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Submanifolds of a Euclidean space with homothetic Gauss map

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§ 1. Introduction.

In another paper of the present author some properties of the Gauss map of a submanifold in a Euclidean space were studied [8]. In that paper following topics were treated: (i) Gauss-critical submanifolds, (ii) Submanifolds such that the sectional curvature of the Grassmann manifold in each tangent plane of the Gauss image totally vanishes.

Let M be a compact orientable C^{∞} manifold of dimension m and $i: M \rightarrow E^{n}$ be an immersion such that the Gauss map $\Gamma: iM \rightarrow G(m, n-m)$ is regular. When the Grassmann manifold $G(m, n-m)$ is endowed with the standard Riemannian metric $\tilde{g}$ [7], we can consider the volume of the Gauss image $\Gamma_{i}(M)=\Gamma(iM)$ and letting i move in a certain domain we can define the Gauss-critical immersion. If i is a Gauss-critical immersion, $iM=M'$ is called a Gauss-critical submanifold.

Let g_{i} be the Riemannian metric induced from the standard Riemannian metric of E^{n} on the m-dimensional submanifold $M'=iM$ and G_{i} be the Riemannian metric induced from the standard Riemannian metric $\tilde{g}$ of $G(m, n-m)$ on the Gauss image $\Gamma_{i}(M)=\Gamma(iM)$. If G_{i} is homothetic to g_{i} by Γ, namely, $\Gamma: (iM, g_{i}) \rightarrow (\Gamma(iM), G_{i})$ is a homothetic mapping, then the Gauss map is said to be homothetic.

The purpose of the present paper is to study some properties of a submanifold in E^{n} or of an immersion into E^{n} such that the Gauss map is homothetic and satisfies some other conditions.

Some years ago D. Ferus [2], [3], [4] studied immersions with parallel second fundamental form and J. Vilms [13] studied the relation between totally geodesic Gauss maps and the second fundamental form of immersed manifolds. These results overlap some part of our present study when the Gauss image is assumed to be totally geodesic. But it is to be noticed that, if the Gauss map is not assumed to be homothetic, there exist submanifolds of Euclidean space with totally geodesic Gauss image but without parallel second fundamental form [2].

Our main results are the following.

THEOREM I. *Let i be an immersion of an m-dimensional compact orientable C^∞ manifold M into a Euclidean space E^n such that the Gauss map $\Gamma_i : M \to G(m, n-m)$ is homothetic, namely $G_i = c^2 g_i$ where c is a positive constant. Consider the following three conditions* (i), (ii), (iii). *Then* (ii) *is satisfied if* (i) *is satisfied,* (iii) *is equivalent to* (ii). *If moreover (iM, g_i) is Einsteinian, i is a minimal immersion into a hypersphere of E^n when* (iii) *is satisfied.*

(i) *The Gauss image is totally geodesic.*

(ii) *The associated Gauss map is harmonic.*

(iii) *i is Gauss-critical.*

THEOREM II. *Let (M, g) be a Riemannian manifold and i be an isometric minimal immersion of (M, g) into a hypersphere of E^n. Then the associated Gauss map Γ_i is harmonic and the following two conditions* (i), (ii) *are mutually equivalent.*

(i) *(M, g) is an Einstein manifold.*

(ii) *Γ_i is homothetic.*

If (M, g) is a compact orientable Einstein manifold, i is as stated above and the second fundamental form vanishes nowhere, then i is a Gauss-critical immersion (into E^n).

THEOREM III. *Let (M, g) be an m-dimensional compact orientable flat Riemannian manifold and i be an isometric minimal immersion of (M, g) into a hypersphere of a Euclidean space E^n such that the associated Gauss map Γ_i is regular and homothetic. Then the following two conditions* (i) *and* (ii) *are mutually equivalent.*

(i) *The Gauss image is totally geodesic.*

(ii) *At any point of the Gauss image the sectional curvature of the Grassmann manifold $G(m, n-m)$ vanishes in every tangential plane direction of the Gauss image.*

THEOREM IV. *Let (S^m, g) be a standard sphere $S^m(1)$. If an isometric immersion of this sphere into a hypersphere of radius r of the E^n is given by n eigenfunctions x^h of the Laplacian on (S^m, g) with the second least positive eigenvalue $\lambda_2 = 2(m+1)$, then the Gauss map is homothetic and the Gauss image is totally geodesic. r is given by $r^2 = m/2(m+1)$. This submanifold is a Veronese manifold* [**1**], [**5**], [**6**].

THEOREM V. *Let V be a Veronese manifold immersed in a Euclidean space E^n where the immersion is full and let A_ξ be defined as usual from the second fundamental form and a unit normal vector field ξ on a domain of M. Then A_ξ vanishes nowhere.*

§2 is devoted to a brief explanation of the Grassmann manifold and the Gauss map. Some results of the previous paper are recollected. In §3 we study submanifolds of E^n with homothetic Gauss map and with totally geodesic Gauss image. In § 4 submanifolds isometric to Einstein manifolds are studied for which the Gauss map is homothetic and the Gauss image is totally geodesic or the immersion is Gauss-critical. § 5 is devoted to harmonic Gauss map. In § 6 some isometric immersion of a flat Riemannian manifold is studied. In § 7 we study an isometric immersion of a sphere into a hypersphere of E^n with the use of eigenfunctions of the Laplacian on the sphere. In § 8 some examples are given.

Some years ago a problem was proposed to me by Professor M. Obata to find the relation between harmonic Gauss map and Gauss-critical immersion. Theorem I contains a partial solution of this problem. I wish to express my hearty thanks to Professor M. Obata and Professor K. Ogiue for their kind suggestions and their informing me of literature.

§ 2. The Grassmann manifold and the Gauss map.

In the present paper we understand by a Grassmann manifold $(G(m, n-m), \tilde{g})$ a space of m-planes of the Euclidean space E^n endowed with the standard Riemannian metric $\tilde{g}$ [7]. This Grassmann manifold is a symmetric space.

In our previous paper [8] a system of local coordinates $(\xi_{\alpha x})$ was used where the indices run as follows: $\alpha=1, \cdots, m$; $x=m+1, \cdots, n$. Let Π_0 be a point of $G(m, n-m)$ and U a suitable open neighborhood of Π_0. Π_0 being an m-plane in E^n, we can take a fixed orthonormal frame (e_α, e_x) of E^n where e_α are vectors lying in Π_0 and e_x are vectors normal to Π_0. For each point Π in U we take an orthonormal frame (f_α, f_x) of E^n where f_α lie in the m-plane Π and f_x are normal to Π. Our choice is such that (f_α, f_x) is a differentiable field on U satisfying

$$\langle f_\alpha, e_\beta\rangle=\langle f_\beta, e_\alpha\rangle, \qquad \langle f_x, e_y\rangle=\langle f_y, e_x\rangle \tag{2.1}$$

and such that, when Π coincides with Π_0, (f_α, f_x) coincides with (e_α, e_x). Then we can put

$$\begin{aligned} f_\alpha&=\textstyle\sum_\beta(\delta_{\alpha\beta}+\xi_{\alpha\beta})e_\beta+\sum_y\xi_{\alpha y}e_y\,, \\ f_x&=\textstyle\sum_\beta\xi_{x\beta}e_\beta+\sum_y(\delta_{xy}+\xi_{xy})e_y\,, \end{aligned} \tag{2.2}$$

where $\xi_{\alpha\beta}=\xi_{\beta\alpha}$, $\xi_{xy}=\xi_{yx}$ and the range of indices are as follows:

$$\alpha, \beta, \gamma, \delta, \cdots=1, \cdots, m\,;\ x, y, z, u, \cdots=m+1, \cdots, n\,.$$

U is assumed to be such that the coefficients ξ_{ji} in (2.2) satisfy

$$|\xi_{\alpha\beta}|<\varepsilon, \qquad |\xi_{xy}|<\varepsilon, \qquad |\xi_{\alpha x}|<\varepsilon, \qquad |\xi_{x\alpha}|<\varepsilon$$

where ε is a small positive number. We get

$$\begin{aligned} &\xi_{\alpha\beta}=-\frac{1}{2}\Sigma_y\xi_{\alpha y}\xi_{\beta y}+O(\varepsilon^3)\,, \\ (2.3)\qquad &\xi_{x\alpha}=-\xi_{\alpha x}+O(\varepsilon^3)\,, \\ &\xi_{xy}=-\frac{1}{2}\Sigma_\alpha\xi_{\alpha x}\xi_{\alpha y}+O(\varepsilon^3) \end{aligned}$$

which shows that $m(n-m)$ numbers $\xi_{\alpha x}$ can be used as local coordinates in U.

