J. Math. Soc. Japan
Vol. 31, No. 2, 1979

Comparison theorems for Banach spaces of solutions of $\Delta u=Pu$ on Riemann surfaces

By Takeyoshi SATŌ

(Received Aug. 26, 1977)

§ 1. Introduction.

Let R be an open Riemann surface and P a density on R, that is, a non-negative Hölder continuous function on R which depends on the local parameter $z=x+iy$ in such a way that the partial differential equation

$$\Delta u=Pu\,, \quad \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2\,, \tag{1.1}$$

is invariantly defined on R. A real valued function f is said to be a *P-harmonic* function in an open set U of R, if f has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on U. The totality of bounded P-harmonic functions on R is denoted by $PB(R)$. Then, $PB(R)$ is a Banach space with the uniform norm

$$\|f\|=\sup_{z\in R}|f(z)|\,. \tag{1.2}$$

H. L. Royden [1] studied the comparison problem of Banach space structures of $PB(R)$ for different choices of densities P on a hyperbolic Riemann surface R and proved the following comparison theorem: If P and Q are non-negative densities on R such that there is a constant $c\geqq 1$ with

$$c^{-1}Q\leqq P\leqq cQ \tag{1.3}$$

outside some compact subset of R, then the Banach spaces $PB(R)$ and $QB(R)$ are isomorphic. On the other hand, concerning this comparison problem M. Nakai [1] gave a different criterion for $PB(R)$ and $QB(R)$ to be isomorphic and proved the following theorem: If two densities P and Q on R satisfy the condition

$$\int_R |P(z)-Q(z)|\{G^P(z, w_1)+G^Q(z, w_0)\}\,dxdy<+\infty \tag{1.4}$$

for some points w_0 and w_1 in R, where $G^P(z, w)$ and $G^Q(z, w)$ are Green's functions of R associated with (1.1) and the equation $\Delta u=Qu$ respectively, then Banach spaces $PB(R)$ and $QB(R)$ are isomorphic.

A. Lahtinen [1] considered the equation (1.1) for densities P which he called acceptable densities. Acceptable densities can also have negative values, and so, P-harmonic functions do not obey the usual maximum principle. Lahtinen gave generalizations of Nakai's comparison theorem for acceptable densities and also showed, in Lahtinen [2], that for non-negative densities Royden's condition (1.3) is a special case of Nakai's condition (1.4). Recently, M. Nakai [4] and M. Glasner [1] gave, simultaneously, a necessary and sufficient condition for the existence of an isomorphism T between $PB(R)$ and $QB(R)$ such that $|f-T(f)|$ is bounded by a potential on R.

$PX(R)$ is the space consisting of P-harmonic functions f on R with a certain boundedness property X. As for X we can take D to mean the finiteness of the Dirichlet integral

$$D(f)=\int_R\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}dxdy<+\infty\,,$$

E the finiteness of the energy integral

$$E(f)=D(f)+\int_R f^2(z)P(z)dxdy<+\infty\,,$$

B the finiteness of the supremum norm (1.2), and their non-trivial combinations BD and BE. In the connection with Royden's comparison theorem, Nakai [3] discussed whether the condition (1.3) is also sufficient for $PX(R)$ and $QX(R)$ to be isomorphic for $X=D$, E, BD and BE, and he actually showed that the answer to this question is affirmative.

In this paper we consider the equation (1.1) with $P\not\equiv 0$ on R, and give a new boundedness property H_p' $(1\leqq p<+\infty)$ to P-harmonic functions so that the space $PH_p'(R)$, which consists of P-harmonic functions with this boundedness property, may have the comparison theorem. Hardy spaces on Riemann surfaces have been studied by M. Parreau [1], and in the general context of harmonic spaces by L. L. Naim [1]. The Hardy space for the equation (1.1), which is denoted by $PH_p(R)$ in this paper, falls within the framework of Naim [1]. By Naim, a P-harmonic function f belongs to the Hardy space $PH_p(R)$ for the equation (1.1), if and only if $|f|^p$ has a P-harmonic majorant on R. We denote by ${}_pf$ the smallest P-harmonic majorant of $|f|^p$ on R, and take H_p' to mean the finiteness of the expression

$$\|\!|f|\!\|_p^P=\sup_{w\in R}\left\{\frac{1}{2\pi}\int_R {}_pf(z)G^P(z,w)P(z)dxdy\right\}^{1/p}, \tag{1.5}$$

where $G^P(z,w)$ is the Green function of the equation (1.1) on R. Then, we have that, for $1\leqq p<+\infty$,

$$PB(R)\subset PH_p'(R)\subset PH_p(R)\,.$$

In § 2, we show that, for $1\leqq p<+\infty$, $PH_p'(R)$ is a Banach space under the norm (1.5), and, in § 3, that $PH_p'(R)$ is determined by the behavior of the density P near the ideal boundary of R. In § 4, it is proved that the condition (1.3) is also sufficient for $PH_p'(R)$ and $QH_p'(R)$ to be isomorphic.

For the properties of P-harmonic functions we refer to Myrberg's fundamental works (Myrberg [1], [2]), and for the theory of Green potentials with kernel $G^P(z, w)$ to Nakai [2].

§ 2. Definition of the Banach spaces $PH_p'(R)$.

Let R be a connected Riemann surface and let N be the set $\{0, 1, 2, \cdots\}$. By $\{R_n\}_{n\in N}$ we denote an exhaustion of R, which has the following properties: (1) R_n is a regular region, that is, an open set whose closure $\bar{R}_n$ is compact and whose relative boundary ∂R_n consists of a finite number of closed analytic curves, (2) $\bar{R}_n \subset R_{n+1}$ for $n\in N$, (3) $R=\bigcup_{n=0}^{\infty} R_n$. By the solvability of Dirichlet problem on the regular region R_n with continuous boundary values, for any continuous function f on ∂R_n there exists a unique continuous function P_f^n on $\bar{R}_n$ such that $P_f^n=f$ on ∂R_n and P_f^n is a P-harmonic function on R_n. Let z_0 be a fixed point on R_n. Since the mapping $f\rightarrow P_f^n(z_0)$ of the space of all finitely continuous functions f on ∂R_n is a non-negative linear functional on this space of functions on ∂R_n, there exists a non-negative Radon measure μ_{n,z_0}^P on ∂R_n such that

$$\int f d\mu_{n,z_0}^P = P_f^n(z_0)$$

for all finitely continuous functions f on ∂R_n. This measure is the P-harmonic measure on ∂R_n relative to $z_0 \in R_n$ and R_n.

DEFINITION 2.1. A P-harmonic function f on R belongs to the space $PH_p(R)$, $1\leqq p<+\infty$, if and only if there exists a constant $m(z_0)$ independent of $n\in N$ such that

$$\|f\|_{p,n}^P(z_0)\leqq m(z_0)$$

for all $n\in N$, where $z_0\in R$ and

$$\|f\|_{n,z_0}^P(z_0)=\left\{\int |f|^p d\mu_{n,z_0}^P\right\}^{1/p}.$$

This space $PH_p(R)$ has been studied in the general context of harmonic spaces by Lumer-Naim [1]. Hence the results contained therein may be applicable to our studies of the space $PH_p(R)$. For convenience, some results of Naim [1] are quoted in the following. A P-harmonic function f belongs to the space $PH_p(R)$, $1\leqq p<+\infty$, if and only if $|f|^p$ has a P-harmonic majorant on R. By this proposition the definition of $PH_p(R)$ is independent of the choice of $z_0\in R$ and the particular exhaustion $\{R_n\}$ of R. Any P-harmonic function $f\in PH_p(R)$ is the difference of two positive P-harmonic functions in $PH_p(R)$,

$1\leqq p<+\infty$, and conversely. For $1\leqq p<+\infty$, $PH_p(R)$ is a Banach space under the norm

$$\|f\|_p^p=\sup_{n\in N}\|f\|_{p,n}^p(z_0)\,.$$

This norm equals $\{_pf(z_0)\}^{1/p}$, where $_pf$ denotes the smallest P-harmonic majorant of $|f|^p$ in R.

In the theory of $PH_p(R)$ we admit the case $P\equiv 0$, but we assume $P\not\equiv 0$ on R in the following. The *P-Green function* for R_n is an extended real valued function $G^P(R_n, z, w)$ on $R_n\times R_n$ such that for each $w\in R_n$, (1) $G^P(R_n, z, w)$ is P-harmonic on $R_n-\{w\}$; (2) $G^P(R_n, z, w)+\log|w-z|$ is bounded in a neighborhood of w ; (3) $\lim_{z\to b}G^P(R_n, z, w)=0$ for every $b\in\partial R_n$. The increasing sequence $\{G^P(R_n, z, w)\}$ converges uniformly on every compact subset of R to a function $G^P(z, w)$ which we call the *P-Green function* on R. $G^P(z, w)$ is the smallest function of $u(z, w)$ such that (1) $u(z, w)$ is a non-negative P-harmonic function on $R-\{w\}$; (2) $u(z, w)+\log|z-w|$ is bounded in a neighborhood of w. For these and other properties of the P-Green function we refer to Myrberg [**1**] and [**2**]. An inequality which is a result of Myrberg [**2**] is quoted here as it is useful in the following:

$$\int_R G^P(z, w)P(z)dxdy\leqq 2\pi \tag{2.1}$$

for every $w\in R$.

Now, we make some preliminaries on P-superharmonic functions. For any disk V on R we have the P-harmonic measure $\mu_z^{P,V}$ on the boundary ∂V of V with respect to $z\in V$ satisfying

$$P_f^V(z)=\int fd\mu_z^{P,V}$$

for any continuous function f on ∂V, where P_f^V is a continuous function on the closure $\bar{V}$ of V such that $P_f^V=f$ on ∂V and P_f^V is P-harmonic on V. A P-superharmonic function s on an open set of R is then defined as a function with the following properties:

a) $s(z)>-\infty$ at each $z\in S$; $s\not\equiv+\infty$ on any component of S;

b) s is lower semi-continuous on S;

c) For any disk V such that $\bar{V}\subset S$,

$$s(z)\geqq\int sd\mu_z^{P,V}$$

for all $z\in V$.

If s and $-s$ are P-superharmonic on an open set S of R, then s is P-harmonic on S.

If $-s$ is P-superharmonic on S, then s is said to be P-subharmonic on S. For example, if f is P-harmonic on an open set S of R, then $|f|^p$, $1\leqq p<+\infty$, is P-subharmonic on S, and $\max(f, 0)$, $-\min(f, 0)$ are P-subharmonic on S. The following well-known fact is called the maximum principle and used repeatedly in proofs in this paper. Let u be a P-subharmonic function on G, and f a P-harmonic function on G with continuous boundary values. If G is a relative compact set of R and

$$\lim\sup_{z\to b}u(z)\leqq\lim_{z\to b}f(z)$$

for all $b\in\partial G$, then $u<f$ on G or $u\equiv f$ on G. This principle is a consequence of the general theory on harmonic space. In the case of a continuous P-subharmonic function it is given in Myrberg [3].

DEFINITION 2.2. A P-harmonic function f on a connected Riemann surface R belongs to the space $PH_p'(R)$, $1\leqq p<+\infty$, if and only if there exists a constant M independent of $n\in N$ such that

$$\int_{R_n}\{\|f\|_{p,n}^P(z)\}^p G^P(R_n, z, w)P(z)dxdy\leqq M, \quad w\in R_n ,$$

for all $n\in N$.

We shall see that this space $PH_p'(R)$ is independent of the exhaustion $\{R_n\}$ of R.

From now on in this section we shall give properties of our space $PH_p'(R)$ of P-harmonic functions on a connected Riemann surface R.

THEOREM 2.1. *A P-harmonic function f on R belongs to the space $PH_p'(R)$, $1\leqq p<+\infty$, if and only if $|f|^p$ has a P-harmonic majorant u on R such that*

$$\int_R u(z)G^P(z, w)P(z)dxdy\leqq M \tag{2.2}$$

for every $w\in R$, where M is a positive constant.

