Remarks on the L^2-cohomology of singular algebraic surfaces

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§ 1. Introduction.

Let X be a normal singular algebraic surface (over C) embedded in the projective space $P^N(C)$ and let S be its singularity set, which consists of isolated singular points. By restricting the Fubini-Study metric of $P^N(C)$ to $\mathcal{X} = X-S$, we obtain an incomplete Riemannian manifold $(\mathcal{X}, g)$. Then Hsiang-Pati asserted in [9] that the L^2-cohomology $H^i_{(2)}(\mathcal{X})$ is naturally isomorphic to the dual of the middle intersection homology $IH^{\overline{m}}_i(X)$, which is a special case of the conjecture due to Cheeger, Goresky and MacPherson [5, §4, Conjecture C] that it holds for any algebraic variety. However their proof has a certain gap. In this paper we will fill it. Our main result is therefore the reassertion.

THEOREM 1. *For the X, we have*

$$(1.1) \qquad\qquad H^i_{(2)}(\mathcal{X}) \cong (IH^{\overline{m}}_i(X))^* .$$

As for the "non-normal" case, it can obviously be proved in the same way as Theorem 1 (in the "normal" case) by making its normalization, as asserted in [9, Theorem A'] —— see also Remark 3.3 in this paper.

In order to prove (1.1), we will make a good resolution $\pi : \tilde{X} \to X$ according to [9] and investigate the metric π^*g near the $\pi^{-1}(S) = \bigcup D_j$, (irreducible components), which is the first step. It is here that the gap seems to occur: though they regard the metric near the intersection points of the D_j as of the same type as the metric near the non-intersection points, the former one is dominated by the $W(+)$, not by the $W(-)$ which dominates the latter one (and is called "of Cheeger type" in [9]): see Types $(\pm)$ in §2. And, because of the complexity of $W(+)$, we need some argument much subtler than that in [9].

Besides Theorem 1, there still remains the following problem, which has a close relation with Theorem 1. Let d_i be the exterior derivative d acting on the smooth i-forms on $\mathcal{X}$ which and whose images by the d are both square-integrable. Also let $d_{c,i}$ be its restriction to the compactly supported smooth i-forms. Then their closures $\bar{d}_i$ and $\bar{d}_{c,i}$ must be equal to each other, that is,

Conjecture 1. $\bar{d}_i = \bar{d}_{c,i}$.

If we define $\delta_i = -*d_{3-i}*$, the restriction of the formal adjoint of the d, and denote its closure by $\bar{\delta}_i$, then Conjecture 1 says that, with respect to the inner product $\langle \alpha, \beta \rangle = \int_{\mathcal{X}} \alpha \wedge *\beta$, we have

(1.2) $\langle \bar{d}_i \alpha, \beta \rangle = \langle \alpha, \bar{\delta}_i \beta \rangle$

for $\alpha \in \mathrm{dom}\, \bar{d}_i$ and $\beta \in \mathrm{dom}\, \bar{\delta}_i$. Namely Conjecture 1 asserts that the *Stokes' theorem in the L^2-sense* ([3, §1]) holds for the $\mathcal{X}$. So far we can prove only the following.

Proposition 1.1 ([14, Assertion A]).   *Conjecture 1 is true for $i=0, 3, 4$.*

(Note that the cases $i=0$ and $i=3$ are equivalent and the case $i=4$ is obvious.) The cases $i=1$ and $i=2$ (which are equivalent) are still not proved. The author's examination of Theorem 1 started when he got the following comment (on Theorem 1 and Conjecture 1) from J. Cheeger, "It shouldn't be too difficult to generalize $\bar{d} = \bar{d}_c$ for functions to i-forms. Since Hsiang-Pati ([9]) used the argument in [3] without verifying it, your result (in [14]) seems to fill partially that gap in their proof". The fact is, the gap lies in their investigation of the metric near the intersection points (as explained, it is dominated by the $W(+)$) and, because of the complexity of $W(+)$, we cannot so far generalize the argument in [14, Assertion A] to 1- and 2-forms —— refer to the Appendix in this paper for more explanation. (If there were no $W(+)$, Conjecture 1 could be proved in the same way as [3, Theorem 2.2] as Hsiang-Pati might have "proved".) As to the proof of Theorem 1 for $i=2$ (or Proposition 3.1 for $i=2$), in which Conjecture 1 for $i=2$ would play an important role if it could be proved, we will attack it by using an L^2-version of long exact sequence.

The author cannot refrain from expressing his admiration for Cheeger's nice insight (certainly the difficulty lies around the conjecture), as well as the beautiful argument by W. C. Hsiang and V. Pati. Theorem 1 is, of course, largely theirs. The author's contribution lies in checking their resolution, investigating the metric near the intersection points of the divisors carefully ([14, §2]) and verifying that their argument can be extended there provided $i \neq 2$ and that the case $i=2$ can be proved even now when we do not have the Stokes' theorem in the L^2-sense for $i=2$ (and 1).

In this paper, we assume that the L^2-cohomology and the intersection homology are well-known: refer to [1], [3], [5], [7], [8], [11], [12]. Moreover we assume $S = \{p\}$, the one-point set, which causes no loss of generality.

§ 2. Review of some parts of [14] (and [9]).

In general, the quasi-isometric transformation is effective in proving such a theorem as Theorem 1 or also in investigating the property of the heat operator ([13], [14]). Here the *quasi-isometry* means the diffeomorphism $f:(Y_1, g_1) \rightarrow (Y_2, g_2)$ with a constant $C > 0$ satisfying $C^{-1}g_1 \leqq f^*g_2 \leqq Cg_1$. Paying attention to how the objects (or the properties) under consideration are transformed by the quasi-isometry, we discuss them on a Riemannian manifold less-complicated than and quasi-isometric to the original one: this is our guiding principle throughout [9], [14] and this paper.

EXAMPLE 2.1. On the real n-dimensional (possibly, non-compact) Riemannian manifolds, the Dirichlet or Neumann type Laplacian Δ has the following quasi-isometric invariant property: "Spec $(\Delta) = \{(0 \leqq) \lambda_0 \leqq \lambda_1 \leqq \cdots \uparrow \infty\}$ and there exists a constant $K > 0$ such that Tr $(e^{-\Delta t}) = \sum e^{-\lambda_j t} \leqq K t^{-n/2}$ for $0 < t \leqq t_0$" ([4, Lemma 7.1], [14, Principle]). Using the fact, we can say, for example, that the Laplacian on the interval $(0, 1)$ with the metric $(2 + \sin(1/x))dx^2$ has the above property, because the Laplacian on it however with the standard metric dx^2, which is quasi-isometric to the above one, has obviously the property.

Now the L^2-cohomology $H^i_{(2)}(\mathcal{X}) = \text{Ker } \bar{d}_i / \text{Range } \bar{d}_{i-1}$ is obviously of quasi-isometric invariant. Hence it is wise to discuss Theorem 1 also on a less-complicated Riemannian manifold: however, our X has the singular point p and its neighborhood (in $\mathcal{X}$) is quite different from the neighborhood of a regular point (which is quasi-isometric to an open subset of $\boldsymbol{R}^4$ with standard metric). Under such a circumstance, we should follow the procedure: decompose the neighborhood suitably and then search for the manifolds less-complicated than and quasi-isometric to the parts thus gotten. Following such a procedure, we will explain briefly how the metric is near the singular point: refer to [14, § 2 and § 5] (and [9]) for further details.

