J. Math. Soc. Japan
Vol. 27, No. 3, 1975

# A test of Picard principle for rotation free densities

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(Received June 4, 1974)

Consider a nonnegative locally Hölder continuous function $P(z)$ on the punctured unit disk $0<|z|\leqq 1$. Such a function will be referred to as a *density*. To describe the potential theoretic singular behavior of a density $P(z)$ at $z=0$ we consider the *elliptic dimension*, $\dim P$ in notation, which is defined to be the dimension of the half module of nonnegative solutions of the equation $\Delta u=Pu$ on $0<|z|<1$ having the continuous boundary values zero on $|z|=1$. After Bouligand we say that the *Picard principle* is valid for a density $P$ at $z=0$ if $\dim P=1$. One of the central theme of the study of elliptic dimensions is to determine the range of the mapping $P\to\dim P$ and in particular to determine $\{P; \dim P=1\}$, the family of densities for which the Picard principle is valid. Although we have various results on this subject obtained by many authors listed in the references at the end of this paper, the study is quite far from being complete. As an experimental study we considered in our former paper [**13**] *rotation free* densities $P(z)$ in the sense that $P(z)=P(|z|)$ for every $z$ in $0<|z|\leqq 1$. We showed that the Martin compactification $\Omega_P^*$ of the punctured open unit disk $\Omega: 0<|z|<1$ with respect to a rotation free density $P$ on $0<|z|\leqq 1$ is homeomorphic to a closed annulus, i.e.

$$\Omega_P^* \approx (\alpha(P)\leqq |z|\leqq 1) \tag{1}$$

where $\alpha(P)$, referred to as the *singularity index* of $P$, is the proper quantity in $[0, 1)$ associated with $P$ determined as follows: The ordinary differential equation

$$(ru'(r))'=r(P(r)+j^2/r^2)u(r)$$

has a unique bounded solution $e_j(r)$ on $(0, 1]$ with the initial condition $e_j(1)=1$ $(j=0, 1)$ and the function $e_1(r)/e_0(r)$ has a limit as $r\to 0$ which is defined to be $\alpha(P)$:

$$\alpha(P)=\lim_{r\to 0} e_1(r)/e_0(r)\in[0, 1)\,.$$

Therefore in particular we have

$$\dim P=1+\alpha(P)\cdot\mathfrak{c} \tag{2}$$

where $\mathfrak{c}$ is the cardinal number of continuum, and there really can occur both cases $\alpha(P)=0$ and $\alpha(P)>0$. Thus the problem may be said to be settled to a considerable extent for tractable densities $P$ which are rotation free. Practically speaking, however, to compute $\alpha(P)$ is not easy and in fact almost impossible for most cases.

The purpose of this paper as a continuation of [**13**] is to provide a practical test for $\alpha(P)=0$ and $\alpha(P)>0$ which is wishfully expected to supply informations to proceed to general densities. Typical examples of our test ($T$-tests) given in the section 4 are: If

$$\limsup_{r\to 0} P(r)/r^{-2}\cdot\Big(\prod_{j=1}^{n-1}\log_j r^{-1}\Big)^2\cdot(\log_n r^{-1})^2=0 \tag{3}$$

for an integer $n>0$ then $\alpha(P)=0$, and if

$$\liminf_{r\to 0} P(r)/r^{-2}\cdot\Big(\prod_{j=1}^{n-1}\log_j r^{-1}\Big)^2\cdot(\log_n r^{-1})^{2+\varepsilon}>0 \tag{4}$$

for an integer $n>0$ and an $\varepsilon>0$ then $\alpha(P)>0$, where $\log_1 t=\log t$ and $\log_{n+1} t=\log(\log_n t)$ $(n=1, 2, \cdots)$, and $\prod_{j=1}^{0}=1$.

If $r^2P(r)$ is increasing as $r\to 0$, or more generally, if $r^2P(r)$ is 'almost increasing' as $r\to 0$ in the sense that there exist a constant $c\in[1, \infty)$ and a $k\in(0, \infty)$ such that

$$r_1^2P(r_1)+k\leqq c(r_2^2P(r_2)+k)$$

for every pair $(r_1, r_2)$ with $0<r_2\leqq r_1\leqq 1$, then we can give in the section 6 a complete criterion: $\alpha(P)=0$ $(>0$, resp.) if and only if

$$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}}=\infty \qquad (<\infty, \text{ resp.}). \tag{5}$$

## §1. Singularity indices.

**1.1.** Throughout this paper our *density* $P(z)$ on $0<|z|\leqq 1$, i.e. a locally Hölder continuous nonnegative function $P(z)$ on $0<|z|\leqq 1$, will be supposed to be *rotation free* in the sense that $P(z)=P(|z|)$ on $0<|z|\leqq 1$ unless otherwise is explicitly stated. Thus a density $P(z)$ may be considered as a function $P(r)$ on $(0, 1]$. The mere continuity of $P(r)$ will be sufficient for the whole discussion in this paper, we assume the Hölder continuity for convenience of references. We briefly recall results obtained in [**13**]. Consider

$$\frac{1}{r}\Big(\frac{d}{dr}\Big(r\frac{d}{dr}u(r)\Big)\Big)=\Big(P(r)+\frac{n^2}{r^2}\Big)u(r) \qquad (n=0, 1, \cdots). \tag{6}$$

For each $n$, (6) has a unique bounded solution $e_n(r)$ on $(0, 1]$ with the initial condition $e_n(1)=1$. We see that

$$0<e_n(r)\leqq r^j \qquad (j=0, 1, \cdots, n) \tag{7}$$

on $(0, 1]$. The functions $e_n(r)/e_{n-1}(r)$ and $e_n(r)/e_0(r)$ $(n=0, 1, \cdots; e_{-1}\equiv 1)$ are decreasing as $r\to 0$. We set

$$\alpha_n(P)=\lim_{r\to 0} e_n(r)/e_0(r)\,, \tag{8}$$

which is referred to as the $n^{th}$ *singularity index* of $P$ at $z=0$, and in particular, $\alpha(P)=\alpha_0(P)$ as *singularity index* of $P$ at $z=0$. We have the following fundamental inequality:

$$0\leqq\alpha(P)<1\,,\qquad (\alpha(P))^{(3^n-1)/2}\leqq\alpha_n(P)\leqq(\alpha(P))^n \tag{9}$$

for $n=0, 1, \cdots$.

**1.2.** Let $\mathcal{P}_P$ be the half module of nonnegative solutions of the equation

$$\Delta u(z)=P(z)u(z) \tag{10}$$

on $0<|z|<1$ with boundary values zero on $|z|=1$. The main result in [**13**] is the existence of $[0, \infty]$-valued continuous function $K(z, \zeta)$ on $(0<|z|<1)\times(\alpha(P)\leqq|\zeta|\leqq 1)$ such that $z\to K(z, \zeta)$ is a solution of (10) on $(0<|z|<1)-\{\zeta\}$ and that there exists a bijective correspondence $u\leftrightarrow\mu$ between $\mathcal{P}_P$ and the class $\{\mu\}$ of regular Borel measures on $|\zeta|=\alpha(P)$ with

$$u(z)=\int_{|\zeta|=\alpha(P)} K(z, \zeta)d\mu(\zeta) \tag{11}$$

on $0<|z|<1$. Since $\dim P=\dim\mathcal{P}_P$ and the cardinal number $\#\{\mu\}=\#(|\zeta|=\alpha(P))=1+\alpha(P)\cdot\mathfrak{c}$, where $\mathfrak{c}$ is the cardinal number of continuum, (11) implies that

$$\dim P=1+\alpha(P)\cdot\mathfrak{c}\,. \tag{12}$$

Therefore it is important to determine whether $\alpha(P)=0$ or $\alpha(P)>0$ for a given $P$. For this reason we will try to give some tests for $\alpha(P)=0$ and $\alpha(P)>0$.