PROOF. If such a majorant u does exist on R, then for each $n\in N$

$$\begin{aligned}\|f\|_{p,n}^P(z)&=\left\{\int |f|^p d\mu_{n,z}^P\right\}^{1/p}\\ &\leqq\{P_u^n(z)\}^{1/p}\\ &=\{u(z)\}^{1/p},\ z\in R_n ,\end{aligned}$$

that is, $f\in PH_p(R)$. Furthermore,

$$\begin{aligned}&\int_{R_n}\{\|f\|_{p,n}^P(z)\}^p G^P(R_n, z, w)P(z)dxdy\\ &\leqq\int_{R_n}u(z)G^P(R_n, z, w)P(z)dxdy\\ &\leqq\int_R u(z)G^P(z, w)P(z)dxdy\\ &\leqq M, \qquad w\in R_n ,\end{aligned}$$

for all $n \in N$, from which it follows that f is in the space $PH_p'(R)$.

Next, let $f \in PH_p'(R)$. Since the sequence $\{(\|f\|_{p,n}^P)^p\}_{n \in N}$ of P-harmonic functions is increasing, Definition 2.2 and Harnack's principle imply that

$$\lim_{n\to+\infty}\{\|f\|_{p,n}^P(z)\}^p, \qquad z \in R, \tag{2.3}$$

is P-harmonic by Beppo-Levi's theorem, which is denoted by u. The maximum principle gives that

$$\begin{aligned} |f(z)|^p &\leqq P^n_{|f|^p}(z) \\ &= (\|f\|_{p,n}^P(z))^p, \end{aligned}$$

from which it follows that u is a P-harmonic majorant of $|f|^p$ on R. Since there exists a constant M independent of $n \in N$ such that

$$\int_{R_n}\{\|f\|_{p,n}^P(z)\}^p G^P(R_n, z, w)P(z)dxdy \leqq M, \qquad w \in R_n,$$

for all $n \in N$, it follows from Beppo-Levi's theorem, that

$$\begin{aligned} &\int_R u(z)G^P(z, w)P(z)dxdy \\ &= \lim_{n\to+\infty}\int_{R_n}\{\|f\|_{p,n}^P(z)\}^p G^P(R_n, z, w)P(z)dxdy \\ &\leqq M, \qquad w \in R. \end{aligned}$$

Q. E. D.

THEOREM 2.2. *Every $f \in PH_1'(R)$ is the difference of two positive P-harmonic functions in $PH_1'(R)$, and conversely.*

PROOF. Let $f \in PH_1'(R)$. By Theorem 2.1, there is a P-harmonic majorant u of $|f|$ on R such that

$$\begin{aligned} \int_R u(z)G^P(z, w)P(z)dxdy &\leqq M \\ &< +\infty \end{aligned}$$

for all $w \in R$. The sequences

$$\left\{\int \max(f, 0)d\mu_{n,z}^P\right\}$$

and

$$\left\{\int -\min(f, 0)d\mu_{n,z}^P\right\}$$

are monotone increasing by the maximum principle and bounded as n increases. Then, we can define

$$f_1(z) = \lim_{n\to+\infty}\int \max(f, 0)d\mu_{n,z}^P$$

and

$$f_2(z)=\lim_{n\to+\infty}\int -\min(f, 0)d\mu_{n,z}^P, \qquad z\in R.$$

Here, we have, for $i=1, 2$,

$$\begin{aligned}\int_R f_i(z)G^P(z, w)P(z)dxdy \\ \leqq\int_R u(z)G^P(z, w)P(z)dxdy \\ \leqq M<+\infty, \qquad w\in R,\end{aligned}$$

and

$$\begin{aligned}f(z)&=\lim_{n\to+\infty}\int f d\mu_{z,n}^P \\ &=f_1(z)-f_2(z), \qquad z\in R.\end{aligned}$$

Next, we assume that

$$f(z)=f_1(z)-f_2(z),$$

where f_1 and f_2 are positive P-harmonic functions in $PH_1'(R)$. Let u_i be the P-harmonic majorant of f_i on R, $i=1, 2$, such that, for $w\in R$,

$$\begin{aligned}\int_R u_i(z)G^P(z, w)P(z)dxdy&\leqq M_i \\ &<+\infty, \qquad i=1, 2.\end{aligned}$$

Then,

$$\begin{aligned}|f(z)|&\leqq f_1(z)+f_2(z) \\ &\leqq u_1(z)+u_2(z), \qquad z\in R,\end{aligned}$$

and

$$\begin{aligned}\int_R \{u_1(z)+u_2(z)\}G^P(z, w)P(z)dxdy \\ \leqq M_1+M_2,\end{aligned}$$

for all $w\in R$, which implies, by Theorem 2.1, that $f\in PH_1'(R)$. Q. E. D.

We denote by $PB(R)$ the space consisting of P-harmonic functions on R with finite supremum norms:

$$\|f\|_R=\sup_{z\in R}|f(z)|.$$

THEOREM 2.3. *For any finite* $1\leqq p\leqq q$, *we have the inclusions*

$$PB(R)\subset PH_q'(R)\subset PH_p'(R)\subset PH_1'(R).$$

PROOF. Let $f\in PB(R)$. Since

$$\|f\|_{q,n}^P(z)=\left\{\int |f|^q d\mu_{n,z}^P\right\}^{1/q}$$

$$\leqq \|f\|_R \left\{\int d\mu_{n,z}^P\right\}^{1/q}$$

$$\leqq \|f\|_R, \qquad z \in R_n,$$

we have that $f \in PH_q(R)$. Moreover, the inequality (2.1) implies that, for all $n \in N$,

$$\int_{R_n} \{\|f\|_{q,n}^P(z)\}^q G^P(R_n, z, w) P(z) dxdy$$

$$\leqq (\|f\|_R)^q \int_{R_n} G^P(R_n, z, w) P(z) dxdy$$

$$\leqq 2\pi (\|f\|_R)^q, \qquad w \in R_n.$$

And so, we have $f \in PH_q'(R)$, that is, $PB(R) \subset PH_q'(R)$.

Next, we assume that $1 \leqq p \leqq q$. From the inequality

$$|a|^p \leqq 1 + |a|^q$$

for a real number a, it follows that

$$\{\|f\|_{p,n}^P(z)\}^p = \int |f|^p d\mu_{n,z}^P$$

$$\leqq 1 + \{\|f\|_{q,n}^P(z)\}^q,$$

and that

$$\int_{R_n} \{\|f\|_{p,n}^P(z)\}^p G^P(R_n, z, w) P(z) dxdy$$

$$\leqq \int_{R_n} G^P(R_n, z, w) P(z) dxdy$$

$$+ \int_{R_n} \{\|f\|_{q,n}^P(z)\}^q G^P(R_n, z, w) P(z) dxdy$$

$$\leqq 2\pi + \int_{R_n} \{\|f\|_{q,n}^P(z)\}^q G^P(R_n, z, w) P(z) dxdy, \qquad w \in R.$$

Therefore, we have

$$PH_q'(R) \subset PH_p'(R). \qquad \text{Q.E.D.}$$

THEOREM 2.4. *Any f in $PH_p'(R)$ is the difference of two positive P-harmonic functions in $PH_p'(R)$, and conversely.*

PROOF. We consider the same functions f_1 and f_2 on R as that in the proof of Theorem 2.2, that is,

$$f_1(z) = \lim_{n \to +\infty} \int \max(f, 0) d\mu_{n,z}^P,$$

$$f_2(z) = \lim_{n \to +\infty} \int -\min(f, 0) d\mu_{n,z}^P$$

for $z\in R$. Since $f\in PH_p'(R)$, there exists a P-harmonic majorant u of $|f|^p$ satisfying (2.2) in Theorem 2.1. Then, Hölder's inequality gives that, for p and q satisfying $1<p<+\infty$, $1<q<+\infty$ and $1/p+1/q=1$,

$$\begin{aligned}&\int \max(f,0)d\mu_{n,z}^P\\ &\leqq\left[\int\{\max(f,0)\}^p d\mu_{n,z}^P\right]^{1/p}\left(\int d\mu_{n,z}^P\right)^{1/q}\\ &\leqq\left(\int \max(f,0)^p d\mu_{n,z}^P\right)^{1/p}\\ &\leqq\left(\int u\,d\mu_{n,z}^P\right)^{1/p}\\ &\leqq\{u(z)\}^{1/p},\end{aligned}$$

that is, $f_1(z)^p\leqq u(z)$ on R. And, similarly, we have $f_2(z)^p\leqq u(z)$ on R. Then, we complete the proof of the first assertion.

Let $f=f_1-f_2$, where f_1 and f_2 are positive P-harmonic functions in $PH_p'(R)$. By Theorem 2.1 there exists P-harmonic majorants u_1 and u_2 of $f_1{}^p$ and $f_2{}^p$ on R, respectively, which satisfy the condition (2.2) in Theorem 2.1. Then, the inequality

$$(a+b)^p\leqq 2^p(a^p+b^p),\ 1\leqq p<+\infty,$$

gives

$$\begin{aligned}|f|^p&\leqq(f_1+f_2)^p\\ &\leqq 2^p(f_1{}^p+f_2{}^p)\\ &\leqq 2^p(u_1+u_2),\end{aligned}$$

and

$$\int_R(u_1(z)+u_2(z))G^P(z,w)P(z)dxdy\leqq M+M$$

for all $w\in R$, where M is a constant independent of $w\in R$. Therefore, Theorem 2.1 implies $f\in PH_p'(R)$. Q.E.D.

THEOREM 2.5. *Let R be a connected Riemann surface on which $P\not\equiv 0$. And, let*

$$(2.4)\qquad |||f|||_p^P=\sup_{w\in R}\left\{\lim_{n\to+\infty}\frac{1}{2\pi}\int_{R_n}\{\|f\|_{p,n}^P(z)\}^p G^P(R_n,z,w)P(z)dxdy\right\}^{1/p}$$

for $f\in RH_p'(R)$. Then, for $1\leqq p<+\infty$, $PH_p'(R)$ is a Banach space under the norm $|||f|||_p^P$, $f\in PH_p'(R)$. This norm equals

$$(2.5)\qquad \sup_{w\in R}\left\{\frac{1}{2\pi}\int_R {}_pf(z)G^P(z,w)P(z)dxdy\right\}^{1/p},$$

where ${}_pf$ denotes the smallest P-harmonic majorant of $|f|^p$ in R.

PROOF. The function u defined by (2.3) in the proof of Theorem 2.1, that is,

$$u(z)=\lim_{n\to+\infty}\{\|f\|_{p,n}^{P}(z)\}^{p}, \qquad z\in R,$$

is the smallest P-harmonic majorant of $|f|^{p}$ in R, since, for any P-harmonic majorant s of $|f|^{p}$ in R, we have

$$\begin{aligned}\{\|f\|_{p,n}^{P}(z)\}^{p}&=P^{n}_{|f|^{p}}(z)\\ &\leqq P_{s}^{n}(z)=s(z), \qquad z\in R_{n},\end{aligned}$$

which gives $u(z)\leqq s(z)$ on R. By Definition 2.2 and ${}_{p}f=u$, Lebesgue's monotone convergence theorem shows that

$$\begin{aligned}&\frac{1}{2\pi}\int_{R}{}_{p}f(z)G^{P}(z,w)P(z)dxdy\\ &=\lim_{n\to+\infty}\frac{1}{2\pi}\int_{R_{n}}\{\|f\|_{p,n}^{P}(z)\}^{p}G^{P}(R_{n},z,w)P(z)dxdy,\end{aligned}$$

from which the expression (2.5) of $|||f|||_{p}^{P}$ follows.

