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Rate of decay at high energy of local spectral projections associated with Schrödinger operators

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§1. Introduction.

Let $\tilde{\mathfrak{H}}=-(1/2)\Delta+V(x)$, $\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$, be a Schrödinger operator on $\boldsymbol{R}^n$, $n\geqq 1$. We assume that the potential $V(x)$ satisfies the following assumption for some $m\geqq 0$.

ASSUMPTION $(\mathrm{A})_m$. $V(x)$ is a real-valued C^∞-function of x and for any multi-index α,

$$|\partial_x^\alpha V(x)| \leqq C_\alpha(1+|x|)^{m-|\alpha|}, \quad x\in\boldsymbol{R}^n. \tag{1.1}$$

Then the operator $\tilde{\mathfrak{H}}$ with the domain $\mathfrak{D}(\tilde{\mathfrak{H}})=\mathcal{S}(\boldsymbol{R}^n)$, the space of rapidly decreasing functions, is real symmetric in the Hilbert space $L^2(\boldsymbol{R}^n)$. We let $\mathfrak{H}$ be any one of its selfadjoint extensions and $\{E_{\mathfrak{H}}(I), I\in\mathfrak{B}^1\}$ the associated spectral measure. $\mathfrak{B}^1$ is the σ-field of Borel subsets of $\boldsymbol{R}^1$.

The purpose of this paper is to study the spectral projections $E_{\mathfrak{H}}(I)$ at high energy and to prove, in particular, the following theorem. We denote $\tilde{m}=\max(m, 2)$ and $\langle x\rangle=(1+x^2)^{1/2}$.

THEOREM 1.1. *Let $V(x)$ satisfy the assumption $(\mathrm{A})_m$, $m\geqq 0$ and let $\mathfrak{H}$ be a selfadjoint extension of $-(1/2)\Delta+V(x)|_{\mathcal{S}(\boldsymbol{R}^n)}$ in the Hilbert space $L^2(\boldsymbol{R}^n)$. Then for any $q>1/2$ and $\rho>0$ there exists a constant $C>0$ such that*

$$\|\langle x\rangle^{-q}E_{\mathfrak{H}}([\lambda-\rho\lambda^{1/2-1/\tilde{m}}, \lambda+\rho\lambda^{1/2-1/\tilde{m}}])\langle x\rangle^{-q}\| \leqq Cd_m(\lambda) \tag{1.2}$$

for all $\lambda\geqq 0$. Here $d_m(\lambda)=\langle\lambda\rangle^{-1/\tilde{m}}$ and $\langle\lambda\rangle=(1+\lambda^2)^{1/2}$.

REMARK 1.2. (1) When $V(x)\equiv 0$, it is well-known that the decay rate $\lambda^{-1/2}$ is optimal. Thus the theorem implies the invariance of the decay rate of "local spectral measure" $\langle x\rangle^{-q}E_{\mathfrak{H}}([\lambda-\rho, \lambda+\rho])\langle x\rangle^{-q}$ for $m\leqq 2$.

(2) When $V(x)$ is singular, $V\in L^p_{\mathrm{loc}}$, $p>n/2$, a weaker version of (1.2) appears in Section 6.

COROLLARY 1.3. *Let $\phi_j(x)$ be the normalized eigenfunction of $\mathfrak{H}$ associated with the eigenvalue $\lambda_j\geqq 0$. Then for $q>1/2$,* (1.3) $$\|\langle x\rangle^{-q}\phi_j(x)\|_{L^2} \leqq Cd_m(\lambda_j)^{1/2},$$

where C *is independent of* $\lambda_j \geqq 0$.

COROLLARY 1.4. *Let* $R(z)=(\mathfrak{H}-z)^{-1}$. *Then for any* $\varepsilon>0$ *and* $q>1/2$,

(1.4) $$\|\langle x\rangle^{-q}R(\lambda\pm i\varepsilon)\langle x\rangle^{-q}\| \leqq C_{q\varepsilon}d_m(\lambda)\log(\lambda+2), \qquad \lambda\geqq 0.$$

The study of the spectral theory for Schrödinger or for general elliptic operators has a long history and it has a huge body of literature (cf. Hörmander [**8**], Reed-Simon [**16**], Simon [**18**] and references therein). Among the topics intensively studied are the high energy asymptotics of eigenvalues and the asymptotic behavior at $|x|\to\infty$ of the eigenfunctions. Compared to these, the asymptotic behavior at high energy of eigenfunctions is less intensively studied and even the simplest results like (1.2) do not seem to exist in the literature except for the case $n=1$ or the case $m<0$. Titchmarsh [**20**] showed a slightly stronger results than (1.3) for the case $n=1$ when $V(x)$ is increasing in $|x|$ and $V''(x)>0$. Gel'fand-Levitan [**6**] and their colleagues studied in detail the asymptotic behavior as $\lambda\to\infty$ of solutions of $-u''+V(x)u=\lambda u$ and the associated spectral function on compact subsets of $x\in\boldsymbol{R}^1$ (cf. Levitan-Sargsjan [**14**]). As for the case $m<0$, (1.2) is known for a long time in the scattering theory (cf. Agmon [**1**] for the case $m<-1$; for $m<0$ see Saito [**17**] or Isozaki-Kitada [**10**]; see also Vainberg [**21**]).

In fact, the motivation for this work was originated from the stability problem for time periodic Schrödinger equations

(1.5) $$\frac{i\partial u}{\partial t} = -\frac{1}{2}\Delta u+V(x)u+V_1(t, x)u, \qquad V_1(t+1, x)=V_1(t, x).$$

We ask the following question: Suppose $\mathfrak{H}=-(1/2)\Delta+V(x)$ has purely discrete spectrum $\{\lambda_j: j=1, 2, \cdots\}$ with eigenfunctions $\{\phi_j(x)\}$. Then all solutions of the initial value problem for

$$\frac{i\partial u}{\partial t} = -\frac{1}{2}\Delta u+V(x)u$$

may be written in the form $\sum c_j e^{-i\lambda_j t}\phi_j(x)$ and they are bound states in the following sense:

$$\lim_{R\to\infty}\sup_t\|u(t)\|_{L^2(|x|>R)}=0,$$

$$\lim_{\lambda\to\infty}\sup_t\|E_{\mathfrak{H}}((\lambda, \infty))u(t)\|=0.$$

Do all solutions of the initial value problem for (1.5) remain as bound states after a small periodic perturbation $V_1(t, x)$ is turned on? or do some solutions gain more and more energy from $V_1(t, x)$ via the resonance and fly away from any compact subset of x-space?

It is well-known ([**22**], [**23**]) that this problem is virtually equivalent to the spectral problem for the Floquet operator $U(1, 0)$, $U(t, s)$ being the evolution operator for (1.5). Moreover the spectral properties of $U(1, 0)$ can be inferred from those of explicit operator $\mathfrak{K}=-i\partial/\partial t-(1/2)\Delta+V(x)+V_1(t, x)$ on the extended Hibert space $L^2(\boldsymbol{T}\times\boldsymbol{R}^n)$, $\boldsymbol{T}=\boldsymbol{R}/\boldsymbol{Z}$. We regard $\mathfrak{K}$ a perturbation of $\mathfrak{K}_0=-i\partial/\partial t+H$. Clearly $\mathfrak{K}_0$ has only pure point spectrum $\{2n\pi+\lambda_j;\ n=0, \pm 1, \pm 2, \cdots, j=1, 2, \cdots\}$ which, however, is dense in the reals in generic. Thus for dealing with $\mathfrak{K}$ by the perturbation technique, we need overcome the problem of small divisors. This requires some detailed analysis of eigenfunctions at high energy and this work should be regarded as a first step toward such direction (cf. Arnold [2], Bellissard [4], Moser [15], Gallavotti [5], Howland [9]).

The rest of the paper is devoted to the proof of the theorem. Our approach is rather standard and basically simple (cf. [7], [19] and [21]). We decompose the phase space $T^*\boldsymbol{R}^n=\boldsymbol{R}^n\times\boldsymbol{R}^n$ into two parts, the on-shell part where $|\xi^2/2+V(x)-\lambda|<C\lambda$ and the off-shell part $|\xi^2/2+V(x)-\lambda|>C\lambda$, $C>0$. Since $x\in\boldsymbol{R}^n$ can be localized to a ball $B(\rho_0\lambda^{1/\tilde{m}})=\{x: |x|\leqq\rho_0\lambda^{1/\tilde{m}}\}$, this splitting can be accomplished by cutting the ξ-space: $\Phi_0(\xi, \lambda)+\Phi_1(\xi, \lambda)=1$ where $\Phi_j(\xi, \lambda)\in C^\infty(\boldsymbol{R}^n)$ $(j=0, 1)$ is such that $0\leqq\Phi_j(\xi, \lambda)\leqq 1$, $\Phi_0(\xi, \lambda)=1$ for $1/2\leqq\xi^2/2\lambda\leqq 2$ and $\Phi_0(\xi, \lambda)=0$ for $\xi^2/2\lambda\notin[1/3, 3]$. $\chi\in C_0^\infty(\boldsymbol{R}^n)$, $0\leqq\chi(x)\leqq 1$, is a cut-off function such that

$$\chi(x)=1 \quad \text{for } |x|\leqq 9/10 \qquad \text{and} \qquad \chi(x)=0 \quad \text{for } |x|\geqq 1. \tag{1.6}$$

$I=I(\rho, \lambda)=[\lambda-\rho\lambda^{1/2-1/\tilde{m}}, \lambda+\rho\lambda^{1/2-1/\tilde{m}}]$.

In Section 2 we study the off-shell part $E_{\mathfrak{H}}(I)\Phi_1(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$ using the expression by the resolvent:

$$E_{\mathfrak{H}}(I)=-\frac{1}{2\pi i}\int_\Gamma(\mathfrak{H}-\zeta)^{-1}d\zeta. \tag{1.7}$$

We study the operator $(\mathfrak{H}-\zeta)^{-1}\Phi_1(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$ by constructing the parametrix in terms of a pseudo-differential operator. In Sections 3~5, we treat the on-shell part $E_{\mathfrak{H}}(I)\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$. Because of the difficulty near the reals, we evade (1.7) and exploit the expression through the propagator:

$$g(\mathfrak{H}-\lambda)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\check{g}(t)e^{-it(\mathfrak{H}-\lambda)}dt, \tag{1.8}$$

where $\check{g}$ is the inverse Fourier transform of g. We investigate $e^{-it\mathfrak{H}}\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$ by constructing its approximation for $|t|\leqq\rho_0\lambda^{-(1/2-1/\tilde{m})}$ in terms of a Fourier integral operator which has the phase linear in the time variable:

$$F(t)f(x)=(2\pi)^{-n}\int e^{i\phi(x,\xi,y)-it(\xi^2/2+V(y))}A(t, x, \xi, y)f(y)dyd\xi. \tag{1.9}$$

The operator of this form was first introduced by Hörmander [7] for studying the spectral properties of elliptic operators and has been proved to be quite efficient ([**19**], [**21**]). In Section 3 the phase function ϕ is constructed and necessary estimates are proved by solving the Hamilton-Jacobi equation associated with $\mathfrak{H}$. In Section 4 we construct $A(t, x, \xi, y)$ and show that $F(t)$ of (1.9) is indeed a good approximation to $e^{-it\mathfrak{H}}\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$ for $|t|\leqq\rho_0\lambda^{-1/2+1/\tilde{m}}$. This reduces the estimation of $E_{\mathfrak{H}}(I)\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})$ to that of the integral operator of the form

$$(2\pi)^{-n}\int e^{i\phi(x,\xi,y)}B(\lambda^{-1/2+1/\tilde{m}}(\xi^2/2+V(y)-\lambda),\ x,\ \xi,\ y)f(y)dyd\xi .$$

We study this operator in Section 5 and complete the proof of Theorem 1.1 and Corollaries 1.3~1.4. Section 6 contains a brief discussion for non-smooth potentials.

We list here some of the notation which will be used throughout the paper. For $x=(x_1, x_2, \cdots, x_n)\in \boldsymbol{R}^n$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$. When we write $x=(x_1, \underline{x})\in\boldsymbol{R}^n$, $\underline{x}=(x_2, \cdots, x_n)\in\boldsymbol{R}^{n-1}$ and for $\tau=(\tau_1, \cdots, \tau_{n-1})\in\boldsymbol{R}^{n-1}$, $\underline{\tau}=(0, \tau_1, \cdots, \tau_{n-1})\in\boldsymbol{R}^n$. For $p\geqq 1$ and $r\geqq 1$, $L^{p,r}(\boldsymbol{R}^n)$ stands for

$$L^r(\boldsymbol{R}^1, L^p(\boldsymbol{R}^{n-1}))=\left\{f:\left(\int_{-\infty}^{\infty}\left\{\int_{\boldsymbol{R}^{n-1}}|f(x_1, \underline{x})|^p d\underline{x}\right\}^{r/p}dx_1\right)^{1/r}=\|f\|_{p,r}<\infty\right\}.$$

$L^{2,2}(\boldsymbol{R}^n)=L^2(\boldsymbol{R}^n)$ and its norm is denoted simply by $\|\ \ \|$. This symbol is also used to denote the norm of operators from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$. We also use the function space

$$L_w^{2,1}=L_w^1(\boldsymbol{R}^1: L^2(\boldsymbol{R}^{n-1}))=\{f:\sup_{\lambda}\lambda\mu(\{x_1: \|f(x_1, \cdot)\|_{L^2}>\lambda\})=\|f\|_{L_w^{2,1}}<\infty\},$$

where μ stands for the 1-dimensional Lebesgue measure. $\partial_x=(\partial_{x_1}, \cdots, \partial_{x_n})$ and for multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$. $|\alpha|=\alpha_1+\cdots+\alpha_n$. $\alpha<\beta$ means $\alpha\neq\beta$ and $\alpha_j\leqq\beta_j$ for $j=1, 2, \cdots, n$. $\boldsymbol{e}_j=(0, \cdots, 1, \cdots, 0)$ is the j-th standard unit vector. For some class of functions $p(x, \xi, y)$, the pseudo-differential operator $P(x, D, x)$ with the symbol $p(x, \xi, y)$ is defined as

$$P(x, D, x)f(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}p(x, \xi, y)f(y)dyd\xi$$

where the integral is taken in the sense of oscillatory integrals (cf. Hörmander [**7**] or Kumano-go [**13**]). Symbol $f(x)$ may also stand for the multiplication operator by the function $f(x)$. For $a\in\boldsymbol{R}$, $a_+=\max(0, a)$. Hess ϕ is the Hessian matrix $\{\partial^2\phi/\partial x_i\partial x_j\}$ and $\partial_x\partial_y\phi$ etc. stands for the matrix $\{\partial^2\phi/\partial x_i\partial y_j\}$ etc. The same symbol C may stand for different constants in various contexts when no confusions are feared.