By means of such local coordinates the covariant components of the fundamental tensor take the form

$$\tilde{g}_{\beta y,\alpha x}=\delta_{yx}\delta_{\beta\alpha}+\xi_{\alpha y}\xi_{\beta x}+O(\varepsilon^4)$$

and the contravariant components

$$\tilde{g}^{\beta y,\alpha x}=\delta_{yx}\delta_{\beta\alpha}-\xi_{\alpha y}\xi_{\beta x}+O(\varepsilon^4)\,.$$

We get for the Christoffel symbols

$$(2.4)\qquad \left\{ {\alpha x \atop \gamma z,\ \beta y} \right\}=\frac{1}{2}(-\delta_{\gamma\alpha}\delta_{yx}\xi_{\beta z}-\delta_{\beta\alpha}\delta_{zx}\xi_{\gamma y}+\delta_{\gamma\alpha}\delta_{zy}\xi_{\beta x} +\delta_{\gamma\beta}\delta_{zx}\xi_{\alpha y}+\delta_{\gamma\beta}\delta_{yx}\xi_{\alpha z}+\delta_{\beta\alpha}\delta_{zy}\xi_{\gamma x})+O(\varepsilon^3)\,,$$

and for the curvature tensor and the Ricci tensor

$$(2.5)\qquad \tilde{K}_{\delta u,\gamma z,\beta y,\alpha x}=(\delta_{\delta\alpha}\delta_{\gamma\beta}-\delta_{\delta\beta}\delta_{\gamma\alpha})\delta_{uz}\delta_{yx} +\delta_{\delta\gamma}\delta_{\beta\alpha}(\delta_{ux}\delta_{zy}-\delta_{uy}\delta_{zx})+O(\varepsilon^2)\,,$$

$$(2.6)\qquad \tilde{K}_{\gamma z,\beta y}=(n-2)\delta_{\gamma\beta}\delta_{zy}+O(\varepsilon^2)\,.$$

Let M be an m-dimensional differentiable manifold and take an immersion $i: M\to E^n$ which is given in each suitable neighborhood $V\subset M$ by differentiable functions

$$x^h=x^h(y^1,\cdots,y^m)\,.$$

Here and in the sequel x^h $(h=1,\cdots,n)$ are rectangular coordinates of E^n and y^κ $(\kappa=1,\cdots,m)$ are local coordinates of M in V. Define $B_\lambda{}^h$ by $B_\lambda{}^h=\partial x^h/\partial y^\lambda$. The tangent plane $i(M_p)$, $p\in V$, of iM can be considered after a suitable parallel displacement as a point $\Gamma_i(p)$ of the Grassmann manifold $G(m, n-m)$, and from this fact we get naturally a mapping $\Gamma_i: M\to G(m, n-m)$, namely, $\Gamma: iM \to G(m, n-m)$. Γ_i is called the Gauss map associated with the immersion i and $\Gamma_i(M)$ the Gauss image of M. We consider only the case of regular mapping.

Let us assume that $V\subset M$ is a neighborhood of a fixed point $p\in M$ whose local coordinates satisfy $y^\kappa=0$. Let (e_α, e_x) be a fixed orthonormal frame of

E^n such that e_α are vectors of $i(M_p)$ and e_x are normal to $i(M_p)$. For each point $q\in V$ let (f_α, f_x) be an orthonormal frame of E^n where f_α are vectors of $i(M_q)$ and f_x are normal to $i(M_q)$ and such that in V (f_α, f_x) is a differentiable frame satisfying

$$\langle f_\alpha, e_\beta\rangle=\langle f_\beta, e_\alpha\rangle, \qquad \langle f_x, e_y\rangle=\langle f_y, e_x\rangle \tag{2.7}$$

and $f_\alpha(0)=e_\alpha$, $f_x(0)=e_x$. If $f_\alpha{}^h$ are the components of the vector f_α, we can put

$$f_\alpha{}^h=\gamma_\alpha{}^\kappa B_\kappa{}^h \tag{2.8}$$

where, here and in the sequel, summation convention is used for Greek letters κ, λ, μ, $\cdots$ when the same letter appears once in the superscript and once in the subscript. The matrix $(\gamma_\alpha{}^\kappa)$ satisfies

$$\gamma_\beta{}^\mu\gamma_\alpha{}^\lambda g_{\mu\lambda}=\delta_{\beta\alpha}, \qquad g_{\mu\lambda}=\sum_h B_\mu{}^h B_\lambda{}^h \tag{2.9}$$

where $g_{\mu\lambda}$ are the components of the first fundamental form g_i of iM. Then the following identities are immediate,

$$\sum_\alpha \gamma_\alpha{}^\mu\gamma_\alpha{}^\lambda=g^{\mu\lambda} \quad \text{where} \quad g^{\mu\lambda}g_{\mu\kappa}=\delta_\kappa^\lambda. \tag{2.10}$$

The components of the second fundamental form of iM are

$$H_{\mu\lambda}{}^h=\partial_\mu B_\lambda{}^h-\left\{ {\kappa \atop \mu\lambda} \right\} B_\kappa{}^h \tag{2.11}$$

where $\left\{ {\kappa \atop \mu\lambda} \right\}$ are the Christoffel symbols derived from $g_{\mu\lambda}$.

For each point $q\in V$ the image $\Gamma_i(q)$ is the m-plane spanned by $f_1, \cdots, f_m$. The distance $d\sigma$ between the two points $\Gamma_i(q)$ and $\Gamma_i(q+dq)$ is given by

$$(d\sigma)^2=\sum_{\alpha,x}\langle df_\alpha, f_x\rangle^2=\sum_h g^{\lambda\kappa}H_{\nu\lambda}{}^h H_{\mu\kappa}{}^h dy^\nu dy^\mu$$

where dy^κ are the difference between the local coordinates of the points $q+dq$ and q. From this formula we can deduce the Riemannian metric G_i of the Gauss image $\Gamma_i(M)$,

$$G_{\mu\lambda}=\sum_h g^{\sigma\rho}H_{\mu\sigma}{}^h H_{\lambda\rho}{}^h \tag{2.12}$$

since Γ_i is assumed to be a regular mapping.

As the Gauss image is an immersion in the Grassmann manifold and its volume in the usual sense may be difficult to treat, we define its volume by the integral

$$\mathrm{Vol}^*(\Gamma_i(M))=\int_M (\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}\mu(g_i) \tag{2.13}$$

where $\mathfrak{G}_i=\det(G_{\mu\lambda})$, $\mathfrak{g}_i=\det(g_{\mu\lambda})$ and $\mu(g_i)$ is the volume form of (iM, g_i). This integral defines a mapping $\mathrm{Vol}^*: I_M \to R$ where I_M is the space of all immersions i of M into E^n such that Γ_i is regular. The equation of a critical point of this mapping is given by

$$\nabla_\mu \nabla_\lambda \{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu} H_\nu{}^{\lambda h}\} + \nabla_\lambda \{(\mathfrak{G}_i/\mathfrak{g}_i)^{1/2}(G^{-1})^{\nu\mu} \textstyle\sum_j H_\nu{}^{\lambda j} H_\mu{}^{\kappa j} B_\kappa{}^h\} = 0 \tag{2.14}$$

where ∇ stands for the covariant differentiation with respect to the Riemannian metric g_i, $((G^{-1})^{\mu\lambda})$ is the inverse matrix of $(G_{\mu\lambda})$ and $H_\nu{}^{\lambda h} = H_{\nu\rho}{}^h g^{\lambda\rho}$ [8].

§ 3. Submanifolds of E^n with homothetic Gauss map and totally geodesic Gauss image.

In this paragraph (M, g) means an m-dimensional complete differentiable submanifold of E^n. With this understanding we can delete the subscript i in the foregoing formulas so that the Gauss map is denoted Γ: $(M, g) \to (G(m, n-m), \tilde{g})$. Assume that Γ is homothetic, hence

$$G_{\mu\lambda} \equiv g^{\sigma\rho} \textstyle\sum_h H_{\mu\sigma}{}^h H_{\lambda\rho}{}^h = c^2 g_{\mu\lambda} \tag{3.1}$$

where c is a positive constant. Taking the local coordinates of M and of $\Gamma(M)$ in such a way that any point $q \in V \subset M$ and its image $\Gamma(q) \in \Gamma(V)$ have the coordinates (y^κ) in common, we get for the Christoffel symbols $\left\{ {\kappa \atop \mu\lambda} \right\}$ of (M, g) and those ${}^G\left\{ {\kappa \atop \mu\lambda} \right\}$ of $(\Gamma(M), G)$

$$\left\{ {\kappa \atop \mu\lambda} \right\} = {}^G\left\{ {\kappa \atop \mu\lambda} \right\}. \tag{3.2}$$

We have defined the local coordinates $(\xi_{\alpha x})$ of the Grassmann manifold valid in U. Let us assume that $\Gamma(V)$ is contained in U. Then we have

$$\xi_{\alpha x} = \langle f_\alpha, e_x \rangle = \textstyle\sum_h \gamma_\alpha{}^\kappa B_\kappa{}^h e_x{}^h \tag{3.3}$$

where $e_x{}^h$ are the components of the normal vector e_x at p. (3.3) gives a local expression $\xi_{\alpha x} = \xi_{\alpha x}(y)$ of the immersion $\Gamma(M)$ in $G(m, n-m)$. From (3.3) we get

$$\tilde{B}_\lambda{}^{\alpha x} = \partial \xi_{\alpha x}/\partial y^\lambda = \textstyle\sum_h (\nabla_\lambda \gamma_\alpha{}^\kappa B_\kappa{}^h + \gamma_\alpha{}^\kappa H_{\lambda\kappa}{}^h) e_x{}^h \tag{3.4}$$

where, here and in the sequel, ∇ represents covariant differentiation with respect to the Christoffel symbols derived from g and ∇AB stands for $(\nabla A)B$. $\tilde{B}_\lambda{}^{\alpha x}$ are the components of a tensor connecting the Gauss image $\Gamma(M)$ and the ambient manifold $G(m, n-m)$.