**1.3.** Before proceeding to the above mentioned theme we pause here to state two simple consequences of the mere definition of $\alpha(P)>0$ in this and the next nos. By twice applications of the Cauchy mean value theorem we can find an $f(r)\in(0, r)$ for each $r\in(0, 1]$ such that

$$\frac{(f(r)e_1'(f(r)))'}{(f(r)e_0'(f(r)))'}=\frac{e_1(r)}{e_0(r)}\,.$$

On using (6) we then deduce

$$\frac{P(f(r))+f(r)^{-2}}{P(f(r))}\cdot\frac{e_1(f(r))}{e_0(f(r))}=\frac{e_1(r)}{e_0(r)}\,.$$

Therefore, by (8), we see that

$$\alpha(P)\cdot\Big(1+\liminf_{r\to 0}\frac{1}{r^2P(r)}\Big)\leqq\alpha(P)\,.$$

In particular, if $\alpha(P)>0$, then we must have

(13) $$\limsup_{r\to 0} r^2P(r)=\infty\,.$$

By this simple observation we obtain the following:

*The Picard principle is valid for* $P$, *i.e.* $\alpha(P)=0$, *if*

(14) $$\limsup_{r\to 0} r^2P(r)<\infty\,.$$

In connection with this we naturally ask whether the condition (14) is necessary for $\alpha(P)=0$. This is one of the motivations of the present study. Of course the situation is not so simple. We proved in our former paper [**14**] that if

(15) $$\int_{(0<|z|<1)-E} P(z)\log\frac{1}{|z|}\,dxdy<\infty$$

for a *general* (i.e. not necessarily rotation free) density $P$, where $E$ is a closed subset of $0<|z|<1$ thin at $z=0$, then the Picard principle is valid for $P$. Obviously we can find $P(r)$ with (15) but without (14), and thus (14) is not necessary for $\alpha(P)=0$. Hence our concern is to analyze to what extent the condition (14) can be weakend.

**1.4.** We say that the *unicity principle* is valid for $P$ if a solution $u$ of (10) on $0<|z|<1$ satisfies

(16) $$\lim_{z\to 0}|z|^{-n}u(z)=0 \qquad (n=0, 1, \cdots)$$

then $u$ must be identically zero. It is well known that if the degree of singularity of $P$ at $z=0$ is not so high, in particular if $P\equiv 0$, then the unicity principle is valid for $P$ (cf. e.g. Brelot [**1**]). Fix an arbitrary $n=0, 1, \cdots$ and choose an integer $k>n$. By (8) and (9),

$$e_0(r)\leqq 2(\alpha(P))^{(1-3^k)/2}e_k(r)$$

for every $r\in(0, \rho)$ with a certain $\rho\in(0, 1)$. Thus by (7)

$$r^{-n}e_0(r)\leqq 2(\alpha(P))^{(1-3^k)/2}r^{k-n}\,,$$

and if $\alpha(P)>0$, then (16) is valid for $u(z)=e_0(|z|)>0$, a solution of (10). Therefore the unicity principle is invalid for $P$ with $\alpha(P)>0$. In other words,

*The Picard principle is valid for* $P$, *i.e.* $\alpha(P)=0$, *if the unicity principle is valid for* $P$.

We shall see that e.g. the density $P(r)=r^{-2}(\log r)^2$ satisfies $\alpha(P)=0$ (cf. (3)). It is readily seen that $u(z)=e_0(|z|)$ for this $P$ is a nonzero solution of (10). Observe that

$$\Delta(|z|/\rho)^n=(n^2/|z|^2)\cdot(|z|/\rho)^n\leqq P(z)\cdot(|z|/\rho)^n$$

for an arbitrary integer $n>0$ on $(0, \rho]$ if $\rho>0$ is sufficiently small. Therefore $s_\varepsilon(z)=(|z|/\rho)^n+\varepsilon\log(\rho/|z|)-e_0(|z|)/e_0(\rho)$ satisfies $(\Delta-P(z))s_\varepsilon(z)\leqq 0$ for any

$\varepsilon>0$ and $\lim_{z\to\zeta}s_{\varepsilon}(z)\geqq 0$ for every $\zeta$ in the boundary of $0<|z|<\rho$. The maximum principle yields (cf. Lemma 2.2 below) $s_{\varepsilon}(z)\geqq 0$ on $0<|z|<\rho$. On making $\varepsilon\to 0$, we conclude that $e_0(r)\leqq(e_0(\rho)/\rho^n)r^n$ on $(0, \rho]$. Thus (16) is valid for $e_0(|z|)$. Therefore the validity of the unicity principle is not necessary for the validity of the Picard principle.

## § 2. Reduction to Riccati equations.

**2.1.** In dealing with the equation (6) we change the variable $r\in(0, 1]$ to $t\in[0, \infty)$ by $r=e^{-t}$. Then $u$ is a solution of (6) on $(0, 1]$ if and only if $v(t)=u(e^{-t})$ is a solution of

$$\frac{d^2}{dt^2}v(t)=(Q(t)+n^2)v(t), \qquad Q(t)=e^{-2t}P(e^{-t}) \tag{17}$$

on $[0, \infty)$. A positive function $v(t)$ is a solution of (17) on $[0, \infty)$ if and only if $\phi(t)=-\log v(t)$ is a solution of

$$-\frac{d^2}{dt^2}\phi(t)+\left(\frac{d}{dt}\phi(t)\right)^2=Q(t)+n^2 \tag{18}$$

on $[0, \infty)$. Therefore (18) possesses a unique nonnegative solution $\phi_n(t)=-\log(e_n(e^{-t}))$ on $[0, \infty)$ with the initial condition $\phi_n(0)=0$. Consider

$$b(t)=\phi_1(t)-\phi_0(t). \tag{19}$$

Then since $e^{-b(t)}=e_1(r)/e_0(r)$ $(r=e^{-t})$, we see that $b(0)=0$, $b(t)$ is increasing, and the limit

$$\beta(P)=\lim_{t\to\infty}b(t)\in(0, \infty] \tag{20}$$

exists and $\alpha(P)=e^{-\beta(P)}$. Moreover $b(t)$ is a unique positive solution of

$$-\frac{d^2}{dt^2}w(t)+2\frac{d}{dt}\phi_0(t)\cdot\frac{d}{dt}w(t)+\left(\frac{d}{dt}w(t)\right)^2=1 \tag{21}$$

with the initial condition $b(0)=0$. Therefore the question of whether $\alpha(P)=0$ or $\alpha(P)>0$ is equivalent to determine whether $\beta(P)=\infty$ or $\beta(P)<\infty$ for the positive solution $b(t)$ of (21) with $b(0)=0$. Since the property of the solution $b(t)$ is determined only by

$$a(t)=\frac{d}{dt}\phi_0(t), \tag{22}$$

we need to know the property of $a(t)$ which is a nonnegative (cf. no. 1.1.) solution of

$$-\frac{d}{dt}a(t)+a(t)^2=Q(t) \qquad (Q(t)\geqq 0). \tag{23}$$

**2.2.** We have thus to study the equation (23) of Riccati type. We shall see that (23) has a unique nonnegative solution $a_Q$ on $[0, \infty)$. Our main concern is how the asymptotic behavior of $Q(t)$ reflects on that of $a_Q(t)$ as $t\to\infty$. Intuitively we are tempted to say that $a_Q(t)\sim\sqrt{Q(t)}$ as $t\to\infty$. We shall see that this intuition is, to a certain extent, true.