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§ 2. Off-shell estimate.

We let $\phi_j \in C^\infty(\boldsymbol{R}^1)$ $(j=0, 1)$ be such that $0 \leqq \phi_j(t) \leqq 1$ and

$$\phi_0(t)=1 \quad \text{for } 2^{-1} \leqq t \leqq 2, \qquad \phi_0(t)=0 \quad \text{for } t \notin [1/3, 3], \tag{2.1}$$

$$\phi_0(t)+\phi_1(t) = 1, \qquad t \in \boldsymbol{R}^1 \tag{2.2}$$

and set $\Phi_j(\xi, \lambda)=\phi_j(\xi^2/2\lambda)$. Clearly

$$|\partial_\xi^\alpha \Phi_j(\xi, \lambda)| \leqq C_{\alpha,N} \lambda^{-|\alpha|/2} \langle \xi/\sqrt{\lambda} \rangle^{-N}, \qquad |\alpha| \geqq 1,\ N \geqq 0. \tag{2.3}$$

Decomposing the spectral projection

$$E_{\mathfrak{H}}(I(\lambda, \rho)) = E_{\mathfrak{H}}(I(\lambda, \rho))\Phi_0(D, \lambda)+E_{\mathfrak{H}}(I(\lambda, \rho))\Phi_1(D, \lambda), \tag{2.4}$$

we treat, in this section, the off-shell part $E_{\mathfrak{H}}(I(\lambda, \rho))\Phi_1(D, \lambda)$.

THEOREM 2.1. *Let $V(x)$ satisfy* $(\mathrm{A})_m$. *Then for any $\sigma \geqq 0$, there exists a constant $C>0$ such that for $\lambda \geqq 1$*

$$\|E_{\mathfrak{H}}(I(\rho, \lambda))\Phi_1(D, \lambda)\langle x \rangle^{-\sigma}\| \leqq C(\lambda^{-1+(m-1)_+/m-1/\tilde{m}}+\lambda^{-\sigma/m}). \tag{2.5}$$

In the rest of this section, we devote ourselves to proving Theorem 2.1. We let $\Gamma(\lambda, \rho)$ be the complex rectangular contour with the vertexes $\lambda \pm 2\rho\lambda^{1/2-1/\tilde{m}} \pm \sqrt{-1}$ and take $\lambda_0>0$ and $\varepsilon>0$ such that

$$\lambda/5-2\rho\lambda^{(1/2-1/m)_+} \geqq 10 \qquad \text{for } \lambda \geqq \lambda_0, \tag*{(2.6)$_1$}$$

$$|V(x)| \leqq 10^{-6}\lambda \qquad \text{for } |x| \leqq 2\varepsilon\lambda^{1/m},\ \lambda \geqq \lambda_0. \tag*{(2.6)$_2$}$$

It is enough to prove the theorem for $\lambda \geqq \lambda_0$. We set

$$\begin{aligned} P(z, \lambda)f(x) &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi} p(\xi, y; z, \lambda)f(y)dyd\xi, \\ p(\xi, y; z, \lambda) &= (\xi^2/2+V(y)-z)^{-1}\Phi_1(\xi, \lambda)\chi(y/\varepsilon\lambda^{1/m}). \end{aligned} \tag{2.7}$$

LEMMA 2.1. *Let $\lambda \geqq \lambda_0$ and $z \in \Gamma(\lambda, \rho)$. Then $P(z, \lambda)$ maps $\mathcal{S}(\boldsymbol{R}^n)$ into $\mathcal{S}(\boldsymbol{R}^n)$ and with a constant $C>0$ independent of f and z*

$$\|P(z, \lambda)f\| \leqq C\lambda^{-1}\|f\|. \tag{2.8}$$

PROOF. By virtue of (2.3) and (2.6), we have for $(\xi, y) \in \operatorname{supp} p$ and $z \in \Gamma(\lambda, \rho)$ that $|\xi^2/2+V(y)-z| \geqq \max(\lambda/5, \xi^2/10)$. Hence

$$|\partial_y^\beta \partial_\xi^\gamma p(\xi, y; z, \lambda)| \leqq C_{\beta\gamma}\lambda^{(m-|\beta|)_+/m-1}\min(\lambda^{-1-|\gamma|/2}, \langle \xi \rangle^{-2-|\gamma|}). \tag{2.9}$$

It follows ([8]) that $P(z, \lambda)$ is continuous both in $\mathcal{S}(\boldsymbol{R}^n)$ and $L^2(\boldsymbol{R}^n)$ and that the estimate (2.8) is satisfied.

By Lemma 2.1, we may compute for $f\in\mathcal{S}(\boldsymbol{R}^n)$:

$$(2.10)\quad (\mathfrak{H}-z)P(z, \lambda)f(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}(\xi^2/2+V(x)-z)p(\xi, y; z, \lambda)f(y)dyd\xi$$
$$=\Phi_1(D, \lambda)\chi(x/\varepsilon\lambda^{1/m})f(x)+W(z, \lambda)f(x),$$

$$(2.11)\quad W(z, \lambda)f(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}w(x, \xi, y)f(y)dyd\xi,$$
$$w(x, \xi, y)=(V(x)-V(y))p(\xi, y; z, \lambda).$$

LEMMA 2.3. *There exists a constant $C>0$ independent of $z\in\Gamma(\lambda, \rho)$, $\lambda\geqq\lambda_0$ and $f\in\mathcal{S}(\boldsymbol{R}^n)$ such that*

$$(2.12)\quad \|W(z, \lambda)f\|\leqq C\lambda^{-3/2+(m-1)_+/m}\|f\|.$$

PROOF. Since $e^{i(x-y)\cdot\xi}[V(x)-V(y)]=-i\partial_\xi e^{i(x-y)\cdot\xi}\cdot V^{(1)}(x, y)$, $V^{(1)}(x, y)=\int_0^1\partial_x V(\theta x+(1-\theta)y)d\theta$, integration by parts yields

$$(2.13)\quad W(z, \lambda)f(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}V^{(1)}(x, y)\cdot\partial_\xi p(\xi, y; z, \lambda)f(y)dyd\xi.$$

By virtue of (2.9) and (1.1),

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\chi(x/4\varepsilon\lambda^{1/m})V^{(1)}(x, y)\partial_\xi p(\xi, y; z, \lambda)|\leqq C_{\alpha\beta\gamma}\lambda^{-3/2+(m-|\alpha|-|\beta|-1)_+/m}.$$

Hence the Calderón-Vaillancourt theorem implies

$$(2.14)\quad \|\chi(x/4\varepsilon\lambda^{1/m})W(z, \lambda)f\|\leqq C\lambda^{-3/2+(m-1)_+/m}\|f\|.$$

When $|x|\geqq 3\varepsilon\lambda^{1/m}$ and $(\xi, y)\in\operatorname{supp}p$, then $|x-y|\geqq\varepsilon\lambda^{1/m}$ and after integrating by parts with respect to ξ-variables we have,

$$W(z, \lambda)f(x)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}|x-y|^{-2N}V^{(1)}(x, y)(-\Delta_\xi)^N\partial_\xi p(\xi, y; z, \lambda)f(y)dyd\xi.$$

It follows that

$$|(1-\chi(x/4\varepsilon\lambda^{1/m}))W(z, \lambda)f(x)|\leqq C_N\lambda^{-N/2+(m-1)_+/m}\int\langle x-y\rangle^{-N}|f(y)|dy,$$
$$N>n+m,$$

and the Young inequality implies for $N=1, 2, \cdots$

$$(2.15)\quad \|(1-\chi(x/4\varepsilon\lambda^{1/m}))W(z, \lambda)f\|\leqq C_N\lambda^{-N/2+(m-1)_+/m}\|f\|.$$

Summing up (2.14) and (2.15), we obtain (2.12).

PROOF OF THEOREM 2.1. Multiply both sides of (2.10) by $(2\pi i)^{-1}(\mathfrak{H}-z)^{-1}\times E_{\mathfrak{H}}(I(\lambda, \rho))$ and integrate the resulting equation with respect to $z\in\Gamma(\lambda, \rho)$. By Cauchy's theorem, it follows that

$$E_{\mathfrak{H}}(I(\lambda, \rho))\Phi_1(D, \lambda)\chi(x/\varepsilon\lambda^{1/m})f=(2\pi i)^{-1}\int_{\Gamma(\lambda, \rho)}(\mathfrak{H}-z)^{-1}E_{\mathfrak{H}}(I(\lambda, \rho))W(z, \lambda)f\,dz$$

and (2.12) implies

$$\|E_{\mathfrak{L}}(I(\lambda,\ \rho))\Phi_1(D,\ \lambda)\chi(x/\varepsilon\lambda^{1/m})f\| \leqq C\lambda^{-1+(m-1)_+/m-1/\hat{m}}. \tag{2.16}$$

Combining (2.16) with the obvious estimate

$$\|E_{\mathfrak{H}}(I(\lambda,\ \rho))\Phi_1(D,\ \lambda)(1-\chi(x/\varepsilon\lambda^{1/m}))\langle x\rangle^{-\sigma}\| \leqq C\lambda^{-\sigma/m},$$

we conclude the proof of Theorem 2.1.

§ 3. Hamilton-Jacobi equation.

We consider the following boundary value problem for Hamilton-Jacobi equation

$$\frac{1}{2}(\partial_x\phi)^2(x,\ \xi,\ y)+V(x) = \frac{1}{2}\xi^2+V(y), \tag{3.1}$$

$$\phi(x,\ \xi,\ y) = 0 \qquad \text{for } (x-y)\cdot\xi=0, \tag{3.2}$$

$$\partial_x\phi(x,\ \xi,\ y) = \xi \qquad \text{for } x=y. \tag{3.3}$$

We solve the equations (3.1)$\sim$(3.3) by the standard method of bicharacteristics ([**11**]). We find it convenient to parametrize the plane $\{x:(x-y)\cdot\xi=0\}$ via ξ-independent variables (cf. [**19**]): Take a finite covering $\{\hat{S}_j\}_{j=1}^N$ of the unit sphere $S^{n-1}=\{\hat{\xi}:|\hat{\xi}|=1\}$ by open subsets $\hat{S}_j=\{\hat{\xi}\in S^{n-1}:|\hat{\xi}-\hat{\xi}^j|<10^{-n}\}$, $\hat{\xi}^j\in S^{n-1}$, and choose for each j a smooth orthogonal matrix-valued function $O_j(\hat{\xi})$ for $\hat{\xi}\in\hat{S}_j$ such that

$$O_j(\hat{\xi})\hat{\xi} = \boldsymbol{e}_1 = (1,\ 0,\ \cdots,\ 0), \qquad \hat{\xi}\in\hat{S}_j. \tag{3.4}$$

We extend the domain of definition of O_j to the cone $S_j=\{\xi:\hat{\xi}=\xi/|\xi|\in\hat{S}_j\}$ spanned by $\hat{S}_j$ by homogeneity: $O_j(\xi)=O_j(\hat{\xi})$, $\xi\in S_j$ and parameterize the plane $\{x:(x-y)\cdot\xi=0\}$ as $\{O_j(\xi)^*\underline{\tau}+y:\tau\in\boldsymbol{R}^{n-1}\}$ where $\underline{\tau}=(0,\ \tau)$. When S_j is fixed and only $\xi\in S_j$ are considered, the subscript j will be often omitted.

For (3.1)$\sim$(3.3) the corresponding bicharacteristic equations are, after scaling the time by the factor $\lambda^{1/2}$,

$$\begin{aligned} &dq/dt = \lambda^{-1/2}p(t), && dp/dt = -\lambda^{-1/2}\partial_x V(q(t)), \\ &q(0) = y+O(\xi)^*\underline{\tau}, && p(0) = (\xi^2+2(V(y)-V(y+O(\xi)^*\underline{\tau})))^{1/2}\hat{\xi}. \end{aligned} \tag{3.5}$$

We consider (3.5) for the parameter $(\tau,\ \xi,\ y)$ in the domain

$$\begin{aligned} D_j(\lambda,\ m\,;\ \rho) = \{&(\tau,\ \xi,\ y)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^n\times\boldsymbol{R}^n: \\ &|\tau|<3\rho\lambda^{1/\hat{m}},\ \xi^2/2\lambda\in(1/4,\ 4),\ \xi\in S_j,\ |y|<10^{-2}\rho\lambda^{1/\hat{m}}\}, \end{aligned} \tag{3.6}$$

and write its solutions as $(q(t,\ \tau,\ \xi,\ y),\ p(t,\ \tau,\ \xi,\ y))$, though some of the variables are often suppressed. $R=|t|+\langle\tau\rangle+\langle y\rangle$ and $k^*=\max(1,\ k)$.

PROPOSITION 3.1. *Let* $(\mathrm{A})_m$ *be satisfied with* $m>1$. *Then there exist*

$0<\rho_1<(10n)^{-2}$ *and a decreasing function* $\lambda_1(\rho)$ *of* $0<\rho\leqq\rho_1$ *such that the following statements are satisfied for* $|t|\leqq 3\rho\lambda^{1/\tilde{m}}$ *and* $(\tau, \xi, y)\in D_j(\lambda, m; \rho)$, $\lambda\geqq\lambda_1(\rho)$, $0<\rho\leqq\rho_1$:

(1) $2^{-1}\leqq\lambda^{-1/2}|p(t)|\leqq 3$.