REMARK. Though α, β, $\cdots$ run the range $\{1, \cdots, m\}$, these are not used as covariant indices, hence we have

$$\nabla_\mu \gamma_\alpha{}^\kappa = \partial_\mu \gamma_\alpha{}^\kappa + \left\{ {\kappa \atop \mu\lambda} \right\} \gamma_\alpha{}^\lambda .$$

The components of the second fundamental form of $(\Gamma(M), G)$ in $(G(m, n-m), \tilde{g})$ are given by

$$\tilde{H}_{\mu\lambda}{}^{\alpha x} = \partial^2 \xi_{\alpha x}/\partial y^\mu \partial y^\lambda - {}^G\left\{ {\kappa \atop \mu\lambda} \right\} \partial \xi_{\alpha x}/\partial y^\kappa \tag{3.5}$$

$$+ \left\{ {\alpha x \atop \gamma z,\ \beta y} \right\} (\partial \xi_{\gamma z}/\partial y^\mu)(\partial \xi_{\beta y}/\partial y^\lambda)$$

in $\Gamma(V)$. Let us evaluate them at $\Gamma(p)$, that is, at $y^\kappa = 0$.

First we have

$$(\partial \xi_{\alpha x}/\partial y^\lambda)_0 = \sum_h (\gamma_\alpha{}^\kappa H_{\lambda\kappa}{}^h)_0 e_x{}^h$$

and

$$\left\{ {\alpha x \atop \gamma z,\ \beta y} \right\}_0 = 0$$

because of $\sum_h (B_\lambda{}^h)_0 e_x{}^h = 0$, (2.4) and (3.3). On the other hand we have the quite favorable relation (3.2). Hence we get

$$\begin{aligned}(\tilde{H}_{\mu\lambda}{}^{\alpha x})_0 &= (\nabla_\mu \tilde{B}_\lambda{}^{\alpha x})_0 \\ &= \sum_h (\nabla_\mu \nabla_\lambda \gamma_\alpha{}^\kappa B_\kappa{}^h + \nabla_\lambda \gamma_\alpha{}^\kappa H_{\mu\kappa}{}^h + \nabla_\mu \gamma_\alpha{}^\kappa H_{\lambda\kappa}{}^h + \gamma_\alpha{}^\kappa \nabla_\mu H_{\lambda\kappa}{}^h)_0 e_x{}^h \\ &= \sum_h [(\nabla_\lambda \gamma_\alpha{}^\kappa)_0 (H_{\mu\kappa}{}^h)_0 + (\nabla_\mu \gamma_\alpha{}^\kappa)_0 (H_{\lambda\kappa}{}^h)_0 + (\gamma_\alpha{}^\kappa)_0 (\nabla_\mu H_{\lambda\kappa}{}^h)_0] e_x{}^h .\end{aligned}$$

But we can prove as below $(\nabla_\lambda \gamma_\alpha{}^\kappa)_0 = 0$, hence we get

$$(\tilde{H}_{\mu\lambda}{}^{\alpha x})_0 = (\gamma_\alpha{}^\kappa)_0 \sum_h (\nabla_\mu H_{\lambda\kappa}{}^h)_0 e_x{}^h . \tag{3.6}$$

PROOF OF $(\nabla_\lambda \gamma_\alpha{}^\kappa)_0 = 0$. From (2.7) and (2.8) we get

$$\sum_h \gamma_\alpha{}^\kappa B_\kappa{}^h e_\beta{}^h = \sum_h \gamma_\beta{}^\kappa B_\kappa{}^h e_\alpha{}^h .$$

Differentiating this we get, in view of $(\sum_h H_{\mu\lambda}{}^h e_\alpha{}^h)_0 = 0$,

$$(\nabla_\lambda \gamma_\alpha{}^\kappa)_0 \sum_h (B_\kappa{}^h)_0 e_\beta{}^h = (\nabla_\lambda \gamma_\beta{}^\kappa)_0 \sum_h (B_\kappa{}^h)_0 e_\alpha{}^h$$

which becomes

$$(\nabla_\lambda \gamma_\alpha{}^\kappa g_{\kappa\mu} \gamma_\beta{}^\mu)_0 = (\nabla_\lambda \gamma_\beta{}^\kappa g_{\kappa\mu} \gamma_\alpha{}^\mu)_0$$

because of (2.8), (2.9) and $f_\alpha(0) = e_\alpha$. On the other hand we can deduce from (2.9)

$$(\nabla_\nu \gamma_\beta{}^\mu \gamma_\alpha{}^\lambda + \gamma_\beta{}^\mu \nabla_\nu \gamma_\alpha{}^\lambda) g_{\mu\lambda} = 0 .$$

Hence we get $(\nabla_\nu \gamma_\beta{}^\mu \gamma_\alpha{}^\lambda g_{\mu\lambda})_0=0$ which is equivalent to

$$(\nabla_\lambda \gamma_\alpha{}^\kappa)_0=0 .$$

As we can take an arbitrary point of M as the point p and as the vanishing of $(\tilde{H}_{\mu\lambda}{}^{\alpha x})_0$ is according to (3.6) equivalent to

$$\sum_h N^h(\nabla_\mu H_{\lambda\kappa}{}^h)_0=0$$

for every normal vector N to M_p, we immediately get the following theorem.

THEOREM 3.1. *A necessary and sufficient condition for a submanifold (M, g) in E^n for which the Gauss map $\Gamma: (M, g)\to(\Gamma(M), G)$ is homothetic to have a totally geodesic image $\Gamma(M)$ in $(G(m, n-m), \tilde{g})$ is that the second fundamental form of M satisfy*

$$\sum_h N^h \nabla_\nu H_{\mu\lambda}{}^h=0 \tag{3.7}$$

for every normal vector N^h of (M, g).

If the Gauss map Γ of (M, g) is homothetic and the image $\Gamma(M)$ is totally geodesic, then as (3.7) shows there exists a tensor field $S_{\nu\mu\lambda}{}^\kappa$ on (M, g) such that

$$\nabla_\nu H_{\mu\lambda}{}^h=S_{\nu\mu\lambda}{}^\kappa B_\kappa{}^h . \tag{3.8}$$

Put $S_{\nu\mu\lambda\kappa}=S_{\nu\mu\lambda}{}^\rho g_{\rho\kappa}$. Then we have

$$S_{\nu\mu\lambda\kappa}=\sum_h \nabla_\nu H_{\mu\lambda}{}^h B_\kappa{}^h=-\sum_h H_{\nu\kappa}{}^h H_{\mu\lambda}{}^h \tag{3.9}$$

in view of $g_{\mu\lambda}=\sum_h B_\mu{}^h B_\lambda{}^h$ and $\sum_h H_{\mu\lambda}{}^h B_\kappa{}^h=0$. From (3.9) we get

$$S_{\nu\mu\lambda\kappa}g^{\nu\lambda}=-G_{\mu\kappa}=-c^2 g_{\mu\kappa} . \tag{3.10}$$

On the other hand we get from (3.8)

$$\nabla_\lambda H^{\mu\lambda h}=S_{\nu\sigma\lambda}{}^\kappa g^{\nu\lambda} B_\kappa{}^h g^{\sigma\mu}$$

hence

$$\nabla_\lambda H^{\mu\lambda h}=-c^2 g^{\sigma\mu} B_\sigma{}^h \tag{3.11}$$

because of (3.10). From (3.11) we get

$$\nabla_\mu \nabla_\lambda H^{\mu\lambda h}+mc^2 H^h=0 \tag{3.12}$$

where H^h is the mean curvature vector of (M, g). It is easy to see that, if Γ is homothetic with respect to (M, g), then (2.14) is equivalent to (3.12) [8]. Hence we get the following theorem.

THEOREM 3.2. *Let (M, g) be a compact differentiable submanifold of dimension m in E^n such that the Gauss map $\Gamma: (M, g)\to(G(m, n-m), \tilde{g})$ is homothetic* *and the image $\Gamma(M)$ is totally geodesic. Then (M, g) is a Gauss-critical submanifold. Furthermore (M, g) is a locally symmetric Riemannian manifold.*

PROOF. To prove the last part of the theorem we need only to differentiate the equation of Gauss

$$K_{\nu\mu\lambda\kappa}=\sum_h H_{\nu\kappa}{}^h H_{\mu\lambda}{}^h-\sum_h H_{\mu\kappa}{}^h H_{\nu\lambda}{}^h \tag{3.13}$$

and use (3.8). The following lemma also proves it directly.

LEMMA 3.3. *A totally geodesic submanifold of a locally symmetric Riemannian manifold is a locally symmetric Riemannian manifold.*

We also have the following theorem.

THEOREM 3.4. *Assume that an m-dimensional submanifold (M, g) of E^n is not a locally decomposable Riemannian manifold. If (3.8) is satisfied and the Gauss map is regular, then Γ is homothetic and the image $\Gamma(M)$ is totally geodesic.*

PROOF. As $H_{\mu\lambda}{}^h$ satisfies (3.8) we get

$$\nabla_\nu G_{\mu\lambda}=\nabla_\nu(\textstyle\sum_h H_{\mu\rho}{}^h H_\lambda{}^{\rho h})=0\,.$$

As we assume that (M, g) is not a locally decomposable Riemannian manifold, there must exist a constant c such that $G_{\mu\lambda}=\pm c^2 g_{\mu\lambda}$. As Γ is regular we get $G_{\mu\lambda}=c^2 g_{\mu\lambda}$. $\Gamma(M)$ is totally geodesic in view of Theorem 3.1.

REMARK. That Γ is homothetic implies that Γ is affine. As (3.7) is equivalent to saying that the second fundamental form is parallel, Theorem 3.1 is covered by the result obtained by J. Vilms [**13**]. As for Theorem 3.4 see D. Ferus' papers [**2**], [**3**], [**4**].