Next, we have to show that $PH_{p}'(R)$, $1\leqq p<+\infty$, is a vector space with respect to the usual definitions of addition and scalar multiplication of real numbers, and that the non-negative real valued function (2.4) is a norm on $PH_{p}'(R)$. Minkowski's inequality gives that, for f and g in $PH_{p}'(R)$,

$$\begin{aligned}&\left[\int_{R_{n}}\{\|f+g\|_{p,n}^{P}(z)\}^{p}G^{P}(R_{n},z,w)P(z)dxdy\right]^{1/p}\\ &\leqq\left[\int_{R_{n}}\{\|f\|_{p,n}^{P}(z)\}^{p}G^{P}(R_{n},z,w)P(z)dxdy\right]^{1/p}\\ &\quad+\left[\int_{R_{n}}\{\|g\|_{p,n}^{P}(z)\}^{p}G^{P}(R_{n},z,w)P(z)dxdy\right]^{1/p},\end{aligned}$$

which implies that $f+g\in PH_{p}'(R)$ and

$$|||f+g|||_{p}^{P}\leqq|||f|||_{p}^{P}+|||g|||_{p}^{P}.$$

It is clear that, for $f\in PH_{p}'(R)$ and a real number α, $\alpha f\in PH_{p}'(R)$ and

$$|||\alpha f|||_{p}^{P}=|\alpha|\ |||f|||_{p}^{P}.$$

If $f\in PH_{p}'(R)$ satisfies the condition $|||f|||_{p}^{P}=0$, then the smallest P-harmonic majorant ${}_{p}f$ of f satisfies that ${}_{p}f=0$ everywhere on R, since $P\not\equiv 0$ on R. So, $f=0$ everywhere on R.

To prove that $PH_{p}'(R)$ is complete with respect to the norm (2.4), let $\{f_{j}\}$ be a Cauchy sequence in $PH_{p}'(R)$ with respect to the norm (2.4). Then, we can find a subsequence $\{f_{j(i)}\}$, $j(1)<j(2)<\cdots$, of $\{f_{j}\}$ such that

$$|||f_{j(i+1)}-f_{j(i)}|||_{p}^{P}<1/2^{i}, \quad i=1,2,\cdots.$$

Hölder's inequality and the inequality (2.1) give that, for $p>1$,

$$\frac{1}{2\pi}\int_R\{_p(f_{j(i+1)}-f_{j(i)})(z)\}^{1/p}G^P(z, w)P(z)dxdy$$

$$\leqq\left\{\frac{1}{2\pi}\int_R {}_p(f_{j(i+1)}-f_{j(i)})(z)G^P(z, w)P(z)dxdy\right\}^{1/p}$$

$$=|||f_{j(i+1)}-f_{j(i)}|||_p^P,$$

which is evident for $p=1$. Therefore, since

$$\frac{1}{2\pi}\int_R \sum_{i=1}^{k} \{_p(f_{j(i+1)}-f_{j(i)})(z)\}^{1/p}G^P(z, w)P(z)dxdy$$

$$\leqq \sum_{i=1}^{k} 1/2^i<1$$

for every positive integer k, Lebesgue's monotone convergence theorem implies that the series

$$\sum_{i=1}^{\infty} \{_p(f_{j(i+1)}-f_{j(i)})\}^{1/p} \tag{2.6}$$

converges almost everywhere on the support of P.

Let z_0 be a point of the support of the density P at which (2.6) converges. Then, from the inequality

$$\|f_{j(l)}-f_{j(k)}\|_p^P=\|\sum_{i=k}^{l-1}(f_{j(i+1)}-f_{j(i)})\|_p^P$$

$$\leqq \sum_{i=k}^{l-1}\|f_{j(i+1)}-f_{j(i)}\|_p^P$$

$$= \sum_{i=k}^{l-1}\{_p(f_{j(i+1)}-f_{j(i)})(z_0)\}^{1/p}$$

for $k<l$, it follows that the sequence $\{f_{j(i)}\}$ is a Cauchy sequence in $PH_p(R)$, for the series (2.6) converges at z_0. So, there exists a function f in $PH_p(R)$ such that

$$\lim_{i\to+\infty}\|f_{j(i)}-f\|_p^P=0,$$

which implies that the sequence $\{f_{j(i)}\}$ converges, uniformly on every compact subset of R, to f (L. L. Naim [1]).

We now have to prove that f is contained in $PH_p'(R)$ and

$$\lim_{j\to+\infty}|||f_j-f|||_p^P=0.$$

Since

$$f_{j(k)}=\sum_{i=1}^{k-1}(f_{j(i+1)}-f_{j(i)})+f_{j(1)},$$

Fatou's lemma gives that

$$\left[\frac{1}{2\pi}\int_{R_n}\{\|f-f_{j(l)}\|_p^p(z)\}^p G^p(R_n, z, w)P(z)dxdy\right]^{1/p}$$

$$\leqq\left[\liminf_{k\to+\infty}\frac{1}{2\pi}\int_{R_n}\{\|f_{j(k)}-f_{j(l)}\|_p^p(z)\}^p\right.$$

$$\left.\times G^p(R_n, z, w)P(z)dxdy\right]^{1/p}$$

$$\leqq\liminf_{k\to+\infty}|||f_{j(k)}-f_{j(l)}|||_p^p$$

$$\leqq\sum_{i=l}^{\infty}|||f_{j(i+1)}-f_{j(i)}|||_p^p$$

$$\leqq\sum_{i=l}^{\infty}1/2^i=1/2^{l-1},$$

and so,

$$|||f-f_{j(l)}|||_p^p<1/2^{l-1}. \tag{2.7}$$

We can conclude from this inequality that $f-f_{j(l)}$ is in $PH_p'(R)$. Hence, f is in $PH_p'(R)$, since

$$f=(f-f_{j(l)})+f_{j(l)}.$$

And, furthermore it follows, from (2.7), that

$$\lim_{l\to+\infty}|||f-f_{j(l)}|||_p^p=0,$$

which gives that

$$\lim_{j\to+\infty}|||f-f_j|||_p^p=0,$$

for $\{f_j\}$ is a Cauchy sequence in $PH_p'(R)$. Q.E.D.

It will be necessary to consider a disconnected Riemann surface in § 3 and § 4. Let

$$R=\bigcup_{k=1}^{K}W^k$$

be the decomposition of R into connected components W^k of R. We can assume, without loss of generality, that the density P on R satisfies $P\not\equiv 0$ on $W^1, W^2, \cdots, W^L$, $1\leqq L\leqq K$, and $P\equiv 0$ on W^{L+1}, $W^{L+2}, \cdots, W^K$. Since $P\equiv 0$ on W^k, $L<k\leqq K$, $PH_p(W^k)$, $L<k\leqq K$, is the space of harmonic functions on W^k such that $|f|^p$ has a harmonic majorant on W^k, that is, $PH_p(W^k)$, $L<k\leqq K$, is the Hardy space of harmonic functions on W^k. This space of harmonic functions on W^k is denoted by $H_p(W^k)$. It is a result of Parreau [1] that the space $H_p(W^k)$ is a Banach space under the norm

$$\|f\|_p=\{_pf(z_0)\}^{1/p},\quad f\in H_p(W^k),$$

where z_0 is a point in W^k. Now, we define the space $PH_p'(R)$ for the disconnected Riemann surface R as follows.

DEFINITION 2.3. A P-harmonic function f on the disconnected Riemann surface R belongs to the space $PH_p'(R)$, $1 \leqq p < +\infty$, if and only if each restriction $f|W^k$ to W^k of f belongs to $PH_p'(W^k)$ or $H_p(W^k)$ according as $1 \leqq k \leqq L$ or $L < k \leqq K$.

THEOREM 2.6. *Let R be the disconnected Riemann surface on which $p \not\equiv 0$. And, let*

$$\|\!|f|\!\|_p^p = \sum_{k=1}^{L} \|\!|f|W^k|\!\|_p^p + \sum_{k=L+1}^{K} \|f|W^k\|_p \tag{2.8}$$

for $f \in PH_p'(R)$. Then, for $1 \leqq p < +\infty$, $PH_p'(R)$ is a Banach space under the norm (2.8). *This norm equals*

$$\sum_{k=1}^{L} \sup_{w \in W^k} \left\{ \frac{1}{2\pi} \int_{W^k} {}_p(f|W^k)(z) G^P(W^k, z, w) P(z) dxdy \right\}^{1/p}$$

$$+ \sum_{k=L+1}^{K} \{{}_p(f|W^k)(z^k)\}^{1/p},$$

where ${}_p(f|W^k)$, $1 \leqq k \leqq K$, denotes the smallest P-harmonic majorant of $|f|W^k|^p$ on W^k and z^k, $L < k \leqq K$, is a point in W^k.

PROOF. This is clear by the preceding lemma. Q. E. D.

In the following of this section we consider the relation between two Banach spaces $PH_p(R)$ and $PH_p'(R)$ under the assumption that the density P vanishes outside a compact subset of the connected Riemann surface R.

LEMMA 2.7. *If the density P vanishes outside a compact subset of R, then $PH_p'(R) = PH_p(R)$ and there exists a positive constant C such that*

$$\|\!|f|\!\|_p^p \leqq C \|f\|_p^p$$

for every $f \in PH_p(R)$.

PROOF. We assume that P vanishes outside a compact subset K of R. Let z_0 be a point of R with $z_0 \notin K$. Then, there exists, by Harnack's theorem (Myrberg [1]), a constant c such that

$${}_pf(z) \leqq c \times {}_pf(z_0)$$

for every $z \in K$ and every $f \in PH_p(R)$. Therefore, the inequality (2.1) gives that

$$\begin{aligned} &\frac{1}{2\pi} \int_R {}_pf(z) G^P(z, w) P(z) dxdy \\ &= \frac{1}{2\pi} \int_K {}_pf(z) G^P(z, w) P(z) dxdy \\ &\leqq \frac{1}{2\pi} c \times {}_pf(z_0) \int_R G^P(z, w) P(z) dxdy \\ &\leqq c \times {}_pf(z_0), \end{aligned}$$

and so,

$$|||f|||_p^P \leqq (c)^{1/p} \times \|f\|_p^P,$$

which completes the proof. Q. E. D.

Theorem 2.6. *If the density P vanishes outside a compact subset of R, then the Banach space $(PH_p'(R), |||\cdot|||_p^P)$ is isomorphic to the Banach space $(PH_p(R), \|\cdot\|_p^P)$.*

Proof. The identity map of $(PH_p(R), \|\cdot\|_p^P)$ onto $(PH_p'(R), |||\cdot|||_p^P)$ is a one-to-one continuous linear transformation and so must be an isomorphism by the open mapping theorem. Q. E. D.

§ 3. The structure of $PH_p'(R)$.

Let W be a connected or disconnected open subset of R whose complement is a regular region. Hereafter we always use W for such a subset of R. To show that the Banach space structure of $PH_p'(R)$ is determined by the behavior of the density P on a neighborhood of the ideal boundary of R, we define the subset $PH_p'(W; \partial W)$ of $PH_p'(R)$ as follows.

Definition 3.1. $PH_p'(W; \partial W)$, $1 \leqq p < +\infty$, is the class of all functions f in $PH_p'(W)$ such that there exists a continuous extension of f to the closure $\overline{W}$ of W whose restriction to the boundary ∂W of W vanishes.