(2) $2^{-1}(|t|+|\tau|)\leqq|q(t)-y|\leqq 3(|t|+|\tau|)$.

(3) *For* $k=0, 1, 2, \cdots$, *and multi-indeces* $\tilde{\alpha}$, β *and* γ,

$$(3.7)\quad |\partial_t^k\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma(q(t)-y-O(\xi)^*\underline{\tau}-t\lambda^{-1/2}\xi)| \leqq C_{\alpha\beta\gamma k}\lambda^{-1-|\gamma|/2}|t|^{(1-k)_+}R^{(m-|\tilde{\alpha}+\beta+\gamma|-k^*+1)_++|\gamma|}(\log R)^{\varepsilon(m,|\tilde{\alpha}+\beta+\gamma|)},$$

where $\varepsilon(m, j)=1$ *if* $m\in\boldsymbol{N}$, $m\geqq 2$ *and* $j\geqq m$, *and* $\varepsilon(m, j)=0$ *otherwise.*

PROOF. By the assumption (1.1) it is possible to find $0<\rho_2<1$ and a decreasing function $\lambda_1(\rho)$ such that

$$(3.8)\quad |V(x)|+10^2\rho\lambda^{1/\tilde{m}}|\partial_xV(x)|\leqq 10^{-2}\lambda \qquad \text{for } |x|\leqq 10^2\rho\lambda^{1/\tilde{m}}$$

$$(3.9)\quad 10^2(\rho_2+1)\leqq\lambda_1(\rho_2)^{1/\tilde{m}}, \qquad \lim_{\rho\to 0}\rho\lambda_1(\rho)^{1/\tilde{m}}=\infty$$

for any $0<\rho\leqq\rho_2$ and $\lambda\geqq\lambda_1(\rho)$. Hereafter we assume $\lambda\geqq\lambda_1(\rho)$ and $0<\rho\leqq\rho_2$. Let $t^*=\inf\{|t|: |q(t)-y|>10\rho\lambda^{1/\tilde{m}}\}$. Then for $|t|\leqq t^*$, (3.8) implies

$$(3.10)\quad p^2(t)/\lambda=\{\xi^2+2V(y)-2V(q(t))\}/\lambda\in[1/3, 9],$$

$$(3.11)\quad 2^{-1}(d/dt)^2(q(t)-y)^2=\lambda^{-1}\{\xi^2+2V(y)-2V(q(t))-(q(t)-y)\partial_xV(q(t))\} \in[1/3, 9],$$

and hence

$$(3.12)\quad t^2/3\leqq(q(t)-y)^2-\tau^2\leqq 9t^2.$$

Therefore $10^2\rho^2\lambda^{2/\tilde{m}}-9\rho^2\lambda^{2/\tilde{m}}\leqq 9t^{*2}$ and (3.10)$\sim$(3.12) imply the statements (1)$\sim$(2). For proving statement (3) we need prepare several lemmas. First three lemmas, Lemma 3.2$\sim$3.4 are easy consequences of elementary estimates and we omit the proof.

LEMMA 3.2. *Let* $p_0=(\xi^2+2V(y)-2V(y+O^*(\xi)\underline{\tau}))^{1/2}\hat{\xi}$ *and* $q_0=y+O(\xi)^*\underline{\tau}+t\lambda^{-1/2}p_0$. *Then for* $(\tau, \xi, y)\in D_j(\lambda, m; \rho)$ *and* $t\in\boldsymbol{R}^1$,

$$|\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma(p_0-\xi)|\leqq C\langle\xi\rangle^{-1-|\gamma|}(\langle\tau\rangle+\langle y\rangle)^{(m-|\tilde{\alpha}+\beta+\gamma|)_++|\gamma|},$$

$$|\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma(q_0-y-O^*(\xi)\underline{\tau}-t\lambda^{-1/2}\xi)|\leqq C|t|\langle\xi\rangle^{-2-|\gamma|}(\langle\tau\rangle+\langle y\rangle)^{(m-|\tilde{\alpha}+\beta+\gamma|)_++|\gamma|}.$$

LEMMA 3.3. *Suppose that* $q(t)=(q_1(t), \cdots, q_n(t))$ *satisfies* (3.7). *Let* $\theta_j=(k_j, \tilde{\alpha}_j, \beta_j, \gamma_j)$, $j=1, 2, \cdots, l$, $l\geqq 2$ *be such that* $|\theta_j|\geqq 1$ *and* $\theta_1+\cdots+\theta_l=\theta=(k, \tilde{\alpha}, \beta, \gamma)$. *Then for any choice of* $1\leqq i_1, i_2, \cdots, i_l\leqq n$ *we have*

$$(3.13)\quad \left|\prod_{j=1}^l\partial_t^{k_j}\partial_\tau^{\tilde{\alpha}_j}\partial_y^{\beta_j}\partial_\xi^{\gamma_j}q_{i_j}(t, \tau, y, \xi)\right| \leqq C\lambda^{-|\gamma|/2}\min(R^{|\gamma|}, R^{l-|\tilde{\alpha}+\beta|-k}+\lambda^{-1}R^{(m-|\theta|)_++|\gamma|+l})$$

for $|t|\leqq 3\rho\lambda^{1/\tilde{m}}$, $(\tau, \xi, y)\in D(\lambda, m; \rho)$.

LEMMA 3.4. *Let σ, $\tilde{\sigma}\geqq 0$. Suppose that $q(t, \tau, y, \xi)$ satisfies* (3.7) *and that $W(x, \xi, y)$ satisfies*

(3.14) $$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma W(x, \xi, y)| \leqq C\lambda^{-|\gamma|/2}(\langle x\rangle+\langle y\rangle)^{(m-|\alpha+\beta+\gamma|-\sigma)_+ +|\gamma|+\tilde{\sigma}}(\log(\langle x\rangle+\langle y\rangle))^{\varepsilon(m,|\alpha+\beta+\gamma|)}.$$

Then, with $\partial_{(t,\tau)}^\alpha=\partial_t^k\partial_\tau^{\tilde{\alpha}}$ for $\alpha=(k, \tilde{\alpha})$ and $R=|t|+\langle\tau\rangle+\langle y\rangle$,

(3.15) $$|\partial_{(t,\tau)}^\alpha \partial_y^\beta \partial_\xi^\gamma W(q(t, \tau, \xi, y), \xi, y)| \leqq C\lambda^{-|\gamma|/2}R^{(m-|\alpha+\beta+\gamma|-\sigma)_+ +|\gamma|+\tilde{\sigma}}(\log R)^{\varepsilon(m,|\alpha+\beta+\gamma|)}.$$

Moreover (3.15) *remains valid after replacing m by $\tilde{m}$; if the* log *term is absent in* (3.14), *then the factor* $(\log R)^{\varepsilon(m,|\alpha+\beta+\gamma|)}$ *in* (3.15) *may be replaced by* $(\log R)^{\varepsilon(m,|\tilde{\alpha}+\beta+\gamma|)}$.

LEMMA 3.5. *There exists a constant $C_\sigma>0$ independent of $(\tau, \xi, y)\in D(\lambda, m\,;\rho)$ and $|t|\leqq 3\rho\lambda^{1/\tilde{m}}$ such that*

(3.16) $$\int_0^{|t|}\langle q(s)\rangle^\sigma ds \leqq \begin{cases} C_\sigma, & \sigma<-1, \\ C_\sigma \log R, & \sigma=-1, \\ C_\sigma R^{1+\sigma}, & \sigma>-1. \end{cases}$$

PROOF. For $\sigma\geqq 0$, (3.16) follows from the statement (2) of the proposition. Let $\sigma<0$. As in (3.11), it is easy to see that $(1/2)(d/dt)^2 q(t)\geqq 1/3$. By integration $q(t)^2\geqq(t-t_{\min})^2/6+q(t_{\min})^2$ where $t_{\min}\in\boldsymbol{R}^1$ is such that $\min q(t)^2=q(t_{\min})^2$, so $\langle q(s)\rangle^\sigma\leqq C_\sigma'\langle s-t_{\min}\rangle^\sigma$. This implies (3.16).

CONTINUATION OF THE PROOF OF PROPOSITION 3.1. We rewrite (3.5) into the equivalent integral equation

(3.17) $$q(t) = q_0(t)-\lambda^{-1}\int_0^t (t-s)\partial_x V(q(s))ds.$$

and prove (3) by induction. Since $|\partial_x V(q(t))|\leqq C\langle q(t)\rangle^{m-1}$ and $m>1$, Lemma 3.2 and 3.5 imply the estimate (3.7) for the case $(k, \tilde{\alpha}, \beta, \gamma)=0$. Suppose now that (3.7) is satisfied for $k=0$ and $|\tilde{\alpha}+\beta+\gamma|\leqq h-1$, $h\geqq 1$ and let us prove it for $k=0$ and $|\tilde{\alpha}+\beta+\gamma|=h$. Denote as $\eta=(t, \tau, \xi, y)$, $\theta=(k, \tilde{\alpha}, \beta, \gamma)$ and $Q_\theta(t)=\partial_\eta^\theta(q(t)-q_0(t))$. Differentiating (3.17), we have, for $k=0$,

(3.18) $$Q_\theta(t) = R_\theta(t)-\lambda^{-1}\int_0^t (t-s)\,\mathrm{Hess}\,V(q(s))Q_\theta(s)ds,$$

where $R_\theta(t)=R_\theta^1(t)+R_\theta^2(t)$ and

$$R_\theta^1(t) = -\lambda^{-1}\int_0^t (t-s)\,\mathrm{Hess}\,V(q(s))\partial_\eta^\theta q_0(s)ds,$$

$$R_\theta^2(t) = -\lambda^{-1}\sum_{l=2}^{|\theta|}\sum_{|\delta|=l}\sum_{\theta_1+\cdots+\theta_l=\theta}\mathrm{Const}\int_0^t (t-s)\partial_x^\delta\partial_x V(q(s))\partial_\eta^{\theta_1}q_{j_1}(s)\cdots\partial_\eta^{\theta_l}q_{j_l}(s)ds.$$

By (3.12) and Lemma 3.2,

$$|R_\theta^1(t)| \leqq C\lambda^{-1-|\gamma|/2}|t|R^{(m-|\theta|)_+ +|\gamma|}. \tag{3.19}$$

Using the estimate (3.13), we have for $m=\tilde{m}\geqq|\delta|+1=l+1$ ($\geqq 3$),

$$|\partial_x^\delta\partial_x V(q(t))\prod_{k=1}^{l}\partial_\eta^{\theta_k}q_{j_k}(t)| \leqq CR^{(m-|\theta|)_+ +|\gamma|-1}\lambda^{-|\gamma|/2}; \tag{3.20$_1$}$$

for ($2\leqq$) $l<\tilde{m}<l+1$ and $|\theta|>\tilde{m}$,

$$|\text{LHS of } (3.20)_1| \leqq C\lambda^{-|\gamma|/2}(R^{|\gamma|}\langle q(t)\rangle^{m-|\theta|-1}+\lambda^{-1}R^{|\gamma|+l}\langle q(t)\rangle^{m-l-1}); \tag{3.20$_2$}$$

for $l=|\theta|<\tilde{m}<l+1$ and $|\theta|\leqq\tilde{m}$,

$$|\text{LHS of } (3.20)_1| \leqq CR^{|\gamma|}\langle q(t)\rangle^{m-|\theta|-1}\lambda^{-|\gamma|/2}; \tag{3.20$_3$}$$

for $l\geqq\tilde{m}$,

$$|\partial_x^\delta\partial_x V(q(t))\prod^{l}\partial_\eta^{\theta_k}q_{j_k}(t)| \leqq C\lambda^{-|\gamma|/2}R^{|\gamma|}\langle q(t)\rangle^{m-l-1}. \tag{3.20$_4$}$$

Combining $(3.20)_1\sim(3.20)_4$ and (3.16), we obtain

$$|R_\theta^2(t)| \leqq C\lambda^{-1-|\gamma|/2}|t|R^{(m-|\theta|)_+ +|\gamma|}(\log R)^{\varepsilon(m,|\theta|)}. \tag{3.21}$$

Now we let $\rho_1\leqq\rho_2$ be sufficiently small so that

$$\left|\lambda^{-1}\int_0^t|(t-s)\text{Hess } V(q(s))|ds\right| \leqq 1/2$$

for any $(\tau, \xi, y)\in D(\lambda, m, \rho)$, $\rho\leqq\rho_1$ and $|t|\leqq 3\rho_1\lambda^{1/\tilde{m}}$. Then we have

$$\sup_{|s|\leqq|t|}|Q_\theta(s)| \leqq C\lambda^{-1-|\gamma|/2}|t|R^{(m-|\theta|)_+ +|\gamma|}(\log R)^{\varepsilon(m,|\theta|)}.$$

This proves (3.7) for all $(k, \tilde{\alpha}, \beta, \gamma)$ with $k=0$. For proving (3.7) for the case $k=1$, we differentiate

$$\partial_t q(t) = \partial_t q_0(t)-\lambda^{-1}\int_0^t\partial_x V(q(s))ds$$

by $\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma$ and proceed exactly in the same way as above. This proves the case $k=1$. For $k\geqq 2$, we differentiate

$$\partial_t^2 q(t) = -\partial_x V(q(t, \tau, \xi, y))\lambda^{-1}$$

and apply Lemma 3.4 inductively.

Using Proposition 3.1 we prove the following lemma. Hereafter we assume, without loss of generality, $m>1$.