Let the singular point be at $[1, 0, \cdots, 0] \in P^N(\boldsymbol{C})$ and let us take the local coordinates, $[w_0, w_1, \cdots, w_N] \mapsto (z_1, \cdots, z_N) = (w_1/w_0, \cdots, w_N/w_0)$. First, according to [9], we shall make a good resolution

$$(2.1) \qquad\qquad \pi : \tilde{X} \longrightarrow X$$

with the property; near any point of $\pi^{-1}(0)$, we can take a local parametrization (of π) of the standard form. That is, taking a suitable pair, a permutation σ and a local coordinate neighborhood $(U, (u, v))$ around the point, we can write the π on the U as follows:

$$z_{\sigma(1)} = u^{n_1}v^{m_1}, \qquad\qquad\qquad (n_1, m_1) \neq (0, 0),$$

$$z_{\sigma(2)} = f_2(z_{\sigma(1)}) + u^{n_2}v^{m_2}g_2(u, v), \quad \det\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \neq 0, \quad g_2(0, 0) \neq 0,$$

$$\vdots$$

$$(2.2) \qquad z_{\sigma(l)} = f_l(z_{\sigma(1)}) + u^{n_l}v^{m_l}g_l(u, v), \quad \det\begin{pmatrix} n_1 & m_1 \\ n_l & m_l \end{pmatrix} \neq 0, \quad g_l(0, 0) \neq 0,$$

$$z_{\sigma(l+1)} = f_{l+1}(z_{\sigma(1)}),$$

$$\vdots$$

$$z_{\sigma(N)} = f_N(z_{\sigma(1)}),$$

satisfying that $f_j(z) = \sum a_{jn}z^{\varepsilon_n}$ with $\varepsilon_n \geq 1$, and

$$(2.3) \qquad n_1 \leq n_2 \leq \cdots \leq n_l, \qquad m_1 \leq m_2 \leq \cdots \leq m_l.$$

On the U, $\pi^{-1}(0) =$ "$u=0$, $v=0$ or $uv=0$" and accordingly, "$n_1 > m_1 = 0$, $0 = n_1 < m_1$ or $n_1 m_1 > 0$". If $\pi^{-1}(0) =$ "$u=0$", then, for the divisor (determined by) "$u=0$", the number n_2/n_1 is uniquely determined (i.e., it does not depend on the choice of a generic point on the divisor and also the choice of the $(U, (u, v))$ satisfying (2.2) with (2.3)) and is called its *exponent*; if $\pi^{-1}(0) =$"$v=0$", its exponent is m_2/m_1. Next, moreover according to [9], we choose a function R with the domain V, a neighborhood of $\pi^{-1}(0)$ ($\subset \tilde{X}$), and with the range in $[0, \infty)$, satisfying;

(i) $R|\pi^{-1}(0) = 0$,

(ii) $R|V \setminus \pi^{-1}(0)$ is smooth and positive,

(iii) $V \setminus \pi^{-1}(0) = R^{-1}(0, 1]$,

(iv) using the R and certain appropriate flow lines (in $V \setminus \pi^{-1}(0)$), we can define a product structure

$$(2.4) \qquad R^{-1}(0, 1] = (0, 1] \times R^{-1}(1), \quad x \mapsto (r, \cdots), \quad r = R(x).$$

What we have in mind as an example of R is not the distance function from the $\pi^{-1}(0)$ (defined by the metric π^*g). Precisely we think of (and really can take) the R additionally with the following property:

To simplify the description, we set $\mathcal{X}(\varepsilon) = R^{-1}(0, \varepsilon]$ and $\dot{\mathcal{X}}(\varepsilon) = R^{-1}(\varepsilon)$, $\varepsilon > 0$. Let us decompose the $\dot{\mathcal{X}}(1)$ suitably into finite parts, nonoverlapping except on their boundaries, and, using the product structure (2.4), let us decompose $\mathcal{X}(1)$ itself accordingly (see Figure 2.1);

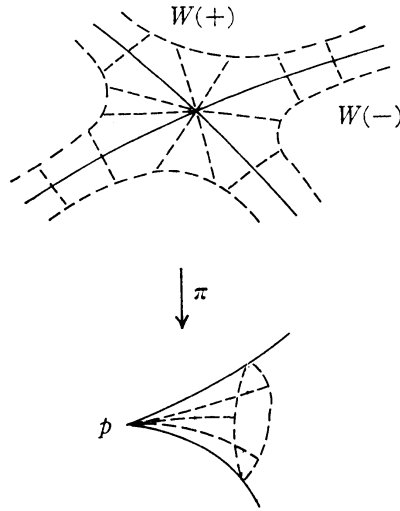$$(2.5) \qquad \mathcal{X}(1) = \bigcup_\gamma \mathcal{W}_\gamma.$$

Figure 2.1.

Then we have the quasi-isometries

(2.6)
$$I_r: \quad (\mathcal{W}_r, \pi^*g) \cong W_r,$$
$$I_r = (r, \cdots), \quad r = R(x),$$

where the W_r are the following Riemannian manifolds $W=W(\pm)$:

TYPE($-$): Fix $c \geq 1$. Let Y be a compact polygon in $\mathbf{R}^2$ and the $\tilde{g}$ be the standard metric on Y. Then we set

$$W = W(-) = \text{``}(0, 1] \times [0, 1] \times Y \ (\ni(r, \theta, y)) \quad \text{with} \quad dr^2 + r^2 d\theta^2 + r^{2c}\tilde{g}(y)\text{''}.$$

TYPE($+$): Fix $b > 0$ and $c \geq 1$. Let $f(r)$ be a smooth function on $(0, 1]$ satisfying $f'(r) \geq 0$ for any $r > 0$, $f(r) = r^b$ for small $r > 0$ and $f(r) = 1/2$ for large $r \leq 1$. Also let $l(x)$ be a smooth function on $[0, \infty)$ satisfying $l'(x) \geq 0$ and $l''(x) \geq 0$ for any $x \geq 0$, $l(x) = 1$ for $0 < x \leq 1 - \varepsilon$ and $l(x) = x$ for $x \geq 1 + \varepsilon$. Set $h(r, s) = f(r)l(s/f(r))$. Then we set

$$W = W(+) = \text{``}(0, 1] \times [0, 1]^3 \ (\ni (r, \theta, s, \Theta))$$

$$\text{with} \quad dr^2 + r^2 d\theta^2 + r^{2c}(ds^2 + h^2(r, s)d\Theta^2)\text{''}.$$

Here we can assume that the $\mathcal{W}_r$ is contained in a sufficiently small local coordinate neighborhood $(U, (u, v))$ satisfying (2.2) with (2.3). Then if $\pi^{-1}(0)$ is "$u=0$" or "$v=0$", the corresponding W_r is the $W(-)$ with the c, the exponent of the divisor $\pi^{-1}(0)$. If $\pi^{-1}(0) = $"$uv=0$", then the corresponding W_r is the $W(+)$, the c is the exponent smaller than another one $\tilde{c}$ and $b = \tilde{c} - c$. For further details, refer to [14, §2].

LEMMA 2.2. *In Type(+),*

(1) *the $h(r, s)$ can be changed into r^b+s (that is, the quasi-isometric class does not change even if we make such a change),*

(2) *we can assume $0<b<1$.*

PROOF. The (1) is a consequence of a straightforward computation ([**14**, (5.11)]). As for (2): After making the resolution (2.1) with (2.2) and (2.3), perform further a blowing-up at an intersection of the divisors. Then the new resolution (gotten by the composition) has also such a good property as the old one (that is, (2.2) with (2.3)). And the old list $\{(n_i, m_i)\}$ at the intersection produces the new lists $\{(n_i+m_i, m_i)\}$ and $\{(n_i, n_i+m_i)\}$ at the two (new) intersections. As for $b>0$, the old b produces $bn_1/(n_1+m_1)$ and $bm_1/(n_1+m_1)$. Therefore, by composing such blowing-ups as many times as we need, we can make the $b>0$ as small as we need.   Q. E. D.

From now on, we assume $0<b<1$ according to Lemma 2.2(2). Additionally, since we do not need to distinguish $(\mathfrak{X}, g)$ and $(\pi^{-1}(\mathfrak{X}), \pi^*g)$ from our viewpoint, we unify them and use the expression $(\mathfrak{X}, g)$ in both senses.

REMARK 2.3. While studying [**14**] and the problem in this paper, the author did not know that [**9**] had already been published and was using its preprint. In the preprint, to make the R and the appropriate flow lines ((iv)), Hsiang-Pati used "the smooth vector field", not "the piecewise smooth vector field" which was adopted in the published version. Certainly there is no problem even if we adopt the smooth vector field (or it is easy to rewrite §2 and [**14**, §5] by using "the piecewise smooth one").

§3. The idea of the proof.

In this section, we intend to reduce the proof of Theorem 1 to those of certain assertions, denoted by Assertions A, B, C. Assertions A, B will be reduced moreover to those on parts $\mathcal{W}_\gamma$ (denoted by Assertions $A(\gamma)$, $B(\gamma)$) and Assertion C will be also reduced to certain estimates on $\mathcal{W}_\gamma$.

First, according to [**3**], [**11**], [**12**], in order to prove Theorem 1, we have only to prove the following.