THEOREM 3.5. *Let (M, g) be a submanifold of E^n with respect to which the Gauss map Γ is homothetic. If $\Gamma(M)$ is totally umbilical, then $\Gamma(M)$ is totally geodesic.*

PROOF. If the assumption of the theorem is satisfied, we have

$$\textstyle\sum_h \gamma_\alpha{}^\kappa \nabla_\mu H_{\lambda\kappa}{}^h N_x{}^h=\tilde{H}_{\mu\lambda}{}^{\alpha x}=G_{\mu\lambda}C^{\alpha x}$$

where N_x $(x=m+1, \cdots, n)$ are orthonormal normal vectors of (M, g) and $C^{\alpha x}$ are some local functions. Hence there exist some local functions $D_\kappa{}^h$ such that

$$\nabla_\mu H_{\lambda\kappa}{}^h-\textstyle\sum_i \nabla_\mu H_{\lambda\kappa}{}^i B_\sigma{}^i B_\rho{}^h g^{\sigma\rho}=G_{\mu\lambda}D_\kappa{}^h\,. \tag{3.14}$$

As the first member is symmetric in λ and κ, we get

$$G_{\mu\lambda}D_\kappa{}^h-G_{\mu\kappa}D_\lambda{}^h=0\,.$$

If $D_\kappa{}^h\neq 0$ there exist some local functions A_μ and B_λ satisfying $G_{\mu\lambda}=A_\mu B_\lambda$. As we have assumed that Γ is homothetic this is impossible. Hence we get $D_\kappa{}^h=0$. Then from (3.14) we get (3.7).

§ 4. Submanifolds isometric to Einstein manifolds.

Let (M, g) be a submanifold of E^n with respect to which the Gauss map Γ is homothetic. Then we get from (3.1) and (3.13)

$$\sum_h H^h H_{\mu\lambda}{}^h = (1/m)(c^2 g_{\mu\lambda} + K_{\mu\lambda}) \tag{4.1}$$

where $K_{\mu\lambda}$ is the Ricci tensor. We observe in passing that, if (M, g) is pseudo-umbilical and its Gauss map is homothetic, then (M, g) is Einsteinian, and also that, if (M, g) is Einsteinian and the Gauss map is homothetic, then (M, g) is pseudo-umbilical. This is a special case of more extensive results obtained by M. Obata [**9**].

REMARK. A symmetric tensor field $a_{\mu\lambda}$ on (M, g) satisfying $\nabla_\nu a_{\mu\lambda} = 0$ is equal to the fundamental tensor multiplied by a constant if (M, g) is not a locally decomposable Riemannian manifold. Hence a locally symmetric Riemannian manifold is Einsteinian if it is not a locally decomposable Riemannian manifold.

Now we study the case where (M, g) is Einsteinian. Then we get from (4.1)

$$\sum_h H^h H_{\mu\lambda}{}^h = (1/m)(c^2 + K/m) g_{\mu\lambda} .$$

If, furthermore, $\Gamma(M)$ is totally geodesic, we get from (3.8) and (3.9)

$$\nabla_\mu H^h = -\sum_i H^i H_{\mu\sigma}{}^i g^{\sigma\rho} B_\rho{}^h .$$

Hence we have

$$\nabla_\mu H^h = -(1/m)(c^2 + K/m) \nabla_\mu x^h , \tag{4.2}$$

and integrating this

$$H^h = Z^h - (1/m)(c^2 + K/m) x^h$$

where Z^h is a constant vector of E^n. After a suitable parallel displacement of (M, g) in E^n we get

$$H^h = -(1/m)(c^2 + K/m) x^h , \tag{4.3}$$

hence

$$\nabla_\mu \nabla^\mu x^h + (c^2 + K/m) x^h = 0 . \tag{4.4}$$

On the other hand we get from (4.2)

$$\nabla_\mu \nabla^\mu H^h = -(1/m)(c^2 + K/m) H_\mu{}^{\mu h} = -(c^2 + K/m) H^h .$$

This proves the following theorem.

THEOREM 4.1. *Assume that (M, g) is an m-dimensional differentiable submani-*

fold of E^n isometric to an Einstein manifold and such that the Gauss map Γ is homothetic and the image $\Gamma(M)$ is totally geodesic. Then the components of the mean curvature vector are eigenfunctions of the Laplacian on (M, g) belonging to an eigenvalue

$$\lambda=c^2+K/m\,.$$

Moreover, after a suitable parallel displacement, the rectangular coordinates x^h of the point of (M, g) are also eigenfunctions with the same eigenvalue λ.

This theorem holds if (M, g) is replaced by an isometric immersion of a Riemannian manifold (M, g).

From (4.2) we see that the length of the mean curvature vector is constant and in view of (4.3) the submanifold M lies in a hypersphere with the origin as center. Then by T. Takahashi's theorem we get the following theorem [**12**].

THEOREM 4.2. *Let (M, g) be a submanifold of E^n satisfying the assumption of Theorem* 4.1. *Then (M, g) is a minimal submanifold in a hypersphere of E^n of radius $r=m(K+mc^2)^{-1/2}$.*

r is obtained from (4.4).

If the multiplicity of the eigenvalue $\lambda=c^2+K/m$ is p and φ^a $(a=1, \cdots, p)$ are linearly independent eigenfunctions in the eigenspace $\mathcal{P}_\lambda(M, g)$, then we have

$$x^h=\textstyle\sum_a A_a{}^h\varphi^a$$

where $A_a{}^h$ are constants. Hence (M, g) lies in some p-plane. If $p<m$, M can not be m-dimensional. If $p=m$ then M lies in an m-plane and the Gauss map is not regular. Hence $p\geqq m+1$. If $p=m+1$, then (M, g) is a hypersphere of E^{m+1}. This proves the following theorem.

THEOREM 4.3. *Let (M, g) be a submanifold as assumed in Theorem* 4.1. *Then the Laplacian on (M, g) has at least one eigenvalue $\lambda>K/m$ of multiplicity $\geqq m+1$. Especially, if (M, g) is such that the immersion is full in E^n, there exists at least one eigenvalue $\lambda>K/m$ of multiplicity $\geqq n$.*

REMARK. Though obtained independently, these results are widely overlapped by D. Ferus [**4**].

We have observed that, if (M, g) is a compact orientable and Gauss-critical submanifold with homothetic Gauss map, then (3.12) holds. Furthermore we have

$$\nabla_\mu\nabla^\mu H^h+(c^2+K/m)H^h=0 \tag{4.5}$$

since (M, g) is assumed to be Einsteinian [8].

Suppose $c^2+K/m=0$. Then we get $H^h=$constant from $\nabla_\mu\nabla^\mu H^h=0$. From $\nabla_\mu\nabla^\mu x^h=$constant we get $\nabla_\mu\nabla^\mu x^h=0$ by integration. But this leads to

$$0=\int \sum_h x^h \nabla_\mu \nabla^\mu x^h \mu(g)$$

$$=-\int \sum_h \nabla_\mu x^h \nabla^\mu x^h \mu(g) = -m \int \mu(g)$$

which is impossible. Hence we have $c^2+K/m>0$.

As (4.5) is equivalent to

$$\nabla_\mu \nabla^\mu [(c^2+K/m)x^h+mH^h]=0\,,$$

we can put

$$(c^2+K/m)x^h+mH^h=Z^h$$

where Z^h is a constant vector of E^n. This proves that, after a suitable parallel displacement of the immersion, we can put

$$x^h=-m(c^2+K/m)^{-1}H^h\,.$$

Hence x^h satisfies

$$\nabla_\mu \nabla^\mu x^h+\lambda x^h=0$$

where $\lambda=c^2+K/m$ and

$$\sum_h x^h x^h=r^2=\text{constant}.$$

This proves that such an immersion is a minimal immersion into a hypersphere by Takahashi's theorem.

Thus we have obtained the following theorem.

THEOREM 4.4. *Let (M, g) be an m-dimensional compact orientable Einstein manifold and i be an isometric immersion of (M, g) into the Euclidean space E^n. If i is Gauss-critical and the associated Gauss map is homothetic, then i is a minimal immersion into a hypersphere of E^n. Moreover we have the same relation as in Theorem* 4.1 *between λ, c^2 and K.*

§ 5. Homothetic and harmonic Gauss maps.

Harmonic Gauss maps were studied by E. A. Ruh and J. Vilms [**11**]. A necessary and sufficient condition that a submanifold of a Euclidean space has a harmonic Gauss map is that the second fundamental tensor satisfies

$$\sum_h N^h g^{\mu\lambda} \nabla_\mu H_{\lambda\kappa}{}^h=0 \tag{5.1}$$

for every normal vector N^h. Obviously (5.1) is equivalent to

$$\sum_h N^h \nabla_\mu H^h=0$$

or

$$g^{\mu\lambda} \nabla_\nu H_{\mu\lambda}{}^h=-\sum_i H_\nu{}^{\kappa i} H_{\mu\lambda}{}^i g^{\mu\lambda} B_\kappa{}^h\,.$$

If the associated Gauss map is homothetic, we get in view of (4.1)

$$\nabla_\nu H^h = -(1/m)(K_\nu{}^\kappa + c^2\delta_\nu^\kappa)B_\kappa{}^h . \tag{5.2}$$

If moreover the submanifold is Einsteinian, we get

$$\nabla_\nu H^h = -(1/m)(c^2 + K/m)B_\nu{}^h ,$$

namely (4.2).

Thus we have proved the following theorem.