LEMMA 3.6. *If $\rho_1>0$ is sufficiently small, the following statement is satisfied for every $0<\rho\leqq\rho_1$ and $\lambda\geqq\lambda_1(\rho)$: For every (ξ, y) with $|y|\leqq 10^{-2}\rho\lambda^{1/\tilde{m}}$, $1/4\leqq\xi^2/2\lambda\leqq 4$, $\xi\in S_j$, $q(t, \tau, \xi, y)$ is diffeomorphic from*

$$T(3\rho)=\{(t, \tau): \ |t|<3\rho\lambda^{1/\tilde{m}}, \ |\tau|\leqq 3\rho\lambda^{1/\tilde{m}}\}$$

to its image and

(3.22) $$q(T(\rho/9), \xi, y)\subset X(\rho)=\{x: |x|<\rho\lambda^{1/\tilde{m}}\}\subset q(T(3\rho), \xi, y).$$

PROOF. By estimate (3.7), we have

(3.23) $$\left\|(\partial_t, \partial_\tau)O_j(\xi)q(t, \tau, \xi, y)-\begin{pmatrix}\lambda^{-1/2}|\xi| & 0\\ 0 & 1\end{pmatrix}\right\|\leqq C\lambda^{-1}(|t|+\langle\tau\rangle+\langle y\rangle)^{\tilde{m}}$$

for $|t|\leqq 3\rho\lambda^{1/\tilde{m}}$, $(\tau, \xi, y)\in D_j(\lambda, m; \rho)$, $0<\rho\leqq\rho_1$ and $\lambda\geqq\lambda(\rho_1)$. Here the constants C in (3.7) is independent of $0<\rho\leqq\rho_1$ and if we choose ρ_1 sufficiently small the RHS of (3.23) is smaller than $10^{-6}/2n$ for $|t|\leqq 3\rho_1\lambda^{1/\tilde{m}}$, $(\tau, \xi, y)\in D_j(\lambda, m; \rho_1)$. Then the lemma follows by the implicit function theorem.

We put for $\lambda\geqq\lambda_1(\rho_1)$,

$$Y_j(\rho_1)=\{(\xi, y): \ 1/4\leqq\xi^2/2\lambda\leqq 4, \ \xi\in S_j, \ |y|\leqq 10^{-2}\rho_1\lambda^{1/\tilde{m}}\}$$

and denote the inverse of the map $(t, \tau)\rightarrow x=q(t, \tau, \xi, y)$ as $(t, \tau)=(t(x, \xi, y), \tau(x, \xi, y))$ for $(x, \xi, y)\in X(\rho_1)\times Y_j(\rho_1)$. For $(t, \tau, \xi, y)\in T(3\rho_1)\times Y_j(\rho_1)$, we set

(3.24) $$u_j(t, \tau, \xi, y)=\lambda^{-1/2}\int_0^t p(s, \tau, \xi, y)^2 ds$$

and define for $(x, \xi, y)=X(\rho_1)\times Y_j(\rho_1)$, $j=1, \cdots, N$,

$$\phi_j(x, \xi, y)=u_j(t(x, \xi, y), \tau(x, \xi, y), \xi, y).$$

It is clear that $\phi_k=\phi_j$ on $X(\rho_1)\times(Y_k(\rho_1)\cap Y_j(\rho_1))$ and

$$\phi(x, \xi, y)=\phi_j(x, \xi, y), \qquad (x, \xi, y)\in X(\rho_1)\times Y_j(\rho_1)$$

defines $\phi(x, \xi, y)\in C^\infty(\Omega(\rho_1))$ on

$$\Omega(\rho_1)=\{(x, \xi, y): \ |x|<\rho_1\lambda^{1/\tilde{m}}, 4^{-1}<\xi^2/2\lambda<4, \ |y|<10^{-2}\rho_1\lambda^{1/\tilde{m}}\}.$$

PROPOSITION 3.7. *For every* $(\xi, y)\in\bigcup_{j=1}^N Y_j(\rho_1)$, $\phi(x, \xi, y)$ *satisfies the Hamilton-Jacobi equation* (3.1) *on* $X(\rho_1)$ *with the boundary conditions* (3.2)~(3.3). *Moreover it satisfies*

(3.25) $$\partial_x\phi(q(t, \tau, \xi, y), \xi, y)=p(t, \tau, \xi, y),$$

(3.26) $$(\partial_\xi\phi)(q(t, \tau, \xi, y), \xi, y)=t\lambda^{-1/2}\xi+|\xi|^{-1}|p_0|O(\xi)^*\underline{\tau},$$

(3.27) $$(\partial_y\phi)(q(t, \tau, \xi, y), \xi, y)=t\lambda^{-1/2}\partial_y V(y)-p_0(\tau, \xi, y).$$

PROOF. The statements are well known and are easy consequences of the theory of first order partial differential equations (cf. [**11**]). Hence the proof is omitted.

Using the statement (3) of Proposition 3.1, we prove the following

PROPOSITION 3.8. *Let $\phi(x, \xi, y)$ be as above. Then for $(x, \xi, y)\in\Omega(\rho_1)$, $\phi_D(x, \xi, y)=\phi(x, \xi, y)-(x-y)\cdot\xi$ satisfies*

$$\phi_D(y, \xi, y)=0, \quad (\partial_x\phi_D)(y, \xi, y)=0, \tag{3.28}$$

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\partial_x\phi_D(x, \xi, y)| \leqq C_{\alpha\beta\gamma}\lambda^{-(1+|\gamma|)/2}L^{(m-|\alpha+\beta+\gamma|)_+ +|\gamma|}(\log L)^{\varepsilon(m,|\alpha+\beta+\gamma|)}, \tag{3.29}$$

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\partial_\xi\phi_D(x, \xi, y)| \leqq C_{\alpha\beta\gamma}\lambda^{-(2+|\gamma|)/2}L^{(m-|\alpha+\beta+\gamma|)_+ +|\gamma|+1}(\log L)^{\varepsilon(m,|\alpha+\beta+\gamma|)}, \tag{3.30}$$

where $L=\langle x\rangle+\langle y\rangle$ and $C_{\alpha\beta\gamma}$ are independent of $\lambda\geqq\lambda_1(\rho_1)$ and $(x, \xi, y)\in\Omega(\rho_1)$.

PROOF. Equations (3.28) are obvious by (3.2)~(3.3). We denote $\eta=(t, \tau, \xi, y)$, $R=|t|+\langle\tau\rangle+\langle y\rangle$ and treat

$$\partial_x\phi_D(x, \xi, y)=\partial_x\phi(x, \xi, y)-\xi\equiv\tilde{\phi}(x, \xi, y)$$

first. Differentiating (3.25), we have for $|\alpha+\beta+\gamma|\geqq 1$

$$\begin{aligned}(3.31)\quad & \partial_{(t,\tau)}^\alpha\partial_y^\beta\partial_\xi^\gamma[p(t, \tau, \xi, y)-\xi] \\ &= \sum_{i_1,\dots,i_{|\alpha|}=1}^{n}(\partial_{x_{i_1}}\cdots\partial_{x_{i_{|\alpha|}}}\partial_y^\beta\partial_\xi^\gamma\tilde{\phi})(q, \xi, y)\partial_{\mu_{j_1}}q_{i_1}(t)\cdots\partial_{\mu_{j_{|\alpha|}}}q_{i_{|\alpha|}}(t) \\ &\quad +\sum\sum \mathrm{Const}(\partial_x^\delta\partial_\xi^{\beta^1}\partial_x^{\gamma^1}\tilde{\phi})(q, \xi, y)\partial_\eta^{\theta_1}q_{i_1}\cdots\partial_\eta^{\theta_l}q_{i_l}.\end{aligned}$$

Here $\mu=(t, \tau)$, $\alpha=(k, \tilde{\alpha})$ and in the second member in the RHS the first sum is taken over β^1, γ^1 and δ such that $\beta^2=\beta-\beta^1\geqq 0$, $\gamma^2=\gamma-\gamma^1\geqq 0$, $1\leqq|\delta|\leqq|\alpha|+|\beta^2|+|\gamma^2|$ and such that either $|\beta^1|+|\gamma^1|\leqq|\beta|+|\gamma|-1$ or $|\delta|\leqq|\alpha|-1$; the second over $\theta_j=(\alpha_j, \beta_j^2, \gamma_j^2)$ and i_j, $1\leqq j\leqq l=|\delta|$ such that $|\theta_j|\geqq 1$, $\theta_1+\cdots+\theta_l=(\alpha, \beta^2, \gamma^2)$ and $e_{i_1}+\cdots+e_{i_l}=\delta$. If we fix $N\geqq 0$ and let α run in (3.31) over multi-indeces of length $|\alpha|=N$, then by virtue of (3.23), the resulting equations can be solved for $\{\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\tilde{\phi}: |\alpha|=N\}$ and we have

$$\begin{aligned}(3.32)\quad & \sum_{|\alpha|=N}|(\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\tilde{\phi})(q(t, \tau, \xi, y), \xi, y)| \\ &\leqq C\Big(\sum_{|\alpha|=N}|\partial_{(t,\tau)}^\alpha\partial_y^\beta\partial_\xi^\gamma\{p(t, \tau, \xi, y)-\xi\}| \\ &\quad +\sum\sum|(\partial_x^\delta\partial_y^{\beta^1}\partial_\xi^{\gamma^1}\tilde{\phi})(q, \xi, y)\partial_\eta^{\theta_1}q_{i_1}(t)\cdots\partial_\eta^{\theta_l}q_{i_l}(t)|\Big)\end{aligned}$$

where the second sum in the RHS is taken as in (3.31) but here we also let α run all over $|\alpha|=N$. We prove (3.29) by induction. We treat the case $\beta=\gamma=0$ first. When $|\alpha|\leqq 1$, the second summand in the RHS of (3.32) is absent and (3.7) implies (3.29). Suppose that (3.29) is satisfied for every α with $|\alpha|\leqq N-1$ and $\beta=\gamma=0$, $N\geqq 2$. Then by Lemma 3.4 and Proposition 3.1 (3)

$$\sum_{|\alpha|=N}|\partial_x^\alpha\tilde{\phi}(q(t, \tau, \xi, y), \xi, y)|\leqq C\lambda^{-1/2}R^{(m-N)_+}(\log R)^{\varepsilon(m,N)}. \tag{3.33}$$

Since $C^{-1}\leqq(\langle q(t, \tau, \xi, y)\rangle+\langle y\rangle)/R\leqq C$ by Proposition 3.1 (2), (3.33) proves (3.29) for all α if $\beta=\gamma=0$ by induction. Thus for completing the induction argument it suffices to show (3.29) for (α, β, γ) with $|\alpha+\beta+\gamma|=M\geqq 1$ under the condition that it is satisfied for all $(\alpha', \beta', \gamma')$ such that either $|\alpha'+\beta'+\gamma'|\leqq M-1$ or $|\alpha'+\beta'+\gamma'|=M$ and $(\beta', \gamma')<(\beta, \gamma)$. Since either $|\delta|+|\beta^1|+|\gamma^1|\leqq M-1$ or $(\beta^1, \gamma^1)<(\beta, \gamma)$ in the second sum in the RHS of (3.32), the induction hypothesis and Lemma 3.4 imply (3.29) for $|\alpha+\beta+\gamma|=M$. By (3.26), (3.7) and (3.15)

$$\begin{aligned}(3.34)\qquad &|\partial_{(t,\tau)}^{\alpha}\partial_y^{\beta}\partial_\xi^{\gamma}\{\partial_\xi\phi(q(t, \tau, \xi, y), \xi, y)-(q(t, \tau, \xi, y)-y)\}| \\ &\leqq C_{\alpha\beta\gamma}\lambda^{-1-|\gamma|/2}R^{(m-|\alpha+\beta+\gamma|)_{+}+|\gamma|+1}(\log R)^{\varepsilon(m,|\alpha+\beta+\gamma|)}.\end{aligned}$$

Starting with (3.34), we repeat the argument for proving (3.29) virtually word by word and obtain (3.30).

It is sometimes desirable that the function ϕ is defined everywhere. We take $\phi_2\in C_0^\infty(\boldsymbol{R}^1)$ such that

$$\phi_2(t)=1 \qquad \text{for } 13/48\leqq t\leqq 15/4, \quad \operatorname{supp}\phi_2\subset(1/4, 4)$$

and define

$$(3.35)\qquad \tilde{\phi}_D(x, \xi, y)=\chi(x/\rho_1\lambda^{1/\tilde{m}})\phi_2(\xi^2/2\lambda)\chi(y/10^{-2}\rho_1\lambda^{1/\tilde{m}})\phi_D(x, \xi, y)$$

$$(3.36)\qquad \tilde{\phi}(x, \xi, y)=(x-y)\xi+\tilde{\phi}_D(x, \xi, y).$$

It should be clear that

$$\tilde{\phi}(x, \xi, y)=\phi(x, \xi, y), \qquad (x, \xi, y)\in\Omega_1(\rho_1),$$

where $\Omega_1(\rho_1)=\{(x, \xi, y): |x|\leqq(1/2)\rho_1\lambda^{1/\tilde{m}}, 13/48<\xi^2/2\lambda<15/4, |y|\leqq(10^{-2}/2)\rho_1\lambda^{1/\tilde{m}}\}$ and $\tilde{\phi}(x, \xi, y)$ satisfies (3.28)$\sim$(3.30) of Proposition 3.8.