PROPOSITION 3.1. *Naturally we have*

(3.1)
$$H^i_{(2)}(\mathfrak{X}(1)) \cong \begin{cases} H^i_{DR}(\mathfrak{X}(1)) & ;\ i\leq 1, \\ 0 & ;\ i\geq 2. \end{cases}$$

Letting d_i, $\bar{d}_i$ be the intrinsic operators on $\mathfrak{X}(1)="R^{-1}(0, 1]$ with the metric g restricted", naturally we have

(3.2) $H^i_{(2)}(\mathcal{X}(1)) = \mathrm{Ker}\, \bar{d}_i / \mathrm{Range}\, \bar{d}_{i-1} = \mathrm{Ker}\, d_i / \mathrm{Range}\, d_{i-1}$.

That is, there is no difference between the definitions with $\{\bar{d}_i\}$ and $\{d_i\}$: see [3, (1.5)]. To define the isomorphism (3.1) with $i \leq 1$, it is convenient to use the $\{d_i\}$-type; that is, we can definitely say that the isomorphism is given by the map

(3.3) $[\phi + dr \wedge \omega] \longmapsto [\phi(1)]$,

where $\phi + dr \wedge \omega \in \mathrm{Ker}\, d_i$ and ϕ, ω do not involve dr. It is obviously well-defined.

REMARK 3.2. ·The intersection homology with the (lower) middle perversity $\bar{m} = (0, 0, 1)$ for $X(1) = R^{-1}[0, 1]$ $(\subset X)$ has naturally the isomorphism

(3.4) $IH^{\bar{m}}_i(X(1)) \cong \begin{cases} H_i(\mathcal{X}(1)) & ;\ i \leq 1, \\ 0 & ;\ i \geq 2. \end{cases}$

Observing (3.1) and (3.4), we can notice that there exists the relation (1.1) for $X(1)$: connect $H^i_{DR}(\mathcal{X}(1))$ with $H_i(\mathcal{X}(1))$ by the de Rham pairing. If we have such a local relation near the singular point, we can conclude from the sheaf theoretic viewpoint ([12, (4.2)]) that the (1.1) holds for our X: if we want to express the isomorphism explicitly, compare the L^2-version and the intersection homology version of the long exact sequences for $(\mathcal{X}, \mathcal{X}(1))$ and $(X, X(1))$ ([11]). Additionally, if we decompose X into $X(1) \cup M$, naturally we have

(3.5) $IH^{\bar{m}}_i(X) \cong \begin{cases} H_i(M) \cong H_i(\mathcal{X}) & ;\ i \leq 1, \\ \iota(H_i(M)) \cong \iota(H_i(\mathcal{X})) & ;\ i = 2, \\ H_i(M, \partial M) \cong H_i(X) & ;\ i \geq 3, \end{cases}$

where the ι mean the maps induced from the inclusion maps, $(M, \varnothing) \hookrightarrow (M, \partial M)$ and $\mathcal{X} \hookrightarrow X$: see [3, (6.5)] and [5, §2.2].

REMARK 3.3. Let us explain briefly the case where X is not normal. As explained in [9, Theorem A′], we make the normalization $\hat{\pi} : \hat{X} \to X$ and then make a good resolution $\tilde{\pi} : \tilde{X} \to \hat{X}$ (as explained in §2). For simplicity we may assume that the singularity set of $\hat{X}$ consists of one point p. Let the D_j be the irreducible components of $\tilde{\pi}^{-1}(p)$. Then we may assume that the proper transform of the singularity set S of X by $\pi = \hat{\pi} \circ \tilde{\pi}$, denoted by $\tilde{S}$, intersects with $\cup D_j$ transversely and does not intersect with the intersection points of the D_j. Now, regarding $\mathcal{X}(1)$, $\hat{\mathcal{X}}(1)$ as to be in X (not in $\tilde{X}$) and setting $\dot{S}(1) = \mathcal{X}(1) \cap S$, we can prove (by the very same way as the proof of Proposition 3.1) the existence of the natural isomorphism
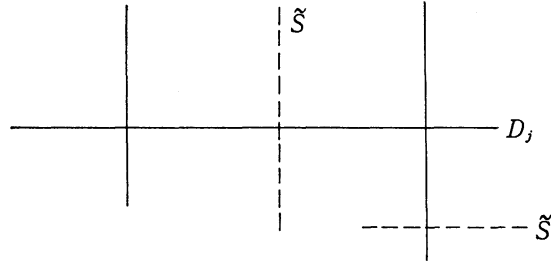
Figure 3.1 ([**9**, Figure 4]).

$$(3.6) \qquad H^i_{(2)}(\mathscr{X}(1)-S) \cong \begin{cases} H^i_{(2)}(\mathscr{X}(1)-\dot{S}(1)) & ; \ i \leq 1, \\ 0 & ; \ i \geq 2. \end{cases}$$

This fact, which is the calculation of the local L^2-cohomology at $\hat{\pi}(p)$, and its calculation at $x \in S - \hat{\pi}(p)$, which is omitted here, imply the (1.1) for the non-normal X (according to the same argument as in Remark 3.2 (the normal case)). Notice that we have

$$(3.7) \qquad \begin{aligned} IH^{\overline{p}}_i(X(1)) &\cong IH^{\overline{p}}_i(\mathscr{X}(1)) \ (\cong IH^{\overline{p}}_i(\hat{\mathscr{X}}(1)) \cong H_i(\hat{\mathscr{X}}(1))) \quad ; \ i \leq 1, \\ IH^{\overline{p}}_i(X(1)) &\cong 0 \quad ; \ i \geq 2, \end{aligned}$$

where we put $\hat{\mathscr{X}}(1) = \hat{\pi}^{-1}(\mathscr{X}(1))$, which has no singularity. We can carry out, in the category of [**3**] and [**11**], the proof of the existence of the natural isomorphism between the right hand sides of (3.6) and (3.7) for $i \leq 1$. Remark that the intersection homology is invariant under normalization ([**7**, § 4.2]), however, it seems to be still an open problem for the L^2-cohomology. Of course, for our X, (even if we do not use the relation (1.1)) it can be easily verified in the category of [**3**], [**11**]. For a general (algebraic) variety V (with the normalization $\hat{\pi} : \hat{V} \to V$), it would be sufficient to prove $\bar{d} = \bar{d}_c$ on the $\hat{V}$ with $\hat{\pi}^{-1}$ ("the singularity set") delated: see also [**5**, § 3.5].

Now let us discuss the way of the proof of Proposition 3.1. Let $\Lambda^i(\mathscr{X}(1))$ and $L^2\Lambda^i(\mathscr{X}(1))$ be the spaces of smooth i-forms and of square-integrable i-forms on $\mathscr{X}(1)$ respectively and let us denote the inner product on the second space by $\langle \, , \rangle_{\mathscr{X}(1)}$: for general U, $\Lambda^i(U)$, $L^2\Lambda^i(U)$ and $\langle \, , \rangle_U$ always mean those for the U. Also let $\bar{d}_i$, $\tilde{d}_i$, etc. be the intrinsic operators on $\mathscr{X}(1)$. We first introduce two assertions.

ASSERTION A. *Consider the case $i \leq 1$ and take $\phi \in L^2\Lambda^i(\mathscr{X}(1))$. Then, regarding ϕ naturally as a form on $\mathscr{X}(1)$, $\phi(r, \tilde{x}) = \phi(1, \tilde{x})$, we have $\phi \in L^2\Lambda^i(\mathscr{X}(1))$.*

Moreover, for $\alpha = \phi + dr \wedge \omega \in \Lambda^i(\mathscr{X}(1))$, we set

$$(3.8) \qquad (\kappa\alpha)(r, \tilde{x}) = \begin{cases} \int_a^r \omega(r_1, \tilde{x})dr_1 & ; \quad i\leq 2, \\ \int_0^r \omega(r_1, \tilde{x})dr_1 & ; \quad i\geq 3, \end{cases}$$

where the a, $0<a\leq 1$, is not specified so far. Then we have

ASSERTION B. *The κ defines the bounded operators*

$$(3.9) \qquad \kappa: L^2\Lambda^i(\mathcal{X}(1)) \longrightarrow L^2\Lambda^{i-1}(\mathcal{X}(1)).$$

If we assume the assertions (and Proposition 1.1) hold, then we can prove the following corollary (from which the name "homotopy operator" of the κ comes).