THEOREM 5.1. *Let (M, g) be an m-dimensional Einstein manifold and i an isometric immersion of (M, g) into E^n such that the Gauss map associated with i is homothetic and harmonic. Then i is a minimal immersion of (M, g) into a hypersphere of E^n.*

Now let us consider an isometric immersion i of a Riemannian manifold (M, g) into E^n satisfying

$$\Delta x^h = \lambda x^h , \qquad \lambda > 0 .$$

Then substituting this into $\nabla_\mu(\Delta x^h) = -m\nabla_\mu H^h$ we get

$$\nabla_\mu H^h = (-\lambda/m)B_\mu{}^h , \tag{5.3}$$

which proves the following theorem by virtue of Takahashi's theorem.

THEOREM 5.2. *Let (M, g) be a Riemannian manifold and i be an isometric minimal immersion of (M, g) into a hypersphere of E^n. Then the associated Gauss map where i is considered as $i: M \to E^n$ is a harmonic map.*

REMARK. When we consider a Gauss map in this paper we always regard i as $i: M \to E^n$.

From $\sum_h B_\nu{}^h H_{\mu\lambda}{}^h = 0$ we get

$$\sum_h (H_{\nu\kappa}{}^h H_{\mu\lambda}{}^h + B_\nu{}^h \nabla_\kappa H_{\mu\lambda}{}^h) = 0$$

hence

$$K_{\nu\mu\lambda\kappa} + \sum_h H_{\mu\kappa}{}^h H_{\nu\lambda}{}^h + \sum_h B_\nu{}^h \nabla_\kappa H_{\mu\lambda}{}^h = 0 .$$

Transvecting $g^{\mu\lambda}$ we get

$$K_{\nu\kappa} + G_{\nu\kappa} + \sum_h B_\nu{}^h \nabla_\kappa H_\mu{}^{\mu h} = 0$$

which proves in view of (5.3)

$$K_{\nu\kappa} + G_{\nu\kappa} = \lambda \sum_h B_\nu{}^h B_\kappa{}^h = \lambda g_{\nu\kappa} ,$$

and the following theorem.

THEOREM 5.3. *Let (M, g) and i be those as stated in Theorem* 5.2. *Then the following conditions* (i) *and* (ii) *are mutually equivalent.*

(i) *(M, g) is an Einstein manifold.*

(ii) *The associated Gauss map is homothetic.*

From (5.3) we get

$$\nabla_\lambda H^{\mu\lambda h}=(K^{\mu\lambda}-\lambda g^{\mu\lambda})B_\lambda{}^h$$

hence

$$\nabla_\mu\nabla_\lambda H^{\mu\lambda h}+m(\lambda-K/m)H^h=0$$

if (M, g) is Einsteinian. This proves the following theorem.

THEOREM 5.4. *Let (M, g) and i be those as stated in Theorem 5.2. If (M, g) is Einsteinian and $\lambda-K/m>0$, then the associated Gauss map is homothetic and i is a Gauss-critical immersion.*

Now let (M, g) be a compact orientable Riemannian manifold of dimension m and i be an isometric immersion $i:(M, g)\to E^n$ such that the associated Gauss map is homothetic. If the associated Gauss map is harmonic, we have

$$\nabla_\lambda H^{\mu\lambda h}=-\sum_i H_\lambda{}^{\nu i}H^{\mu\lambda i}B_\nu{}^h=-c^2g^{\nu\mu}B_\nu{}^h$$

hence

$$\nabla_\mu\nabla_\lambda H^{\mu\lambda h}=-c^2g^{\nu\mu}H_{\nu\mu}{}^h \tag{5.4}$$

which proves that i is Gauss-critical. If the Gauss image is totally geodesic, we get in view of (3.8) and (3.9)

$$\nabla_\mu H_\sigma{}^{\sigma h}=-\sum_i H_\mu{}^{\nu i}H_\sigma{}^{\sigma i}B_\nu{}^h$$

which proves that the Gauss map is harmonic.

Let us consider the integral

$$\begin{aligned} I&=\int\sum_h(\nabla_\lambda H^{\mu\lambda h}+c^2g^{\mu\lambda}B_\lambda{}^h)(\nabla_\nu H_\mu{}^{\nu h}+c^2B_\mu{}^h)\mu(g)\\ &=\int(\sum_h\nabla_\lambda H^{\mu\lambda h}\nabla_\nu H_\mu{}^{\nu h}+2c^2g^{\mu\lambda}\sum_h B_\lambda{}^h\nabla_\nu H_\mu{}^{\nu h}+mc^4)\mu(g)\\ &=\int[\sum_h\nabla_\lambda H^{\mu\lambda h}(\nabla_\mu H_\nu{}^{\nu h}+K_\mu{}^\nu B_\nu{}^h)\\ &\qquad -2c^2g^{\mu\lambda}\sum_h H_{\nu\lambda}{}^h H_\mu{}^{\nu h}+mc^4]\mu(g)\,. \end{aligned}$$

Then applying Green's theorem we get

$$I=\int(-\sum_h\nabla_\mu\nabla_\lambda H^{\mu\lambda h}H_\nu{}^{\nu h}-K_\mu{}^\nu\sum_h H_{\nu\lambda}{}^h H^{\mu\lambda h}-mc^4)\mu(g)\,.$$

If (5.4) is satisfied, we get

$$I=c^2\int(\sum_h H_\nu{}^{\nu h}H_\mu{}^{\mu h}-K-mc^2)\mu(g)\,.$$

But we have

$$\sum_h H_\nu{}^{\nu h}H_\mu{}^{\mu h}=\sum_h H_\mu{}^{\nu h}H_\nu{}^{\mu h}+K_{\nu\mu}{}^{\mu\nu}=mc^2+K\,,$$

hence $I=0$. This proves that, if (5.4) is satisfied, then

$$\nabla_\lambda H^{\mu\lambda h}+c^2 g^{\mu\lambda}B_\lambda{}^h=0\,.$$

Thus we have the following theorem.

THEOREM 5.5. *Let i be an isometric immersion of an m-dimensional compact orientable C^∞ Riemannian manifold (M, g) into a Euclidean space E^n such that the associated Gauss map is homothetic. Consider the following three conditions* (i), (ii), (iii):

(i) *The Gauss image is totally geodesic.*

(ii) *The associated Gauss map is harmonic.*

(iii) *i is Gauss-critical.*

Then (ii) *is satisfied if* (i) *is satisfied.* (ii) *and* (iii) *are mutually equivalent. If moreover (M, g) is Einsteinian, then i is a minimal immersion into a hypersphere of E^n when* (ii) *is satisfied.*

§ 6. Some minimal immersions into a hypersphere of a Euclidean space.

As we have stated above a minimal immersion into a hypersphere of a Euclidean space has interesting properties. But the associated Gauss map has not in general totally geodesic image. We consider now an isometric minimal immersion of a compact orientable Riemannian manifold (M, g) into a hypersphere such that the associated Gauss map is homothetic and study the integral

$$I=\int \sum_h(\nabla^\nu H^{\mu\lambda h}+\sum_j H^{\nu\sigma j}H^{\mu\lambda j}B_\sigma{}^h) \times(\nabla_\nu H_{\mu\lambda}{}^h+\sum_i H_\nu{}^{\rho i}H_{\mu\lambda}{}^i B_\rho{}^h)\mu(g)\,. \tag{6.1}$$

It is easy to see that

$$I=-\int \sum_h H^{\mu\lambda h}\nabla_\nu\nabla^\nu H_{\mu\lambda}{}^h\mu(g) -\int \sum_i H^{\nu\kappa j}H^{\mu\lambda j}\sum_i H_{\nu\kappa}{}^i H_{\mu\lambda}{}^i\mu(g)\,.$$

As we get, in view of $H_\nu{}^{\nu h}=-\lambda x^h$,

$$\nabla_\nu\nabla^\nu H_{\mu\lambda}{}^h=-\lambda H_{\mu\lambda}{}^h+K_\lambda{}^\kappa H_{\mu\kappa}{}^h+K_\mu{}^\kappa H_{\lambda\kappa}{}^h -2K^\nu{}_{\mu\lambda}{}^\kappa H_{\nu\kappa}{}^h+(\nabla_\mu K_\lambda{}^\kappa-\nabla^\nu K_{\nu\mu\lambda}{}^\kappa)B_\kappa{}^h\,,$$

we get

$$I=m\lambda c^2\int\mu(g)-2c^2\int K\mu(g)$$

$$+\int K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\mu(g)$$

$$-\int \sum_j H^{\nu\kappa j}H^{\mu\lambda j}\sum_i H_{\nu\kappa}{}^i H_{\mu\lambda}{}^i\mu(g)$$

where we have used

$$\sum_h H_{\mu\sigma}{}^h H_\lambda{}^{\sigma h}=c^2 g_{\mu\lambda}\,,$$

$$2K^{\nu\mu\lambda\kappa}\sum_h H_{\nu\kappa}{}^h H_{\mu\lambda}{}^h=K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\,.$$

On the other hand we have

$$\begin{aligned}&\sum_{j,i}(H^{\nu\kappa j}H^{\mu}{}_{\kappa}{}^{i}-H^{\nu\kappa i}H^{\mu}{}_{\kappa}{}^{j})(H_\nu{}^{\lambda j}H_{\mu\lambda}{}^i-H_\nu{}^{\lambda i}H_{\mu\lambda}{}^j)\\&=2G^{\lambda\kappa}G_{\lambda\kappa}-2\sum_j H^{\nu\kappa j}H^{\mu\lambda j}\sum_i H_{\nu\lambda}{}^i H_{\mu\kappa}{}^i\\&=2G^{\lambda\kappa}G_{\lambda\kappa}-2\sum_j H^{\nu\kappa j}H^{\mu\lambda j}\sum_i H_{\nu\kappa}{}^i H_{\mu\lambda}{}^i+K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\,,\end{aligned}$$

from which follows

$$\begin{aligned}&-\sum_j H^{\nu\kappa j}H^{\mu\lambda j}\sum_i H_{\nu\kappa}{}^i H_{\mu\lambda}{}^i\\&\qquad=-mc^4-(1/2)K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}+(1/2)\sum_{j,i}A^{\nu\mu ji}A_{\nu\mu}{}^{ji}\end{aligned}$$

where

$$A_{\nu\mu}{}^{ji}=H_\nu{}^{\sigma j}H_{\mu\sigma}{}^i-H_\nu{}^{\sigma i}H_{\mu\sigma}{}^j\,. \tag{6.2}$$

This proves

$$\begin{aligned}I=&\,mc^2(\lambda-c^2)\int\mu(g)-2c^2\int K\mu(g)\\&+(1/2)\int K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\mu(g)\\&+(1/2)\int\sum_{j,i}A^{\nu\mu ji}A_{\nu\mu}{}^{ji}\mu(g)\,.\end{aligned} \tag{6.3}$$

Thus we get in view of (3.8) and (3.9) the following lemma.