Let $\kappa=\kappa_m=(1/2-1/\tilde{m})/2$. The change of the scale $(x, \xi, y)\to(\lambda^{-\kappa}x, \lambda^{\kappa}\xi, \lambda^{-\kappa}y)$ will be often used in what follows. We set as

$$\tilde{\phi}_\kappa(x, \xi, y)=\tilde{\phi}(\lambda^{-\kappa}x, \lambda^{\kappa}\xi, \lambda^{-\kappa}y),$$

$$D(\tilde{\phi}_\kappa)(x, \xi, y)=\begin{pmatrix}\partial_x\partial_y\tilde{\phi}_\kappa & \partial_\xi\partial_y\tilde{\phi}_\kappa\\ \partial_x\partial_\xi\tilde{\phi}_\kappa & \partial_\xi\partial_\xi\tilde{\phi}_\kappa\end{pmatrix}.$$

LEMMA 3.9. *If* $0<\rho_1$ *is sufficiently small, then for* $\lambda\geqq\lambda_1(\rho_1)$,

$$(3.37)\qquad \left\|D(\tilde{\phi}_\kappa)(x, \xi, y)-\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\|\leqq 10^{-2}n^{-2},$$

$$\begin{aligned}(3.38)\qquad &|\partial_x^{\alpha}\partial_y^{\beta}\partial_\xi^{\gamma}D(\tilde{\phi}_\kappa)(x, \xi, y)| \\ &\leqq C\lambda^{-1+1/\tilde{m}-\kappa|\alpha+\beta+\gamma|+(m-|\alpha+\beta+\gamma|-1)_{+}/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\alpha+\beta+\gamma|+1)}.\end{aligned}$$

PROOF. On $\operatorname{supp}\tilde{\phi}_D$, $\langle x\rangle+\langle y\rangle\leqq 10\rho_1\lambda^{1/\tilde{m}}$. Hence (3.37)$\sim$(3.38) follow from (3.29)$\sim$(3.30) by changing ρ_1 small, if necessary.

§4. Parametrix.

We take ρ_1 small enough so that the properties of Section 3 are satisfied for $0<\rho\leqq\rho_1$ and set $\rho_0=10^{-3}\rho_1$. By virtue of (3.37), the mapping

$$\xi \longrightarrow \eta(x, \xi, y) = \int_0^1 \partial_x \phi(\sigma x+(1-\sigma)y, \xi, y)d\sigma \tag{4.1}$$

defines for each $|x|<10^3\rho_0\lambda^{1/\tilde{m}}$ and $|y|<10\rho_0\lambda^{1/\tilde{m}}$ a diffeomorphism from the annulus $\{\xi: 1/4<\xi^2/2\lambda<4\}$ to its image which contains $\{\eta: 7/24<\eta^2/2\lambda<7/2\}$ such that the inverse image of $\{\eta: 1/3<\eta^2/2\lambda<3\}$ is contained in $\{\xi: 7/24<\xi^2/2\lambda<7/2\}$. We denote the inverse of (4.1) by $\xi=\xi(x, \eta, y)$.

We choose $\phi_3\in C_0^\infty(\boldsymbol{R}^1)$ such that $\phi_3(t)=1$ for $7/24\leqq t\leqq 7/2$ and $\operatorname{supp}\phi_3\subset[13/48, 15/4]$. We have $\phi_2(t)\phi_3(t)=\phi_3(t)$ and, by the preceeding remark $\phi_3(\xi(x, \eta, y)^2/2\lambda)\Phi_0(\eta, \lambda)=\Phi_0(\eta, \lambda)$. We let for $\lambda\geqq\lambda_1(\rho_0)$,

$$A_0(x, \xi, y) = \chi(x/10\rho_0\lambda^{1/\tilde{m}})\Phi_0(\xi, \lambda)\chi(y/\rho_0\lambda^{1/\tilde{m}}), \tag{4.2}$$

$$F_0 f(x) = (2\pi)^{-n}\int e^{i\phi(x,\xi,y)}A_0(x, \xi, y)f(y)dyd\xi. \tag{4.3}$$

Note that the change of ϕ to $\tilde{\phi}$ in the integrand of (4.3) does not affect the definition of F_0.

LEMMA 4.1. *There exists a constant $C>0$ such that*

$$\|F_0 f-\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})f\| \leqq C\lambda^{2/\tilde{m}-3/2+(\tilde{m}-3)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}\|f\|, \tag{4.4}$$

for all $f\in\mathcal{S}$, $\lambda\geqq\lambda_1(\rho_0)$.

PROOF. We set

$$\tilde{A}_0(x, \xi, y) = \chi(x/10\rho_0\lambda^{1/\tilde{m}})\chi(y/\rho_0\lambda^{1/\tilde{m}})\phi_3(\xi^2/2\lambda)\Phi_0(\eta(x, \xi, y), \lambda)\det(\partial\eta(x, \xi, y)/\partial\xi)$$

and let $\tilde{F}_0 f$ be defined by (4.3) with $\tilde{A}_0$ replacing A_0. Writing as $\phi(x, \xi, y)=(x-y)\cdot\eta(x, \xi, y)$ and making the change of variables $\xi\to\xi(x, \eta, y)$, we have that

$$\tilde{F}_0 f(x) = (2\pi)^{-n}\int e^{i(x-y)\cdot\eta}A_0(x, \eta, y)f(y)dyd\eta.$$

After integrating by parts, we have for any $N\geqq 0$,

$$\begin{aligned}&|\tilde{F}_0 f(x)-\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})f(x)|\\&= (2\pi)^{-n}\left|\int e^{i(x-y)\cdot\xi}(1-\chi(x/10\rho_0\lambda^{1/\tilde{m}}))\Phi_0(\xi, \lambda)\chi(y/\rho_0\lambda^{1/\tilde{m}})f(y)dyd\xi\right|\\&\leqq C_N\int(|x-y|+\rho_0\lambda^{1/\tilde{m}})^{-2N}\langle\xi\rangle^{-N}\lambda^{-N/2}|f(y)|dyd\xi.\end{aligned}$$

Hence by the Young inequality

$$\|\tilde{F}_0 f(x)-\Phi_0(D, \lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})f\| \leqq C_N\lambda^{-N/2}\|f\|. \tag{4.5}$$

On the other hand this same change of variables leads to

$$\begin{aligned} F_0f(x)-\tilde{F}_0f(x) = (2\pi)^{-n}\int & e^{i(x-y)\cdot\eta}\chi(x/10\rho_0\lambda^{1/\tilde{m}})\chi(y/\rho_0\lambda^{1/\tilde{m}}) \\ & \times\{\Phi_0(\xi(x, \eta, y), \lambda)\det(\partial\xi(x, \eta, y)/\partial\eta)-\Phi_0(\eta, \lambda)\}f(y)dyd\eta. \end{aligned} \tag{4.6}$$

Since $K(x, \eta, y)\equiv\Phi_0(\xi(x, \eta, y), \lambda)\det(\partial\xi(x, \eta, y)/\partial\eta)$ satisfies

$$K(x, \eta, y)-\Phi_0(\eta, \lambda) = (x-y)\cdot\int_0^1\partial_x K(\sigma x+(1-\sigma)y, \eta, y)d\sigma$$

by (3.2) and (3.3), we may rewrite (4.6) as

$$\begin{aligned} F_0f(x)-\tilde{F}_0f(x) = (2\pi)^{-n}\int & e^{i(x-y)\cdot\eta}\chi(x/10\rho_0\lambda^{1/\tilde{m}})\chi(y/\rho_0\lambda^{1/\tilde{m}}) \\ & \times\left(\int_0^1 i(\partial_\eta\cdot\partial_x)K(\sigma x+(1-\sigma)y, \eta, y)d\sigma\right)f(y)dyd\eta. \end{aligned}$$

By (3.29) we have

$$\begin{aligned} & |\partial_x^\alpha\partial_y^\beta\partial_\eta^\gamma(\partial_\eta\cdot\partial_x)K(\sigma x+(1-\sigma)y, \eta, y)\chi(x/10\rho_0\lambda^{1/\tilde{m}})\chi(y/\rho_0\lambda^{1/\tilde{m}})| \\ & \leqq C_{\alpha\beta\gamma}\lambda^{-3/2+2/\tilde{m}+(\tilde{m}-3)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}. \end{aligned}$$

Thus by Calderon-Vaillancourt theorem

$$\|F_0f-\tilde{F}_0f\| \leqq C\lambda^{2/\tilde{m}-3/2+(\tilde{m}-3)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}\|f\|. \tag{4.7}$$

Combining (4.7) with (4.5), we obtain (4.4).

Let $|y|\leqq\rho_0\lambda^{1/\tilde{m}}$ and $1/3\leqq|\xi|^2/2\lambda\leqq3$. Then by (3.22), for every $|t|\leqq3\rho_0\lambda^{1/\tilde{m}-1/2}$ the map $Q: T(2\rho_1)\ni(s, \tau)\to q(\lambda^{1/2}t+s, \tau, \xi, y)$ is a diffeomorphism such that $Q(T(2\rho_1))\supset X(150\rho_0)\supset Q(T(30\rho_0))$. Hence it is clear that the equation

$$\begin{aligned} & A(t, q(\lambda^{1/2}t+s, \tau, \xi, y), \xi, y)(\det\partial q(\lambda^{1/2}t+s, \tau, \xi, y)/\partial(s, \tau))^{1/2} \\ & \quad = A_0(q(s, \tau, \xi, y), \xi, y)(\det\partial q(s, \tau, \xi, y)/\partial(s, \tau)))^{1/2} \end{aligned} \tag{4.8}$$

uniquely defines a smooth function $A(t, x, \xi, y)\in C_0^\infty(\{(x, \xi, y): |x|<150\rho_0\lambda^{1/\tilde{m}}$, $|\xi|^2/2\lambda\in[1/3, 3]$, $|y|<\rho_0\lambda^{1/\tilde{m}}\})$ for every $|t|\leqq3\rho_0\lambda^{1/\tilde{m}-1/2}$. Note that $\phi(x, \xi, y)$ $=\tilde{\phi}(x, \xi, y)$ on $\operatorname{supp} A(t, \cdot)\subset\Omega_1(\rho_1)$.

LEMMA 4.2. *Let $A(t, x, \xi, y)$ be defined as above. Then:*

$$A(0, x, \xi, y) = A_0(x, \xi, y). \tag{4.9}$$

(4.10) *For $|t|\leqq3\rho_0\lambda^{1/\tilde{m}-1/2}$ and $(x, \xi, y)\in\boldsymbol{R}^{3n}$,*

$$\partial_t A+\sum_{i=1}^{n}\partial_{x_i}\tilde{\phi}\cdot\partial_{x_i}A+(1/2)\Delta_x\tilde{\phi}\cdot A=0.$$

(4.11) $|\partial_t^k\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma A(t, x, \xi, y)|$

$$\leqq C\lambda^{-1-(1/2-1/\tilde{m})(|\gamma|-k)+k^*/\tilde{m}+(\tilde{m}-|\alpha+\beta+\gamma|-k^*)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\alpha+\beta+\gamma|+1)}.$$

PROOF. Equation (4.9) is obvious. By (3.25), $t\rightarrow q(\sqrt{\lambda}\,t+s, \tau, \xi, y)$ for each fixed (s, τ, ξ, y) is an integral curve for the vector field $\partial_x\phi(x, \xi, y)$. Since $\phi=\tilde{\phi}$ on all the integral curves starting from $\operatorname{supp}A_0$, $|t|\leqq 3\rho_0\lambda^{1/\tilde{m}-1/2}$, it is well-known that the equation (4.8) determines a unique solution of (4.9)$\sim$(4.10). We prove (4.11). We set, for $\theta=(k, \tilde{\alpha}, \beta, \gamma)$,

$$\sigma(\theta)=-1-(1/2-1/\tilde{m})|\gamma|+(\tilde{m}-|\tilde{\alpha}+\beta+\gamma|-k^*)_+/\tilde{m}+(k^*-k)/\tilde{m}.$$

For $R=|t|+\langle\tau\rangle+\langle y\rangle\leqq C\lambda^{1/\tilde{m}}$, (3.7) and Lemma 3.4 imply that

(4.12) $|\partial_s^k\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma\det(\partial q(\lambda^{1/2}t+s, \tau, \xi, y)/\partial(s, \tau))|\leqq C\lambda^{\sigma(\theta)}(\log\lambda)^{\varepsilon(m,|\tilde{\alpha}+\beta+\gamma|+1)}$,

$$|\partial_t^k\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma A_0(q(t, \tau, \xi, y), \xi, y))|$$
$$\leqq C\lambda^{-1-(1/2-1/\tilde{m})|\gamma|+(\tilde{m}-|\theta|)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\tilde{\alpha}+\beta+\gamma|)}.$$

Since $|\det\partial q(\lambda^{1/2}t+s, \tau, \xi, y)/\partial(s, \tau)|\geqq 1/2$, (4.12) remains valid if LHS is replaced by $|\partial_s^k\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma(\det\partial q(\lambda^{1/2}t+s, \tau, \xi, y)/\partial(s, \tau))^{-1}|$. Hence

(4.13) $|\partial_s^k\partial_\tau^{\tilde{\alpha}}\partial_y^\beta\partial_\xi^\gamma A(t, q(\lambda^{1/2}t+s, \tau, \xi, y), \xi, y)|\leqq C\lambda^{\sigma(\theta)}(\log\lambda)^{\varepsilon(m,|\tilde{\alpha}+\beta+\gamma|+1)}$

and we obtain (4.11) for the case $k=0$. For general $k\geqq 1$, we inductively use (4.10) and obtain (4.11).

We define for $|t|<3\rho_0\lambda^{1/\tilde{m}-1/2}$

(4.14) $$F(t)f(x)=(2\pi)^{-n}\int e^{i\phi(x,\xi,y)-it(\xi^2/2+V(y))}A(t, x, \xi, y)f(y)dyd\xi.$$

It is clear that for $f\in L^1_{loc}(\boldsymbol{R}^n)$, $F(t)f(x)\in C_0^\infty(\boldsymbol{R}^n)$ and is smooth in the t-variable.

PROPOSITION 4.3. *There exists a constant* $\rho_0>0$ *such that for* $|t|<3\rho_0\lambda^{1/\tilde{m}-1/2}$ *and* $f\in L^2(\boldsymbol{R}^n)$,

(4.15) $\|F(t)f-e^{-it\mathfrak{H}}\Phi_0(D, \lambda)\chi(y/\rho_0\lambda^{1/\tilde{m}})f\|\leqq C\lambda^{2/\tilde{m}-3/2+(\tilde{m}-3)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}\|f\|$.