COROLLARY 3.4. *Let $\alpha=\phi+dr\wedge\omega\in L^2\Lambda^i(\mathcal{X}(1))$.*

(1) *In the case $i=0$ or 1, if $\alpha\in\mathrm{dom}\,d_i$, then $\kappa\alpha\in\mathrm{dom}\,d_{i-1}$ and we have*

$$(3.10) \qquad d\kappa\alpha+\kappa d\alpha = \alpha-\phi(a).$$

(2) *In the case $i=3$ or 4, if $\alpha\in\mathrm{dom}\,\bar{d}_i$, then $\kappa\alpha\in\mathrm{dom}\,\bar{d}_{i-1}$ and we have*

$$(3.11) \qquad \bar{d}\kappa\alpha+\kappa\bar{d}\alpha = \alpha.$$

PROOF OF (1). By an easy calculation, we have

$$(3.12) \qquad d\kappa\alpha+\kappa d\alpha = \alpha-\phi(a).$$

And Assertions A, B imply that $\kappa d\alpha$ and $\phi(a)$ (which must be regarded naturally as a form on $\mathcal{X}(1)$) belong to $L^2\Lambda^i(\mathcal{X}(1))$. Hence (by (3.12)) $d\kappa\alpha$ also belongs to $L^2\Lambda^i(\mathcal{X}(1))$. That is, $\kappa\alpha\in\mathrm{dom}\,d_{i-1}$ and (3.10) holds. Q. E. D.

PROOF OF (2). Clearly it holds for $\alpha\in\mathrm{dom}\,d_i$ with $\alpha(r, \tilde{x})\equiv 0$ for r small enough. Next let us consider a general $\alpha\in\mathrm{dom}\,\bar{d}_i$. Proposition 1.1 (for $i=3, 4$) implies the existence of the sequence $\alpha_j\in\mathrm{dom}\,d_i$ such that $\alpha_j(r, \tilde{x})\equiv 0$ for r small enough (not uniformly with respect to j) and

$$(3.13) \qquad \lim_{j\to\infty}\alpha_j = \alpha, \qquad \lim_{j\to\infty}d\alpha_j = \bar{d}\alpha.$$

Let β be a smooth i-form on $\mathcal{X}(1)$ with $\mathrm{supp}\,\beta\subset\mathcal{X}(\eta_1, \eta_2)=R^{-1}[\eta_1, \eta_2]$, $0<\eta_1<\eta_2<1$, that is, an element of $\mathrm{dom}\,\delta_{c,i-1}$, where $\delta_{c,i-1}=-*d_{c,4-i}*$ and the $d_{c,4-i}$ is the exterior derivative acting on the compactly supported $(4-i)$-forms on $\mathcal{X}(1)-\dot{\mathcal{X}}(1)$. Then Assertion B and the (2) for α_j imply

$$(3.14) \qquad \langle\kappa\alpha, \delta\beta\rangle_{\mathcal{X}(1)} = \lim_{j\to\infty}\langle\kappa\alpha_j, \delta\beta\rangle = \lim_{j\to\infty}\langle d\kappa\alpha_j, \beta\rangle$$
$$= \lim_{j\to\infty}\langle\alpha_j-\kappa d\alpha_j, \beta\rangle = \langle\alpha-\kappa\bar{d}\alpha, \beta\rangle.$$

That is, $\kappa\alpha$ belongs to $\mathrm{dom}\,\bar{\delta}_{c,i-1}^*=\mathrm{dom}\,\bar{d}_{i-1}$ and (3.11) holds. Q. E. D.

Now we can prove Proposition 3.1 for $i \neq 2$ by using Corollary 3.4.

PROOF OF PROPOSITION 3.1 FOR $i \neq 2$. Consider first the case $i \leq 1$. If $\alpha \in \text{Ker } d_i$, then Corollary 3.4(1) with $a=1$ implies $\kappa\alpha \in \text{dom } d_{i-1}$ and $\alpha = \phi(1) + d\kappa\alpha$. Here, if $[\phi(1)] = 0$, that is, if we have $\psi \in \text{dom } \bar{d}_{i-1}$ such that $\phi(1) = \bar{d}\psi$, then the ψ extended naturally to $\mathcal{X}(1)$ belongs to $\text{dom } d_{i-1}$ because of Assertion A and moreover we have $\alpha = d(\psi + \kappa\alpha)$. Hence the map (3.3) is injective. On the other hand, if we extend $\phi \in \text{Ker } \bar{d}_i$ naturally to $\mathcal{X}(1)$, then the extended ϕ belongs to $\text{Ker } d_i$ (because of Assertion A). That is, the map (3.3) is surjective. Next, the case $i \geq 3$ clearly holds because of Corollary 3.4(2).   Q. E. D.

Next we will prove Proposition 3.1 for $i=2$. First let us introduce another assertion. Let $\hat{d}_i$, $\check{d}_i$ be the exterior derivatives (on $\mathcal{X}(1)$) with the following domains;

$$(3.15) \qquad \begin{aligned} \text{dom } \hat{d}_i &= \{\alpha \in \text{dom } d_i \mid \alpha(r, \tilde{x}) \equiv 0 \quad \text{for } r \text{ small}\}, \\ \text{dom } \check{d}_i &= \{\alpha \in \text{dom } d_i \mid \alpha(r, \tilde{x}) \equiv 0 \quad \text{for } r \text{ near } r=1\}. \end{aligned}$$

In the same way we define $\hat{\delta}_i$, $\check{\delta}_i$, that is, we set $\hat{\delta}_i = -*\hat{d}_{3-i}*$ and $\check{\delta}_i = -*\check{d}_{3-i}*$. Their closures are denoted by $\bar{\hat{d}}_i$, $\bar{\check{d}}_i$, etc. Note that we have $\bar{\hat{\delta}}_i^* = \bar{\check{d}}_i$, $\bar{\check{\delta}}_i^* = \bar{\hat{d}}_i$, etc.

ASSERTION C.

(1) $\text{Ker } \bar{\hat{d}}_2 / \text{Range } \bar{\check{d}}_1 = 0$,

(2) $\text{Ker } \bar{\hat{d}}_1 / \text{Range } \bar{\hat{d}}_0 \cong H^1_{DR}(\mathcal{X}(1))$.

The isomorphism at (2) can be defined in the same way as (3.3). We will leave the proof of the assertion (and also the detailed explanation of the isomorphism) for a while and prove Proposition 3.1 for $i=2$.

PROOF OF PROPOSITION 3.1 FOR $i=2$. Let us set $V = R^{-1}[1/2, 1]$ $(= \mathcal{X}(1/2, 1))$ and consider the exterior derivative $d_{i, (\mathcal{X}(1), V)}$ with the following domain;

$$(3.16) \qquad \text{dom } d_{i, (\mathcal{X}(1), V)} = \{\alpha \in \text{dom } d_i \mid \alpha \text{ restricted to } V \text{ is equal to } 0\}.$$

Also let $d_{i, V}$ be the usual exterior derivative on V (acting on the smooth i-forms on V): note that V is compact. Their closures are denoted by $\bar{d}_{i, (\mathcal{X}(1), V)}$ and $\bar{d}_{i, V}$. Then we have the short exact sequence (refer to [11, § 3]),

$$(3.17) \qquad 0 \longrightarrow \text{dom } \bar{d}_{i, (\mathcal{X}(1), V)} \longrightarrow \text{dom } \bar{d}_i \longrightarrow \text{dom } \bar{d}_{i, V} \longrightarrow 0.$$

Hence we have the L^2-version of long exact sequence,

$$(3.18) \qquad \begin{aligned} \cdots &\longrightarrow H^2_{(2)}(\mathcal{X}(1), V) \longrightarrow H^2_{(2)}(\mathcal{X}(1)) \longrightarrow H^2_{(2)}(V) \\ &\longrightarrow H^3_{(2)}(\mathcal{X}(1), V) \longrightarrow H^3_{(2)}(\mathcal{X}(1)) \longrightarrow \cdots. \end{aligned}$$

Here $H_{(2)}^i(\mathscr{X}(1), V)$ is the (relative L^2-) cohomology defined by the cochain complex $\{\mathrm{dom}\,\bar{d}_{i,(\mathscr{X}(1),V)}\}$. Moreover we have clearly the isomorphism,

$$(3.19) \qquad H_{(2)}^i(\mathscr{X}(1), V) \cong \mathrm{Ker}\,\bar{d}_i/\mathrm{Range}\,\bar{d}_{i-1}.$$