For convenience we state properties of solutions of the equations (17), (18), (21) and (23) in a selfcontained manner without referring to the results in §1. Consider the following second order ordinary differential operator

$$L_A v = v''(t) - g(t)\cdot v'(t) - A(t)\cdot v(t) = 0 \qquad (g(t),\ A(t) \geqq 0) \tag{24}$$

on $[0, \infty)$ where $g$ and $A$ are continuous on $[0, \infty)$. Let $s$ be a continuous function on $[a, b]\subset[0, \infty)$. The *maximum principle* (minimum principle) says that if $L_A s\leqq 0$ on $(a, b)$ and $s(a)$, $s(b)\geqq 0$, then $s(t)\geqq 0$ on $[a, b]$. Another formulation of the maximum principle is that if $L_A s\geqq 0$ on $(a, b)$ with $s(a)$, $s(b)\geqq 0$, then $s(t)\leqq\max(s(a), s(b))$ on $[a, b]$. As a consequence of the maximum principle, we have the following *comparison principle*: if $v_1$ and $v_2$ are continuous on $[a, b]$ such that $L_{A_1}v_1 = L_{A_2}v_2 = 0$ on $(a, b)$, $v_1(a)\geqq v_2(a)$, and $v_1(b)\geqq v_2(b)$, then $A_1\leqq A_2$ on $[a, b]$ implies that $v_1\geqq v_2$ on $[a, b]$. To state the properties of solutions of (17), (18), (21) and (23), the following is fundamental:

LEMMA. *There exists a unique bounded solution $v_A$ of (24) on $[\varepsilon, \infty)$ $(\varepsilon\geqq 0)$ with the initial condition $v_A(\varepsilon)=1$. The solution $v_A$ is decreasing on $[\varepsilon, \infty)$. If $A_1\leqq A_2$, then $v_{A_1}\geqq v_{A_2}$ on $[\varepsilon, \infty)$.*

The lemma can also be applied to the corresponding equations on $(0, \eta]$ $(\eta\in(0, 1])$ by the change of variable $t=-\log r$. For each integer $j>0$ there exists a solution $v_j$ of (24) on $[\varepsilon, j]$ such that $v_j(\varepsilon)=1$ and $v_j(j)=0$. This is obtained e.g. as a linear combination of two solutions of Cauchy problems on $[\varepsilon, j]$ with suitable initial conditions. By the maximum principle, $0\leqq v_j\leqq v_{j+1}\leqq 1$ on $[\varepsilon, j]$ and $v_j$ is decreasing on $[\varepsilon, j]$. By the Harnack principle, $v=\lim_{j\to\infty}v_j$ is a solution of (24) on $[\varepsilon, \infty)$ such that $0\leqq v\leqq 1$, $v(\varepsilon)=1$, and $v$ is decreasing on $[\varepsilon, \infty)$. To see the uniqueness of $v_A$, let $v$ be a bounded solution of (24) on $[\varepsilon, \infty)$ with $v(\varepsilon)=0$. Observe that $L_A t=-(g(t)+A(t)\cdot t)\leqq 0$ on $[\varepsilon, \infty)$. Therefore, for any $\eta>0$, $L_A(v(t)-\eta t)\geqq 0$ on $[\varepsilon, \infty)$. Since $v(t)-\eta t\leqq 0$ for $t=0$ and $t=\tau$ if $\tau$ is sufficiently large, $v(t)-\eta\cdot t\leqq 0$ on $[\varepsilon, \tau]$. Hence $v(t)\leqq\eta t$ on $[\varepsilon, \infty)$ for any $\eta>0$. A fortiori $v\equiv 0$ on $[\varepsilon, \infty)$. The last assertion can be verified by the limiting process $v_A=\lim v_j$ on using the comparison principle.

**2.3.** On choosing $g(t)\equiv 0$ and $A(t)\equiv Q(t)$, a nonnegative *continuous* function on $[0, \infty)$, (24) reduces to (17) with $n=0$. Thus the lemma 2.2 can be restated as follows:

LEMMA. *There exists a unique bounded solution $v_Q$ of $v''=Qv$ $(Q\geqq 0)$ on*

$[\varepsilon, \infty)$ $(\varepsilon \geqq 0)$ *with the initial condition* $v_Q(\varepsilon)=1$. *The solution* $v_Q$ *is decreasing on* $[\varepsilon, \infty)$. *If* $0 \leqq Q_1 \leqq Q_2$, *then* $v_{Q1} \geqq v_{Q2}$ *on* $[\varepsilon, \infty)$.

**2.4.** We turn to the equation (18). By the correspondence $\phi=-\log v$ $(v>0)$, $\phi$ is a solution of $-\phi''+\phi'^2=Q$ if and only if $v$ is a solution of $v''=Qv$; $\phi$ is nonnegative if and only if $v \leqq 1$. Thus the lemma in 2.3 takes the following form:

LEMMA. *There exists a unique nonnegative solution* $\phi_Q$ *of* $-\phi''+\phi'^2=Q$ $(Q \geqq 0)$ *on* $[\varepsilon, \infty)$ $(\varepsilon \geqq 0)$ *with the initial condition* $\phi_Q(\varepsilon)=0$. *The solution* $\phi_Q$ *is increasing on* $[\varepsilon, \infty)$. *If* $0 \leqq Q_1 \leqq Q_2$, *then* $\phi_{Q1} \leqq \phi_{Q2}$ *and* $\phi'_{Q_1} \leqq \phi'_{Q_2}$ *on* $[\varepsilon, \infty)$.

Only the last assertion is beyond the Lemma 2.3. Let $v=e^{-(\phi_{Q_2}-\phi_{Q_1})}$, which is a bounded solution of

$$v''-2\phi'_{Q_1}\cdot v'-(Q_2-Q_1)v=0$$

on $[\varepsilon, \infty)$ with $v(\varepsilon)=1$. Since $\phi_{Q_1}$ is increasing, $2\phi'_{Q_1} \geqq 0$. Therefore Lemma 2.2 is applicable to deduce that $v$ is decreasing, i. e. $v'=(\phi'_{Q_1}-\phi'_{Q_2})v \leqq 0$ and a fortiori $\phi'_{Q_1} \leqq \phi'_{Q_2}$ on $[\varepsilon, \infty)$.

**2.5.** Let $\phi_Q$ and $\phi_{Q+1}$ be as in Lemma 2.4 on $[\varepsilon, \infty)$. Observe that $w=\phi_{Q+1}-\phi_Q$ is a nonnegative solution of

$$(21)' \qquad -w''+2\phi'_Q\cdot w'+w'^2=1$$

on $[\varepsilon, \infty)$ with the initial condition $w(\varepsilon)=0$. Conversely assume that $w$ is a nonnegative solution of (21)$'$ on $[\varepsilon, \infty)$ with the initial condition $w(\varepsilon)=0$. Then $\phi=w+\phi_Q$ satisfies the equation $-\phi''+\phi'^2=Q+1$ on $[\varepsilon, \infty)$ with $\phi(\varepsilon)=0$. Therefore $\phi=\phi_{Q+1}$, i. e. $w=\phi_{Q+1}-\phi_Q$. Thus we have seen that (21)$'$ has a unique positive solution $w_Q$ on $[\varepsilon, \infty)$ with the initial condition $w_Q(\varepsilon)=0$. By Lemma 2.4, $w'_Q=\phi'_{Q+1}-\phi'_Q \geqq 0$. Therefore $w_Q$ is increasing and the limit

$$(25) \qquad \beta_\varepsilon(Q)=\lim_{t\to\infty} w_Q(t) \in (0, \infty]$$

exists. Observe that $\exp(-\phi_Q(-\log r))=e_0(r)$ and $\exp(-\phi_{Q+1}(-\log r))=e_1(r)$ where $e_n(r)$ is the unique solution of (6) with $P(r)=r^{-2}Q(-\log r)$ on $(0, e^{-\varepsilon}]$ with $e_n(e^{-\varepsilon})=1$ $(n=0, 1)$. Therefore $\exp(-w_Q(-\log r))=e_1(r)/e_0(r)$ and if we set

$$(26) \qquad \alpha_{(\varepsilon)}(P)=\lim_{r\to 0} e_1(r)/e_0(r)\,, \qquad \alpha_{(0)}(P)=\alpha(P)\,,$$

then we have

$$(27) \qquad \alpha_{(\varepsilon)}(P)=e^{-\beta_\varepsilon(Q)}\,.$$

Let $w_{Q,\varepsilon}$ be for the interval $[\varepsilon, \infty)$ and $w_Q$ the interval $[0, \infty)$. Then $w_{Q,\varepsilon}(t)=w_Q(t)-w_Q(\varepsilon)$ and thus

$$(28) \qquad \beta_\varepsilon(Q)=\beta(Q)-w_Q(\varepsilon)$$

where $\beta(Q)=\beta_0(Q)$. This means that the fact $\alpha(P)>0$ or $\alpha(P)=0$ depends only on the behavior of $P$ in any small vicinity of $z=0$. We have

PROPOSITION. *There exists a unique positive solution* $w_Q$ *of* $-w''+2\psi_Q'w'+w'^2=1$ *on* $[\varepsilon,\infty)$ $(\varepsilon\geqq 0)$ *with the initial condition* $w_Q(\varepsilon)=0$. *The solution* $w_Q$ *is increasing on* $[\varepsilon,\infty)$. *The singularity index* $\alpha(P)$ *is zero or positive* $(Q(t)=e^{-2t}P(e^{-t})\geqq 0)$ *if and only if* $\beta_\varepsilon(Q)=\lim_{t\to\infty}w_Q(t)$ *is infinite or finite for one and hence for every* $\varepsilon\in[0,\infty)$.

