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On a class of type I solvable Lie groups II

By Morikuni GOTO

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§ 1. Introduction.

This is a continuation of "On a class of type I solvable Lie groups I", J. Math. Soc. Japan, **30** (1978), which will be quoted as { I } in this paper, and we shall retain the terminology and notation there.

Let E be a connected Lie subgroup of $GL(m, \boldsymbol{R})$. If no element of the Lie algebra of E has non-zero purely imaginary eigenvalue, E is said to be *of exponential type*. When E is of exponential type, E is closed in $GL(m, \boldsymbol{R})$, simply connected and solvable. Let L be a connected Lie group, and let l be the Lie algebra of L. When the exponential map $\exp : l \to L$ is a surjective diffeomorphism, L is called an *exponential group*. If the adjoint group $Ad(L)$ is of exponential type, then L is an exponential group, and conversely, cf. Dixmier [**12**] and Saito [**13**], [**14**]. Here we shall extend the notion of exponential groups as follows:

DEFINITION. A connected solvable Lie group G is said to be an *(EA)-group* if the adjoint group $Ad(G)$ contains a normal subgroup E of exponential type such that the factor group $Ad(G)/E$ is a toral group.

REMARK. A more general definition of (EA)-groups will be given in § 4 below.

Exponential groups are of course (EA). In § 4 we shall see that a connected adjoint semi-algebraic group is an (EA)-group. The purpose of this paper is to prove the following theorem.

THEOREM. *A simply connected, solvable (EA)-group is of type* I.

As in { I }, we shall prove the orbit condition and the integrability condition, due to Auslander and Kostant [**1**], for such groups.

In § 4 we shall outline the relations between classes of linear groups discussed in this series of papers. The details may be given elsewhere.

§ 2. Lemmas.

The following lemma must be known. Because no suitable reference could be located, we shall give a proof.

LEMMA 1. *Let A and B be pre-algebraic groups in $GL(m, \boldsymbol{R})$. If $AB=BA$, then $G=AB$ is pre-algebraic.*

PROOF. Since A and B are pre-algebraic, $G=AB$ is a locally compact set. On the other hand, a locally compact subgroup of a topological group is closed. Hence G is a closed subgroup. Since A and B are finitely connected, so is G. Let a and b be the Lie algebras of A and B, respectively. Then the Lie algebra of AB is $a+b$. So it suffices to prove the existence of an algebraic group with Lie algebra $a+b$.

We pick an algebraic group A_1 in $GL(m, \boldsymbol{R})$ with Lie algebra a. The set of polynomials defining A_1 defines an algebraic group $A_1^{\boldsymbol{C}}$ in $GL(m, \boldsymbol{C})$, and the Lie algebra of $A_1^{\boldsymbol{C}}$ is the complex linear span $a^{\boldsymbol{C}}$ of a in $gl(m, \boldsymbol{C})$. In a similar way, the complex linear span $b^{\boldsymbol{C}}$ of b is the Lie algebra of a suitable algebraic group in $GL(m, \boldsymbol{C})$. Since $a+b$ is a Lie algebra, so is $a^{\boldsymbol{C}}+b^{\boldsymbol{C}}=(a+b)^{\boldsymbol{C}}$.

By a theorem in Chevalley [**11**], a Lie subalgebra h of $gl(m, \boldsymbol{C})$ corresponds to an algebraic group if and only if we can pick a basis $\{x_1, \cdots, x_n\}$ of h such that $\exp(\boldsymbol{C}X_i)$ is an algebraic group for $i=1, \cdots, n$. Hence $a^{\boldsymbol{C}}+b^{\boldsymbol{C}}$ is the Lie algebra of an algebraic group, say $G^{\boldsymbol{C}}$. Then $G^{\boldsymbol{C}}\cap GL(m, \boldsymbol{R})$ is an algebraic group with Lie algebra $a+b$.