Now Assertion C(1) and (3.19) imply $H_{(2)}^2(\mathscr{X}(1), V)=0$ and Proposition 3.1 for $i=3$ implies $H_{(2)}^3(\mathscr{X}(1))=0$. Hence we have only to prove the following (because of (3.18));

$$(3.20) \qquad \dim H_{(2)}^3(\mathscr{X}(1), V) = \dim H_{(2)}^2(V).$$

We will prove it by using Assertion C(2). First we have

$$(3.21) \qquad H_{(2)}^2(V) \cong H_{DR}^2(\mathscr{X}(1)),$$

(see, for example, [3, Lemma 3.1]). Hence, by (3.19) and (3.21), it suffices to prove

$$(3.22) \qquad \dim \mathrm{Ker}\,\bar{d}_3/\mathrm{Range}\,\bar{d}_2 = \dim H_{DR}^2(\mathscr{X}(1)).$$

Let us consider the following two kinds of Hodge decompositions;

$$(3.23) \qquad L^2\Lambda^i(\mathscr{X}(1)) = \overline{\mathrm{Range}\,\bar{d}_{i-1}} \oplus \mathscr{H}_i \oplus \overline{\mathrm{Range}\,\bar{\delta}_i},$$

$$(3.24) \qquad L^2\Lambda^i(\mathscr{X}(1)) = \overline{\mathrm{Range}\,\bar{d}_{i-1}} \oplus \check{\mathscr{H}}_i \oplus \overline{\mathrm{Range}\,\bar{\delta}_i},$$

where $\mathscr{H}_i = \mathrm{Ker}\,\bar{d}_i \cap \mathrm{Ker}\,\bar{\delta}_{i-1}$ and $\check{\mathscr{H}}_i = \mathrm{Ker}\,\bar{d}_i \cap \mathrm{Ker}\,\bar{\delta}_{i-1}$. Since $H_{(2)}^3(\mathscr{X}(1), V)$ and $\mathrm{Ker}\,\bar{d}_1/\mathrm{Range}\,\bar{d}_0$ are finite dimensional, $\mathrm{Range}\,\bar{d}_2$ and $\mathrm{Range}\,\bar{d}_0$ are closed in $L^2\Lambda^3(\mathscr{X}(1))$ and $L^2\Lambda^1(\mathscr{X}(1))$ respectively. In this way we got the natural isomorphisms,

$$(3.25) \qquad \mathrm{Ker}\,\bar{d}_3/\mathrm{Range}\,\bar{d}_2 \cong \check{\mathscr{H}}_3, \qquad \mathrm{Ker}\,\bar{d}_1/\mathrm{Range}\,\bar{d}_0 \cong \mathscr{H}_1.$$

Moreover, clearly we have

$$(3.26) \qquad *\check{\mathscr{H}}_3 = \mathscr{H}_1.$$

Thus, combined with Assertion C(2), they imply the isomorphism,

$$(3.27) \qquad \mathrm{Ker}\,\bar{d}_3/\mathrm{Range}\,\bar{d}_2 \cong H_{DR}^1(\mathscr{X}(1)).$$

Since $H_{DR}^1(\mathscr{X}(1)) \cong H_{DR}^2(\mathscr{X}(1))$ (the usual Poincaré duality theorem: note that $\mathscr{X}(1)$ is a three dimensional closed manifold), (3.22) was proved. $\qquad$ Q.E.D.

In this way the proof of Proposition 3.1 could be reduced to those of Assertions A, B, C. As for Assertions A, B, obviously we have only to prove them on each part $\mathscr{W}_\gamma$. That is, setting

$$(3.28) \qquad \dot{\mathscr{W}}_\gamma(\varepsilon) = \mathscr{W}_\gamma \cap \dot{\mathscr{X}}(\varepsilon),$$

we should prove

ASSERTION A(γ). *Consider the case* $i \leq 1$ *and take* $\phi \in L^2 \Lambda^i(\mathcal{W}_\gamma(1))$. *Then,* *regarding* ϕ *naturally as a form on* $\mathcal{W}_\gamma$, *we have* $\phi \in L^2 \Lambda^i(\mathcal{W}_\gamma)$.

Also, defining the operator κ_γ on each $\mathcal{W}_\gamma$ in the same way as (3.8), we should prove

ASSERTION B(γ). *The* κ_γ *defines the bounded operators*

$$(3.29) \qquad\qquad \kappa_\gamma \colon\ L^2 \Lambda^i(\mathcal{W}_\gamma) \longrightarrow L^2 \Lambda^{i-1}(\mathcal{W}_\gamma).$$

Assertion C will be proved by using Assertions A, B and by some calculation on the $\mathcal{W}_\gamma$.

The purposes of the following two sections are their proofs.

§4. The proofs of Assertions A(γ), B(γ).

Since we are going to discuss them on each $\mathcal{W}_\gamma$, we omit the subscript γ to simplify the description. First the I $(=I_\gamma)$ given at (2.6) defines the bounded operators

$$(4.1) \qquad\qquad L^2 \Lambda^i(\mathcal{W}) \overset{I_*}{\underset{I^*}{\rightleftarrows}} L^2 \Lambda^i(W).$$

Let us define $\dot{W}(\varepsilon) = \{(r, \tilde{x}) \in W \mid r = \varepsilon\}$ (in the same way as (3.28)) for each $\varepsilon > 0$. Then the I also defines the bounded operators

$$(4.2) \qquad\qquad L^2 \Lambda^i(\dot{\mathcal{W}}(\varepsilon)) \overset{I(\varepsilon)_*}{\underset{I(\varepsilon)^*}{\rightleftarrows}} L^2 \Lambda^i(\dot{W}(\varepsilon)),$$

whose operator norms are *bounded uniformly* with respect to $0 < \varepsilon \leq 1$. (The properties "bounded" and "bounded uniformly" can be obviously deduced from the fact that the I is quasi-isometric.) Hence we have only to prove the assertions with $(\mathcal{W}, \dot{\mathcal{W}}(1))$ replaced by $(W, \dot{W}(1))$.

Let $\|\cdot\|$ $(=\|\cdot\|_W)$ and $\|\cdot\|_\varepsilon$ $(=\|\cdot\|_{\dot{W}(\varepsilon)})$ be the L^2-norms on W and $\dot{W}(\varepsilon)$ respectively. Remark that, for a form ϕ on W which does not involve dr, $\|\phi(r)\|_\varepsilon$ means the L^2-norm of $\phi(r)$ (which is a form on $\dot{W}(r)$) regarded naturally as a form on $\dot{W}(\varepsilon)$.

We will first give the relation between $\|\phi(r)\|_\varepsilon$ and $\|\phi(r)\|_1$. Let us decompose the space of i-forms on W as follows:

$$(4.3) \qquad \begin{aligned} \Lambda^i(W(-)) &= \sum \Lambda^{i,p,q}(-) = \sum \{f(dr)^i \wedge (d\theta)^p \wedge (dy)^q\}, \\ \Lambda^i(W(+)) &= \sum \Lambda^{i,p,q,P}(+) = \sum \{f(dr)^i \wedge (d\theta)^p \wedge (ds)^q \wedge (d\Theta)^P\}, \end{aligned}$$

where, for example, $(dr)^0 \wedge (d\theta)^1 \wedge (ds)^0 \wedge (d\Theta)^1$ means $d\theta \wedge d\Theta$ and $(dy)^q$ means

1 ($q=0$), dy_1 or dy_2 ($q=1$) and $dy_1 \wedge dy_2$ ($q=2$). And, for the $c>0$ of $W(\pm)$ and $p, q \in \mathbf{Z}$, we set

$$(4.4) \qquad k(p, q) = (1-2p)+2c(1-q).$$

Then, by an easy calculation, we get

LEMMA 4.1. (1) *For* $\phi \in \Lambda^{0, p, q}(-)$, *set* $k=k(p, q)$, *then*

$$(4.5) \qquad \|\phi(r)\|_\varepsilon = \varepsilon^{k/2}\|\phi(r)\|_1.$$

(2) *For* $\phi \in \Lambda^{0, p, q, P}(+)$, *set* $k=k(p, q+P)$, *then*

$$(4.6) \qquad \|\phi(r)\|_\varepsilon = \varepsilon^{k/2}\|\{h(\varepsilon, s)/h(1, s)\}^{1/2-P}\phi(r)\|_1.$$

The followings are the cases we are going to discuss:

$$(4.7) \quad \begin{aligned} (p, q: k(p, q)) &= \text{``}(0, 0: 2c+1), (1, 0: 2c-1), (0, 1: 1), \\ &\quad (1, 1: -1), (0, 2: -2c+1), (1, 2: -2c-1)\text{''}, \\ (p, q, P: k(p, q+P)) &= \text{``}(0, 0, 0: 2c+1), (1, 0, 0: 2c-1), (0, 1, 0: 1), \\ &\quad (0, 0, 1: 1), (1, 1, 0: -1), (1, 0, 1: -1), \\ &\quad (0, 1, 1: -2c+1), (1, 1, 1: -2c-1)\text{''}. \end{aligned}$$

Using the lemma, we can prove Assertions A(γ), B(γ) for $W(\pm)$.