**2.6.** From the view point of the practical application it is convenient to reformulate the proposition 2.5 as follows. Suppose that $b$ is a nonnegative $C^2$ function satisfying $-b''+2\psi_Q'b'+b'^2\geqq 1$ $(\leqq 1$, resp.) on $[\varepsilon,\infty)$ $(\varepsilon\geqq 0)$. On putting $\psi=b+\psi_Q$, we see that $-\psi''+\psi'^2\geqq Q+1$ $(\leqq Q+1$, resp.) on $[\varepsilon,\infty)$. Let $v=e^{-(\psi-\psi(\varepsilon))}$. Then $v''\geqq(Q+1)v$ $(\leqq(Q+1)v$, resp.) on $[\varepsilon,\infty)$. The maximum principle applied to $v_{Q+1}(t)+\eta t-v(t)$ $(v(t)+\eta t-v_{Q+1}(t)$, resp.) $(\eta>0)$ for the operator $Lf=f''-(Q+1)f$ yields $v_{Q+1}(t)+\eta t\geqq v(t)$ $(v(t)+\eta t\geqq v_{Q+1}(t)$, resp.) on $[\varepsilon,\tau]$ for sufficiently large $\tau>\varepsilon$. Thus we can conclude that $v\leqq v_{Q+1}$ $(v\geqq v_{Q+1}$, resp.) on $[\varepsilon,\infty)$. Since $v_{Q+1}=e^{-\psi_{Q+1}}$, we see that

$$\psi-\psi(\varepsilon)\geqq\psi_{Q+1}\qquad(\psi-\psi(\varepsilon)\leqq\psi_{Q+1},\ \text{resp.})$$

on $[\varepsilon,\infty)$. Therefore

$$b-b(\varepsilon)\geqq w_Q\qquad(b-b(\varepsilon)\leqq w_Q,\ \text{resp.})$$

on $[\varepsilon,\infty)$. Then $\beta_\varepsilon(Q)<\infty$ $(\beta_\varepsilon(Q)=\infty$, resp.) if

$$\liminf_{t\to\infty} b(t)<\infty\qquad(\limsup_{t\to\infty} b(t)=\infty,\ \text{resp.})$$

We summarize this observation in the following

THEOREM. *The singularity index* $\alpha(P)=0$ *if and only if there exists a nonnegative* $C^2$ *function* $b(t)$ *on* $[\varepsilon,\infty)$ $(\varepsilon\geqq 0)$ *such that*

$$-b''(t)+2\psi_Q'(t)b'(t)+b'(t)^2\leqq 1\qquad(Q(t)=e^{-2t}P(e^{-t}))\tag{29}$$

*on* $[\varepsilon,\infty)$ $(\varepsilon\geqq 0)$ *and*

$$\limsup_{t\to\infty} b(t)=\infty\,.\tag{30}$$

*The singularity index* $\alpha(P)>0$ *if and only if there exists a nonnegative* $C^2$ *function* $b(t)$ *on* $[\varepsilon,\infty)$ $(\varepsilon\geqq 0)$ *such that*

$$-b''(t)+2\psi_Q'(t)b'(t)+b'(t)^2\geqq 1\tag{31}$$

*on* $[\varepsilon,\infty)$ *and*

$$\liminf_{t\to\infty} b(t)<\infty\,.\tag{32}$$

**2.7.** In view of the preceding theorem we need to know the property of $\psi_Q'$ in terms of $Q$. By Lemma 2.4, $a=\psi_Q'$ is a nonnegative solution of $-a'+a^2=Q$ on $[\varepsilon,\infty)$. Conversely assume that $a$ is a nonnegative solution of $-a'+a^2$

$=Q$ on $[\varepsilon, \infty)$. Then

$$\psi(t)=\int_{\varepsilon}^{t} a(s)ds$$

is a nonnegative solution of $-\psi''+\psi'^2=Q$ on $[\varepsilon, \infty)$ with $\psi(\varepsilon)=0$. Thus again by Lemma 2.4, $\psi=\psi_Q$ and a fortiori $a=\psi'_Q$. Therefore the Lemma 2.4 implies the following

LEMMA. *There exists a unique nonnegative solution* $a_Q$ *of* $-a'+a^2=Q$ $(Q\geqq 0)$ *on* $[\varepsilon, \infty)$ $(\varepsilon\geqq 0)$. *If* $0\leqq Q_1\leqq Q_2$, *then* $a_{Q_1}\leqq a_{Q_2}$.

## § 3. Asymptotic behaviors.

**3.1.** We are interested in the question how the asymptotic behavior of $Q(t)$ as $t\to\infty$ reflects on that of $a_Q$, the unique nonnegative solution of

$$-a'(t)+a(t)^2=Q(t) \qquad (Q\geqq 0)$$

on $[\varepsilon, \infty)$ $(\varepsilon\geqq 0)$. Specifically we ask to what extent the relation $a(t)\sim\sqrt{Q(t)}$ $(t\to\infty)$ is true. Consider a positive $C^1$ function $T(t)$ on $[\varepsilon, \infty)$ $(\varepsilon\geqq 0)$ satisfying either of the following conditions:

$$\limsup_{t\to\infty} T'(t)/T(t)^2<\infty\,; \tag{33}$$

$$\liminf_{t\to\infty} T'(t)/T(t)^2>-\infty\,. \tag{34}$$

To describe the asymptotic behavior of $Q$ in terms of $T$ consider the following conditions:

$$\limsup_{t\to\infty} Q(t)/T(t)^2<\infty\,; \tag{35}$$

$$\liminf_{t\to\infty} Q(t)/T(t)^2>0\,. \tag{36}$$

Similarly we consider the following conditions for $a_Q$:

$$\limsup_{t\to\infty} a_Q(t)/T(t)<\infty\,; \tag{37}$$

$$\liminf_{t\to\infty} a_Q(t)/T(t)>0\,. \tag{38}$$

As an answer to the above question we state the following

PROPOSITION. *For a* $T$ *with* (33) ((34), *resp.*) *if* $Q$ *satisfies* (35) ((36), resp.), *then* $a_Q$ *satisfies* (37) ((38), *resp.*).

**3.2.** For the proof first we assume that $T$ satisfies (33). On replacing $T$ by $aT+b$ with suitable positive constants $a$ and $b$, we may assume that $T(t)\geqq 1$ and that (35) implies

$$Q(t)\leqq T(t)^2$$

on $[\varepsilon, \infty)$ for a sufficiently large $\varepsilon>0$. In view of (33) we can find a positive constant $k$ so large that

$$Q_0(t)=-kT'(t)+k^2T(t)^2 \geqq T(t)^2$$

on $[\varepsilon, \infty)$. Observe that $a_{Q_0}=kT$, and $Q_0 \geqq Q$. Therefore the lemma in 2.7 implies that $a_{Q_0} \geqq a_Q$, i. e.

$$a_Q(t)/T(t) \leqq k$$

and a fortiori we conclude that $a_Q$ satisfies (37).