LEMMA 2. *Let G be a connected Lie subgroup of $GL(m, \boldsymbol{R})$. Suppose that there exist a compact subgroup K and a connected normal Lie subgroup E in G such that $G=KE$. Let v be in $\boldsymbol{R}^m$ such that the orbit Ev is locally compact. Then Gv is locally compact, and Ev is closed in Gv.*

PROOF. Let $\mathcal{A}(E)$ denote the pre-algebraic hull of E. Then K normalizes $\mathcal{A}(E)$ and $K\mathcal{A}(E)=\mathcal{A}(E)K$ is a group. Since a compact subgroup of $GL(m, \boldsymbol{R})$ is algebraic, we have that $K\mathcal{A}(E)$ is pre-algebraic by Lemma 1. Hence $K\mathcal{A}(E)$ coincides with the pre-algebraic hull of G : $\mathcal{A}(G)=K\mathcal{A}(E)$.

Let I denote the isotropy subgroup of $\mathcal{A}(G)$ at v. Since the orbit $\mathcal{A}(G)v$ is locally compact, it can be identified with the factor space $\mathcal{A}(G)/I=\{xI\,;\, x\in\mathcal{A}(G)\}$. Let ξ be the natural map

$$\mathcal{A}(G)\ni x \mapsto \xi(x)=xv\in\mathcal{A}(G)v.$$

Since Ev is locally compact, so is $\xi^{-1}(Ev)=EI$. Because the normalizer of E in $GL(m, \boldsymbol{R})$ is algebraic and contains G, we see that E is a normal subgroup of $\mathcal{A}(G)$, and EI is a group. Hence EI is a closed subgroup of $\mathcal{A}(G)$. Since K is compact, $\xi^{-1}(Gv)=GI=K(EI)$ is closed, which implies that Gv is locally compact.

Since NI is closed, Nv is closed in Gv.

§ 3. Proof of Theorem.

Let $\tilde{G}$ be a connected, simply connected solvable Lie group, and g its Lie algebra. Let φ denote the adjoint representation of $\tilde{G}$:

$$\tilde{G} \ni x \mapsto \varphi(x) = Ad(x) \in Ad(g) \subset GL(g).$$

Suppose that $\tilde{G}$ is an (EA)-group. Let E be a normal subgroup of exponential type in $Ad(g)$ such that $Ad(g)/E$ is a toral group. Let $\tilde{L}$ denote the identity component of $\varphi^{-1}(E)$. Then $\tilde{L}$ is a closed normal subgroup in $\tilde{G}$ and $\varphi(\tilde{L})=E$. Let l denote the Lie algebra of $\tilde{L}$. Then for X in l, $ad(X) \in ad(l)$ is the restriction of $d\varphi(X)$ into l. Since $d\varphi(X)$ has no non-zero purely imaginary eigenvalue, the same is true for $ad(X)$. Hence $\tilde{L}$ is an exponential group.

Let μ denote the coadjoint representation of $\tilde{G}$. Since μ is the dual representation of φ, for any Y in g, $d\mu(Y)$ and $-d\varphi(Y)$ share the set of eigenvalues. Hence in particular, $\mu(\tilde{L})$ is of exponential type. Thus $\tilde{L}$ is an exponential group, and $\mu(\tilde{L})$ is of exponential type. By Pukanszky [**9**], for any $f \in g^*$, $\mu(\tilde{L})f$ is locally compact and the isotropy subgroup

$$\tilde{L}(f) = \{x \in \tilde{L}\,;\, \mu(x)f = f\}$$

is connected.

Since $\mu(\tilde{G})/\mu(\tilde{L})$ is a toral group, we can find a toral subgroup K of $\mu(\tilde{G})$ such that $\mu(\tilde{G}) = K\mu(\tilde{L})$, see {I} Proposition 2. By Lemma 2, the orbit $\mu(\tilde{G})f$ is locally compact, and the orbit condition is satisfied.