Then, $PH_p'(W; \partial W)$ is a vector space with respect to the usual definitions of addition and scalar multiplication of real numbers. And, $PH_p'(W; \partial W)$ is a subspace of the Banach space $PH_p'(W)$ with the norm (2.8) in Theorem 2.6:

Theorem 3.1. *$PH_p'(W; \partial W)$ is a closed linear subspace of $PH_p'(W)$.*

Proof. Let $f \in PH_p'(W)$ be the limit of a sequence $\{f_n\}$ in $PH_p'(W; \partial W)$:

$$\lim_{n \to +\infty} |||f - f_n|||_p^P = 0.$$

It is sufficient to show that $f | W^k$ has a continuous extension to $\overline{W}^k$ whose restriction to ∂W^k vanishes for each connected component W^k of W. If $P \not\equiv 0$ on W^k, then there exists a subsequence $\{f_{n(i)}\}$ of $\{f_n\}$ which converges, uniformly on every compact subset of W^k, to f, by the proof of Theorem 2.5. If $P \equiv 0$ on W^k, the existence of such a subsequence $\{f_{n(i)}\}$ follows from the fact

$$\lim_{n \to +\infty} \|f | W^k - f_n | W^k\|_p^P = 0.$$

Let G^k be a regular region which contains the boundary of W^k, and let w be a continuous function on the closure of $G^k \cap W^k$ such that w is P-harmonic on $G^k \cap W^k$ and w have $w | \partial G^k = m^k$, $w | \partial W^k = 0$, where

$$m^k = \sup_{z \in G^k \cap W^k} |f | W^k(z)| + 1.$$

Then, by the maximum principle we have that

$$|f_{n(i)}(z)| \leqq w(z), \quad z \in G^k \cap W^k$$

for sufficiently large $i\in N$, and so,

$$|f(z)|=\lim_{i\to+\infty}|f_{n(i)}(z)| \leqq w(z), \quad z\in G^{k}\cap W^{k}.$$

This shows that $\lim_{z\to b}f(z)=0$ for all $b\in\partial W^{k}$, that is, if we extend f on ∂W^{k} so that $f(b)=0$ for $b\in\partial W^{k}$, then f belongs to $PH_{p}'(W;\partial W)$, which complets the proof. Q. E. D.

LEMMA 3.2. *Let f be in $PH_{p}'(W;\partial W)$. Then, the smallest P-harmonic majorant ${}_{p}f$ of $|f|^{p}$ has a continuous extension to $\overline{W}$ whose restriction to ∂W vanishes.*

PROOF. It is sufficient to prove only that ${}_{p}f|W^{k}$ have this property. The sequence $\{(\|f|W^{k}\|_{p,n}^{P}(z))^{p}\}$, which is a monotone increasing sequence of P-harmonic functions on $R_{n}\cap W^{k}$, converges to ${}_{p}f|W^{k}$. Harnack's principle implies that the convergence is locally uniform in W^{k}. Let G^{k} be the same subset of R as that in Theorem 3.1, and let w be the P-harmonic function on $G^{k}\cap W^{k}$ which have a continuous extension to the closure of $G^{k}\cap W^{k}$ such that $w|\partial W^{k}=0$ and $w|\partial G^{k}=1$. Then, by the same way as that in the proof of Theorem 3.1, we can show that

$$\{\|f|W^{k}\|_{p,n}^{P}(z)\}^{p}\leqq\beta^{k}w(z), \quad z\in W^{k}\cap G^{k},$$

for sufficiently large $n\in N$, where

$$\beta^{k}=\sup_{z\in\partial G^{k}\cap W^{k}} {}_{p}f(z).$$

Therefore,

$${}_{p}f|W^{k}(z)\leqq\beta^{k}w(z), \quad z\in W^{k}\cap G^{k},$$

which implies the conclusion. Q. E. D.

In Rodin and Sario [1] they discussed the problem of finding on a given harmonic space a harmonic function which imitates the behavior of a given harmonic function on a neighborhood of the ideal boundary of the harmonic space. We quote from Chapter VII of Rodin and Sario [1] the method of finding a P-harmonic function which imitates the behavior of a given P-harmonic function on a neighborhood of the ideal boundary of the connected Riemann surface R. This problem of finding such a P-harmonic function on R can be stated as the following: Given a continuous function f on the closure $\overline{W}$ of W which is P-harmonic on W, find a P-harmonic function F on R with

$$\sup_{z\in W}|F(z)-f(z)|<+\infty,$$

where W is a neighborhood of the ideal boundary of R: in particular, an open subset of R whose complement is a regular region of R.

Let $\{R_{n}\}$ be an exhaustion of R with $\partial R_{n}\subset(W-\partial W)$. Then, we can find a unique continuous function $B_{n}(f)$ on the closure of $R_{n}\cap(W-\partial W)$ which is

P-harmonic on $R_n \cap (W-\partial W)$ and which takes the boundary values f and 0 on the boundaries ∂W and ∂R_n, respectively. Since $\lim_{n\to+\infty} B_n(f)$ exists, an operator $f \to B(f)$ from the space of all continuous functions on ∂W into the space of continuous functions on $\overline{W}$ which is P-harmonic on $W-\partial W$ is defined by

$$B(f)=\lim_{n\to+\infty} B_n(f).$$

The operator B has the following properties:

(B1) $B(f+g)=B(f)+B(g)$, $B(cf)=cB(f)$,

(B2) $B(f)|\partial W=f$,

(B3) $\min(0, \min_{\partial W} f) \leqq B(f) \leqq \max(0, \max_{\partial W} f)$,

where f and g are continuous functions on ∂W and c is a real number.

Since the density P of our equation (1.1) does not vanish constantly, the harmonic space defined by the equation (1.1) is hyperbolic, that is, $B(1) \not\equiv 1$ for some choice of $W \subset R$, or there is an open set in R on which the constant function 1 is not P-harmonic. Therefore, as a special case of principal function problem solved by Nakai, we have the following existence theorem; Let f be a continuous function on $\overline{W}$ which is P-harmonic on W. Then there always exists a unique (f, B)-principal function, that is, a P-harmonic function F on R with

$$B(F|\partial W-f|\partial W)=F|W-f \quad \text{on } W.$$

By reformulation this theorem we obtain the complete solution of the above problem.

To show that the Banach spaces $PH_p'(R)$ and $PH_p'(W;\partial W)$ are isomorphic we define an operator λ_P^W as follows. Let $P(R)$ be the space of all P-harmonic function on R. And, consider the linear space $P(W;\partial W)$ of continuous functions on $\overline{W}$ which are P-harmonic on W and whose restriction to ∂W vanish constantly.

DEFINITION 3.2. We define an operator λ_P^W by

$$\lambda_P^W(f)=\lim_{n\to+\infty} P_f^n$$

for $f \in P(W;\partial W)$ which is the difference of two non-negative functions in $P(W;\partial W)$, where P_f^n is the solution of Dirichlet problem of the equation (1.1) with the boundary value f on ∂R_n.

To see that the operator λ_P^W is well-defined for such a f in $P(W;\partial W)$, let

$$f=f_1-f_2,\ f_i \in P(W;\partial W),\ f_i \geqq 0, \quad i=1,2.$$

We can find, by the existence theorem of the principal function problem, P-harmonic functions F_1, F_2 defined on R satisfying

$$\sup_{z\in W} |F_i(z)-f_i(z)| < +\infty, \quad i=1,2.$$

These supremums are denoted by m_1 and m_2, respectively. Since

$$F_i+m_i \geqq P^n_{f_i} \quad \text{on } R_n \ (i=1,2)$$

for every $n \in N$ and the sequences $\{P^n_{f_1}\}$ and $\{P^n_{f_2}\}$ are monotone increasing sequences of P-harmonic functions, the $\lim_{n\to+\infty} P^n_{f_i}$ $(i=1,2)$ is a P-harmonic function by Harnak's theorem. Therefore, we have

$$\lim_{n\to+\infty} P^n_f = \lim_{n\to+\infty} P^n_{f_1} - \lim_{n\to+\infty} P^n_{f_2},$$

that is, $\lambda^W_P(f)$ is well-defined for any difference $f=f_1-f_2$ of two non-negative functions in $P(W;\partial W)$ and is a P-harmonic function on R.

This operator λ^W_P is referred to as the canonical extension, and was defined by Nakai [3] on the smaller domain than that of our definition. The domain in his definition was the class $PB(W;\partial W)$ of bounded continuous functions on $\overline{W}$ P-harmonic on W and vanishing on ∂W.

Since the P-Green function $G^P(z, W)$ is strictly positive, symmetric and continuous on $R\times R$ and is finite unless $z=w$, $G^P(z,w)$ is taken as a kernel in the sense of potential theory. If μ is a measure on R and

$$G^P(z,\mu)=\int_R G^P(z,w)\,d\mu(w)$$

is P-superharmonic on R, then $G^P(z,\mu)$ is called the P-Green potential of μ. The P-Green potentials are quite similar to the harmonic Green potentials. Since the potential theoretic method is a powerful tool for the study of the operator λ^W_P and is extensively used in this section, we list some important potential theoretic principles in the following. The theory of P-Green potentials is developed in Nakai [2].

FROSTMAN'S MAXIMUM PRINCIPL. If the inequality $G^P(z,\mu)\leqq 1$ holds on the compact support S_μ of μ, then the same inequality holds on the whole space R.

EEQUILIBRIUM PRINCIPLE. For an arbitrary compact subset K of R there always exists a unique measure called equilibrium measure of K satisfying $S_\mu \subset K$ and $G^P(z,\mu)=1$ on K except for a subset of ∂K of capacity zero and $G^P(z,\mu)\leqq 1$ on R.

To show that the range $\lambda^W_P(PH'_p(W;\partial W))$ of λ^W_P is contained in $PH'_p(R)$, we shall prepare three lemmas.

LEMMA 3.3. *Let S and T be open subsets of R and H a non-negative function on $S\times T$. If (a) for each $w\in T$, $H(\cdot, w)$ is continuous on S, (b) for each $z\in S$, $H(z,\cdot)$ is P-harmonic on T and (c)*

$$h(w)=\int_S H(z,w)\,d\mu(z)<+\infty$$

for each $w\in T$, then h is P-harmonic on T.

PROOF. It can be shown that $H(z,w)$ is a non-negative measurable function on $S\times T$ to which Fubini's theorem can be applied. Then, for any disk

V such that $\bar{V}\subset T$

$$\int h\,d\mu_w^{P,V}=\int_S\left\{\int_{\partial V}H(z,\cdot)d\mu_w^{P,V}\right\}d\mu(z),$$

where $\mu_w^{P,V}$ is the P-harmonic measure with respect to V and $w\in V$. This shows that h is P-harmonic on T. Q. E. D.

The following lemma gives a relation between P-Green's potentials for different regions, when one is a subset of the other. For the harmonic case, this fact is stated in Helmes [1]. So we only restate it for our case.

LEMMA 3.4. *Let S and T be regular regions such that $S\supset T$, and let μ be a measure on S such that $\mu(S-T)=0$ and $G^P(S,z,\mu)$ is a finite P-Green's potential. Then, there is a non-negative P-harmonic function h on T which satisfies*

$$G^P(S,z,\mu)=G^P(T,z,\mu|T)+h(z)$$

on T, where $\mu|T$ is the restriction of μ on T and $G^P(S,z,w)$ is the P-Green's function of S.

PROOF. For $z,w\in T$ with $z\neq w$, let

$$H(z,w)=G^P(S,z,w)-G^P(T,z,w),$$

which is positive. Then, for each $z\in T$, $H(z,w)$ is a P-harmonic function on T, since z is a removable singular point, and so, $H(z,\cdot)$ is a continuous function for each $z\in T$. Also, $H(\cdot,w)$ is a P-harmonic function for each $w\in T$, for $H(z,w)$ is symmetric. Since $G^P(S,z,\mu)\geqq G^P(T,z,\mu|T)$ on T by $G^P(S,z,w)\geqq G^P(T,z,w)$ on $T\times T$,

$$G^P(S,z,\mu)-G^P(T,z,\mu|T)=\int_T H(z,w)d\mu(w)<+\infty,$$

where the last integral is a P-harmonic function on T by the preceding lemma. Q. E. D.

Let W be an open subset of R whose complement is a regular region. We assume that $P\not\equiv 0$ on $W^1,W^2,\cdots,W^L,(1\leqq L\leqq K)$ and $P\equiv 0$ on $W^{L+1},W^{L+2},\cdots,W^K$, where

$$W=\bigcup_{i=1}^{K}W^i$$

is the decomposition of W into connected components $W^1,W^2,\cdots,W^K$.