**3.3.** Next suppose that $T$ satisfies (34). On replacing $T$ by $aT$ with a suitable positive constant $a$, we may assume that (36) implies

$$Q(t)>T(t)^2 \tag{39}$$

on $[\varepsilon, \infty)$ for a sufficiently large $\varepsilon>0$. In view of (36) we can find a positive constant $k$ so small that

$$Q_0(t)=-kT'(t)+k^2T(t)^2 \leqq T(t)^2$$

on $[\varepsilon, \infty)$. With (39) we then have

$$Q_0(t)<Q(t) \tag{40}$$

on $[\varepsilon, \infty)$. If $Q_0(t)$ were nonnegative, then Lemma 2.7 would imply the desired conclusion as in 3.2. However this assumption may not valid and thus we need an extra discussion as follows. Consider the function

$$f(t)=kT(t)-a_Q(t)$$

on $[\varepsilon, \infty)$. We maintain that $f(t) \leqq 0$ on $[t_0, \infty)$ for some $t_0 \geqq \varepsilon$, from which the desired conclusion (38) follows. Suppose the assertion is false. Then there occur two cases: $f(t)$ is not of constant sign on $[\tau, \infty)$ for any large $\tau \geqq \varepsilon$; $f(t) \geqq 0$ on $[\tau, \infty)$ for some $\tau \geqq \varepsilon$. First suppose the former is the case, i. e. there exists an interval $[a, b] \subset [\varepsilon, \infty)$ such that $f(a)=f(b)=0$ and $f(t) \geqq 0$ on $[a, b]$. By the mean value theorem there exists a $c \in (a, b)$ such that $f'(c)=0$. Then $a_Q'(c)=kT'(c)$ and $a_Q(c) \leqq kT(c)$. A fortiori

$$Q(c)=-a_Q'(c)+a_Q(c)^2 \leqq -kT''(c)+k^2T(c)^2=Q_0(c)\,.$$

This contradicts (40). Next we consider the case where there exists a $t_0 \geqq \varepsilon$ such that $f(t) \geqq 0$ on $[t_0, \infty)$. We treat this case in the following two exclusive situations. First suppose there exists a $c \in [t_0, \infty)$ such that $f'(c) \leqq 0$. Then $a_Q'(c) \geqq kT'(c)$ and $a_Q(c) \leqq kT(c)$. By the similar consideration as above we deduce $Q(c) \leqq Q_0(c)$, a contradiction. Otherwise $f'(t)>0$ on $[t_0, \infty)$. Let $\psi_Q$ be for the interval $[t_0, \infty)$ and set

$$\psi(t)=\int_{t_0}^t kT(s)ds\,.$$

The maximum principle applied to $e^{-\phi(t)}+\eta t-v_Q(t)$ ($v_Q=e^{-\phi_Q}$, $\eta>0$) for the operator $Lg=g''-Qg$ on the interval $[t_0, \tau]$ for sufficiently large $\tau>0$ yields $e^{-\phi(t)}+\eta t-v_Q(t)\geqq 0$ on $[t_0, \tau)$ and hence on $[t_0, \infty)$. Here we have used the following:

$$L(e^{-\phi(t)}+\eta t-v_Q(t))=-\eta tQ(t)+(Q_0(t)-Q(t))e^{-\phi(t)}\leqq 0$$

on $[t_0, \infty)$. On making $\eta\to 0$, we conclude that $e^{-\phi(t)}\geqq e^{-\phi_Q(t)}$ on $[t_0, \infty)$, i. e.,

$$\phi(t)\leqq\phi_Q(t) \tag{41}$$

on $[t_0, \infty)$. On the other hand,

$$-\phi_Q''(t)+kT'(t)=-a_Q'(t)+kT'(t)=f'(t)>0$$

on $[t_0, \infty)$, and the integration of the above inequality over $[t_0, t]$ yields

$$\phi_Q'(t)\leqq kT(t)-c$$

where $c=-\phi_Q'(t_0)+kT(t_0)=-a_Q(t_0)+kT(t_0)=f(t_0)\geqq 0$. Hence we have

$$\phi_Q'(t)\leqq kT(t)$$

on $[t_0, \infty)$. Again by the integration of the above over $[t_0, t]$ we deduce

$$\phi_Q(t)\leqq\phi(t)$$

on $[t_0, \infty)$. This with (41) implies that $\phi=\phi_Q$ on $[t_0, \infty)$, i. e.

$$Q_0(t)=-\phi''(t)+\phi'(t)^2=-\phi_Q''(t)+\phi_Q'(t)^2=Q(t)$$

on $[t_0, \infty)$, a contradiction.

## § 4. $T$-tests.

**4.1.** Based on the foregoing discussions we are now able to state one of our main results in this paper: tests (sufficient conditions) for $\alpha(P)=0$ and also for $\alpha(P)>0$. Our tests will use auxiliary functions $T$ and thus we shall refer them as $T$-tests. We start with a test for $\alpha(P)=0$. For convenience we shall call a nonnegative $C^1$ function $T$ on $[\tau, \infty)$ $(\tau\geqq 0)$ an *upper tester* if the following two conditions are satisfied:

$$\limsup_{t\to\infty}\frac{T'(t)}{T(t)^2}<\infty\,; \tag{42}$$

$$\int_\varepsilon^\infty\frac{dt}{T(t)}=\infty \tag{43}$$

for any $\varepsilon\geqq 0$. If $T$ is an upper tester, then $T+c$ with a nonnegative constant $c$ is again an upper tester. We maintain the following *upper $T$-test*:

THEOREM. *The Picard principle is valid for a density $P$, i.e. $\alpha(P)=0$, if* *there exists an upper tester $T$ such that*

$$\limsup_{r\to 0}\frac{P(r)}{r^{-2}T(\log r^{-1})^2}<\infty\,. \tag{44}$$

**4.2.** The proof is by the reduction to Theorem 2.6. On setting $Q(t)=e^{-2t}P(e^{-t})$, (44) takes the form

$$\limsup_{t\to\infty} Q(t)/T(t)^2<\infty\,.$$

By the proposition in 3.1, we have

$$\limsup_{t\to\infty} a_Q(t)/T(t)<\infty$$

with $a_Q=\psi'_Q$. By replacing $T$ by $T+1$, if necessary, we may assume that $T\geqq 1$. Thus, if we choose $\eta>0$ sufficiently small and $\varepsilon>0$ sufficiently large, then

$$\eta T'/T^2+2\eta\psi'_Q/T+\eta^2/T^2\leqq 1 \tag{45}$$

on $[\varepsilon,\infty)$. Set

$$b(t)=\eta\int_\varepsilon^t T(s)^{-1}ds\,.$$

Then $b(t)$ is a nonnegative $C^2$ function on $[\varepsilon,\infty)$ and (45) implies that

$$-b''+2\psi'_Q\cdot b'+b'^2\leqq 1$$

and $\limsup_{t\to\infty} b(t)=\eta\int_\varepsilon^\infty T(s)^{-1}ds=\infty$. A fortiori, by Theorem 2.6, we conclude that $\alpha(P)=0$.