LEMMA 6.1. *Let (M, g) be a compact orientable Riemannian manifold and $i: M\to E^n$ be an isometric minimal immersion of (M, g) into a hypersphere of E^n such that the associated Gauss map is homothetic. If moreover i satisfies*

$$\begin{aligned}&mc^2(\lambda-c^2)\int\mu(g)-2c^2\int K\mu(g)\\&\qquad+(1/2)\int K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\mu(g)=0\,,\end{aligned} \tag{6.4}$$

then the following conditions (i) *and* (ii) *are mutually equivalent:*

(i) *The Gauss image is totally geodesic.*

(ii) *The second fundamental form satisfies*

$$H_\nu{}^{\sigma j}H_{\mu\sigma}{}^i = H_\nu{}^{\sigma i}H_{\mu\sigma}{}^j\,.$$

If the immersion satisfies

$$mc^2(\lambda - c^2)\int \mu(g) - 2c^2\int K\mu(g) + (1/2)\int K^{\nu\mu\lambda\kappa}K_{\nu\mu\lambda\kappa}\mu(g) > 0\,,$$

then the Gauss image can not be totally geodesic.

Now we consider that (M, g) is a compact orientable flat Riemannian manifold. Then we have

$$\begin{aligned} m\lambda\int \mu(g) &= -\lambda\int \textstyle\sum_h x^h \nabla_\mu \nabla^\mu x^h \mu(g) \\ &= \int \textstyle\sum_h \nabla_\mu \nabla^\mu x^h \nabla_\lambda \nabla^\lambda x^h \mu(g) \\ &= \int \textstyle\sum_h \nabla_\mu \nabla_\lambda x^h \nabla^\mu \nabla^\lambda x^h \mu(g) \\ &= \int G_{\mu\lambda} g^{\mu\lambda} \mu(g) = mc^2 \int \mu(g)\,. \end{aligned}$$

Hence $\lambda = c^2$ in this case and we have

$$I = (1/2)\int \textstyle\sum_{j,i} A^{\nu\mu ji} A_{\nu\mu}{}^{ji} \mu(g)\,. \tag{6.5}$$

In view of Lemma 5.1 of [8] we get the following theorem.

THEOREM 6.2. *Let (M, g) be an m-dimensional compact orientable flat Riemannian manifold and $i: M \to E^n$ be an isometric minimal immersion of (M, g) into a hypersphere of E^n such that the associated Gauss map is homothetic. Then the following conditions* (i) *and* (ii) *are mutually equivalent:*

(i) *The Gauss image is totally geodesic.*

(ii) *The sectional curvature of the Grassmann manifold vanishes in every tangential plane direction of the Gauss image.*

§7. Some minimal submanifolds of a hypersphere of E^n isometric to a sphere—Veronese manifolds.

By virtue of T. Takahashi's theorem [12] a minimal submanifold of a hypersphere of E^n can be obtained with the use of eigenfunctions of the Laplacian Δ of the manifold to be immersed isometrically. In the present paper we study an immersion of a standard sphere (S^m, g) with the use of eigenfunctions of Δ with the eigenvalue $\lambda_2 = 2(m+1)$.

Let φ^{α} $(\alpha, \beta, \cdots =1, \cdots, m+1)$ be the eigenfunctions of Δ on (S^m, g) satisfying

$$\Sigma_{\alpha}(\varphi^{\alpha})^2=1\,, \tag{7.1}$$

$$\nabla_{\mu}\varphi^{\beta}\nabla^{\mu}\varphi^{\alpha}=\delta^{\beta\alpha}-\varphi^{\beta}\varphi^{\alpha}\,, \tag{7.2}$$

$$\nabla_{\mu}\nabla_{\lambda}\varphi^{\alpha}=-g_{\mu\lambda}\varphi^{\alpha} \tag{7.3}$$

where $g_{\mu\lambda}$ are the components of the standard Riemannian metric of S^m. Let us call the system (φ^{α}) the fundamental system of eigenfunctions.

Let x^h $(h, i, j, \cdots =1, \cdots, n)$ be the components of the position vector of the point of the immersed sphere and assume that they are given by

$$x^h=F_{\beta\alpha}{}^h\varphi^{\beta}\varphi^{\alpha} \tag{7.4}$$

where $F_{\beta\alpha}{}^h=F_{\alpha\beta}{}^h$ and

$$\Sigma_{\alpha}F_{\alpha\alpha}{}^h=0\,. \tag{7.5}$$

Here and in the sequel we adopt the summation convention for indices such as $\alpha, \beta, \gamma, \cdots$ which run from 1 to $m+1$ as well as $\kappa, \lambda, \mu, \cdots$ which run from 1 to m when and only when the same letter appears twice, once as a superscript and once as a subscript.

x^h are eigenfunctions of Δ with eigenvalue $\lambda_2=2(m+1)$.

We assume moreover that the immersion is isometric, that the immersed submanifold lies in a hypersphere $S^{n-1}(r)$ of E^n and that the immersion is full, namely, the eigenfunctions $x^1, \cdots, x^n$ are linearly independent. Then we have $n\leqq m(m+3)/2$ and

$$\Sigma_h(x^h)^2=r^2\,,$$

namely,

$$\Sigma_h F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h\varphi^{\delta}\varphi^{\gamma}\varphi^{\beta}\varphi^{\alpha}=r^2\,.$$

As φ^{α} satisfy (7.1) this is equivalent to

$$\begin{aligned}\Sigma_h(F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h+F_{\delta\beta}{}^h F_{\gamma\alpha}{}^h+F_{\delta\alpha}{}^h F_{\gamma\beta}{}^h)&\\ =r^2(\delta_{\delta\gamma}\delta_{\beta\alpha}+\delta_{\delta\beta}\delta_{\gamma\alpha}+\delta_{\delta\alpha}\delta_{\gamma\beta})&\,.\end{aligned} \tag{7.6}$$

From (7.5) and (7.6) we obtain

$$2\Sigma_{h,\gamma}F_{\beta\gamma}{}^h F_{\gamma\alpha}{}^h=(m+3)r^2\delta_{\beta\alpha}\,. \tag{7.7}$$

From (7.4) we get

$$\nabla_{\mu}x^h=2F_{\beta\alpha}{}^h\varphi^{\beta}\nabla_{\mu}\varphi^{\alpha}\,, \tag{7.8}$$

where ∇ stands for covariant differentiation with respect to the standard Riemannian metric of S^m. But as we have assumed that the immersion (7.4) is isometric, we get

$$g_{\mu\lambda}=\sum_h \nabla_\mu x^h \nabla_\lambda x^h . \tag{7.9}$$

On the other hand we have

$$g_{\mu\lambda}=\delta_{\mu\lambda}+\varphi^\mu\varphi^\lambda/(\varphi^{m+1})^2 \tag{7.10}$$

if we use $\varphi^1, \cdots, \varphi^m$ as the local coordinates $y^1, \cdots, y^m$ of $S^m(1)$. Equations (7.5), (7.6) and the equations obtained by substituting (7.8) and (7.9) into (7.10) yield a complete system of equations to be satisfied by $F_{\beta\alpha}{}^h$. It will be given later.