PROOF OF ASSERTION A(γ) FOR $W(-)$: Take $\phi \in \Lambda^{0, p, q}(-)$. In this case, we have $(p, q: k)=\text{``}(0, 0: 2c+1), (1, 0: 2c-1)$ or $(0, 1: 1)\text{''}$. Hence (4.5) implies

$$(4.8) \qquad \|\phi(1)\|^2 = \int_0^1 \|\phi(1)\|_r^2 dr = \int_0^1 r^k \|\phi(1)\|_1^2 dr \leqq \|\phi(1)\|_1^2.$$

Q. E. D.

PROOF OF ASSERTION A(γ) FOR $W(+)$: Take $\phi \in \Lambda^{0, p, q, P}(+)$. In this case, we have $(p, q, P: k)=\text{``}(0, 0, 0: 2c+1), (1, 0, 0: 2c-1), (0, 1, 0: 1)$ or $(0, 0, 1: 1)\text{''}$. We set $h(r, s)=r^b+s$, $0<b<1$, according to Lemma 2.2. Then (4.6) implies

$$(4.9) \qquad \|\phi(1)\|^2 = \int_0^1 \|\phi(1)\|_r^2 dr = \int_0^1 r^k \|\{(r^b+s)/(1+s)\}^{1/2-P}\phi(1)\|_1^2 dr$$

$$\leqq \|\phi(1)\|_1^2 \int_0^1 r^{1-b} dr \leqq \|\phi(1)\|_1^2.$$

Q. E. D.

PROOF OF ASSERTION B(γ) FOR $W(-)$: We assume $\alpha=\phi+dr \wedge \omega \in L^2(W(-))$ with $\omega \in \Lambda^{0, p, q}(-)$. Setting $k=k(p, q)$ and taking $0<\eta, \xi \leqq 1$, (4.5) implies

$$(4.10) \quad \left\| \int_\eta^\xi \omega dr \right\|_r = r^{k/2} \left\| \int_\eta^\xi \omega dr \right\|_1 \le r^{k/2} \left| \int_\eta^\xi \|\omega(r)\|_1 dr \right| \le \|\omega\| r^{k/2} \left| \int_\eta^\xi r^{-k} dr \right|^{1/2}$$

$$= \|\omega\| \begin{cases} |k-1|^{-1} |\xi^{1-k} r^k - \eta^{1-k} r^k|^{1/2} & ; \quad k \ne 1, \\ |r \log \xi - r \log \eta|^{1/2} & ; \quad k = 1. \end{cases}$$

Hence, remembering the definition of the κ, the proof is complete.   Q. E. D.

PROOF OF ASSERTION B(γ) FOR $W(+)$: We assume $\alpha = \phi + dr \wedge \omega \in L^2(W(+))$ with $\omega \in \Lambda^{0, p, q, P}(+)$, and again assume $h(r, s) = r^b + s$, $0 < b < 1$. Setting $k = k(p, q+P)$ and taking $0 < \eta, \xi \le 1$, (4.6) implies

$$(4.11) \quad \left\| \int_\eta^\xi \omega dr \right\|_r = r^{k/2} \left\| \int_\eta^\xi \{(r^b+s)/(1+s)\}^{1/2-P} \omega(r_1) dr_1 \right\|_1$$

$$\le r^{k/2} \left| \int_\eta^\xi r_1^{-k/2} \|\{(r^b+s)/(r_1^b+s)\}^{1/2-P} \omega(r_1)\|_{r_1} dr_1 \right|.$$

Hence, if $p+q+P \le 1$ and $P=0$, then we have

$$(4.12) \quad \|\kappa\alpha\|^2 \le \int_0^a r^k \left\{ \int_r^a r_1^{-k/2} \|\omega(r_1)\|_{r_1} dr_1 \right\}^2 dr + \int_a^1 r^{k+b} \left\{ \int_a^r r_1^{-(k+b)/2} \|\omega(r_1)\|_{r_1} dr_1 \right\}^2 dr$$

$$\le \|\omega\|^2 \left\{ \int_0^a \int_r^a (r/r_1)^k dr_1 dr + \int_a^1 \int_a^r (r/r_1)^{k+b} dr_1 dr \right\},$$

that is, moreover if $k > 1$, then

$$(4.13) \quad \|\kappa\alpha\|^2 \le K \|\omega\|^2 \left\{ \int_0^a (a^{-k+1} r^k - r) dr + \int_a^1 (r - a^{-k-b+1} r^{k+b}) dr \right\},$$

and, if $k = 1$, then

$$(4.14) \quad \|\kappa\alpha\|^2 \le K \|\omega\|^2 \left\{ \int_0^a (r \log a - r \log r) dr + \int_a^1 (r - a^{-b} r^{b+1}) dr \right\}.$$

Thus the proof in this case is complete. Next, if $p+q+P \le 1$ and $P=1$, then we have

$$(4.15) \quad \|\kappa\alpha\|^2 \le K \|\omega\|^2 \left\{ \int_0^a (a^b r^{1-b} - r) dr + \int_a^1 (r \log r - r \log a) dr \right\}.$$

Thus the proof in this case is also complete. Moreover, if $p+q+P \ge 2$ and $P=0$, then

$$(4.16) \quad \|\kappa\alpha\|^2 = \int_0^1 \left\| \int_0^r \omega(r_1) dr_1 \right\|_r^2 dr$$

$$\le \int_0^1 r^{-1+b} \left\{ \int_0^r r_1^{(1-b)/2} \|\omega(r_1)\|_{r_1} dr_1 \right\}^2 dr \le K \|\omega\|^2,$$

and, finally, if $p+q+P \ge 2$ and $P=1$, then

$$(4.17) \quad \|\kappa\alpha\|^2 \le \int_0^1 r^k \left\{ \int_0^r r_1^{-k/2} \|\omega(r_1)\|_{r_1} dr_1 \right\}^2 dr \le K \|\omega\|^2.$$

Thus the proofs in all cases are complete. Q. E. D.

§ 5. The proof of Assertion C.

First we give the proof of (1).

PROOF OF ASSERTION C(1). First consider a smooth $\alpha = \phi + dr \wedge \omega \in \mathrm{Ker}\, \bar{\bar{d}}_2$.
Since $\hat{\bar{\delta}}_2^* = \bar{\bar{d}}_2$, we have, for any $\beta \in \mathrm{dom}\, \hat{\delta}_2$, $0 = \langle \alpha, \delta\beta \rangle = \pm \int_{\dot{\mathcal{X}}(1)} \alpha \wedge *\beta$. Here $*\beta$
restricted to $\dot{\mathcal{X}}(1)$ is arbitrary. Hence

(5.1) $$\phi(1, \tilde{x}) = 0\,.$$

Moreover, since $d\alpha = \bar{d}\phi + dr \wedge (\partial\phi/\partial r - \bar{d}\omega)$, we have $\partial\phi/\partial r = \bar{d}\omega$. Hence

(5.2) $$\phi(r, \tilde{x}) = \int_1^r \frac{\partial\phi}{\partial r_1}(r_1, \tilde{x})dr_1 = \bar{d}\int_1^r \omega(r_1, \tilde{x})dr_1 = \bar{d}\kappa\alpha\,.$$

(Here the κ is the one given at (3.8) with $a=1$.) Therefore $d\kappa\alpha = \alpha$. Moreover
Assertion B implies $\kappa\alpha \in L^2\Lambda^1(\mathcal{X}(1))$ and clearly $(\kappa\alpha)(1, \tilde{x}) = 0$. Thus $\kappa\alpha \in \mathrm{dom}\, \bar{\bar{d}}_1$
and $\alpha = \bar{\bar{d}}\kappa\alpha$.