**4.3.** We turn to a test for $\alpha(P)>0$. Similarly as in 4.1, a nonnegative $C^1$ function $T$ on $[\tau,\infty)$ $(\tau\geqq 0)$ will be referred to as a *lower tester* if the following two conditions are satisfied:

$$\liminf_{t\to\infty}\frac{T'(t)}{T(t)^2}\geqq 0\,; \tag{46}$$

$$\int_\varepsilon^\infty\frac{dt}{T(t)}<\infty \tag{47}$$

for a certain $\varepsilon\geqq\tau$. We assert the following *lower $T$-test*:

THEOREM. *The Picard principle is invalid for a density $P$, i.e. $\alpha(P)>0$, if there exists a lower tester $T$ such that*

$$\liminf_{t\to\infty}\frac{P(r)}{r^{-2}T(\log r^{-1})^2}>0\,. \tag{48}$$

**4.4.** As in 4.2, (48) takes on the following form for $Q(t)=e^{-2t}P(e^{-t})$: $\liminf_{t\to\infty} Q(t)/T(t)^2>0$. By Proposition 3.1, $\liminf_{t\to\infty} a_Q(t)/T(t)>0$ with $a_Q=\psi'_Q$. Choose a sufficiently large $\eta>0$ and an $\varepsilon>0$ such that (47) is valid and

$$\eta T'/T+2\eta\psi'_Q/T+\eta^2/T^2\geqq 1 \tag{49}$$

on $[\varepsilon, \infty)$. Consider the function $b(t)=\eta\int_{\varepsilon}^{t} T(s)^{-1}ds$ which is a nonnegative $C^2$ function on $[\varepsilon, \infty)$ and (49) implies that

$$-b''+2\phi'_Q\cdot b'+b'^2 \geqq 1$$

on $[\varepsilon, \infty)$ and $\liminf_{t\to\infty} b(t)=\eta\int_{\varepsilon}^{\infty} T(s)^{-1}ds<\infty$. By Theorem 2.6, we conclude that $\alpha(P)>0$.

**4.5.** To derive the applications of the above $T$-tests stated in the introduction we consider the iterated logarithms $\log_j t$ $(j=0, 1, \cdots)$ defined by

$$\log_0 t=t\,, \qquad \log_{j+1} t=\log(\log_j t) \qquad (j=0, 1, \cdots)\,.$$

Consider the function

$$T_{n,\mu}(t)=(\prod_{j=0}^{n-1}\log_j t)\cdot(\log_n t)^{1+\mu}$$

where $\mu\geqq 0$ and $n=0, 1, \cdots$. Here the convention $\prod_{j=0}^{-1}=1$ is made, i.e. $T_{0,\mu}(t)=t^{1+\mu}$ and $T_{1,\mu}(t)=t\cdot(\log t)^{1+\mu}$. It is easily checked that $T_{n,\mu}$ is an upper tester (lower tester, resp.) if $\mu=0$ $(\mu>0$, resp.). The condition (44) with $T=T_{n,0}$ or $T=1$, a trivial upper tester, takes the form

$$\limsup_{r\to 0}\frac{P(r)}{r^{-2}(\prod_{j=1}^{n}\log_j r^{-1})^2}<\infty \tag{50}$$

for some $n=0, 1, \cdots$. The condition may be restated as

$$\limsup_{r\to 0}\frac{P(r)}{r^{-2}\cdot(\prod_{j=1}^{n-1}\log_j r^{-1})^2\cdot(\log_n r^{-1})^2}=0 \tag{51}$$

for some $n=1, 2, \cdots$. The condition (48) with $T=T_{n,\varepsilon/2}$ $(\varepsilon>0)$ takes the form

$$\liminf_{r\to 0}\frac{P(r)}{r^{-2}\cdot(\prod_{j=1}^{n-1}\log_j r^{-1})^2\cdot(\log_n r^{-1})^{2+\varepsilon}}>0 \tag{52}$$

for some $n=1, 2, \cdots$. Thus we have

THEOREM. *The Picard principle is valid for a density $P$, i.e. $\alpha(P)=0$, if the condition* (50) *or* (51) *is satisfied. The Picard principle is invalid for a density $P$, i.e. $\alpha(P)>0$, if the condition* (52) *is satisfied.*

## § 5. Comparisons.

**5.1.** It seems likely that $\dim cP=\dim P$ $(c>0)$ and $\dim P_1\leqq\dim P_2$ if $P_1\leqq P_2$ even for general densities $P$, $P_1$ and $P_2$ (cf. [**13, 14**]). Although we are unable to settle these questions yet, we shall show that the latter is the case at least for rotation free densities $P_1$ and $P_2$*):

PROPOSITION. *If* $P_1 \leqq P_2$, *then* $\dim P_1 \leqq \dim P_2$.

We only have to show that $\alpha(P_2)=0$ implies that $\alpha(P_1)=0$. As before set $Q_i(t)=e^{-2t}P_i(e^{-t})$ $(i=1, 2)$. By Proposition 2.5, $\alpha(P_2)=0$ implies that

$$-w''_{Q_2}+2\psi'_{Q_2}\cdot w'_{Q_2}+w'^2_{Q_2}=1 \tag{53}$$

on $[\varepsilon, \infty)$ $(\varepsilon \geqq 0)$ and $\lim_{t\to\infty} w_{Q_2}(t)=\infty$. Since $Q_1 \leqq Q_2$, Lemma 2.4 assures that $\psi'_{Q_1} \leqq \psi'_{Q_2}$. Therefore, in view of (53), $b=w_{Q_2}$ satisfies

$$-b''+2\psi'_{Q_1}\cdot b'+b'^2 \leqq 1$$

on $[\varepsilon, \infty)$ and $\lim_{t\to\infty} b(t)=\infty$. By Theorem 2.6, we deduce that $\alpha(P_1)=0$.

**5.2.** If $P$ and $P_0$ are densities, then, by the above, we have the inequality $\dim(P+P_0) \geqq \dim P$. When does the equality hold? We shall use the following in the next section:

PROPOSITION. *If* $P_0 \in L^\lambda(0, 1]$ $(\lambda>1)$ *or* $\limsup_{r\to 0} P_0(r)/r^{-2}<\infty$, *then* $\dim(P+P_0)=\dim P$.

We only have to show that $\dim(P+P_0) \leqq \dim P$, and for this purpose it is sufficient to prove that $\alpha(P)=0$ implies $\alpha(P+P_2)=0$. We first treat the case $P_0 \in L^\lambda$. Let $e_n$ and $\bar{e}_n$ $(n=0, 1, \cdots)$ be as in no. 1.1 for $P$ and $\bar{P}=P+P_0$. Let $p_\varepsilon$ be the unique bounded solution of $r^{-1}(ru')'=P_0 u$ on $(0, \varepsilon]$ with $p_\varepsilon(\varepsilon)=1$. Then (cf. e. g. [**14**])

$$p_\varepsilon(r)=1-\int_0^\varepsilon \min\left(\log\frac{\varepsilon}{r}, \log\frac{\varepsilon}{t}\right)\cdot P_0(t)p_\varepsilon(t)t\,dt\,.$$

In view of $p_\varepsilon(t) \leqq 1$ and $P_0(t) \in L^\lambda(0, 1]$ $(\lambda>1)$, the Hölder inequality assures that the integral on the right can be made arbitrary small by choosing $\varepsilon>0$ sufficiently small. Thus $p_\varepsilon(0)>0$. Set $p=p_1$ and observe that $p_\varepsilon(r)=p(r)/p(\varepsilon)$. Therefore

$$p(0)=\lim_{r\to 0} p(r)>0\,.$$

Since $p'$ and $e'_0$ are nonnegative, the inequality

$$r^{-1}(r\cdot(pe_0)')' \geqq \bar{P}\cdot(pe_0)$$

follows from the identity $r^{-1}(r\cdot(pe_0)')'=\bar{P}\cdot(pe_0)+2p'\cdot e'_0$. Thus by the comparison principle (cf. Lemma 2.2; apply the lemma by changing variables $t=-\log r$), we deduce

$$pe_0 \leqq \bar{e}_0$$

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*) After the completion of this work the author found that the former is also the case for rotation free densities $P$, i. e. $\dim cP=\dim P$ $(c>0)$. The proof is based on the equivalence of $\int_0^\infty \frac{dt}{a_Q(t)+1}=\infty$ $(Q(t)=e^{-2t}P(e^{-t}))$ and $\dim P=1$. The detail will be discussed elsewhere.

on (0, 1]. Observe that $\bar{P}+r^{-2}\geqq P+r^{-2}$. Again by the comparison principle, $\bar{e}_1\leqq e_1$. Then

$$\bar{e}_1/\bar{e}_0\leqq e_1/pe_0=(e_1/e_0)/p$$

and a fortiori $\alpha(\bar{P})\leqq\alpha(P)/p(0)$. Since $p(0)>0$ and $\alpha(P)=0$, we deduce that $\alpha(P+P_0)=\alpha(\bar{P})=0$.