From (7.8) we get

$$\nabla_\mu\nabla_\lambda x^h=-2g_{\mu\lambda}x^h+2F_{\beta\alpha}{}^h\nabla_\mu\varphi^\beta\nabla_\lambda\varphi^\alpha \tag{7.11}$$

and

$$\begin{aligned}\sum_h \nabla_\mu\nabla^\rho x^h\nabla_\lambda\nabla_\rho x^h=4[&r^2 g_{\mu\lambda}-2\sum_h x^h F_{\beta\alpha}{}^h\nabla_\mu\varphi^\beta\nabla_\lambda\varphi^\alpha \\ &+\sum_h F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h(\delta^{\gamma\alpha}-\varphi^\gamma\varphi^\alpha)\nabla_\mu\varphi^\delta\nabla_\lambda\varphi^\beta] .\end{aligned} \tag{7.12}$$

As we have (7.7) and

$$\sum_h x^h\nabla_\mu\nabla_\lambda x^h=-\sum_h \nabla_\mu x^h\nabla_\lambda x^h=-g_{\mu\lambda} ,$$

$$\begin{aligned}\sum_\alpha \nabla_\mu\varphi^\alpha\nabla_\lambda\varphi^\alpha&=\nabla_\mu(\sum_\alpha\varphi^\alpha\nabla_\lambda\varphi^\alpha)-\sum_\alpha\varphi^\alpha\nabla_\mu\nabla_\lambda\varphi^\alpha \\ &=g_{\mu\lambda}\end{aligned}$$

we get from (7.12)

$$\begin{aligned}\sum_h \nabla_\mu\nabla^\rho x^h\nabla_\lambda\nabla_\rho x^h&=4r^2 g_{\mu\lambda}-4\sum_h x^h(\nabla_\mu\nabla_\lambda x^h+2g_{\mu\lambda}x^h) \\ &\quad+2(m+3)r^2\delta_{\beta\alpha}\nabla_\mu\varphi^\beta\nabla_\lambda\varphi^\alpha-\sum_h\nabla_\mu x^h\nabla_\lambda x^h \\ &=(2(m+1)r^2+3)g_{\mu\lambda} ,\end{aligned}$$

hence

$$G_{\mu\lambda}=(2(m+1)r^2+3)g_{\mu\lambda} . \tag{7.13}$$

This proves that the Gauss map is homothetic.

From (7.11) we get

$$\begin{aligned}\nabla_\nu\nabla_\mu\nabla_\lambda x^h&=-2g_{\mu\lambda}\nabla_\nu x^h-2F_{\beta\alpha}{}^h\varphi^\beta\nabla_\lambda\varphi^\alpha g_{\nu\mu}-2F_{\beta\alpha}{}^h\nabla_\mu\varphi^\beta\varphi^\alpha g_{\nu\lambda} \\ &=-2g_{\mu\lambda}\nabla_\nu x^h-g_{\nu\mu}\nabla_\lambda x^h-g_{\nu\lambda}\nabla_\mu x^h .\end{aligned}$$

This proves that the Gauss image is totally geodesic.

Thus we get the following lemma.

LEMMA 7.1. *Let (S^m, g) be the standard sphere $S^m(1)$. If an isometric im-*

mersion of this sphere into a hypersphere of radius r *of the Euclidean* n*-space is given by* n *eigenfunctions* x^h *of* Δ *of the form* (7.4), *then the Gauss map is homothetic satisfying* (7.13) *and the Gauss image is totally geodesic.*

The complete system of equations to be satisfied by $F_{\beta\alpha}{}^h$ is

$$\Sigma_h F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h = -\frac{1}{2(m+1)}\delta_{\delta\gamma}\delta_{\beta\alpha} + \frac{1}{4}(\delta_{\delta\beta}\delta_{\gamma\alpha} + \delta_{\delta\alpha}\delta_{\gamma\beta}) . \tag{7.14}$$

This is proved in Appendix. At the same time we get

$$r^2 = m/2(m+1) \tag{7.15}$$

and (7.13) becomes

$$G_{\mu\lambda} = (m+3)g_{\mu\lambda} . \tag{7.16}$$

We also prove in Appendix that

$$n = m(m+3)/2 . \tag{7.17}$$

Thus we get the following theorem.

THEOREM 7.2. *If*

$$x^h = F_{\beta\alpha}{}^h \varphi^\beta \varphi^\alpha , \qquad \Sigma_\beta F_{\beta\beta}{}^h = 0 ,$$

where φ^α *are the fundamental eigenfunctions of the Laplacian* Δ *on the standard sphere* (S^m, g), *gives an isometric minimal immersion of* (S^m, g) *into a hypersphere* $S^{n-1}(r)$ *of the Euclidean* n*-space* E^n, *then* $F_{\beta\alpha}{}^h$ *satisfy* (7.14) *and* $r^2 = m/2(m+1)$. x^h *are eigenfunctions of* Δ *on* (S^m, g). *The least possible value of* n *is* $n_2 = m(m+3)/2$ *and if* $n > n_2$ *the immersion is not full. If* $n = n_2$, *then* E^n *is a space of irreducible representation of* $SO(m+1)$. *The Gauss map of the immersed manifold in* E^n *is homothetic with* $c^2 = m+3$. *The Gauss image is totally geodesic.*

REMARK. Minimal immersions of spheres into hyperspheres of Euclidean spaces have been completely obtained by M. P. do Carmo and N. R. Wallach [**1**]. The result obtained in the present paper is some differential geometric supplement. That the immersion obtained in § 7 is isotropic is easily seen from (7.6) and (7.11). For isotropic immersions see T. Itoh and K. Ogiue [**6**] and B. O'Neill [**10**].

Now we prove the following

THEOREM 7.3. *Let* V *be a Veronese manifold immersed in a Euclidean space* E^n *and let the immersion be full. Then* A_ξ *defined as usual from the second fundamental form* h *and a unit normal vector field* ξ *on a domain of* M *vanishes nowhere.*

We prove that, if a vector X^h of E^n satisfies

(7.18) $$\sum_h H_{\mu\lambda}{}^h X^h = 0$$

at a point p of V, then X^h satisfies at p

$$X^h = t^\nu B_\nu{}^h$$

for some vector t^ν of V_p.

We use results obtained in the foregoing part of § 7, but we consider only at p. If we put

$$X_{\beta\alpha} = \sum_h F_{\beta\alpha}{}^h X^h, \qquad X = \sum_h x^h X^h$$

we have

$$X = X_{\beta\alpha}\varphi^\beta\varphi^\alpha.$$

From (7.11) and (7.18) we get

$$X_{\beta\alpha}\nabla_\mu\varphi^\beta\nabla_\lambda\varphi^\alpha - g_{\mu\lambda}X = 0.$$

Transvecting $\nabla^\mu\varphi^\delta\nabla^\lambda\varphi^\gamma$ we get in view of (7.2)

(7.19) $$X_{\delta\gamma} - X_{\beta\gamma}\varphi^\beta\varphi^\delta - X_{\beta\delta}\varphi^\beta\varphi^\gamma - X\delta_{\delta\gamma} + 2X\varphi^\delta\varphi^\gamma = 0.$$

Putting $\delta=\gamma$ and summing up we get $\sum_\gamma X_{\gamma\gamma} = (m+1)X$, but as we have (7.5) we get $X=0$. Hence (7.19) takes the form

$$X_{\beta\alpha} = X_\beta\varphi^\alpha + X_\alpha\varphi^\beta, \qquad X_\beta = X_{\beta\alpha}\varphi^\alpha.$$

If X^h is normal to V_p, we have

$$\sum_h X^h F_{\beta\alpha}{}^h \varphi^\beta\nabla_\lambda\varphi^\alpha = 0,$$

hence

$$X_\alpha\nabla_\lambda\varphi^\alpha = 0.$$

Transvecting $\nabla^\lambda\varphi^\beta$ we get

$$X_\alpha(\delta^{\beta\alpha} - \varphi^\beta\varphi^\alpha) = 0.$$

As we have proved $X_\alpha\varphi^\alpha=0$, we get $X_\alpha=0$, which proves $X_{\beta\alpha}=0$ and consequently

$$\sum_h F_{\beta\alpha}{}^h X^h = 0.$$

So far we considered only at p. But as we have (7.4) $\sum_h x^h X^h$ vanishes everywhere on V which cannot occur since the immersion is full. Now, if X^h is a vector of E^n, we can put $X^h = N^h + T^h$ where N^h is normal to V_p and T^h is parallel to V_p. As we have $\sum_h H_{\mu\lambda}{}^h T^h = 0$ at p, $\sum_h H_{\mu\lambda}{}^h X^h = 0$ is equivalent to $\sum_h H_{\mu\lambda}{}^h N^h = 0$ which can be satisfied only when $N^h = 0$. This proves Theorem 7.3.

REMARK. This theorem can also be proved by using the fact that the immersion is isotropic [**10**].

§ 8. Examples.

For each of the following submanifolds the Gauss map is homothetic. For 1° and 2° the Gauss image is totally geodesic.

1°. A torus T^2 in E^4 given by

$$x^1=r\cos u,\qquad x^2=r\sin u,\qquad x^3=r\cos v,\qquad x^4=r\sin v\,.$$

x^h are eigenfunctions of the Laplacian on the flat torus and the immersed torus lies in a hypersphere of radius $\sqrt{2}\,r$, as a minimal submanifold.

2°. The Veronese surface in E^5 given by

$$x^1=vw,\qquad x^2=wu,\qquad x^3=uv,\qquad x^4=(u^2-v^2)/2\,,$$

$$x^5=(u^2+v^2-2w^2)/2\sqrt{3}\,,\qquad u^2+v^2+w^2=1\,.$$

This is an example of immersions given in § 7 such that $m=2$, $n=5$, $r^2=1/3$.

3°. A torus T^3 in E^8 given by

$$x^1=\cos u\cos v\cos w\,,\qquad x^2=\cos u\cos v\sin w\,,$$

$$x^3=\cos u\sin v\cos w\,,\qquad x^4=\cos u\sin v\sin w\,,$$

$$x^5=\sin u\cos v\cos w\,,\qquad x^6=\sin u\cos v\sin w\,,$$

$$x^7=\sin u\sin v\cos w\,,\qquad x^8=\sin u\sin v\sin w\,.$$

x^h are eigenfunctions of the Laplacian on the flat torus and the immersion is into a hypersphere of radius 1. As $\partial^3x^h/\partial u\partial v\partial w$ is a normal vector of the immersed torus, the Gauss image is not totally geodesic.