LEMMA 3.5. *If a nonnegative P-harmonic function f in $P(W;\partial W)$ satisfies that, for every i, $1\leqq i\leqq L$,*

$$\sup_{w\in W^i}\int_{W^i}f|W^i(z)G^P(W^i,z,w)P(z)dxdy<+\infty,$$

then

$$\sup_{w\in R}\int_R \lambda_P^W(f)(z)G^P(z, w)P(z)dxdy<+\infty .$$

PROOF. Let $\{R_n\}$ be an exhaustion of R such that $\partial R_0\subset W$. Then, since the sequence $\{P_f^n\}$ converges increasingly to $\lambda_P^W(f)$ on R, the maximum principle gives that

$$P_f^n\leqq\max_{\partial W}\lambda_P^W(f)+f$$

on $R_n\cap W^i$ for each $n\in N$. Therefore, for $1\leqq i\leqq L$,

$$\begin{aligned}&\int_{R_n\cap W^i}P_f^n(z)G^P(R_n\cap W^i, z, w)P(z)dxdy\\ &\leqq\max_{\partial W}\lambda_P^W(f)\times\int_{R_n\cap W^i}G^P(R_n\cap W^i, z, w)P(z)dxdy\\ &\quad+\int_{R_n\cap W^i}f(z)G^P(R_n\cap W^i, z, w)P(z)dxdy\\ &\leqq 2\pi\times\max_{\partial W}\lambda_P^W(f)+\sup_{w\in W^i}\int_{W^i}f(z)G^P(W^i, z, w)P(z)dxdy\\ &<+\infty .\end{aligned}$$

Let

$$M^i=\sup_{w\in W^i}\int_{W^i}\lambda_P^W(f)\mid W^i(z)G^P(W^i, z, w)P(z)dxdy . \tag{3.1}$$

Then, Lebesgue's monotone convergence theorem gives that

$$\begin{aligned}&\int_{W^i}\lambda_P^W(f)\mid W^i(z)G^P(W^i, z, w)P(z)dxdy\\ &=\lim_{n\to+\infty}\int_{R_n\cap W^i}P_f^n(z)G^P(R_n\cap W^i, z, w)P(z)dxdy\\ &\leqq 2\pi\times\max_{\partial W}\lambda_P^W(f)+\sup_{w\in W^i}\int_{W^i}f\mid W^i(z)G^P(W^i, z, w)P(z)dxdy ,\end{aligned}$$

from which it follows that $M^i<+\infty$, $1\leqq i\leqq L$.

To show that the integral

$$\int_R\lambda_P^W(f)(z)G^P(z, w)P(z)dxdy \tag{3.2}$$

is a P-Green's potential, that is, $\not\equiv+\infty$, let α be a number such that

$$\sup_{z\in\partial R_0}G^P(z, w_0)<\alpha ,$$

and let

$$\beta^i=\inf_{z\in\partial R_0\cap W^i}G^P(W^i, z, w_0) ,$$

where w_0 is a fixed point in $(W^i-\partial W^i)\cap R_0$. Since the sequence $\{G^P(R_n, z, w)\}$ converges increasingly to $G^P(z, w)$ on R, we have

$$\sup_{z\in\partial R_0} G^P(R_n, z, w_0) < \alpha$$

for every $n \in N$. Then, the maximum principle gives that

$$G^P(R_n, z, w_0) \leqq \delta^i G^P(W^i, z, w_0)$$

on $(R_n - \bar{R}_0) \cap W^i$, where $\delta^i = \alpha/\beta^i$. So, we have

$$\begin{aligned} G^P(z, w_0) &= \lim_{n\to+\infty} G^P(R_n, z, w_0) \\ &\leqq \delta^i G^P(W^i, z, w_0) \end{aligned} \tag{3.3}$$

on $(R - R_0) \cap W^i$. Since (3.1) and (3.3) give that

$$\begin{aligned} &\int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w_0) P(z) dxdy \\ &\leqq \delta^i \times \sup_{w\in W^i} \int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(W^i, z, w_0) P(z) dxdy \\ &\leqq \delta^i M^i < +\infty , \end{aligned}$$

which shows that

$$\int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy, \quad 1 \leqq i \leqq L,$$

is a P-Green potential. Then,

$$\begin{aligned} &\int_{R-R_0} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy \\ &= \sum_{i=1}^{L} \int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy \end{aligned}$$

is a P-Green potential. And, since

$$\begin{aligned} &\int_{\bar{R}_0} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy \\ &\leqq \sup_{\bar{R}_0} \lambda_P^W(f) \times \int_R G^P(z, w) P(z) dxdy \\ &\leqq 2\pi \times \sup_{\bar{R}_0} \lambda_P^W(f) \\ &< +\infty , \end{aligned}$$

the integral (3.2) is a P-Green potential.

To show that the P-Green potential (3.2) is finite everywhere on R, let w be any point in R, and let V be a disc with center at w. Then, since the P-Green potential

$$\int_{R-V} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$

is P-harmonic on V: continuous on V, the inequality

$$\int_V \lambda_P^W(f)(z)G^P(z,w)P(z)dxdy$$
$$\leqq \sup_{\bar V}\lambda_P^W(f)\times\int_R G^P(z,w)P(z)dxdy$$
$$\leqq 2\pi\times\sup_{\bar V}\lambda_P^W(f)$$
$$<+\infty$$

implies that the P-Green potential (3.2) is finite everywhere on R.

The integral

$$\int_{(R_n-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(R_{n+1},z,w)P(z)dxdy\,,\quad 1\leqq i\leqq L\,,$$

is a finite P-Green potential on R_{n+1}, for this integral is smaller than the integral (3.2). So, Lemma 3.4 implies that there exists a P-harmonic function u_n^i on $W^i\cap R_{n+1}$ such that

$$\int_{(R_n-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(R_{n+1},z,w)P(z)dxdy \tag{3.4}$$
$$=\int_{(R_n-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(R_{n+1}\cap W^i,z,w)P(z)dxdy+u_n^i(w)$$

for $w\in W^i\cap R_{n+1}$. Since $u_n^i|\partial R_{n+1}\cap W^i=0$ and, for any $w_0\in\partial W^i$,

$$u_n^i(w_0)=\int_{(R_n-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(R_{n+1},z,w_0)P(z)dxdy$$
$$\leqq \sup_{w\in\partial W^i}\int_{R-R_0}\lambda_P^W(f)(z)G^P(z,w)P(z)dxdy$$
$$<+\infty\,,$$

denoting by ε^i the above supremum the maximum principle gives

$$u_n^i\leqq\varepsilon^i\quad\text{on } R_{n+1}\cap W^i\,.$$

Since, by (3.3) and (3.4), the Lebesgue's monotone convergence theorem implies that

$$\int_{(R-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(z,w)P(z)dxdy$$
$$=\lim_{n\to+\infty}\int_{(R_n-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(R_{n+1},z,w)P(z)dxdy$$
$$=\int_{(R-R_0)\cap W^i}\lambda_P^W(f)(z)G^P(W^i,z,w)P(z)dxdy+\lim_{n\to+\infty}u_n^i(w)$$
$$\leqq M^i+\varepsilon^i\,,\quad w\in W^i\,,$$

the Frostman's maximum principle shows that the inequality

$$\int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$
$$\leqq M^i + \varepsilon^i$$

holds on R, for the support of the measure of the P-Green potential

$$\int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$

is contained in W^i. Therefore, we have

$$\int_R \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$
$$= \sum_{i=1}^{L} \int_{(R-R_0)\cap W^i} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$
$$+ \int_{\bar{R}_0} \lambda_P^W(f)(z) G^P(z, w) P(z) dxdy$$
$$\leqq \sum_{i=1}^{L} (M^i + \varepsilon^i) + 2\pi \times \sup_{\bar{R}_0} \lambda_P^W(f)$$

for every $w \in R$, which completes the proof. Q. E. D.

THEOREM 3.6. $\lambda_P^W(PH_p'(W; \partial W)) \subset PH_p'(R)$, $1 \leqq p < +\infty$.

PROOF. Let f be in $PH_p'(W; \partial W)$. Theorem 2.5 states that the smallest P-harmonic majorant ${}_p(f|W^i)$ of $|f|W^i|^p$ on W^i satisfies

$$\sup_{w\in W^i} \int_{W^i} {}_p(f|W^i)(z) G^P(W^i, z, w) P(z) dxdy < +\infty\,, \tag{3.5}$$

for i, $1 \leqq i \leqq L$. By Definition 3.2 and Lemma 3.2, the maximum principle shows that

$$\lambda_P^W({}_pf) \geqq {}_pf \quad \text{on } W.$$

Then, since $\{\lambda_P^W({}_pf)\}^{1/p}$ is a P-superharmonic function on R by Hölder's inequality, we have

$$|P_f^n| \leqq \{\lambda_P^W({}_pf)\}^{1/p} \quad \text{on } R_n\,,$$

from which it follows that

$$|\lambda_P^W(f)|^p = |\lim_{n\to+\infty} P_f^n|^p$$
$$\leqq \lambda_P^W({}_pf) \quad \text{on } R.$$

That is, $\lambda_P^W({}_pf)$ is a P-harmonic majorant of $|\lambda_P^W(f)|^p$ on R. And, by (3.5), Lemma 3.5 shows that

$$\sup_{w\in R} \int_R \lambda_P^W({}_pf)(z) G^P(z, w) P(z) dxdy < +\infty\,.$$

Therefore, by Theorem 2.1, $\lambda_P^W(f)$ belongs to the space $PH_p'(R)$. Q. E. D.

Let $\{R_n\}$ be an exhaustion such that $R_0\supset\partial W$. For a given function g on W, let g_n be a function defined on $\partial R_n\cup\partial W$ such that

$$g_n|\partial W=0 \quad\text{and}\quad g_n|\partial R_n=g.$$

If g is a non-negative P-harmonic function on R, the sequence $\{P_{g_n}^{R_n\cap W}\}$ is a monotone decreasing sequence of P-harmonic functions. Then,

$$\lim_{n\to+\infty}P_{g_n}^{R_n\cap W}$$

exists and is a P-harmonic function on R. Now, if g is the difference of two non-negative P-harmonic functions, then we can define an operator μ_P^W, which was referred to as the canonical restriction by Nakai ([3], [4]), as follows:

DEFINITION 3.3. *For $g\in P(R)$ which is the difference of two non-negative P-harmonic functions on R,*

$$\mu_P^W(g)=\lim_{n\to+\infty}P_{g_n}^{R_n\cap W}.$$

THEOREM 3.7. *$\mu_P^W\circ\lambda_P^W$ is the identity mapping on $PH_p'(W;\partial W)$.*

PROOF. Let f be in $PH_p'(W;\partial W)$, and suppose $f\geqq 0$ on W. Since

$$P_{(\lambda_P^W(f))_n}^{R_n\cap W}=f+P_{(\lambda_P^W(f)-f)_n}^{R_n\cap W}$$

and

$$\begin{aligned}0&\leqq P_{(\lambda_P^W(f)-f)_n}^{R_n\cap W}\\&\leqq P_{\lambda_P^W(f)-f}^{R_n}\\&=\lambda_P^W(f)-P_f^{R_n}\quad\text{on } R_n\cap W,\end{aligned}$$

we have, by $\lambda_P^W(f)=\lim_{n\to+\infty}P_f^{R_n}$, that

$$\begin{aligned}\mu_P^W\circ\lambda_P^W(f)&=\mu_P^W(\lambda_P^W(f))\\&=\lim_{n\to+\infty}P_{(\lambda_P^W(f))_n}^{R_n\cap W}\\&=f\end{aligned}\tag{3.6}$$

for every $f\in PH_p'(W;\partial W)$ with $f\geqq 0$ on W. From the linearity of λ_P^W and μ_P^W, (3.6) follows for any $f\in PH_p'(W;\partial W)$. Q.E.D.

