard equation and related equations, J. Nonlinear Anal. Theory, Methods, Appl., **4** (1980), 1073-1079.

[11] S. Matsumoto and K. Mochizuki, On blowing up of solutions of Schrödinger equations with cubic convolution, preprint, 1987.

[12] G.P. Menzala, On regular solutions of a nonlinear equations of Choquard's type, Proc. Roy. Soc. Edinburgh, Sect. A, **86** (1980), 291-301.

[13] G.P. Menzala and W.A. Strauss, On a wave equation with cubic convolution, J. Differential Equations, **43** (1982), 93-105.

[14] K. Mochizuki and T. Motai, On small data scattering for some nonlinear wave equations, Patterns and Waves—Qualitative Analysis of Nonlinear Differential Equations—, 1986, pp. 543-560.

[15] H. Pecher, L^p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen I, Math. Z., **150** (1976), 159-183.
[16] H. Pecher, Nonlinear small data scattering for the wave and Klein-Gordon equation, Math. Z., **185** (1984), 261-270.
[17] W. A. Strauss, Nonlinear scattering theory at law energy, J. Funct. Anal., **41** (1981), 110-133, Sequel, **43** (1981), 281-293.

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