4°. An immersion of $S^2(1)$ into E^7 where the position vector x satisfies $\Delta x=\lambda_3 x$, $\lambda_3=12$, or more precisely,

$$x^1=\sqrt{5/2}\,uvw\,,\qquad x^2=\sqrt{5/8}\,(-u^2+v^2)w\,,$$

$$x^3=\sqrt{5/8}\,(-v^2+w^2)u\,,\qquad x^4=\sqrt{5/8}\,(-w^2+u^2)v\,,$$

$$x^5=(1/2\sqrt{6}\,)(5u^3-3u)\,,\qquad x^6=(1/2\sqrt{6}\,)(5v^3-3v)\,,$$

$$x^7=(1/2\sqrt{6}\,)(5w^3-3w)$$

where $u^2+v^2+w^2=1$.

For this immersion we have

$$(x^1)^2+\cdots+(x^7)^2=1/6\,.$$

Taking u, v as the local coordinates y^1, y^2, we get

$$
\begin{array}{ll}
B_1{}^1=\sqrt{5/2}\,(-u^2+w^2)v/w\,, & B_2{}^1=\sqrt{5/2}\,(-v^2+w^2)u/w\,,\\
B_1{}^2=\sqrt{5/8}\,(u^2-v^2-2w^2)u/w\,, & B_2{}^2=\sqrt{5/8}\,(u^2-v^2+2w^2)v/w\,,\\
B_1{}^3=\sqrt{5/8}\,(-2u^2-v^2+w^2)\,, & B_2{}^3=-\sqrt{10}\,uv\,,\\
B_1{}^4=\sqrt{10}\,uv\,, & B_2{}^4=\sqrt{5/8}\,(u^2+2v^2-w^2)\,,\\
B_1{}^5=(3/2\sqrt{6})(5u^2-1)\,, & B_2{}^5=0\,,\\
B_1{}^6=0\,, & B_2{}^6=(3/2\sqrt{6})(5v^2-1)\,,\\
B_1{}^7=-(3/2\sqrt{6})(5w^2-1)u/w\,, & B_2{}^7=-(3/2\sqrt{6})(5w^2-1)v/w
\end{array}
$$

which proves $g_{\mu\lambda}=\delta_{\mu\lambda}+y^\mu y^\lambda/w^2$. After a straightforward calculation we get $G_{\mu\lambda}=11\ g_{\mu\lambda}$, hence the Gauss map is homothetic. As we get $\nabla_1 H_{11}{}^h=\partial^3 x^h/(\partial u)^3-\partial x^h/\partial u$ at the point $u=v=0$, $w=1$, we can see after some calculation that the Gauss image is not totally geodesic.

Appendix.

In order to prove (7.14) and (7.15) we choose φ^λ ($\lambda=1, \cdots, m$) as local coordinates $y^1, \cdots, y^m$ of $S^m(1)$ and write

$$\varphi^0=[1-\textstyle\sum_\lambda(\varphi^\lambda)^2]^{1/2}\,.$$

Then we get

$$g_{\mu\lambda}=\delta_{\mu\lambda}+\varphi^\mu\varphi^\lambda/(\varphi^0)^2\,. \tag{A.1}$$

We also get from (7.6)

$$\textstyle\sum_h F_{00}{}^h F_{00}{}^h=r^2\,, \tag{A.2}$$

$$\textstyle\sum_h F_{00}{}^h F_{0\lambda}{}^h=0\,, \tag{A.3}$$

$$\textstyle\sum_h F_{00}{}^h F_{\mu\lambda}{}^h+2\sum_h F_{0\mu}{}^h F_{0\lambda}{}^h=r^2\delta_{\mu\lambda}\,, \tag{A.4}$$

$$\textstyle\sum_h F_{0\nu}{}^h F_{\mu\lambda}{}^h+\sum_h F_{0\mu}{}^h F_{\nu\lambda}{}^h+\sum_h F_{0\lambda}{}^h F_{\nu\mu}{}^h=0\,, \tag{A.5}$$

$$\begin{aligned}&\textstyle\sum_h F_{\omega\nu}{}^h F_{\mu\lambda}{}^h+\sum_h F_{\omega\mu}{}^h F_{\nu\lambda}{}^h+\sum_h F_{\omega\lambda}{}^h F_{\nu\mu}{}^h\\ &\quad=r^2(\delta_{\omega\nu}\delta_{\mu\lambda}+\delta_{\omega\mu}\delta_{\nu\lambda}+\delta_{\omega\lambda}\delta_{\nu\mu})\,.\end{aligned} \tag{A.6}$$

On the other hand we get from (7.8) and (7.11)

$$(r^2-1/2)g_{\mu\lambda}=\textstyle\sum_h F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h\varphi^\delta\varphi^\gamma\nabla_\mu\varphi^\beta\nabla_\lambda\varphi^\alpha$$

and it follows that $F_{\beta\alpha}{}^h$ satisfy

$$(\varphi^0)^2\textstyle\sum_h F_{\delta\gamma}{}^h F_{\beta\alpha}{}^h\varphi^\delta\varphi^\gamma\nabla_\sigma\varphi^\beta\nabla_\rho\varphi^\alpha=(r^2-1/2)[(\varphi^0)^2\delta_{\sigma\rho}+\varphi^\sigma\varphi^\rho]\,. \tag{A.7}$$

Substituting $\nabla_\nu\varphi^\kappa=\delta^\kappa_\nu$, $\nabla_\nu\varphi^0=-\varphi^\nu/\varphi^0$ into (A.7) and taking (A.3) into account, we get

$$
\begin{aligned}
(\text{A.8}) \qquad & (\varphi^0)^4\textstyle\sum_h F_{00}{}^h F_{\sigma\rho}{}^h+2(\varphi^0)^3\sum_h F_{0\mu}{}^h F_{\sigma\rho}{}^h\varphi^\mu \\
& +(\varphi^0)^2\textstyle\sum_h[F_{00}{}^h F_{00}{}^h\varphi^\sigma\varphi^\rho \\
& \qquad -2F_{0\mu}{}^h\varphi^\mu(F_{0\rho}{}^h\varphi^\sigma+F_{0\sigma}{}^h\varphi^\rho)+F_{\nu\mu}{}^h\varphi^\nu\varphi^\mu F_{\sigma\rho}{}^h] \\
& -\varphi^0\textstyle\sum_h F_{\nu\mu}{}^h\varphi^\nu\varphi^\mu(F_{0\rho}{}^h\varphi^\sigma+F_{0\sigma}{}^h\varphi^\rho) \\
& +\textstyle\sum_h F_{\nu\mu}{}^h\varphi^\nu\varphi^\mu F_{00}{}^h\varphi^\sigma\varphi^\rho \\
& =(r^2-1/2)(\varphi^0)^2\delta_{\sigma\rho}+(r^2-1/2)\varphi^\sigma\varphi^\rho\,.
\end{aligned}
$$

As (A.8) is an identity in $\varphi^1, \cdots, \varphi^m$ when $(\varphi^0)^2=1-\sum_\lambda(\varphi^\lambda)^2$ is substituted into it, we get from the second and the fourth terms of the first member

$$2(1-\textstyle\sum_\lambda(\varphi^\lambda)^2)\sum_h F_{0\mu}{}^h\varphi^\mu F_{\sigma\rho}{}^h-\sum_h F_{\nu\mu}{}^h\varphi^\nu\varphi^\mu(F_{0\rho}{}^h\varphi^\sigma+F_{0\sigma}{}^h\varphi^\rho)=0\,,$$

hence

$$(\text{A.9}) \qquad \textstyle\sum_h F_{0\mu}{}^h F_{\sigma\rho}{}^h=0\,.$$

From the remaining terms we get

$$(\text{A.10}) \qquad \textstyle\sum_h F_{00}{}^h F_{\mu\lambda}{}^h=(r^2-1/2)\delta_{\mu\lambda}\,,$$

$$(\text{A.11}) \qquad \textstyle\sum_h F_{0\mu}{}^h F_{0\lambda}{}^h=\delta_{\mu\lambda}/4\,,$$

$$(\text{A.12}) \qquad \textstyle\sum_h F_{\omega\nu}{}^h F_{\mu\lambda}{}^h=(r^2-1/2)\delta_{\omega\nu}\delta_{\mu\lambda}+(\delta_{\omega\mu}\delta_{\nu\lambda}+\delta_{\omega\lambda}\delta_{\nu\mu})/4$$

when we take (A.2), (A.4) and (A.6) into account.

From (7.5) we get in view of (A.2) and (A.10)

$$
\begin{aligned}
0 &= \textstyle\sum_h(F_{00}{}^h\sum_\beta F_{\beta\beta}{}^h) \\
&= \textstyle\sum_h F_{00}{}^h F_{00}{}^h+\sum_h F_{00}{}^h\sum_\lambda F_{\lambda\lambda}{}^h \\
&= r^2+m(r^2-1/2)
\end{aligned}
$$

hence (7.15). From (7.15), (A.2), (A.3), (A.9), (A.10), (A.11) and (A.12) we get (7.14).

To prove (7.17) suppose $n<(m+1)(m+2)/2-1$. Then there exists a symmetric matrix $(k^{\beta\alpha})$ which is not a scalar matrix, has vanishing trace and satisfies $F_{\beta\alpha}{}^h k^{\beta\alpha}=0$. But, then we get $k^{\delta\gamma}=0$ from (7.14) and this proves $n\geqq(m+1)(m+2)/2-1$. As we have $n\leqq m(m+3)/2$, we get (7.17).

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