**5.3.** We next consider the case $\lim_{r\to 0}P_0(r)/r^{-2}<\infty$. For a sufficiently large integer $k>0$, $P_0(r)\leqq(3k)^2/r^{-2}$. Again assuming $\alpha(P)=0$, we shall show that $\alpha(P+P_0)=0$. Since $\dim(P+P_0)\leqq\dim(P+(3k)^2/r^2)$, we only have to prove $\alpha(P+(3k)^2/r^2)=0$. Let $e_n$ and $\bar{e}_n$ be as in no. 1.1 for $P$ and $\bar{P}=P+(3k)^2/r^2$. Observe that $(3k)^2+(4k)^2=(5k)^2$ and thus

$$\bar{e}_0=e_{3k}, \qquad \bar{e}_{4k}=e_{5k}.$$

On the other hand we have

$$e_n/e_{n-1}\geqq e_{n+1}/e_n \qquad (n=1, 2, \cdots)$$

(cf. [13]). Therefore we see that

$$\frac{\bar{e}_{4k}}{\bar{e}_0}=\frac{e_{5k}}{e_{3k}}=\frac{e_{5k}}{e_{5k-1}}\cdot\frac{e_{5k-1}}{e_{5k-2}}\cdots\frac{e_{3k+1}}{e_{3k}}\leqq\left(\frac{e_1}{e_0}\right)^{2k},$$

and thus by (9)

$$\alpha(\bar{P})\leqq(\alpha_{4k}(\bar{P}))^{2/(3^{4k}-1)}\leqq(\alpha(P))^{4k/(3^{4k}-1)}=0.$$

## § 6. Almost increasing densities.

**6.1.** Hereafter we shall consider special rotation free densities $P$ such that $r^2P(r)$ are increasing as $r\to 0$ or more generally 'almost increasing' as $r\to 0$. Here a function $f$ is *almost increasing* as $r\to 0$ on (0, 1] if there exist a constant $c\geqq 1$ and a constant $k\geqq 0$ such that

$$f(r_1)+k\leqq c(f(r_2)+k) \tag{54}$$

for every $0<r_2\leqq r_1\leqq 1$. If we can choose $c$ to be the unity 1, then $f$ is increasing in the usual sense as $r\to 0$. Increasing functions, decreasing functions, bounded functions, and functions which are sums of increasing and bounded functions are simple examples of almost increasing functions. By Theorems 4.1 and 3, we see that if $r^2P(r)$ is 'not so large' as $r\to 0$ then $\alpha(P)=0$ and that if $r^2P(r)$ is 'enough large' as $r\to 0$ then $\alpha(P)>0$. We wish to describe the rate of 'this largeness' exactly. We are able to perform this if $r^2P(r)$ is supposed to be almost increasing as $r\to 0$. Namely we state another of our main results in this paper:

THEOREM. *The Picard principle is valid for a density $P$ such that $r^2P(r)$ is almost increasing as $r\to 0$, i.e. $\alpha(P)=0$, if and only if*

(55) $$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}} = \infty .$$

*In other words the Picard principle is invalid for a density $P$ such that $r^2P(r)$ is almost increasing as $r \to 0$, i.e. $\alpha(P) > 0$, if and only if*

(56) $$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}} < \infty .$$

We are unable to decide whether the condition of almost increasingness is really needed. We rather feel something like (55) would be a complete condition for $\dim P = 1$ for general densities. The proof will be given for increasing $r^2P(r)$ first and then for almost increasing $r^2P(r)$. The divergence of (55) for $P$ is equivalent to that for $P+1$. By Proposition 5.2, $\dim(P+1) = \dim P$, and thus $\alpha(P+1) = 0$ is equivalent to $\alpha(P) = 0$. For this reason we my assume in the proof of the above theorem that

(57) $$P(r) \geqq 1$$

on $(0, 1]$. This condition will be assumed throughout nos. 6.2–6.5.

**6.2.** First we assume that $r^2P(r)$ is increasing as $r \to 0$. Set, as we have been doing repeatedly, $Q(t) = e^{-2t}P(e^{-t}) > 0$. Then since $Q(t) = r^2P(r)$ $(r = e^{-t})$, $Q(t)$ is an increasing function on $[0, \infty)$ as $t \to \infty$. As a result we see that $a'_Q = \psi''_Q$ does not change its sign on a certain $[\varepsilon, \infty)$ $(\varepsilon \geqq 0)$. If this were not the case there would exist an interval $[\varepsilon_1, \varepsilon_2]$ $(\varepsilon_1 < \varepsilon_2)$ such that

$$a'_Q(\varepsilon_1) = a'_Q(\varepsilon_2) = 0 , \qquad a'_Q \leqq 0 , \qquad a'_Q \not\equiv 0$$

on $[\varepsilon_1, \varepsilon_2]$. Then $a_Q(\varepsilon_1) > a_Q(\varepsilon_2)$. On the other hand, since

$$-a'_Q + a_Q^2 = Q ,$$

we have $Q(\varepsilon_1) = a_Q(\varepsilon_1)^2 > a_Q(\varepsilon_2)^2 = Q(\varepsilon_2)$, which contradicts the increasingness of $Q$. By the assumption (57), $Q(0) = P(1) \geqq 1$. Thus the increasingness of $Q$ implies that

(58) $$Q(t) \geqq 1$$

on $[0, \infty)$. By the change of variable $t = -\log r$

$$\int_0^1 \frac{dr}{r\sqrt{r^2P(r)+1}} = \int_0^\infty \frac{dt}{\sqrt{Q(t)+1}} .$$

In view of (58) and the above identity, the condition (55) is then equivalent to the following: for every $\varepsilon > 0$

(59) $$\int_\varepsilon^\infty \frac{dt}{\sqrt{Q(t)}} = \infty .$$

**6.3.** Another consequence of (58) for $a_Q = \psi'_Q$ is the following. Since

$-1'+1^2=1\leqq Q$, Lemma 2.7 assures that $a_Q\geqq 1$. Let $a_Q'$ be of constant sign on $[\varepsilon, \infty)$ $(\varepsilon\geqq 0)$ (cf. 6.2), and consider the function

$$b(t)=\eta\int_\varepsilon^t \frac{ds}{a_Q(s)} \tag{60}$$

which is of class $C^2$ on $[\varepsilon, \infty)$. Observe that

$$\begin{aligned}-b''+2a_Q\cdot b'+b'^2 &= \eta\cdot a_Q'/a_Q^2+2\eta+\eta^2/a_Q^2\\ &= \eta\cdot(1-Q/a_Q^2)+2\eta+\eta^2/a_Q^2\\ &\leqq 3\eta+\eta^2\end{aligned}$$

on $[\varepsilon, \infty)$. We fix an $\eta>0$ so small that $3\eta+\eta^2\leqq 1$, and we have

$$-b''+2\phi_Q'\cdot b'+b'^2\leqq 1$$

on $[\varepsilon, \infty)$. Suppose $a_Q'\leqq 0$ on $[\varepsilon, \infty)$. Then $Q=-a_Q'+a_Q^2\geqq a_Q^2$. Thus

$$\lim_{t\to\infty} b(t)=\eta\int_\varepsilon^\infty \frac{ds}{a_Q(s)}\geqq \eta\int_\varepsilon^\infty \frac{ds}{\sqrt{Q(s)}}=\infty$$

if the condition (59) is assumed. By Theorem 2.6 we deduced that $\alpha(P)=0$. Next suppose $a_Q'\geqq 0$ on $[\varepsilon, \infty)$. Then from

$$\frac{1}{Q}=\frac{1}{a_Q^2}+\frac{1}{Q}\cdot\frac{a_Q'}{a_Q^2}$$

it follows that

$$\begin{aligned}\frac{1}{\sqrt{Q}} &\leqq \frac{1}{a_Q}+\sqrt{\frac{1}{Q}\cdot\frac{a_Q'}{a_Q^2}}\\ &\leqq \frac{1}{a_Q}+\frac{1}{2}\left(\frac{1}{Q}+\frac{a_Q'}{a_Q^2}\right)\\ &\leqq \frac{1}{a_Q}+\frac{1}{2}\left(\frac{1}{\sqrt{Q}}-\left(\frac{1}{a_Q}\right)'\right).\end{aligned}$$