PROOF. It is clear from (4.3) and (4.9) that

(4.16) $$F(0)f(x)=F_0f(x)$$

and that the changes of $\phi(x, \xi, y)$ and $V(y)$ respectively by $\tilde{\phi}(x, \xi, y)$ and $\tilde{V}(y)=\chi(y/2\rho_0\lambda^{1/\tilde{m}})V(y)$ in the integral of (4.14) (so that $|\tilde{V}(y)|<10^{-2}\lambda$) do not change $F(t)f(x)$. After a simple computation using (3.1) and (4.10), we see for $f\in L^2(\boldsymbol{R}^n)$

(4.17) $L(t)f(x)=(-i\partial/\partial t-(1/2)\Delta+V(x))F(t)f(x)$

$$=-2^{-1}(2\pi)^{-n}\int e^{i\tilde{\phi}(x,\xi,y)-it(\xi^2/2+\tilde{V}(y))}(\Delta_x A)(t,\ x,\ \xi,\ y)f(y)dyd\xi\,.$$

We change the scale as $(x, \xi, y)\rightarrow(\lambda^{-\kappa}x, \lambda^{\kappa}\xi, \lambda^{-\kappa}y)$, $\kappa=(1/2-1/\tilde{m})/2$, in (4.17). By (4.11) and Lemma 3.9,

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(\Delta_x A)(t,\ \lambda^{-\kappa}x,\ \lambda^{\kappa}\xi,\ \lambda^{-\kappa}y)|$$

$$\leqq C\lambda^{-1-\kappa|\alpha+\beta+\gamma|+(\tilde{m}-|\alpha+\beta+\gamma|-3)_+/\tilde{m}+1/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\alpha+\beta+\gamma|+3)},$$

$$\left\|D(\tilde{\phi}_\kappa-t(\lambda^{2\kappa}\xi^2+\tilde{V}(\lambda^{-\kappa}y)))-\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\|\leqq 10^{-1}n^{-1},$$

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma D(\tilde{\phi}_\kappa-t(\lambda^{2\kappa}\xi^2+\tilde{V}(\lambda^{-\kappa}y)))|\leqq C_{\alpha\beta\gamma}\,.$$

Hence the L^2-boundedness theorem of oscillatory integral operators ([**3**]) implies

(4.18) $$\|L(t)f\|\leqq C\lambda^{-1+(\tilde{m}-3)_+/\tilde{m}+1/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}\,.$$

Integrating (4.17) and using (4.16) and (4.18), we obtain

(4.19) $$\|e^{-it\mathfrak{H}}F_0f-F(t)f\|\leqq\int_0^t\|L(s)f\|ds$$

$$\leqq C\lambda^{2/\tilde{m}-3/2+(\tilde{m}-3)_+/\tilde{m}}(\log\lambda)^{\varepsilon(m,3)}\|f\|$$

for $|t|\leqq 3\rho_0\lambda^{1/\tilde{m}-1/2}$. (4.19) and (4.4) imply (4.11).

§5. Local spectral projection, proof of Theorem 1.1.

We choose $g\in\mathcal{S}(\boldsymbol{R}^1)$ such that $\operatorname{supp}\check{g}\subset(-1, 1)$ and $g(\zeta)>0$ for $|\zeta|\leqq 1$ and set

(5.1) $$g_F(\lambda,\ \mathfrak{H})f(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{i\lambda t}(\rho_0\lambda^{1/\tilde{m}-1/2})^{-1}\check{g}(t/\rho_0\lambda^{1/\tilde{m}-1/2})F(t)f(x)dt\,.$$

LEMMA 5.1. *Let* $\sigma\geqq 1/2$. *Then for* $\lambda\geqq\lambda_1(\rho_0)$

$$\|\{g(\rho_0^{-1}\lambda^{1/\tilde{m}-1/2}(\mathfrak{H}-\lambda))\Phi_0(D,\ \lambda)-g_F(\lambda,\ \mathfrak{H})\}\langle x\rangle^{-\sigma}f\|$$

$$\leqq C\lambda^{-1/2\tilde{m}}\|f\|,\qquad f\in L^2(\boldsymbol{R}^n).$$

PROOF. By (4.15), we have $\|g_F(\lambda,\ \mathfrak{H})f\|\leqq C\|f\|$ and

(5.2) $$\|\{g(\rho_0\lambda^{1/\tilde{m}-1/2}(\mathfrak{H}-\lambda))\Phi_0(D,\ \lambda)-g_F(\lambda,\ \mathfrak{H})\}\langle x\rangle^{-\sigma}(1-\chi(2x/\rho_0\lambda^{1/\tilde{m}}))f\|$$

$$\leqq C\lambda^{-1/2\tilde{m}}\|f\|$$

for $\sigma\geqq 1/2$. Again by (4.15),

$$\begin{aligned}(5.3)\qquad &\|\{g(\rho_0\lambda^{1/\tilde{m}-1/2}(\mathfrak{H}-\lambda))\Phi_0(D,\lambda)-g_F(\lambda,\mathfrak{H})\}\chi(2x/\rho_0\lambda^{1/\tilde{m}})f\| \\ &=\|(2\pi)^{-1/2}\int_{-\infty}^{\infty}(\rho_0\lambda^{1/\tilde{m}-1/2})^{-1}\check{g}(t/\rho_0\lambda^{1/\tilde{m}-1/2})e^{i\lambda t} \\ &\quad\times\{e^{-it\mathfrak{H}}\Phi_0(D,\lambda)\chi(x/\rho_0\lambda^{1/\tilde{m}})-F(t)\}\chi(2x/\rho_0\lambda^{1/\tilde{m}})f\,dt\| \\ &\leqq C\lambda^{-1/2\tilde{m}}\|f\|.\end{aligned}$$

Combining (5.2) and (5.3), we obtain Lemma 5.1.

Substituting (4.14) into (5.1), we rewrite

$$\begin{aligned}(5.4)\quad & g_F(\lambda,\mathfrak{H})f(x)=(2\pi)^{-n}\int e^{i\phi(x,\xi,y)}\hat{K}_g\Big(\frac{\xi^2/2+\tilde{V}(y)-\lambda}{\lambda^{1/2-1/\tilde{m}}},x,\xi,y\Big)f(y)dyd\xi, \\ & K_g(t,x,\xi,y)=\rho_0^{-1}\check{g}(t/\rho_0)A(\lambda^{1/\tilde{m}-1/2}t,x,\xi,y), \\ & \hat{K}_g(\zeta,x,\xi,y)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-it\zeta}K_g(t,x,\xi,y)dt.\end{aligned}$$

LEMMA 5.2. *$\hat{K}_g\in\mathcal{S}(\boldsymbol{R}^{3n+1})$ and for each fixed ζ, $\hat{K}_g(\zeta,\cdot)\in C_0^\infty(\Omega_1(\rho_1))$. Furthermore $\hat{K}_g$ satisfies for any $l=0,1,2,\cdots$, and $\theta=(\alpha,\beta,\gamma)$,*

$$|\partial_\zeta^l\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma\hat{K}_g(\zeta,x,\xi,y)|\leqq\begin{cases}C_{jl\alpha\beta\gamma}\langle\zeta\rangle^{-\sigma} & \text{for any } \sigma<2 \text{ when } m\leqq2,\\ C_{jl\alpha\beta\gamma}\langle\zeta\rangle^{-[m]}\lambda^{(1/m-1/2)|\gamma|}(\log\lambda)^{\varepsilon(m,|\theta|+1)} & \text{when } m>2.\end{cases}$$

Here $[m]$ is the largest integer not greater than m.

PROOF. As $K_g\in C_0^\infty((-\rho_0,\rho_0)\times\Omega_1(\rho_1))$, the first statement of the lemma is obvious. By (4.11) it is easy to see that

$$\begin{aligned}&|\partial_t^k\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma K_g(t,x,\xi,y)| \\ &\quad\leqq C_{k\alpha\beta\gamma}\lambda^{-(1/2-1/\tilde{m})|\gamma|+((\tilde{m}-|\theta|-k^*)_+-(\tilde{m}-k^*))/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\theta|+1)}.\end{aligned}$$

Hence the integration by parts shows that the quantity in question is bounded for any $j=0,1,2,\cdots$ by

$$(5.5)\qquad C\langle\zeta\rangle^{-j}\lambda^{-(1/2-1/\tilde{m})|\gamma|+((\tilde{m}-|\theta|-j^*)_+-(\tilde{m}-j^*))/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\theta|+1)}.$$

Letting $j=[m]$ in (5.5), we obtain the lemma for the case $m>2$. When $m\leqq2$ we let $j=1$ and $j=2$ in (5.5) and interpolate the resulting estimates.

For estimating $g_F(\lambda,\mathfrak{H})$, we first decompose the integral (5.4) as follows. Take a function $\omega(\hat{\xi},\hat{\eta})\in C^\infty(S^{n-1}\times S^{n-1})$ such that $\operatorname{supp}\omega\subset\{(\hat{\xi},\hat{\eta}):\hat{\xi}\cdot\hat{\eta}>1-10^{-7}\}$ and $\int_{S^{n-1}}\omega(\hat{\xi},\hat{\eta})d\nu(\eta)=1$ for every $\hat{\xi}\in S^{n-1}$ where $d\nu$ is the standard surface measure on S^{n-1}, and define for each $\hat{\eta}\in S^{n-1}$ as

$$\begin{aligned}(5.6)\quad & g_F(\lambda,\mathfrak{H};\hat{\eta})f(x) \\ &=(2\pi)^{-n}\int e^{i\tilde{\phi}(x,\xi,y)}\hat{K}_g\Big(\frac{\xi^2/2+\tilde{V}(y)-\lambda}{\lambda^{1/2-1/\tilde{m}}},x,\xi,y\Big)\omega(\hat{\xi},\hat{\eta})f(y)dyd\xi\end{aligned}$$

where $\hat{\xi}=\xi/|\xi|$. Then

$$g_F(\lambda, \mathfrak{H})f(x) = \int_{S^{n-1}} g_F(\lambda, \mathfrak{H};\ \hat{\eta})d\nu(\eta). \tag{5.7}$$

PROPOSITION 5.3. *Let $q>1/2$. Then there exists a constant $C>0$ independent of $\hat{\eta}\in S^{n-1}$ such that for every $f\in L^2(\boldsymbol{R}^n)$*

$$\|g_F(\lambda, \mathfrak{H};\ \hat{\eta})f\| \leqq C\lambda^{-1/2\tilde{m}}\|\langle x\rangle^{+q}f\|.$$

PROOF. By the rotationary invariance of our assumption it suffices to prove the case $\hat{\eta}=\boldsymbol{e}_1$ only. In this case the proposition follows immediately from the following

LEMMA 5.4. *There exists a constant $C>0$ such that*

$$\|g_F(\lambda, \mathfrak{H};\ \boldsymbol{e}_1)f\|_{2,\infty} \leqq C\lambda^{-1/\tilde{m}}\|f\|_{2,1}, \tag{5.8}$$

$$\|g_F(\lambda, \mathfrak{H};\ \boldsymbol{e}_1)f\|_{2,1,w} \leqq C\|(\log\langle x_1\rangle+1)f\|_{2,1}, \tag{5.9}$$

$$\|g_F(\lambda, \mathfrak{H};\ \boldsymbol{e}_1)f\|_{2,2} \leqq C\lambda^{-1/2\tilde{m}}\|(\log\langle x_1\rangle+1)^{1/2}f\|_{2,1}. \tag{5.10}$$

PROOF OF LEMMA 5.4. By the Marcinkiewicz interpolation theorem for vector valued functions (5.10) follows from (5.8) and (5.9). We prove (5.8) first. If $(x, \xi, y)\in\operatorname{supp}\hat{K}_g\cap\operatorname{supp}\omega(\cdot, \boldsymbol{e}_1)$, then $\xi_1>0$, $\xi^2/2\lambda\in[1/3, 3]$, $|\tilde{V}(y)|\leqq 10^{-2}\lambda$ and $\underline{\xi}^2/\xi^2<10^{-6}$, $\underline{\xi}=(\xi_2, \cdots, \xi_n)$. Hence $\lambda/2+10^2\underline{\xi}^2+2|\tilde{V}(y)|\leqq\xi_1^2\leqq 6\lambda$ and the change of the variable $\xi_1\to(\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}$ is permissible in (5.6). Denote as $\hat{\omega}(\xi)=\omega(\hat{\xi}, \boldsymbol{e}_1)$ and $v_1=(x_1, \xi_1, y_1)$. Taking $\omega_1\in C_0^\infty(S^{n-1})$ such that

$$\omega_1(\hat{\xi})\omega(\hat{\xi}, \boldsymbol{e}_1)=\omega(\hat{\xi}, \boldsymbol{e}_1) \quad \text{and} \quad \operatorname{supp}\omega_1\subset\{\hat{\xi}\in S^{n-1}:\ \hat{\xi}\cdot\boldsymbol{e}_1>1-2\cdot 10^{-7}\}$$

and defining $\hat{\omega}_1(\xi)=\omega_1(\hat{\xi})$, we set

$$\begin{aligned}\phi(v_1, \underline{x}, \underline{\xi}, \underline{y}) &= \tilde{\phi}(x, (\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi}, y)\hat{\omega}_1((\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi})\\ &\quad +(1-\hat{\omega}_1((\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi}))(x-y)\cdot\xi,\\ W(v_1, \underline{x}, \underline{\xi}, \underline{y}) &= \hat{K}_g\Big(\frac{\xi_1^2/2-\lambda}{\lambda^{1/2-1/\tilde{m}}}, x, (\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi}, y\Big)\\ &\qquad \times\xi_1(\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{-1/2}\hat{\omega}((\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi}).\end{aligned}$$

Then performing the change of variable as above, we rewrite (5.6) as

$$\begin{aligned}&g_F(\lambda, \mathfrak{H};\ \boldsymbol{e}_1)f(x)\\ &\quad = (2\pi)^{-1}\int_{\boldsymbol{R}^2} dy_1 d\xi_1\Big\{(2\pi)^{-n+1}\int_{\boldsymbol{R}^{2n-2}} e^{i\phi(x_1, \xi_1, y_1, \underline{x}, \underline{\xi}, \underline{y})}\\ &\qquad\qquad \times W(x_1, \xi_1, y_1, \underline{x}, \underline{\xi}, \underline{y})f(y_1, \underline{y})d\underline{y}d\underline{\xi}\Big\}\end{aligned} \tag{5.11}$$

and regard the inner integral as an operator with parameter $v_1=(x_1, \xi_1, y_1)$ acting on the function spaces over $\boldsymbol{R}^{n-1}$:

$$(5.12)\qquad Z(v_1, \lambda)h(\underline{x}) = (2\pi)^{-n+1}\int e^{i\psi(v_1, \underline{x}, \underline{\xi}, \underline{y})} W(v_1, \underline{x}, \underline{\xi}, \underline{y})h(\underline{y})d\underline{y}d\underline{\xi}.$$

An application of Lemma 5.2 and an elementary estimation yield the following

LEMMA 5.5. *For any multi-index* $\theta=(\alpha, \beta, \gamma)$ *we have*

$$(5.13)\qquad |\partial_{\underline{x}}^{\alpha}\partial_{\underline{y}}^{\beta}\partial_{\underline{\xi}}^{\gamma}W(v_1, \underline{x}, \underline{\xi}, \underline{y})| \leqq \begin{cases} C\langle\zeta\rangle^{-\sigma} & \text{for any } \sigma<2 \text{ when } m\leqq 2, \\ C\langle\zeta\rangle^{-[m]}\lambda^{(1/m-1/2)|\gamma|}(\log\lambda)^{\varepsilon(m, |\theta|+1)} & \text{when } m>2, \end{cases}$$

where $\zeta=(\xi_1^2/2-\lambda)/\lambda^{1/2-1/\tilde{m}}$.