LEMMA 3.8.

$$\mu_P^W(PH_p'(R))\subset PH_p'(W;\partial W).$$

PROOF. It is sufficient to prove this lemma only for a non-negative g in $PH_p'(R)$. Then, from

$$g\geqq P_{g_n}^{R_n\cap W}$$

on $R_n\cap W$, it follows that

$${}_pg\geqq|g|^p$$

$$\geqq |\lim_{n\to+\infty} P_{g_n}^{R_n\cap W}|^p$$

$$= |\mu_P^W(g)|^p$$

on W, that is, ${}_pg|W$ is a P-harmonic majorant of $|\mu_P^W(g)|^p$ on W. Furthermore, Theorem 2.5 shows that

$$\sup_{w\in R}\int_R {}_pg(z)G^P(z, w)P(z)dxdy < +\infty ,$$

which implies, by Theorem 2.1, that $\mu_P^W(g)\in PH_p'(W)$ for every g in $PH_p'(R)$. And, it is shown that $\mu_P^W(g)$ has a continuous extension to the closure $\overline{W}$ of W whose restriction to ∂W vanishes. That is, $\mu_P^W(g)\in PH_p'(W;\partial W)$.

Q. E. D.

A P-potential on R is a non-negative P-superharmonic function on R whose greatest P-harmonic minorant is non-positive. As in the case of classical Green potentials, we can show that any P-harmonic minorant of a P-Green potential is non-positive. Then, a P-Green potential is a P-potential. It is useful to modify a terminology and a lemma which was stated in Nakai [3]. A function f on R will be referred to as a *quasi P-potential* if $|f|$ is majoranted by a P-potential.

LEMMA 3.9. *If f is a continuous quasi P-potential such that $-|f|$ is P-superharmonic on R, then $f\equiv 0$ on R.*

PROOF. Assume that $|f|$ is majorated by a P-potential p. Since

$$0\leqq |f|$$

$$\leqq P_{|f|}^{R_n} \leqq P_p^{R_n} ,$$

from

$$\lim_{n\to+\infty} P_p^{R_n} = 0$$

it follows that $f\equiv 0$ on R. Q. E. D.

THEOREM 3.10. *$\lambda_P^W\circ\mu_P^W$ is the identity mapping on $PH_p'(R)$.*

PROOF. For $f\in PH_p'(R)$, let f_n and f_n' be functions on $\partial R_n\cup\partial W$ such that

$$f_n|\partial R_n = f, \qquad f_n|\partial W = 0$$

and

$$f_n'|\partial R_n = 0, \qquad f_n'|\partial W = f.$$

If $f\geqq 0$ on R, by the equilibrium principle, there exists a P-Green potential $G^P(z,\mu)$ such that

$$G^P(z,\mu)\leqq \sup_{R-W} f, \qquad z\in R,$$

$$G^P(z,\mu) = \sup_{R-W} f, \qquad z\in R-W,$$

and the support of μ is contained in $R-W$. Since

$$0\leqq f(z) - P_{f_n}^{R_n\cap W}(z)$$

$$=P_{f_n}^{R_n\cap W}(z)\leqq G^P(z,\mu),\qquad z\in R_n\cap W,$$

for every $n\in N$, it follows that

$$\begin{aligned}0&\leqq f(z)-\mu_P^W(f)(z)\\&=f(z)-\lim_{n\to+\infty}P_{f_n}^{R_n\cap W}(z)\\&\leqq G^P(z,\mu),\quad z\in W,\end{aligned}$$

which shows that the function $f-\mu_P^W(f)$ is a quasi P-potential on W.

Next, let $g=\lambda_P^W\circ\mu_P^W(f)$, which is contained in $PH_p'(R)$. By

$$\mu_P^W(f)-\lambda_P^W\circ\mu_P^W(f)=\mu_P^W(g)-g\,,$$

the above discussion shows that the function

$$\mu_P^W(f)-\lambda_P^W\circ\mu_P^W(f)$$

is also a quasi P-potential for a non-negative function f in $PH_p'(R)$. Therefore, from

$$|f-\lambda_P^W\circ\mu_P^W(f)|\leqq|f-\mu_P^W(f)|+|\mu_P^W(f)-\lambda_P^W\circ\mu_P^W(f)|,$$

the P-harmonic function $f-\lambda_P^W\circ\mu_P^W(f)$ is a quasi P-potential on W, which shows that $f=\lambda_P^W\circ\mu_P^W(f)$ by Lemma 3.9. And, it is evident that this equality holds for any f in $PH_p'(R)$, since λ_P^W and μ_P^W are linear. Q.E.D.

COROLLARY 3.11. *μ_P^W is a one-to-one map of $PH_p'(R)$ onto $PH_p'(W;\partial W)$, and*

$$\lambda_P^W:\ PH_p'(W;\partial W)\to PH_p'(R)$$

is the inverse of μ_P^W.

PROOF. This corollary follows easily from Theorem 3.7 and Theorem 3.10. Q.E.D.

THEOREM 3.12. *The mapping*

$$\mu_P^W:\ PH_p'(R)\to PH_p'(W;\partial W)$$

is an isomorphism, that is, $PH_p'(R)$ and $PH_p'(W;\partial W)$ are isomorphic.

PROOF. It is clear that μ_P^W is linear on $PH_p'(R)$. Since

$$\begin{aligned}|P_{g_n}^{R_n\cap W}|^p&\leqq P_{(|g|^p)_n}^{R_n\cap W}\\&\leqq P_{(pg)_n}^{R_n\cap W}\leqq {}_pg\,|\,R_n\cap W,\quad n\in N,\end{aligned}$$

for $g\in PH_p'(R)$, as $n\to+\infty$ it is shown that ${}_pg\,|\,W$ is a P-harmonic majorant of $|\mu_P^W(g)|^p$ on W for $g\in PH_p'(R)$. So,

$${}_pg\,|\,W\geqq{}_p(\mu_P^W(g))\,,$$

by which Theorem 2.5 and Definition 2.6 imply that

$$|||g|||_p^P \geqq |||\mu_P^W(g)|||_p^P .$$

Therefore, μ_P^W is a continuous mapping of $PH_p'(R)$.

Since μ_P^W is a continuous linear one-to-one mapping of the Banach space $PH_p'(R)$ onto the Banach space $PH_p'(W; \partial W)$, the open mapping theorem gives that μ_P^W is an open mapping, that is,

$$\mu_P^W : PH_p'(R) \to PH_p'(W; \partial W)$$

is an isomorphism. Q. E. D.

COROLLARY 3.13. *If P and Q are two densities on R such that $P=Q$ outside a compact subset of R, then $PH_p'(R)$ and $QH_p'(R)$ are isomorphic.*

PROOF. Assume that $P=Q$ on $W \subset R$. The Banach spaces $PH_p'(R)$, $QH_p'(R)$ are isomorphic with the Banach space $PH_p'(W; \partial W) = QH_p'(W; \partial W)$. Q. E. D.

§ 4. The comparison theorem.

In the first part of this section we assume R to be connected, and let P and Q be two densities on R. We shall prove that the spaces $PH_p'(R)$ and $QH_p'(R)$ $(1 \leqq p < +\infty)$ are isomorphic providing the existence of a constant $c \geqq 1$ such that

$$c^{-1}Q \leqq P \leqq cQ$$

on R.

LEMMA 4.1. *Let P and Q be densities on R which are not identically zero. If there exists a constant $c \geqq 1$ such that*

$$c^{-1}Q \leqq P \leqq cQ \tag{4.1}$$

on R, then we have

$$G^Q(z, w) = G^P(z, w) + \frac{1}{2\pi}\int_R (P(\zeta) - Q(\zeta)) G^Q(\zeta, w) G^P(\zeta, z) d\xi d\eta \tag{4.2}$$

for every $z, w \in R$ with $z \neq w$, where $\zeta = \xi + i\eta$.

PROOF. The Green's formula implies that, for $z, w \in R_n$ with $z \neq w$,

$$\begin{aligned} G^Q(R_n, z, w) &= G^P(R_n, z, w) \\ &\quad + \frac{1}{2\pi}\int_{R_n} (P(\zeta) - Q(\zeta)) G^Q(R_n, \zeta, w) G^P(R_n, \zeta, z) d\xi d\eta , \end{aligned} \tag{4.3}$$

where $\zeta = \xi + i\eta$.

Let

$$F(z, w, \zeta) = |P(\zeta) - Q(\zeta)| G^Q(\zeta, w) G^P(\zeta, z) .$$

To prove (4.2), we show that, if $z \neq w$, the integral

$$\int_R F(z, w, \zeta) d\xi d\eta$$

is finite. Let U and V be disks with centers z and w, respectively, such that $V\cap U=\emptyset$. Then, since (4.1) implies that

$$|P-Q|\leqq cP,\ |P-Q|\leqq cQ$$

on R, and the maximum principle gives that

$$\sup_{\zeta\in\partial U}G^{P}(\zeta, z)\geqq G^{P}(\zeta, z), \qquad \zeta\in\bar{V},$$

and

$$\sup_{\zeta\in\partial V}G^{Q}(\zeta, w)\geqq G^{Q}(\zeta, w), \qquad \zeta\in R-V,$$

we have

$$\begin{aligned}\int_{\bar{V}}F(z, w, \zeta)d\xi d\eta &\leqq\sup_{\zeta\in\partial U}G^{P}(\zeta, z)\times\int_{R}|P(\zeta)-Q(\zeta)|G^{Q}(\zeta, w)d\xi d\eta\\ &\leqq\sup_{\zeta\in\partial U}G^{P}(\zeta, z)\times c\int_{R}G^{Q}(\zeta, w)Q(\zeta)d\xi d\eta\\ &\leqq 2\pi c\times\sup_{\zeta\in\partial U}G^{P}(\zeta, z)<+\infty\end{aligned}$$

and

$$\begin{aligned}\int_{R-V}F(z, w, \zeta)d\xi d\eta &\leqq\sup_{\zeta\in\partial V}G^{Q}(\zeta, w)\times c\int_{R}P(\zeta)G^{P}(\zeta, z)d\xi d\eta\\ &\leqq 2\pi c\times\sup_{\zeta\in\partial V}G^{Q}(\zeta, w)<+\infty .\end{aligned}$$

Therefore,

$$\int_{R}F(z, w, \zeta)d\xi d\eta=\int_{\bar{V}}F(z, w, \zeta)d\xi d\eta+\int_{R-V}F(z, w, \zeta)d\xi d\eta<+\infty$$

for $z\neq w$ in R.

Since the sequences $\{G^{Q}(R_{n}, z, w)\}$ and $\{G^{P}(R_{n}, z, w)\}$ converge increasingly to $G^{Q}(z, w)$ and $G^{P}(z, w)$, respectively, we have

$$\begin{aligned}&\lim_{n\to+\infty}(P(\zeta)-Q(\zeta))G^{Q}(R_{n}, \zeta, w)G^{P}(R_{n}, \zeta, z)\\ &\quad=(P(\zeta)-Q(\zeta))G^{Q}(\zeta, w)G^{P}(\zeta, z)\end{aligned}$$

and

$$|P(\zeta)-Q(\zeta)|G^{Q}(R_{n}, \zeta, w)G^{P}(R_{n}, \zeta, z)\leqq F(z, w, \zeta)$$

for each $n\in N$. The Lebesgue's theorem of dominated convergence implies that, if $z\neq w$,

$$\begin{aligned}&\lim_{n\to+\infty}\int_{R_{n}}(P(\zeta)-Q(\zeta))G^{Q}(R_{n}, \zeta, w)G^{P}(R_{n}, \zeta, z)d\xi d\eta\\ &\quad=\int_{R}(P(\zeta)-Q(\zeta))G^{Q}(\zeta, w)G^{P}(\zeta, z)d\xi d\eta .\end{aligned}$$

Therefore, (4.2) follows from (4.3). Q.E.D.