Next, let Z denote the center of $\tilde{G}$. Because $\tilde{L}/\tilde{L} \cap Z \cong E$ is simply connected, $\tilde{L} \cap Z$ must be connected, and coincides with the identity component Z_e of Z. Let k denote the dimension of $Ad(g)/E$. Then there exists a free abelian group D of k generators such that $Z = D \times Z_e$. We put $\tilde{G}/D = G$. Then G contains $L = \tilde{L}D/D \cong \tilde{L}$ as a closed normal subgroup such that G/L is a toral group of k dimension. Hence we can find a toral subgroup C of G such that $G = CL$, $C \cap L = \{e\}$. Therefore G is faithfully representable.

Let g^* denote the dual vector space to g, and let f be in g^*. Let $G(f)$ be the isotropy subgroup of G at f. For $x \in G(f)$, we put

$$x = \gamma(x)\lambda(x) \qquad \gamma(x) \in C, \quad \lambda(x) \in L.$$

Then $x \to \gamma(x)$ is a continuous homomorphism from $G(f)$ into C. The kernel of the homomorphism is the isotropy subgroup $L(f)$ of L at f. We recall that $L(f)$ is connected. We put the image $\gamma(G(f)) = C'$, and we shall prove that C' is closed in C.

Let μ_0 denote the coadjoint representation of G. Suppose that $\gamma_i \in C$, $\lambda_i \in L$ $(i=1, 2, \cdots)$ such that $\mu_0(\gamma_i\lambda_i)f = f$. If $\lim \gamma_i = \gamma_0$, then $\lim \mu_0(\lambda_i)f = \mu_0(\gamma_0^{-1})f$.

By the last statement in Lemma 2, $\mu_0(L)f$ is closed in $\mu_0(G)f$, and so we can find a $\lambda_0 \in L$ with $\mu_0(\gamma_0^{-1})f=\mu_0(\lambda_0)f$, i.e., $\mu_0(\gamma_0\lambda_0)f=f$. Hence $\gamma_0 \in C'$, and we have shown that C' is closed. Since $G(f)/N(f) \cong C'$, and C' is finitely connected, so is $G(f)$. By Corollary 1 in {I}, the integrability condition holds for $\tilde{G}$.

§ 4. Remarks on linear Lie groups.

A connected Lie subgroup T of $GL(m, \boldsymbol{R})$ is said to be ($\boldsymbol{R}$-) triangularizable if the eigenvalues of every member of T are all real. A triangularizable T is of exponential type and is an exponential group.

Let G be a connected Lie subgroup of $GL(m, \boldsymbol{R})$, and R the radical of G. By Lie's theorem, there exists an $x \in GL(m, \boldsymbol{C})$ such that xRx^{-1} can be (upper) triangularized simultaneously. Let T_1 be the totality of y in R with all eigenvalues real. Then T_1 is a closed normal subgroup of R. Let T be the identity component of T_1. We shall call T the *triangularizable radical* of G. The triangularizable radical is obviously a normal subgroup of G, but is not characteristic in general. We can also define the triangularizable radical as a maximal triangularizable, normal connected Lie subgroup of G.

In [3], the author defined a faithfully representable connected Lie group H *completely reducible* if the radical of H is a toral group. Notice that "completely reducible" is stronger than "reductive". There it was proved that a connected Lie group is faithfully representable if and only if there exists a completely reducible subgroup H and a closed simply connected solvable normal Lie subgroup L such that

$$(*) \qquad G=HL, \quad H \cap L=\{e\}\ ;$$

the H is determined uniquely up to inner automorphisms and will be called a *completely reducible part* of G. Let us call L a *linear radical* of G, and the decomposition (*) *canonical.*

PROPOSITION. *Let G be a connected Lie subgroup of $GL(m, \boldsymbol{R})$.*

(i) *G is pre-algebraic if and only if the triangularizable radical is pre-algebraic* (*of the form AN, where A is a connected pre-algebraic abelian subgroup composed of semisimple elements, and N the unipotent radical of G*) *and is a linear radical of G.*

(ii) *G is semi-algebraic if and only if the triangularizable radical is a linear radical.*

PROOF. (i) Suppose that a linear radical L of G is pre-algebraic. Let H be a completely reducible part, and H' the commutator subgroup of H. Then H' is a closed connected semisimple subgroup, and there exists a central toral subgroup K with $H=H'K$. By repeated use of Lemma 1, we have that G is pre-algebraic.