As for the phase function, we write as

$$\psi(v_1, \underline{x}, \underline{\xi}, \underline{y}) = (x-y)\cdot\xi + \{(x_1-y_1)(\eta_1-\xi_1)+\tilde{\phi}_D(x, \eta_1, \underline{\xi}, y)\}\hat{\omega}_1(\eta_1, \underline{\xi})$$

where $\eta_1=(\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}$. We set $\kappa=(1/2-1/\tilde{m})/2$ as before. Using the relation (3.35)~(3.36) and Lemma 3.9, we have the following

LEMMA 5.6. *Let* $\psi_\kappa(v_1, \underline{x}, \underline{\xi}, \underline{y})=\psi(v_1, \lambda^{-\kappa}\underline{x}, \lambda^{\kappa}\underline{\xi}, \lambda^{-\kappa}\underline{y})$ *and*

$$\underline{D}(\psi_\kappa)(v_1, \underline{x}, \underline{\xi}, \underline{y}) = \begin{pmatrix} \partial_{\underline{x}}\partial_{\underline{y}}\psi_\kappa & \partial_{\underline{\xi}}\partial_{\underline{y}}\psi_\kappa \\ \partial_{\underline{x}}\partial_{\underline{\xi}}\psi_\kappa & \partial_{\underline{\xi}}\partial_{\underline{\xi}}\psi_\kappa \end{pmatrix}.$$

Then for sufficiently small $\rho_0>0$ *the following statements hold:*

(i) *Each element of the matrix* $\underline{D}(\psi_\kappa)$ *is bounded with their derivatives uniformly w.r.t.* $(v_1, \underline{x}, \underline{\xi}, \underline{y})$ *and* $\lambda\geqq\lambda_1(\rho_0)$.

(ii)
$$\left\|\underline{D}(\psi_\kappa)(v_1, \underline{x}, \underline{\xi}, \underline{y})-\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\| \leqq 10^{-1},$$

for all $(v_1, \underline{x}, \underline{\xi}, \underline{y})$ *and* $\lambda\geqq\lambda_1(\rho_0)$.

We now apply the L^2-boundedness theorem for the oscillatory integrals operators ([**3**]) to (5.12). Since Lemma 5.5 implies that $W_\kappa(v_1, \underline{x}, \underline{\xi}, \underline{y})= W(v_1, \lambda^{-\kappa}\underline{x}, \lambda^{\kappa}\underline{\xi}, \lambda^{-\kappa}\underline{y})$ satisfies

$$|\partial_{\underline{x}}^{\alpha}\partial_{\underline{y}}^{\beta}\partial_{\underline{\xi}}^{\gamma}W_\kappa(v_1, \underline{x}, \underline{\xi}, \underline{y})| \leqq \begin{cases} C\langle\zeta\rangle^{-\sigma} & \text{for any } \sigma<2 \text{ when } m\leqq 2, \\ C\langle\zeta\rangle^{-[m]} & \text{when } m>2, \end{cases}$$

with $\zeta=(\xi_1^2/2-\lambda)\lambda^{1/\tilde{m}-1/2}$, the boundness theorem ([**3**]) yields

$$(5.14)\qquad \|Z(v_1, \lambda)h(\underline{x})\|_{L^2(R^{n-1})} \leqq C\left\langle\frac{\xi_1^2/2-\lambda}{\lambda^{1/2-1/\tilde{m}}}\right\rangle^{-\delta}\|h\|_{L^2(R^{n-1})}.$$

Here, and hereafter, $\delta=[m]$ when $m>2$ and when $m\leqq 2$ δ is an arbitrary number smaller than 2. Hence by (5.10)~(5.11)

$$(5.15)\qquad \|g_F(\lambda, \mathfrak{H}, e_1)f(x_1, \cdot)\|_{2,\infty} \leqq C\int_{\sqrt{\lambda/2}}^{\sqrt{8\lambda}}\left\langle\frac{\xi_1^2/2-\lambda}{\lambda^{1/2-1/\tilde{m}}}\right\rangle^{-\delta}d\xi_1\|f\|_{2,1}.$$

Combining (5.15) with the obvious inequality

$$(5.16)\qquad \int_{\sqrt{\lambda/2}}^{\sqrt{8\lambda}}((\xi_1^2/2-\lambda)^2/\lambda^{1-2/\tilde{m}}+1)^{-\delta/2}d\xi_1 \leqq C\lambda^{-1/\tilde{m}}$$

for $\delta>1$, we obtain (5.8).

For proving (5.9) we return to the expression (5.6). By (3.28) and (3.29) we may write for $\lambda\geqq\lambda_1(\rho_0)$, $\rho_0>0$ small as

$$\partial_\xi\tilde{\phi}_D(x, \xi, y)=(x-y)\cdot\int_0^1\partial_x\partial_\xi\tilde{\phi}_D(\theta x+(1-\theta)y, \xi, y)d\theta$$

and

$$\left\|\int_0^1\partial_x\partial_\xi\tilde{\phi}_D(\theta x+(1-\theta)y, \xi, y)d\theta\right\|\leqq 10^{-1}.$$

Hence $|\partial_\xi\tilde{\phi}(x, \xi, y)|\geqq|x-y|/2$ and after performing integration by parts, we may rewrite as

$$(5.17)\quad g_F(\lambda, \mathfrak{H}; \hat{\eta})f(x)=(2\pi)^{-n}\int e^{i\tilde{\phi}(x,\xi,y)}M\Big(\frac{\xi^2/2+\tilde{V}(y)-\lambda}{\lambda^{1/2-1/\tilde{m}}}, x, \xi, y\Big)f(y)dyd\xi,$$

$$\begin{aligned}M(\zeta, x, \xi, y)=&[(1+|\partial_\xi\tilde{\phi}|^2)^{-1}+i\sum_{j=1}^n\partial_{\xi_j}(\partial_{\xi_j}\tilde{\phi}(1+|\partial_\xi\tilde{\phi}|^2)^{-1})]\hat{K}_g(\zeta, x, \xi, y)\omega(\hat{\xi}, \hat{\eta})\\&+i\sum_{j=1}^n(1+|\partial_\xi\tilde{\phi}|^2)^{-1}\cdot\partial_{\xi_j}\tilde{\phi}\cdot\partial_{\xi_j}\{\hat{K}_g(\zeta, x, \xi, y)\omega(\hat{\xi}, \hat{\eta})\}\\&+i\lambda^{1/\tilde{m}-1/2}(1+|\partial_\xi\tilde{\phi}|^2)^{-1}\Big(\sum_{j=1}^n\xi_j\cdot\partial_{\xi_j}\tilde{\phi}\Big)(\partial_\zeta\hat{K}_g)(\zeta, x, \xi, y)\omega(\hat{\xi}, \hat{\eta}).\end{aligned}$$

LEMMA 5.7. *For any $\theta=(\alpha, \beta, \gamma)$, we have, with $\zeta=\lambda^{1/\tilde{m}-1/2}(\xi^2/2+\tilde{V}(y)-\lambda)$,*

$$(5.18)\quad \begin{aligned}&|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma M(\zeta, x, \xi, y)|\\&\leqq\begin{cases}C\langle x-y\rangle^{-1}\lambda^{-(1/2-1/m)|\gamma|}\langle\zeta\rangle^{-[m]}\lambda^{1/m}(\log\lambda)^{\varepsilon(m,|\theta|+1)}, & m>2;\\ C\langle x-y\rangle^{-1}\langle\zeta\rangle^{-\sigma}\lambda^{1/2} \quad \textit{for any } \sigma<2,\ m\leqq2.\end{cases}\end{aligned}$$

PROOF. By (3.30), we have

$$\begin{aligned}&\langle x-y\rangle|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(1+|\partial_\xi\tilde{\phi}|^2)^{-1}|+|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma(1+|\partial_\xi\tilde{\phi}|^2)^{-1}\partial_\xi\tilde{\phi}|\\&\leqq C\langle x-y\rangle^{-1}\lambda^{-1-|\gamma|/2+(\tilde{m}-|\theta|)_+/\tilde{m}+|\gamma|/\tilde{m}}(\log\lambda)^{\varepsilon(m,|\theta|)}.\end{aligned}$$

Combining this with Lemma 5.2, we obtain (5.18).

Starting from (5.17), we proceed in exactly the same way as before. For $\hat{\eta}=\boldsymbol{e}_1$, we set

$$\tilde{Q}(v_1, \underline{x}, \underline{\xi}, \underline{y})=\hat{M}\Big(\frac{\xi_1^2/2-\lambda}{\lambda^{1/2-1/\tilde{m}}}, x, (\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{1/2}, \underline{\xi}, y\Big)\cdot\xi_1(\xi_1^2-\underline{\xi}^2-2\tilde{V}(y))^{-1/2},$$

$$\tilde{Z}(v_1, \lambda)h(\underline{x})=(2\pi)^{-n+1}\int e^{i\phi(v_1,\underline{x},\underline{\xi},\underline{y})}\tilde{Q}(v_1, \underline{x}, \underline{\xi}, \underline{y})h(\underline{y})d\underline{y}d\underline{\xi}.$$

Then, we have as before,

$$(5.19)\qquad g_F(\lambda, \mathfrak{H}; \boldsymbol{e}_1)f(x)=(2\pi)^{-1}\int_{R^2}\tilde{Z}(v_1, \lambda)f(y_1, \cdot)dy_1d\xi_1.$$

Lemma 5.7 and the argument which lead to (5.14) yield

$$\|\tilde{Z}(v_1, \lambda)h\|_{L^2(R^{n-1})}\leqq C\langle x_1-y_1\rangle^{-1}\lambda^{1/\tilde{m}}\Big\langle\frac{\xi_1^2/2-\lambda}{\lambda^{1/2-1/\tilde{m}}}\Big\rangle^{-\delta}\|h\|,$$

where as before $\delta=[m]$ for $m>2$ and δ is any number in $(0,2)$ for $m\leqq 2$. Thus by (5.16)

$$\|g_F(\lambda, \mathfrak{H}; e_1)f(x_1, \cdot)\| \leqq C\int \langle x_1-y_1\rangle^{-1}\|f(y_1, \cdot)\|dy_1$$

and (5.9) follows. This completes the proof of Lemma 5.4.

COMPLETION OF THE PROOF OF THEOREM 1.1. Let $q>1/2$. By (5.6)~(5.7) and Proposition 5.3 we have

$$\|g_F(\lambda, \mathfrak{H})\langle x\rangle^{-q}f\| \leqq C\lambda^{-1/2\tilde{m}}\|f\|. \tag{5.20}$$

Combining (5.20) with Lemma 5.1, we obtain

$$\|g(\rho_0^{-1}\lambda^{1/\tilde{m}-1/2}(\mathfrak{H}-\lambda))\Phi_0(D, \lambda)\langle x\rangle^{-q}f\| \leqq C\lambda^{-1/2\tilde{m}}\|f\|.$$

Since $g(\xi)\geqq C>0$ for $|\xi|\leqq 1$, it follows that

$$\|E_{\mathfrak{H}}([\lambda-\rho_0\lambda^{1/2-1/\tilde{m}}, \lambda+\rho_0\lambda^{1/2-1/\tilde{m}}])\Phi_0(D, \lambda)\langle x\rangle^{-q}f\| \leqq C\lambda^{-1/2\tilde{m}}\|f\|. \tag{5.21}$$

Adding up (2.5) and (5.21), we obtain

$$\|E_{\mathfrak{H}}([\lambda-\rho_0\lambda^{1/2-1/\tilde{m}}, \lambda+\rho_0\lambda^{1/2-1/\tilde{m}}])\langle x\rangle^{-q}\| \leqq C\lambda^{-1/2\tilde{m}}$$

and it is clear that this remains valid for arbitrary $\rho>0$ replacing ρ_0. Estimate (1.2) then follows by the well-known identity $\|T^*T\|=\|T\|^2$. This completes the proof of Theorem 1.1.

Corollary 1.3 is a trivial consequence of Theorem 1.1.