LEMMA 4.2. *Let P and Q be densities on R which are not identically zero* *on R and which satisfies* (4.1) *on R. Jf a continuous function f on R satisfies the condition*

$$\sup_{w\in R}\int_R |f(z)|G^P(z, w)Q(z)dxdy<+\infty\,, \tag{4.4}$$

then f also satisfies

$$\sup_{w\in R}\int_R |P(z)-Q(z)|G^Q(z, w)|f(z)|dxdy<+\infty\,.$$

And, in this case we have

$$\begin{aligned}\sup_{w\in R}\int_R |P(z)-Q(z)|G^Q(z, w)|f(z)|dxdy \\ \leqq c(c+1)\times\sup_{w\in R}\int_R |f(z)|G^P(z, w)Q(z)dxdy\,.\end{aligned} \tag{4.5}$$

PROOF. Since the inequality (4.1) gives

$$|P-Q|\leqq cP,\ cQ \text{ on } R, \tag{4.6}$$

from Lemma 4.1 it follows that

$$G^Q(z, w)\leqq G^P(z, w)+\frac{c}{2\pi}\int_R Q(\zeta)G^Q(\zeta, w)G^P(\zeta, z)d\xi d\eta\,. \tag{4.7}$$

Then, by the inequalities (2.1) and (4.6),

$$\begin{aligned}&\int_R |P(z)-Q(z)|G^Q(z, w)|f(z)|dxdy \\ &\leqq c\int_R Q(z)G^Q(z, w)|f(z)|dxdy \\ &\leqq c\int_R Q(z)G^P(z, w)|f(z)|dxdy \\ &\quad +\frac{c}{2\pi}\int_R Q(z)|f(z)|\left\{\int_R Q(\zeta)G^Q(\zeta, w)G^P(\zeta, z)d\xi d\eta\right\}dxdy \\ &\leqq c(c+1)\times\sup_{w\in R}\int_R |f(z)|G^P(z, w)Q(z)dxdy\,.\end{aligned}$$

This inequality completes our proof. Q. E. D.

We define an auxiliary transformation T^n_{PQ} of real valued continuous functions f defined on the closure $\bar{R}_n$ of R_n as follows:

$$T^n_{PQ}(f)(w)=f(w)+\frac{1}{2\pi}\int_{R_n}(P(z)-Q(z))G^Q(R_n, z, w)f(z)dxdy\,.$$

LEMMA 4.3. *If f is continuous on $\bar{R}_n$ and P-harmonic on R_n, then $T^n_{PQ}(f)$ is Q-harmonic on R_n and is a continuous function on $\bar{R}_n$ such that*

$$T^n_{PQ}(f)|\partial R_n=f|\partial R_n\,.$$

PROOF. The Green's formula and the properties of Green's function $G^{Q}(R_{n}, z, w)$ imply that $T_{PQ}^{n}(f)$ is the solution of Dirichlet problem with respect to the equation $\Delta u=Qu$ and the domain R_{n} with the boundary value f on ∂R_{n} (see, for example, Nakai [1]).

DEFINITION 4.1. For a real-valued continuous function f defined on the connected Riemann surface R satisfying the condition (4.4) in Lemma 4.2, we define a transformation $T_{PQ}(f)$ as follows:

$$T_{PQ}(f)(w)=f(w)+\frac{1}{2\pi}\int_{R}(P(z)-Q(z))G^{Q}(z, w)f(z)dxdy,$$

which is well defined by Lemma 4.2.

LEMMA 4.4. *Let P and Q be densities on R which are not identically zero, and assume that there is a constant c satisfying* (4.1). *If a continuous function f on R satisfies the condition* (4.4) *in Lemma* 4.2, *then*

$$T_{PQ}(f)=\lim_{n\to+\infty}T_{PQ}^{n}(f).$$

PROOF. Let α be the function

$$z\to c\left\{Q(z)G^{P}(z, w)|f(z)|+\frac{c}{2\pi}\times Q(z)|f(z)|\times\int_{R}Q(\zeta)G^{Q}(\zeta, w)G^{P}(\zeta, z)d\xi d\eta\right\},$$

which satisfies that

$$\int_{R}\alpha(z)dxdy\leqq c(c+1)\times\sup_{w\in R}\int_{R}|f(z)|G^{P}(z, w)Q(z)dxdy<+\infty. \tag{4.8}$$

Since

$$\lim_{n\to+\infty}(P(z)-Q(z))G^{Q}(R_{n}, z, w)f(z)=(P(z)-Q(z))G^{Q}(z, w)f(z)$$

and, by Lemma 4.1 and the inequality (4.6),

$$|P(z)-Q(z)|G^{Q}(R_{n}, z, w)|f(z)|\leqq cQ(z)G^{Q}(z, w)|f(z)|\leqq\alpha(z),$$

Lebesgue's theorem on dominated convergence implies, by (4.8), that

$$\lim_{n\to+\infty}\int_{R_{n}}(P(z)-Q(z))G^{Q}(R_{n}, z, w)f(z)dxdy$$

$$=\int_{R}(P(z)-Q(z))G^{Q}(z, w)f(z)dxdy,$$

from which it follows that

$$\lim_{n\to+\infty}T_{PQ}^{n}(f)(w)=T_{PQ}(f)(w), \qquad w\in R.$$

Q. E. D.

LEMMA 4.5. *Under the assumption of Lemma* 4.4, *$T_{PQ}(f)$ is a Q-harmonic function on R.*

PROOF. Since a sequence $\{f_{n}\}$ of Q-harmonic functions on a domain U of R such that $|f_{n}|\leqq M<+\infty$ has a subsequence which converges uniformly on each compact subset of U to a Q-harmonic function on R (refer to Myrberg [1]), it is sufficient to show that the sequence $\{T_{PQ}^{n}(f)\}$ of Q-harmonic functions is uniformly bounded on a neighborhood V of any $w\in R$. Lemma 4.2 shows that

$$\begin{aligned}|T_{PQ}^{n}(f)(w)| &\leqq \sup_{w\in\bar V}\left\{|f|+\frac{1}{2\pi}\int_{R_n}|P(z)-Q(z)|G^{Q}(R_n, z, w)|f(z)|dxdy\right\}\\ &\leqq \sup_{w\in\bar V}|f|+\sup_{w\in R}\frac{1}{2\pi}\int_{R}|P(z)-Q(z)|G^{Q}(z, w)|f(z)|dxdy\\ &\leqq \sup_{w\in\bar V}|f|+c(c+1)/2\pi\times\sup_{w\in R}\int_{R}|f(z)|G^{P}(z, w)Q(z)dxdy\\ &<+\infty,\quad w\in V.\end{aligned}$$

Q. E. D.

LEMMA 4.6. *Let P and Q be densities on R which are not identically zero, and assume that there exists a constant $c\geqq 1$ satisfying the inequality* (4.1) *on R. If f is in $PH_p'(R)$ $(1\leqq p<+\infty)$, then $T_{PQ}(f)$ is contained in the space $QH_p'(R)$.*

PROOF. From Theorem 2.3, it follows that a function f in $PH_p'(R)$ satisfies the condition in Theorem 4.2, that is, $T_{PQ}(f)$ is defined for f in $PH_p'(R)$. Also, $T_{PQ}({}_pf)$ is defined by Theorem 2.5.

Since it is evident that

$$|T_{PQ}^{n}(f)|^{p}=|f|^{p}\leqq {}_pf=T_{PQ}^{n}({}_pf)$$

on ∂R_n for every $n\in N$, the Q-subharmonic function $|T_{PQ}^{n}(f)|^{p}$ is dominated by the Q-harmonic function $T_{PQ}^{n}({}_pf)$ on R_n for each $n\in N$. Thus, Lemma 4.4 shows that

$$|T_{PQ}(f)|^{p}\leqq T_{PQ}({}_pf)$$

on R, that is, $T_{PQ}({}_pf)$ is a Q-harmonic majorant of $|T_{PQ}(f)|^{p}$ on R.

To prove $T_{PQ}(f)\in QH_p'(R)$, it is sufficient, by Theorem 2.1, to show that

$$\sup_{w\in R}\int_{R}T_{PQ}({}_pf)(z)G^{Q}(z, w)Q(z)dxdy<+\infty .$$

By Definition 4.1, this integral equals to

$$\int_{R}{}_pf(z)G^{Q}(z, w)Q(z)dxdy+\int_{R}\left\{\frac{1}{2\pi}\int_{R}(P(\zeta)-Q(\zeta))G^{Q}(\zeta, z)\times{}_pf(\zeta)d\xi d\eta\right\}G^{Q}(z, w)Q(z)dxdy. \tag{4.9}$$

The first term of (4.9) is dominated by

$$\int_{R}{}_pf(z)G^{P}(z, w)Q(z)dxdy$$
$$+\int_{R}{}_pf(z)\left\{\frac{1}{2\pi}\int_{R}|P(\zeta)-Q(\zeta)|G^{Q}(\zeta, w)\right.$$

$$\times G^{P}(\zeta, z)d\xi d\eta\Big\}Q(z)dxdy$$

$$\leqq\Big\{1+\frac{1}{2\pi}\int_{R}|P(\zeta)-Q(\zeta)|G^{Q}(\zeta, w)d\xi d\eta\Big\}$$

$$\times\sup_{w\in R}\int_{R}{}_{p}f(z)G^{P}(z, w)Q(z)dxdy$$

$$\leqq c(1+c)\times\sup_{w\in R}\int_{R}{}_{p}f(z)G^{P}(z, w)P(z)dxdy,$$

where the inequality $|P-Q|\leqq cQ$ on R and Lemma 4.1 were used. The inequality (4.5) in Lemma 4.2 shows that the second term of (4.9) is dominated by

$$c(c+1)\times\sup_{w\in R}\int_{R}{}_{p}f(\zeta)G^{P}(\zeta, z)Q(\zeta)d\xi d\eta\frac{1}{2\pi}\int_{R}G^{Q}(z, w)Q(z)dxdy$$

$$\leqq c^{2}(c+1)\times\sup_{w\in R}\int_{R}{}_{p}f(z)G^{P}(z, w)P(z)dxdy.$$

Therefore, we have

$$\sup_{w\in R}\int_{R}T_{PQ}({}_{p}f)(z)G^{Q}(z, w)Q(z)dxdy$$

$$\leqq c(c+1)^{2}\times\sup_{w\in R}\int_{R}{}_{p}f(z)G^{P}(z, w)P(z)dxdy$$

$$<+\infty.$$

Q. E. D.

LEMMA 4.7. *Let P and Q be densities which are not identically zero on the connected Riemann surface R. If there exists a constant $c\geqq 1$ satisfying the inequality* (4.1) *on R, then T_{PQ} is a bounded linear transformation from $PH_{p}'(R)$ into $QH_{p}'(R)$, and T_{QP} is a bounded linear transformation from $QH_{p}'(R)$ into $PH_{p}'(R)$.*

PROOF. Since Lemma 4.6 shows that $T_{PQ}(f)$ is well-defined and is contained in the space $QH_{p}'(R)$ for every $f\in PH_{p}'(R)$, it is clear that T_{PQ} is a linear mapping of $PH_{p}'(R)$ into $QH_{p}'(R)$.

Since $T_{PQ}({}_{p}f)$ is a Q-harmonic majorant of $|T_{PQ}(f)|^{p}$ on R (this was shown in the proof of Lemma 4.6), by (4.10) in the proof of Lemma 4.6 and (4.1), we have that

$$\{|||T_{PQ}(f)|||_{p}^{Q}\}^{p}=\sup_{w\in R}\frac{1}{2\pi}\int_{R}{}^{p}(T_{PQ}(f))(z)G^{Q}(z, w)Q(z)dxdy$$

$$\leqq\sup_{w\in R}\frac{1}{2\pi}\int_{R}T_{PQ}({}_{p}f)(z)G^{Q}(z, w)Q(z)dxdy$$

$$\leqq c(c+1)^{2}\sup_{w\in R}\frac{1}{2\pi}\int_{R}{}_{p}f(z)G^{P}(z, w)P(z)dxdy$$

$$=c(c+1)^2\times\{|||f|||_p^P\}^p,$$

that is

$$|||T_{PQ}(f)|||_p^Q\leqq\{c(c+1)^2\}^{1/p}\times|||f|||_p^P \tag{4.11}$$

for every $f\in PH_p'(R)$. This shows that the mapping T_{PQ} is a bounded linear transformation from $PH_p'(R)$ into $QH_p'(R)$. By changing the roles of P and Q we can see that T_{QP} is a bounded linear transformation from $QH_p'(R)$ into $PH_p'(R)$. Q.E.D.