Here the essential use of (58) is made. A fortiori

$$\frac{1}{a_Q}\geqq\frac{1}{2}\frac{1}{\sqrt{Q}}+\frac{1}{2}\left(\frac{1}{a_Q}\right)'.$$

Since $a_Q\geqq 1$ is increasing, $1/a_Q(\infty)=\lim_{t\to\infty} 1/a_Q(t)\leqq 1$ exists and

$$\lim_{t\to\infty} b(t)=\eta\int_\varepsilon^\infty \frac{ds}{a_Q(s)}\geqq\frac{\eta}{2}\int_\varepsilon^\infty\frac{ds}{\sqrt{Q(s)}}+\frac{\eta}{2}\left(\frac{1}{a_Q(\infty)}-\frac{1}{a_Q(\varepsilon)}\right).$$

Therefore $\lim_{t\to\infty} b(t)=\infty$ if the condition (59) is supposed. Again by Theorem 2.6 we conclude that $\alpha(P)=0$.

**6.4.** Our next purpose is to prove the implication of (59) from the assumption $\alpha(P)=0$. First assume that $a_Q'\leqq 0$ on $[\varepsilon, \infty)$. Then $a_Q$ is decreasing and

$a_Q \leqq a_Q(\varepsilon) \equiv k$ on $[\varepsilon, \infty)$. In view of

$$\frac{1}{k^2} \leqq \frac{1}{a_Q^2} = \frac{1}{Q} + \frac{1}{Q} \cdot \frac{-a_Q'}{a_Q^2}$$

on $[\varepsilon, \infty)$ we deduce that

$$\frac{1}{k} \leqq \frac{1}{\sqrt{Q}} + \sqrt{\frac{1}{Q} \cdot \frac{-a_Q'}{a_Q^2}}$$
$$\leqq \frac{1}{\sqrt{Q}} + \frac{1}{2}\left(\frac{1}{\sqrt{Q}} + \left(\frac{1}{a_Q}\right)'\right)$$

on $[\varepsilon, \infty)$ (cf. 6.3). Since $a_Q \geqq 1$ is decreasing, $1/a_Q(\infty) \leqq 1$ exists as in 6.3. Therefore we deduce that

$$\infty = \int_\varepsilon^\infty \frac{ds}{k} \leqq \frac{3}{2}\int_\varepsilon^\infty \frac{ds}{\sqrt{Q(s)}} + \frac{1}{2}\left(\frac{1}{a_Q(\infty)} - \frac{1}{a_Q(\varepsilon)}\right),$$

from which the condition (59) follows. Here observe that the condition $\alpha(P) = 0$ is not used. Actually the condition $a_Q' \leqq 0$ is sufficient both for $\alpha(P) = 0$ and (59). Therefore the essential case is when $a_Q' = \psi_Q'' \geqq 0$ on $[\varepsilon, \infty)$. We once more use the function $b(t)$ given by (60) with $\eta = 1$. Observe that

$$-b'' + 2a_Q \cdot b' + b'^2 = a_Q'/a_Q^2 + 2 + 1/a_Q^2 \geqq 2$$

and in particular

$$-b'' + 2\psi_Q' \cdot b' + b'^2 \geqq 1$$

on $[\varepsilon, \infty)$. We maintain that

$$\int_\varepsilon^\infty \frac{ds}{a_Q(s)} = \infty .$$

If this were not the case, then we would have

$$\lim_{t\to\infty} b(t) = \int_\varepsilon^\infty \frac{ds}{a_Q(s)} < \infty .$$

Then, by Theorem 2.6, we must have $\alpha(P) > 0$, which contradicts our assumption $\alpha(P) = 0$. Since we have $Q = -a_Q' + a_Q^2 \leqq a_Q^2$, we deduce that

$$\int_\varepsilon^\infty \frac{ds}{\sqrt{Q(s)}} \geqq \int_\varepsilon^\infty \frac{ds}{a_Q(s)} = \infty ,$$

i.e. (59) is valid. The theorem is herewith proved when $r^2 P(r)$ is increasing as $r \to 0$.

**6.5.** We proceed to the case when $r^2 P(r)$ is almost increasing as $r \to 0$, i.e. there exist a constant $c \geqq 1$ and a constant $k \geqq 0$ such that

$$r_1^2 P(r_1) + k \leqq c(r_2^2 P(r_2) + k) \tag{61}$$

for every pair $(r_1, r_2)$ with $0 < r_2 \leqq r_1 \leqq 1$. We may assume that $c > 1$. We set

$$P^*(r)=r^{-2}\max_{r\leqq s\leqq 1} s^2P(s)$$

for $r\in(0, 1]$. Then clearly $P^*(z)=P^*(|z|)$ is a rotation free density on $0<|z|\leqq 1$ such that $r^2P^*(r)$ is *increasing* as $r\to 0$ and

$$P(r)\leqq P^*(r)\,,\qquad c^{-1}P^*(r)\leqq P(r)+lr^{-2} \tag{62}$$

with $l=k(c-1)/c$. By the second of the above inequality, the condition (55) for $P$ implies that for $P^*$ and a fortiori $\alpha(P^*)=0$. By the first inequality in (62), Proposition 5.1 assures that $\dim P\leqq \dim P^*$ and thus $\alpha(P)=0$. Conversely assume that $\alpha(P)=0$. By Propositions 5.1 and 2, (62) implies that

$$\dim(c^{-1}P^*)\leqq \dim(P+lr^{-2})=\dim P=1\,,$$

i.e. $\alpha(c^{-1}P^*)=0$. Since $c^{-1}P^*$ is also a density such that $r^2(c^{-1}P^*(r))=c^{-1}r^2P^*(r)$ is increasing as $r\to 0$, the condition (55) for $c^{-1}P^*$ follows from $\alpha(c^{-1}P^*)=0$. This implies the validity of (55) for $P^*$ and then for $P$ by $P\leqq P^*$. The proof of Theorem 6.1 is herewith complete.

**6.6.** Since $\alpha(P)=0$ ($>0$, resp.) if there exists another $P_0\geqq P$ ($\leqq P$, resp.) with $\alpha(P_0)=0$ ($>0$, resp.), we can formulate various criteria for $\alpha(P)=0$ and $\alpha(P)>0$ based on Theorem 6.1. As an example we state the following:

THEOREM. *The Picard principle is valid for a density $P$, i.e. $\alpha(P)=0$, if*

$$\int_0^1 \frac{dr}{r\sqrt{\max_{0\leqq s\leqq r} s^2P(s)+1}}=\infty\,. \tag{63}$$

*The Picard principle is invalid for a density $P$, i.e. $\alpha(P)>0$, if*

$$\int_0^1 \frac{dr}{r\sqrt{\min_{0\leqq s\leqq r} s^2P(s)+1}}<\infty\,. \tag{64}$$

Set $P_1(r)=r^{-2}\max_{r\leqq s\leqq 1} s^2P(s)$ for $r\in(0, 1]$. Then $P_1$ is a density for which $r^2P_1(r)$ is increasing as $r\to 0$. The condition (63) means that $P_1$ satisfies (56) and a fortiori $\alpha(P_1)=0$. Since $P_1\geqq P$, Proposition 5.1 implies that $\alpha(P)=0$. Similarly we set $P_2(r)=r^{-2}\min_{0\leqq s\leqq r} s^2P(s)$ for $r\in(0, 1]$. Clearly $P_2(r)$ is a density such that $r^2P_2(r)$ is increasing as $r\to 0$. The condition (64) assures that $P_2$ satisfies (57) and thus $\alpha(P_2)>0$. Obviously $P_2\leqq P$ and again by Proposition 5.1 we see that $\alpha(P)>0$.

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