PROOF OF COROLLARY 1.4. Note first that $\langle E_{\mathfrak{H}}([n, n+1])\langle x\rangle^{-q}f, \langle x\rangle^{-q}f\rangle\leqq Cd_m(\lambda)\|f\|^2$ for any $n\in[\lambda/2, \lambda]$. Hence,

$$\begin{aligned}
&|(R(\lambda\pm i\varepsilon)f, g)| \\
&\leqq |(R(\lambda\pm i\varepsilon)E_{\mathfrak{H}}(\boldsymbol{R}^1\setminus(\lambda-\lambda^{1/\tilde{m}}, \lambda+\lambda^{1/\tilde{m}}))f, g)|+\left|\int_{\lambda-\lambda^{1/\tilde{m}}}^{\lambda+\lambda^{1/\tilde{m}}}(\mu-\lambda+i\varepsilon)^{-1}(E_{\mathfrak{H}}(d\lambda)f, g)\right| \\
&\leqq \lambda^{-1/\tilde{m}}\|f\|\|g\|+\sum_{n=[\lambda-\lambda^{1/\tilde{m}}]}^{[\lambda+\lambda^{1/\tilde{m}}]+1}\max_{n<\mu<n+1}((\mu-\lambda)^2+\varepsilon^2)^{-1/2}\|E_{\mathfrak{H}}(n, n+1)f\|\cdot\|E_{\mathfrak{H}}(n, n+1)g\| \\
&\leqq C(\lambda^{-1/\tilde{m}}+d_m(\lambda)\log(\lambda+2))\|\langle x\rangle^q f\|\|\langle x\rangle^q g\|
\end{aligned}$$

and the relation (1.4) follows.

§ 6. Singular potentials.

In this section we assume

ASSUMPTION $(\mathrm{B})_m$. For some $m\geqq 0$ and $p\geqq\max(n/2+\varepsilon, 2)$, $\varepsilon>0$

$$\sup_{x\in\boldsymbol{R}^n}\langle x\rangle^{-m}\left(\int_{|x-y|\leqq 1}|V(y)|^p dy\right)^{1/p}=\|V\|_{*, m, p}<\infty. \tag{6.1}$$

By Sobolev embedding theorem, we have for $\sigma\geqq\sigma_0=n/p$

$$\|\langle x\rangle^{-m}V(x)\langle D\rangle^{-\sigma}f\|\leqq C\|V\|_{*,m,p}\|f\| \tag{6.2}$$

and $-(1/2)\Delta+V(x)|_{\mathcal{S}(\boldsymbol{R}^n)}$ is real symmetric on $L^2(\boldsymbol{R}^n)$. We let $\mathfrak{H}$ be any of its selfadjoint extensions.

THEOREM 6.1. *Let Assumption* $(\mathrm{B})_m$ *be satisfied and* $\mathfrak{H}$ *be as above. Then for* $I=[-a, a]$ *and* $R>1$,

$$\|\chi(x/R)E_{\mathfrak{H}}(I+\lambda)\chi(x/R)\|\leqq C\lambda^{-(1-n/2p)/(2m+n/p+3)}, \qquad \lambda\geqq 1. \tag{6.3}$$

We prove (6.3), following the argument of Sections $2\sim5$ but with slight modifications. $\Phi_0(D, \lambda)$ and $\Phi_1(D, \lambda)$ are as in §2 and $\Gamma_1(\lambda, a)$ the contour $\lambda+2a+i\to\lambda-2a+i\to\lambda-2a-i\to\lambda+2a-i\to\lambda+2a+i$. We denote $\chi_L(x)=\chi(x/L)$ and $\mathfrak{H}_0=-(1/2)\Delta$.

LEMMA 6.1. *Let* l, $s\geqq 0$. *Then for any* $N\geqq 0$,

$$\|(1-\chi_{2L})\langle x\rangle^l\langle D\rangle^s(\mathfrak{H}_0-z)^{-1}\Phi_1(D, \lambda)\chi_L\|\leqq C_N\lambda^{-N} \tag{6.4}$$

for $z\in\Gamma_1(\lambda, a)$, $L\geqq 1$.

PROOF. Let $g_s(x)=x^{2s}(x^2/2-z/\lambda)^{-1}\phi_1(x^2/2)$. Then by integrations by parts, we have for $|x|\geqq 2L$

$$\begin{aligned} &|\langle x\rangle^l(-\Delta)^s(\mathfrak{H}_0-z)^{-1}\Phi_1(D, \lambda)\chi_Lf(x)| \\ &=\left|(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\langle x\rangle^l|x-y|^{-2N}\lambda^{s-1-N}(\Delta^Ng_s)(\xi/\sqrt{\lambda})\chi_Lf(y)dyd\xi\right| \\ &\leqq(2\pi)^{-n}16^N\|\chi_L\|_2\|\Delta^Ng_s\|_1\lambda^{s+n/2-1-N}\langle x\rangle^{l-2N}\|f\|_2. \end{aligned} \tag{6.5}$$

Integrating (6.5) and using the interpolation, we obtain (6.4).

LEMMA 6.2. *For* $0\leqq s\leqq 2$ *and* $\lambda\geqq\lambda^*=10(a+1)$,

$$\|\langle D\rangle^s(\mathfrak{H}_0-z)^{-1}\Phi_1(D, \lambda)\|\leqq 4\lambda^{s/2-1}, \qquad z\in\Gamma_1(\lambda, a). \tag{6.6}$$

PROOF. Apply the Plancherel theorem.

LEMMA 6.3. *For* $\lambda\geqq\lambda^*$, $L\geqq 1$ *and* $\sigma_0=n/p$,

$$\|V(\mathfrak{H}_0-z)^{-1}\Phi_1(D, \lambda)\chi_L\|\leqq C\lambda^{\sigma_0/2-1}, \qquad z\in\Gamma_1(\lambda, a). \tag{6.7}$$

PROOF. This is an immediate consequence of (6.2) and (6.4).

PROPOSITION 6.4. *Let* $I=[-a, a]$, $a>0$, $R\geqq 1$ *and* $\lambda\geqq 10(a+1)$. *Then*

$$\|E_{\mathfrak{H}}(I+\lambda)\Phi_1(D, \lambda)\chi_R\|\leqq C\lambda^{\sigma_0/2-1}. \tag{6.8}$$

PROOF. For $f\in\mathcal{S}$ and $z\in\Gamma_1(\lambda, a)$, $(\mathfrak{H}_0-z)^{-1}\Phi_1(D, \lambda)\chi_Rf\in\mathcal{S}(\boldsymbol{R}^n)$ and

$$E_{\mathfrak{H}}(I+\lambda)(\mathfrak{H}_0-z)^{-1}\Phi_1(D,\lambda)\chi_R f = E_{\mathfrak{H}}(I+\lambda)(\mathfrak{H}-z)^{-1}\Phi_1(D,\lambda)\chi_R f$$
$$+E_{\mathfrak{H}}(I+\lambda)(\mathfrak{H}-z)^{-1}V(\mathfrak{H}_0-z)^{-1}\Phi_1(D,\lambda)\chi_R f.$$

Integrate this on $z\in\Gamma(\lambda, a)$ and use (6.7) to obtain (6.8).

For studying $E_{\mathfrak{H}}(I+\lambda)\Phi_0(D,\lambda)\chi_R$ we adopt again the expression (1.8). Here our parametrix is simply $\exp(-it\mathfrak{H}_0)$.

LEMMA 6.5. *Let s, l and $N\geqq 0$. Then there exists a constant $C>0$ such that for any $\lambda\geqq 1$, $L\geqq 1$ and $-\infty<t<\infty$,*

$$\|(1-\chi_{4(\sqrt{\lambda}|t|+L)})\langle x\rangle^l\langle D\rangle^s e^{-it\mathfrak{H}_0}\Phi_0(D,\lambda)\chi_L\| \leqq C\lambda^{-N}(\sqrt{\lambda}\,|t|+L)^{-N}. \tag{6.9}$$

PROOF. We prove (6.9) for $t\geqq 0$. If $|x|\geqq 4(\sqrt{\lambda}t+L)$, $|y|\leqq L$ and $\xi\in\operatorname{supp}\Phi_0(\cdot,\lambda)$, then $|x-t\xi-y|\geqq(|x|+\sqrt{\lambda}t+L)/4$. Hence by integration by parts, we have for any $N\geqq 0$,

$$\begin{aligned}|\langle D\rangle^s e^{-it\mathfrak{H}_0}\Phi_0(D,\lambda)\chi_L f(x)| &= \left|(2\pi)^{-n}\int e^{i(x-y)\xi-it\xi^2/2}\left[\left\{\frac{(x-y-t\xi)}{(x-y-t\xi)^2}\cdot D_\xi\right\}^{*2N}\langle\xi\rangle^s\Phi_0(\xi,\lambda)\right]\chi_L f(y)dyd\xi\right| \\ &\leqq C_{Ns}(|x|+\sqrt{\lambda}t+L)^{-2N}\lambda^{-N+s/2+n/2}L^{n/2}\|f\|.\end{aligned} \tag{6.10}$$

Taking N large enough and integrating (6.10), we obtain (6.9).

LEMMA 6.6. *If $0\leqq 2/l=n(1/2-1/q)<1$, then*

$$\left(\int_{-\infty}^{\infty}\|e^{-it\mathfrak{H}_0}u\|_q^l dt\right)^{1/l} \leqq C_{n,l}\|u\|. \tag{6.13}$$

PROOF. See Lemma 3.1 of [**24**].

By Lemma 6.5 and (6.2),

$$\|V(1-\chi_{4(\sqrt{\lambda}|t|+L)})e^{-it\mathfrak{H}_0}\Phi_0(D,\lambda)\chi_L f\| \leqq C\lambda^{-N}(\sqrt{\lambda}\,|t|+L)^{-N}\|f\|. \tag{6.12}$$

Lemma 6.6 and Hölder inequality imply for $1/p+1/p'=1/2$, $2/l=n/p$ and $1/l+1/l'=1$,

$$\begin{aligned}&\int_0^t\|V\chi_{4(\sqrt{\lambda}s+L)}e^{-is\mathfrak{H}_0}\Phi_0(D,\lambda)\chi_L f\|ds \\ &\quad\leqq\left(\int_0^t\|V\chi_{4(\sqrt{\lambda}s+L)}\|_p^{l'}ds\right)^{1/l'}\left(\int_0^t\|e^{-is\mathfrak{H}_0}\Phi_0(D,\lambda)\chi_L f\|_{p'}^l ds\right)^{1/l} \\ &\quad\leqq C\lambda^{-1/2l'}(\sqrt{\lambda}t+L)^{m+1+(n/2p)}\|f\|, \qquad t\geqq 0.\end{aligned} \tag{6.12}$$

LEMMA 6.7. *Let $\check{g}\in C_0^\infty(-1,1)$ and $\rho\geqq 0$. Then for $L\geqq 1$,*

$$\begin{aligned}&\|\{g((\mathfrak{H}-\lambda)/\lambda^\rho)-g((\mathfrak{H}_0-\lambda)/\lambda^\rho)\}\Phi_0(D,\lambda)\chi_L\| \\ &\quad\leqq C_L\lambda^{-(1-n/2p)/2}(1+\lambda^{(1/2-\rho)(m+n/2p+1)}), \qquad \lambda\geqq 1.\end{aligned} \tag{6.13}$$

PROOF. By (6.1) and (6.12) it follows that

$$\|(e^{-it\mathfrak{H}}-e^{-it\mathfrak{H}_0})\Phi_0(D,\ \lambda)\chi_L f\| = \left\|\int_0^t e^{-i(t-s)\mathfrak{H}} V e^{-is\mathfrak{H}_0}\Phi_0(D,\ \lambda)\chi_L f ds\right\|$$
$$\leqq C(\lambda^{-N}+\lambda^{-1/2l'}(\sqrt{\lambda}|t|+L)^{(m+1+n/2p)}).$$

Thus by the functional calculus

$$\|\{g((\mathfrak{H}-\lambda)/\lambda^{\rho})-g((\mathfrak{H}_0-\lambda)/\lambda^{\rho})\}\Phi_0(D,\ \lambda)\chi_L\|$$
$$\leqq C\int_{-\infty}^{\infty}|\lambda^{\rho}\check{g}(\lambda^{\rho}t)(\lambda^{-N}+\lambda^{-1/2l'}(\sqrt{\lambda}|t|+L)^{m+1+n/2p})|dt$$
$$\leqq C_L(\lambda^{-N}+\lambda^{(1/2-\rho)(m+1+n/2p)-1/2l'}+\lambda^{-1/2l'})\|\check{g}\|_1.$$

LEMMA 6.8. *For* $L\geqq 1$,

$$\|g((\mathfrak{H}_0-\lambda)/\lambda^{\rho})\Phi_0(D,\ \lambda)\chi_L\| \leqq C\lambda^{\rho/2-1/4}\|g\|, \qquad \lambda\geqq 1.$$

PROOF. By the trace theorem

$$\sup_{r>0}\int_{|\xi|=r}|\hat{f}(\xi)|^2 d\xi \leqq C_{\sigma}\|\langle x\rangle^{\sigma} f\|^2, \qquad f\in\mathcal{S}.$$

Hence by using the Minkowski inequality we obtain

$$\|g((\mathfrak{H}_0-\lambda)/\lambda^{\rho})\Phi_0(D,\ \lambda)\chi_L f\|^2 \leqq C\|f\|^2\int_0^{\infty}|g((r^2/2-\lambda)/\lambda^{\rho})|^2 dr$$
$$\leqq C\lambda^{\rho-1/2}\|f\|^2\|g\|^2.$$

PROOF OF THEOREM 6.1. Choose now $\rho=(2m+1+2n/p)/(4m+6+2n/p)$ and $g\in\mathcal{S}(\boldsymbol{R}^1)$ such that $g(x)>0$ for $x\in[-a,\ a]$ and $\check{g}\in C_0^{\infty}(\boldsymbol{R}^1)$. Then by Lemmas 6.7~6.8,

$$\|E_{\mathfrak{H}}(I+\lambda)\Phi_0(D,\ \lambda)\chi_L\| \leqq C\|g((\mathfrak{H}-\lambda)/\lambda^{\rho})\Phi_0(D,\ \lambda)\chi_L\| \tag{6.13}$$
$$\leqq C\lambda^{-(1-n/2p)/(4m+2n/p+6)}.$$

Combining Proposition 6.4 with (6.13), we obtain (6.3).

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