Conversely, suppose that G is pre-algebraic, then so is the radical R. Hence R has an Iwasawa decomposition $R=KAN$, where K is a toral group, A abelian composed of semisimple elements, and N the unipotent radical of R. Let k, a, n, r, denote the Lie algebras of K, A, N and R, respectively. Then $r=k \oplus a \oplus n$, and for $X \in k$, $Y \in a$, and $Z \in n$, the eigenvalues $\alpha_i(X+Y+Z), \cdots, \alpha_m(X+Y+Z)$ are given by $\alpha_i(X+Y+Z)=\alpha_i(X)+\alpha_i(Y)+\alpha_i(Z)$ for $i=1, \cdots, m$, in virtue of Lie's theorem. Since $\alpha_i(X)$ is purely imaginary, $\alpha_i(Y)$ real and $\alpha_i(z)=0$, all $\alpha_i(X+Y+Z)$ are real if and only if $\alpha_i(X)=0$ for $i=1, \cdots, m$, i.e., $X=0$ because X is semisimple. Hence the triangularizable radical is given by AN, which is obviously a linear radical of G.

(ii) Suppose that $GL(m, \boldsymbol{R}) \supset G=HL$ $H \cap L=\{e\}$, where H is a completely reducible part and L is triangularizable. Let $\mathcal{A}(L)$ denote the pre-algebraic hull of L. Then $H\mathcal{A}(L)$ is pre-algebraic and coincides with $\mathcal{A}(G)$: $\mathcal{A}(G)=H\mathcal{A}(L)$. Since L is triangularizable, so is $\mathcal{A}(L)$. $H \cap \mathcal{A}(L)$ is a solvable normal subgroup of H, which is obviously compact and discrete. Hence $H \cap \mathcal{A}(L)$ must be a finite group. But $\mathcal{A}(L)$ contains no element of finite order except e. Hence $H \cap \mathcal{A}(L)=\{e\}$, and $\mathcal{A}(G)/G$ is homeomorphic with $\mathcal{A}(L)/L$ which is a vector group. This implies that G is semi-algebraic.

Conversely, suppose that G is semi-algebraic. Then $\mathcal{A}(G)=H(AN)$ $H \cap AN=\{e\}$. Since $H=H'K$ and G contains all compact subgroups and the commutator subgroup of $\mathcal{A}(G)$, we have that $G \supset H$. Hence $G=H(AN \cap G)$ is a canonical decomposition with linear radical $AN \cap G$ which is of exponential type.

Now we shall give a definition.

DEFINITION. A connected Lie subgroup G of $GL(m, \boldsymbol{R})$ is said to be of *(EA)-type* if there exists a linear radical of exponential type. A connected Lie group is said to be an *(EA)-group* if the adjoint group is of (EA)-type.

REMARK. Let G be a faithfully representable connected Lie group, and K a maximal compact subgroup of G. Then we can find a simply connected solvable Lie subgroup P with

$$G=KP \quad K \cap P=\{e\}, \tag{**}$$

see Goto [3]. For a pre-algebraic group G, we can take $P=AN$ as an Iwasawa subgroup. Hence we shall call (**) a *generalized Iwasawa decomposition* of G. For semi-algebraic G, P can be triangularizable and for G of (EA)-type, P can be of exponential type.

Bibliography

(Continuation to the one in {I})

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Morikuni Goto
Department of Mathematics
University of Pennsylvania
U. S. A.

Added in proof: After this paper was accepted, the author has learned that the main theorem of this paper can also be proved by using machinery due to Mackey. But it seems to the author that to get further results along the way, the method here would be preferable.