Next consider a general $\alpha \in \mathrm{Ker}\, \bar{\bar{d}}_2$. There is a sequence $\alpha_j \in \mathrm{dom}\, \check{d}_2$ such
that $\alpha_j \to \alpha$, $d\alpha_j \to 0$ in the L^2-sense. Let us decompose each α_j according to the
Hodge decomposition (3.24) and denote the $\overline{\mathrm{Range}\, \bar{\bar{d}}_1} \oplus \check{\mathcal{H}}_2$ ($= \mathrm{Ker}\, \bar{\bar{d}}_2$)-part by $\bar{\alpha}_j$,
which is smooth. Since the $\overline{\mathrm{Range}\, \hat{\bar{\delta}}_2}$-part of the α is equal to 0, we have

(5.3) $$\lim_{j \to \infty} \bar{\alpha}_j = \alpha\,.$$

Now, by the above argument (for smooth α), we have $\kappa\bar{\alpha}_j \in \mathrm{dom}\, \bar{\bar{d}}_1$ and $\bar{\alpha}_j = \bar{\bar{d}}\kappa\bar{\alpha}_j$.
Moreover (5.3) and Assertion B imply $\lim_{j \to \infty} \kappa\bar{\alpha}_j = \kappa\alpha$. Thus, also for the α, we
have $\kappa\alpha \in \mathrm{dom}\, \bar{\bar{d}}_1$ and

(5.4) $$\alpha = \bar{\bar{d}}\kappa\alpha\,.$$

Q. E. D.

Next let us prove (2). First we make some preparations. Using the nota-
tions in § 4, we have

LEMMA 5.1. *For a smooth 1-form* $\phi(1, \tilde{x})$ *on* $\dot{W}(1)$, *there exists a constant*
$K > 0$ (*depending on* ϕ) *such that*

(5.5) $$\|\phi(1)\|_\varepsilon \leq K \begin{cases} \varepsilon^{1/2} & ; \quad W = W(-), \\ \varepsilon^{1/2}|\log \varepsilon|^{1/2} & ; \quad W = W(+). \end{cases}$$

PROOF. The case $W = W(-)$ follows from Lemma 4.1(1). Let us consider
the case $W = W(+)$ and assume $h(r, s) = r^b + s$, $0 < b < 1$ (see Lemma 2.2). For

$\phi \in \Lambda^{0,1,0,0}(+)$ or $\phi \in \Lambda^{0,0,1,0}(+)$, set $\phi = \tilde{\phi}d\theta$ or $\phi = \tilde{\phi}ds$ and $k = k(1, 0)$ $(=2c-1)$ or $k = k(0, 1)$ $(=1)$ respectively, then

(5.6) $\qquad \|\phi(1)\|_\varepsilon^2 = \iiint \varepsilon^k(\varepsilon^b+s)\tilde{\phi}(1, \theta, s, \Theta)^2 d\theta ds d\Theta \leqq \sup_{\dot{W}(1)} |\tilde{\phi}|^2 \cdot \varepsilon^k(\varepsilon^b+1)$.

For $\phi \in \Lambda^{0,0,0,1}(+)$, set $\phi = \tilde{\phi}d\Theta$, then

(5.7) $\qquad \|\phi(1)\|_\varepsilon^2 = \iiint \dfrac{\varepsilon}{\varepsilon^b+s} \tilde{\phi}(1, \theta, s, \Theta)^2 d\theta ds d\Theta \leqq \sup_{\dot{W}(1)} |\tilde{\phi}|^2 \cdot \int_0^1 \dfrac{\varepsilon}{\varepsilon^b+s} ds$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \sup_{\dot{W}(1)} |\tilde{\phi}|^2 \cdot \varepsilon\{\log(\varepsilon^b+1) - b\log\varepsilon\}$.

Thus the case $W = W(+)$ was also proved. $\qquad\qquad\qquad\qquad\qquad$ Q. E. D.

Now, for a form α on $\mathcal{X}(1)$ which does not involve dr, we denote the L^2-norm of α restricted to $\mathcal{X}(\varepsilon)$ by $\|\alpha(\varepsilon)\|_{\dot{\mathcal{X}}(\varepsilon)}$. Then we have

COROLLARY 5.2. *Let* $\phi(1, \tilde{x})$ *be a smooth* 1-*form on* $\dot{\mathcal{X}}(1)$ *and let us extend it naturally to a form on* $\mathcal{X}(1)$, *denoted by* α. *Then there exists a constant* $K > 0$ *(depending on* ϕ*) such that, for* $0 < \varepsilon \leqq 1$,

(5.8) $\qquad\qquad\qquad\qquad \|\alpha(\varepsilon)\|_{\dot{\mathcal{X}}(\varepsilon)} \leqq K\varepsilon^{1/2} |\log\varepsilon|^{1/2}$.

PROOF. According to (2.5), we have

(5.9) $\qquad\qquad\qquad\qquad \|\alpha(\varepsilon)\|_{\dot{\mathcal{X}}(\varepsilon)}^2 = \sum_\gamma \|\alpha(\varepsilon)\|_{\mathcal{W}_\gamma(\varepsilon)}^2$.

Since the maps $I(\varepsilon)_*$ and $I(\varepsilon)^*$ (given at (4.2)) are bounded uniformly with respect to $0 < \varepsilon \leqq 1$, in order to estimate $\|\alpha(\varepsilon)\|_{\mathcal{W}_\gamma(\varepsilon)}^2$, it suffices to estimate $\|I(\varepsilon)_*\alpha(\varepsilon)\|_{\dot{W}(\varepsilon)}^2$, which was done at Lemma 5.1. $\qquad\qquad\qquad$ Q. E. D.

COROLLARY 5.3. *Consider* $\phi \in \mathrm{Ker}\,\tilde{d}_1$ *(that is, a closed smooth* 1-*form on* $\dot{\mathcal{X}}(1)$*). Extend it naturally to a form on* $\mathcal{X}(1)$, *denoted by* α. *Then* $\alpha \in \mathrm{Ker}\,\tilde{d}_1$.

PROOF. We have $d\alpha = 0$ and $\alpha \in L^2\Lambda^1(\mathcal{X}(1))$ (because of Assertion A). Hence (since $\tilde{\delta}_1^* = \tilde{d}_1$) we have only to prove the following: for any $\beta \in \mathrm{dom}\,\tilde{\delta}_1$,

(5.10) $\qquad\qquad\qquad\qquad\qquad \langle\alpha, \delta\beta\rangle_{\mathcal{X}(1)} = 0$,

see (1.2). Set $\mathcal{X}(\varepsilon, 1) = R^{-1}[\varepsilon, 1]$, then

(5.11) $\qquad \langle\alpha, \delta\beta\rangle_{\mathcal{X}(\varepsilon,1)} = \int_{\mathcal{X}(\varepsilon,1)} \alpha \wedge *\delta\beta = \int_{\mathcal{X}(\varepsilon,1)} \alpha \wedge d*\beta$

$\qquad\qquad\qquad\qquad\qquad = -\int_{\mathcal{X}(\varepsilon,1)} d(\alpha \wedge *\beta) = \pm\int_{\dot{\mathcal{X}}(\varepsilon)} \alpha \wedge *\beta$.

Therefore, it suffices to find a sequence $\varepsilon_n \downarrow 0$ $(n \to \infty)$ such that

(5.12)
$$\lim_{n\to\infty}\int_{\dot{\mathfrak{X}}(\varepsilon_n)} \alpha\wedge *\beta = 0\,.$$

Now we set $*\beta=\phi+dr\wedge\omega$. Then, by the Schwarz inequality, we have

(5.13)
$$\left|\int_{\dot{\mathfrak{X}}(\varepsilon)} \alpha\wedge *\beta\right| \leq \|\alpha(\varepsilon)\|_{\dot{\mathfrak{X}}(\varepsilon)}\cdot\|\phi(\varepsilon)\|_{\dot{\mathfrak{X}}(\varepsilon)}\,.$$

Here since $\|\phi(r)\|^2_{\dot{\mathfrak{X}}(r)}$, a function of the r, belongs to $L^1(0, 1]$, we have, by [3, Lemma 1.2], the sequence $\varepsilon_n \downarrow 0$ $(n\to\infty)$ satisfying

(5.14)
$$\|\phi(\varepsilon_n)\|_{\dot{\mathfrak{X}}(\varepsilon_n)} = o(\varepsilon_n^{-1/2}|\log\varepsilon_n|^{-1/2})\,.$$

This sequence is the desired one because of Corollary 5.2, (5.13) and (5.14).