LEMMA 4.8. *If P and Q satisfy the same assumption as that in Theorem 4.7, then $T_{QP}\circ T_{PQ}$ is the identity on $PH_p'(R)$, and $T_{PQ}\circ T_{QP}$ is the identity on $QH_p'(R)$.*

PROOF. Since $PH_p'(R)\subset PH_1'(R)$ $(1\leqq p<+\infty)$, any function f in $PH_p'(R)$ satisfies that

$$\begin{aligned}&c^{-1}\int_R|f(z)|G^P(z,w)Q(z)dxdy\\ &\leqq\int_R|f(z)|G^P(z,w)P(z)dxdy\\ &\leqq\int_R{}_1f(z)G^P(z,w)P(z)dxdy\\ &\leqq 2\pi\times|||f|||_1^P<+\infty,\quad w\in R,\end{aligned}$$

which implies, by Lemma 4.2, that

$$\sup_{w\in R}\int_R|P(z)-Q(z)|G^Q(z,w)|f(z)|dxdy<+\infty.$$

Therefore, the last function of the inequality

$$\begin{aligned}&|(Q(z)-P(z))G^P(R_n,z,w)T_{PQ}^n(f)(z)|\\ &\leqq c\Big\{P(z)G^P(z,w)|f(z)|+\frac{1}{2\pi}P(z)G^P(z,w)\\ &\qquad\times\int_{R_n}|P(\zeta)-Q(\zeta)|G^Q(R_n,\zeta,z)|f(\zeta)|d\xi d\eta\Big\}\\ &\leqq c\Big\{P(z)G^P(z,w)|f(z)|+\frac{1}{2\pi}P(z)G^P(z,w)\\ &\qquad\times\int_R|P(\zeta)-Q(\zeta)|G^Q(\zeta,z)|f(\zeta)|d\xi d\eta\Big\}\end{aligned}$$

is integrable for any fixed $w\in R_n$, where this inequality is obtained by the definition of $T_{PQ}^n(f)$ and $|P-Q|\leqq cP$ on R. Since

$$\lim_{n\to+\infty}(Q(z)-P(z))G^P(R_n, z, w)T^n_{PQ}(f)(z)$$
$$=(Q(z)-P(z))G^P(z, w)T_{PQ}(f)(z),$$

Lebesgue's theorem on bounded convergence gives that

$$\lim_{n\to+\infty}\int_{R_n}(Q(z)-P(z))G^P(R_n, z, w)T^n_{PQ}(f)(z)dxdy$$

$$=\int_R(Q(z)-P(z))G^P(z, w)T_{PQ}(f)(z)dxdy,$$

from which it follows that

$$\lim_{n\to+\infty}T^n_{QP}\circ T^n_{PQ}(f)=T_{QP}\circ T_{PQ}(f)$$

on R for $f\in PH'_p(R)$. On the other hand, the maximum principle shows, by Lemma 4.3, that

$$T^n_{QP}\circ T^n_{PQ}(f)=f \text{ on } R_n,$$

for every $n\in N$, and so,

$$T_{QP}\circ T_{PQ}(f)=f \text{ on } R,$$

for any $f\in PH'_p(R)$.

By changing the roles of P and Q we have also that

$$T_{PQ}\circ T_{QP}(g)=g \text{ on } R,$$

for $g\in QH'_p(R)$. Q. E. D.

THEOREM 4.9. *Under the same assumption as that in Lemma* 4.8, T_{PQ} *is an isomorphism between* $PH'_p(R)$ *and* $QH'_p(R)$. *And,* T_{QP} *is the inverse of* T_{PQ}.

PROOF. This follows from Lemma 4.7 and 4.8. Q. E. D.

Now, let R be a disconnected Riemann surface, and let

$$R=\bigcup_{k=1}^{K} W^k$$

be the decomposition of R into connected components W^k, $k=1, 2, \cdots, K$, of R. If the densities satisfy the relation

$$c^{-1}Q\leqq P\leqq cQ \quad \text{on } R\ (c\geqq 1), \tag{4.12}$$

then we can assume that $W^1, W^2, \cdots, W^L$ $(1\leqq L\leqq K)$ are connected components of R on which $P\not\equiv 0$ and $Q\not\equiv 0$, and that $W^{L+1}, W^{L+2}, \cdots, W^K$ are connected components of R on which $P\equiv 0$ and $Q\equiv 0$.

DEFINITION 4.2. If the relation (4.12) holds on the disconnected Riemann surface R, we define the function $T_{PQ}(f)$ on R for $f\in PH'_p(R)$ as follows:

$$T_{PQ}(f)|W^k=T_{PQ}(f|W^k), \quad 1\leqq k\leqq L,$$

and

$$T_{PQ}(f)|W^k=f|W^k, \quad L<k\leqq K.$$

By changing the roles of P and Q we define also $T_{QP}(g)$ for $g\in QH_p'(R)$.

THEOREM 4.10. *Let R be a Riemann surface which may be disconnected, and assume* (4.12). *Then, T_{PQ} is an isomorphism between $PH_p'(R)$ and $QH_p'(R)$. And, T_{QP} is the inverse of T_{PQ}.*

PROOF. Lemma 4.9 gives this theorem. Q.E.D.

Let R be a connected hyperbolic Riemann surface and let P and Q be two densities on R. In the following, we prove the order comparison theorem: If there exists a constant $c\geqq 1$ such that

$$c^{-1}Q\leqq P\leqq cQ \tag{4.13}$$

on R except possibly for a compact subset K of R, then $PH_p'(R)$ and $QH_p'(R)$ are isomorphic.

Let W be an open subset of R such that $R-W\supset K$ and $R-W$ is a regular region. Then, since (4.13) is valid on the whole W, which may be considered a Riemann surface, Lemma 4.10 states that there is the isomorphism between $PH_p'(W)$ and $QH_p'(W)$, which is denoted by T_{PQ}^W in the following.

LEMMA 4.11. *If the inequality* (4.13) *holds on W, then T_{PQ}^W may be considered an isomorphism of $PH_p'(W;\partial W)$ onto $QH_p'(W;\partial W)$.*

PROOF. Since $PH_p'(W;\partial W)$ and $QH_p'(W;\partial W)$ are closed subspaces of $PH_p'(W)$ and $QH_p'(W)$, respectively, it is necessary only to prove that $T_{PQ}^W(f)\in QH_p'(W;\partial W)$ for $f\in PH_p'(W;\partial W)$.

Let $\{R_n\}$ be an exhaustion of R such that $R_n\supset R-W$, $n=0,1,2,\cdots$, and let

$$\alpha=\sup_{w\in\partial R_0}|T_{PQ}^W(f)(w)|.$$

We denotes by ω the continuous function on $\overline{R_0\cap W}$ such that ω is Q-harmonic on $R_0\cap W$ and $\omega|\partial W=0$, $\omega|\partial R_0=1$.

Since Lemma 4.4 states that

$$\lim_{n\to+\infty}T_{PQ}^{W_n}(f)=T_{PQ}^W(f) \text{ on } W,$$

where $T_{PQ}^{W_n}$ is defined for a continuous function on $\overline{R_n\cap W}$ which is Q-harmonic on $W\cap R_n$, for any $\varepsilon>0$ there exists $n_0\in N$ such that

$$|T_{PQ}^{W_n}(f)(w)|\leqq(\alpha+\varepsilon)\omega(w),\quad w\in W\cap R_0$$

for $n>n_0$. So, as $n\to+\infty$, we have

$$|T_{PQ}^W(f)(w)|\leqq(\alpha+\varepsilon)\omega(w),\quad w\in W\cap R_0,$$

from which

$$T_{PQ}^W(f)|\partial W=0,$$

that is,

$$T_{PQ}^W(f)\in QH_p'(W;\partial W)$$

follows. Q.E.D.

THEOREM 4.12 (THE ORDER COMPARISON THEOREM). *Let P and Q be two densities on a connected Riemann surface. If there exists a constant $c\geqq 1$ such that*

$$c^{-1}Q\leqq P\leqq cQ$$

on R except possibly for a compact subset K of R, then $PH_p'(R)$ and $QH_p'(R)$ are isomorphic.

PROOF. Let W be the same open subset of R as that defined before Lemma 4.11. Then, by Theorem 3.12 and Lemma 4.11, the mapping

$$\lambda_Q^W\circ T_{PQ}^W\circ\mu_P^W:\ PH_p'(R)\to QH_p'(R)$$

is an isomorphism. Q.E.D.

References

M. Glasner

[1] Comparison theorems for bounded solutions of $\Delta u=Pu$, Trans. Amer. Math. Soc., **202** (1975), 173-179.

L.L. Helms

[1] Introduction to potential theory, Wiley-Interscience, New-York, 1969.

A. Lahtinen

[1] On the solutions of $\Delta u=Pu$ for acceptable densities on open Riemann surfaces, Ann. Acad. Sci. Fenn. Ser. AI, No.515 (1972).

[2] On the equation $\Delta u=Pu$ and the classification of acceptable densities on Riemann surfaces, Ann. Acad. Sci. Fenn. Ser. AI, No. 533 (1973).

L. Myrberg

[1] Über die Integration der Differentialgleichung $\Delta u=c(P)u$ auf offenen Riemannschen Flächen, Math. Scad., **2** (1954), 142-152.

[2] Über die Existenz der Greenschen Funktion der Gleichung $\Delta u=c(P)u$ auf Riemannschen Flächen, Ann. Acad. Sci. Fenn. Ser. AI, No. 170 (1954).

[3] Über subelliptische Funktionen, Ann. Acad. Sci. Fenn. Ser. AI, No. 290 (1960).

L. Lumer-Naim

[1] $\mathcal{H}^p$-spaces of harmonic functions, Ann. Inst. Fourier (Grenoble), **17** (1967), 425-469.

M. Nakai

[1] The space of bounded solutions of the equation $\Delta u=Pu$ on a Riemann surface, Proc. Japan Acad., **36** (1960), 267-272.

[2] The space of non-negative solutions of the equation $\Delta u=Pu$ on a Riemann surface, Kōdai Math. Sem. Rep., **12** (1960), 151-178.

[3] Order comparisons on canonical isomorphisms, Nagoya Math. J., **50** (1973), 67-87.

[4] Banach spaces of bounded solutions of $\Delta u=Pu\,(P\geqq 0)$ on hyperbolic Riemann surfaces, Nagoya Math. J., **53** (1974), 141-155.

M. Parreau

[1] Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, Ann. Inst. Fourier (Grenoble), **3** (1951), 103-197.

B. Rodin and L. Sario

[1] Principal functions, D. Van Nostrand Company, INC., Princeton, 1968.

H.L. Royden

[1] The equation $\Delta u = Pu$ and the classifications of open Riemann surfaces, Ann. Acad. Fenn. Ser. AI, No. 271 (1959).

L. Sario and M. Nakai

[1] Classification theory of Riemann surfaces, Springer-Verlag, Berlin, 1970.

J.L. Schiff

[1] Isomorphisms between harmonic and P-harmonic Hardy spaces on Riemann surfaces, Pacific J. Math., 62 (1976), 551-560.

Takeyoshi SATŌ
Mathematics Laboratory
Iwamizawa College
Hokkaido University of Education
Iwamizawa, Hokkaido
Japan