Q. E. D.

Moreover, we extend $\phi\in\mathrm{Range}\,\bar{d}_0$ naturally to a form on $\mathfrak{X}(1)$, denoted by α. Then Assertion A implies $\alpha\in\mathrm{Range}\,\bar{d}_0$. Since $\bar{d}_0=\tilde{d}_0$ (by Proposition 1.1), as a result we have $\alpha\in\mathrm{Range}\,\tilde{d}_0$. By this fact and Corollary 5.3, we can make the map (by the natural extension),

(5.15)
$$\tau: H^1_{DR}(\mathfrak{X}(1)) \longrightarrow \mathrm{Ker}\,\tilde{d}_1/\mathrm{Range}\,\tilde{d}_0\,.$$

LEMMA 5.4. *Consider* $\alpha=\phi+dr\wedge\omega\in\mathrm{Ker}\,\tilde{d}_1$. *Then, for almost all* $0<a\leq1$, *we have* $\kappa\alpha\in\mathrm{dom}\,\tilde{d}_0$ *and*

(5.16)
$$\tilde{d}\kappa\alpha = \alpha-\phi(a)\,.$$

PROOF. By Corollary 3.4(1) and the fact $\bar{d}_0=\tilde{d}_0$, if α is smooth, then the lemma holds (for all $0<a\leq1$). Consider next a general $\alpha=\phi+dr\wedge\omega\in\mathrm{Ker}\,\tilde{d}_1$. We have a sequence $\alpha_j\in\mathrm{dom}\,\tilde{d}_1$ such that $\alpha_j\to\alpha$, $d\alpha_j\to0$ in the L^2-sense. Decompose each α_j according to the Hodge decomposition (3.23) and denote its $\overline{\mathrm{Range}\,\tilde{d}_0}\oplus\mathcal{H}_1$ $(=\mathrm{Ker}\,\tilde{d}_1)$-part by $\bar{\alpha}_j=\phi_j+dr\wedge\omega_j$. It is smooth and

(5.17)
$$\lim_{j\to\infty}\bar{\alpha}_j = \alpha\,,$$

because the $\overline{\mathrm{Range}\,\tilde{\delta}_1}$-part of α is equal to 0. Now, (5.17) implies that, for almost all $0<a\leq1$, $\phi_j(a)$ (for any j) and $\phi(a)$ belong to $L^2\Lambda^1(\mathfrak{X}(1))$ and, moreover,

(5.18)
$$\lim_{j\to\infty}\phi_j(a) = \phi(a)$$

with respect to the norm of $L^2\Lambda^1(\mathfrak{X}(1))$. Hence, by Assertion A, the lemma for the $\bar{\alpha}_j$, (5.17) and (5.18), we could prove that, for almost all $0<a\leq1$, we have $\kappa\alpha\in\mathrm{dom}\,\tilde{d}_0$ $(=\mathrm{dom}\,\bar{d}_0)$ and (5.16) holds. Q. E. D.

COROLLARY 5.5. *The map*

(5.19) $\tilde{\pi} : \mathrm{Ker}\, \tilde{d}_1/\mathrm{Range}\, \tilde{d}_0 \longrightarrow H^1_{(2)}(\dot{\mathcal{X}}(1))$

given by $[\alpha] \mapsto [\phi(a)]$ *is well-defined and independent of the a for almost all* $0 < a \leq 1$.

PROOF.   Take $\alpha = \phi + dr \wedge \omega$ and $\beta = \psi + dr \wedge \Omega$ such that $[\alpha] = [\beta] \in \mathrm{Ker}\, \tilde{d}_1/\mathrm{Range}\, \tilde{d}_0$.   There exists $f \in \mathrm{dom}\, \tilde{d}_0$ satisfying

(5.20) $\phi - \psi + dr \wedge (\omega - \Omega) = \tilde{d}f$.

Here, for almost all $0 < a \leq 1$, we have $\phi(a)$, $\psi(a) \in \mathrm{Ker}\, \tilde{\tilde{d}}_1$, $f(a) \in \mathrm{dom}\, \tilde{\tilde{d}}_0$ and moreover

(5.21) $\phi(a) - \psi(a) = \tilde{\tilde{d}}f(a)$.

Thus the map (5.19) is well-defined for almost all $0 < a \leq 1$.   Moreover, if we take a_1 and a_2 satisfying the above and denote the corresponding κ by κ_1, κ_2 respectively, then, on $\dot{\mathcal{X}}(1)$, we have (by (5.16))

(5.22) $\phi(a_1) - \phi(a_2) = d(\kappa_2 \alpha - \kappa_1 \alpha)$.

Here, $\kappa_2 \alpha - \kappa_1 \alpha$ is independent of $0 < r \leq 1$, which means that the $\kappa_2 \alpha - \kappa_1 \alpha$ is the natural extension of some element of $\mathrm{dom}\, \tilde{d}_0$.   That is, $[\phi(a_1)] = [\phi(a_2)]$ as the elements of $H^1_{(2)}(\dot{\mathcal{X}}(1))$.   Q. E. D.

We can now prove Assertion C(2).

PROOF OF ASSERTION C(2).   We have the natural isomorphism

(5.23) $\tilde{p} : H^1_{DR}(\dot{\mathcal{X}}(1)) \cong H^1_{(2)}(\dot{\mathcal{X}}(1))$

(see, for example, (3.2)).   Consider this map and the maps (5.15) and (5.19).   First obviously

(5.24) $\tilde{p}^{-1} \circ \tilde{\pi} \circ \tau = \mathrm{id}$.

Next, let us take $[\alpha] \in \mathrm{Ker}\, \tilde{d}_1/\mathrm{Range}\, \tilde{d}_0$ and set $(\tilde{p}^{-1} \circ \tilde{\pi})([\alpha]) = \tilde{p}^{-1}([\phi(a)]) = [\tilde{\phi}]$.   Then there exists $\tilde{\varphi} \in \mathrm{dom}\, \tilde{d}_0$ such that $\tilde{\phi} - \phi(a) = \tilde{\tilde{d}}\tilde{\varphi}$.   We extend $\tilde{\varphi}$ naturally to a form on $\dot{\mathcal{X}}(1)$, denoted by φ.   Assertion A implies $\varphi \in \mathrm{dom}\, \tilde{d}_0 = \mathrm{dom}\, \tilde{d}_0$ and we have, on $\dot{\mathcal{X}}(1)$,

(5.25) $\tilde{\phi} - \phi(a) = \tilde{d}\varphi$.

Now, Lemma 5.4 and (5.16) imply

(5.26) $\alpha - \tilde{\phi} = \alpha - \phi(a) - \tilde{d}\varphi = \tilde{d}(\kappa \alpha - \varphi)$.

Hence $\tau([\tilde{\phi}]) = [\alpha]$, that is, we get

(5.27) $$\tau \circ \tilde{p}^{-1} \circ \tilde{\pi} = \mathrm{id}.$$

Q. E. D.

§ 6. Appendix.

Let us explain briefly the problem in the proof of Conjecture 1 for $i=1, 2$. If we try to prove it in the same way as [3, Theorem 2.2], we need to extend the result [3, (2.54)] to our case. That is, for $\alpha = \phi + dr \wedge \omega \in \mathrm{dom}\, d_i$ (for $\mathcal{X}(1)$), we must find a sequence $\varepsilon_n \downarrow 0$ $(n \to \infty)$ satisfying

(6.1) $$\lim_{n \to \infty} \|\phi(\varepsilon_n)\|_r = 0.$$

However, in our case, on the $W(+)$, if ϕ is of type $(p, q, P) = (1, 1, 0)$ (see (4.7)), the sequence $\{\varepsilon_n\}$ satisfying the following would be the best of its kind we can find in the same way as [3, (2.54)],

(6.2) $$\|\phi(\varepsilon_n)\|_r = r^{(b-1)/2} o(\varepsilon_n^{-b/2} |\log \varepsilon_n|^{-1/2}).$$

Clearly it is insufficient. Additionally, if the type of ϕ is anything else, we can find the sequence $\{\varepsilon_n\}$ satisfying (6.1) in the same way as [3, (2.54)]. Hence Corollary 3.4(2) could as well be proved in the same way as [3, Lemma 3.2].

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