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Reduction of Monge-Ampère's equations by Imschenetsky transformations

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§ 1. Introduction.

Due to Imschenetsky, we have a method of transforming Monge-Ampère's equations, which is a generalization of Laplace's method of transforming linear hyperbolic equations. Monge-Ampère's equation to which an Imschenetsky transformation can be applied is said to be of Imschenetsky type. Generalizing Monge's method, the author [7], [8] gave a method of integrating Monge-Ampère's equations by integrable systems. Here, applying this method of integration to an equation of Imschenetsky type, we shall prove that the transformed equation is solved by integrable systems of order $n-1$ if and only if the original equation is solved by integrable systems of order n. For an equation of Imschenetsky type, we shall define its invariants H_n $(n\geqq 0)$ and l_n $(n\geqq 1)$, and prove that the given equation can be reduced by n-times applications of the Imschenetsky transformation to an equation solved by Monge's method of integration if and only if $H_n=0$ and $l_1=\cdots=l_n=0$. In the special cases, these results were obtained in [7], [8].

We shall discuss the problem of solving a hyperbolic equation of the second order of the form

$$s+f(x, y, z, p, q)=0 \tag{1.1}$$

by integrating ordinary differential equations along the characteristic $dx=dz-qdy=dp+fdy=0$, where $s=\partial^2 z/\partial x\partial y$, $p=\partial z/\partial x$, $q=\partial z/\partial y$. Monge-Ampère's equation whose two characteristics are different is transformed by a contact transformation to an equation of the form (1.1) if and only if it has an intermediate integral of the first order with respect to each of its two characteristics. The method of integration for solving the Cauchy problem of (1.1) by integrable systems given in [7], [8] is as follows: Consider the Cauchy problem in the space of $x, y, z, p, q_1, \cdots, q_n$ which involves the derivatives of higher order $q_i=\partial^i z/\partial y^i (q_1=q)$ with respect to y. Then it requires us to find a two-dimensional submanifold which satisfies the system of Pfaffian equations

$$(1.2) \qquad dz-pdx-qdy=dq_1+f_0dx-q_2dy$$
$$=dq_2+f_1dx-q_3dy=\cdots=dq_{n-1}+f_{n-2}dx-q_ndy=0$$

and contains a given initial curve satisfying (1.2). Here, f_i is a function of $x, y, z, p, q_1, \cdots, q_{i+1}$ defined inductively by

$$(1.3) \qquad f_i=\Big(G_i-f\frac{\partial}{\partial p}\Big)f_{i-1} \ (i\geqq 1), \qquad f_0=f$$

with

$$(1.4) \qquad G_i=\frac{d}{dy}+\sum_{j=1}^{i} q_{j+1}\frac{\partial}{\partial q_j} (i\geqq 1), \qquad G_0=\frac{d}{dy}=\frac{\partial}{\partial y}+q\frac{\partial}{\partial z}.$$

A system of ordinary differential equations

$$(1.5) \qquad \frac{dx}{0}=\frac{dy}{1}=\frac{dz}{q_1}=\frac{dp}{-f}=\frac{dq_1}{q_2}=\cdots=\frac{dq_{n-1}}{q_n}=\frac{dq_n}{u}$$

with a function u of $x, y, z, p, q_1, \cdots, q_n$ is said to be "integrable" if u is a solution of the following system of two linear partial differential equations

$$(1.6) \qquad \frac{\partial u}{\partial p}=0, \quad \Big(\frac{d}{dx}-\sum_{i=1}^{n} f_{i-1}\frac{\partial}{\partial q_i}\Big)u+\frac{\partial f}{\partial q}u+\Big(G_{n-1}-f\frac{\partial}{\partial p}\Big)f_{n-1}=0,$$

where $d/dx=\partial/\partial x+p\partial/\partial z$. Suppose that an initial curve is given so that it satisfies (1.2) and $dq_n+f_{n-1}dx-udy=0$. Then the surface obtained by integrating (1.5) under the given initial condition satisfies (1.2) and $dq_n+f_{n-1}dx-udy=0$ for each of such initial curves if and only if the system (1.5) is integrable. Hence, if the system (1.5) is integrable, the surface thus obtained gives a solution of the Cauchy problem; For the integral surface $z=\phi(x, y)$, $p=\psi(x, y)$, $q_i=\phi_i(x, y)$, $1\leqq i\leqq n$ of the system (1.5) under the given initial condition satisfies

$$\frac{\partial\phi}{\partial x}=\psi(x, y), \quad \frac{\partial\phi}{\partial y}=\phi_1(x, y), \quad \frac{\partial\phi_i}{\partial y}=\phi_{i+1}, \qquad 1\leqq i<n,$$
$$\frac{\partial\phi_i}{\partial x}=-f_{i-1}(x, y, \phi, \psi, \phi_1, \cdots, \phi_i), \qquad 1\leqq i<n$$

by (1.2), and hence

$$\frac{\partial\psi}{\partial y}=\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial\phi_1}{\partial x}=-f_0=-f,$$
$$\phi_i=\frac{\partial^i\phi}{\partial y^i}, \qquad 1\leqq i\leqq n.$$

Here we assumed that $n>1$. In the case where $n=1$, the integral surface also gives a solution of the Cauchy problem by $dz-pdx-qdy=dq+fdx-udy=0$.

Any system of linear partial differential equations of the first order with one unknown function can be prolonged either to a complete system or to an incompatible system by adding the compatibility conditions. If we get a complete system consisting of r independent equations by this prolongation, then the original system is said to have the rank $m-r$, where m is the number of the independent variables ([7]). Let n be a positive integer. Then equation (1.1) is said to be solved by integrable systems of order n if the system of linear equations (1.6) has its rank greater than zero ([8]). In this case, the Cauchy problem is solved by integrating an integrable system for any initial curve satisfying (1.2), since we can find such a solution u of (1.6) that satisfies $dq_n+f_{n-1}dx-udy=0$ along the given initial curve. If equation (1.1) is solved by Monge's method of integration, then let us say that it is solved by integrable systems of order 0. In this case, equation (1.1) is solved by integrable systems of the first order, and the Cauchy problem can be solved by taking as an integrable system the Lagrange-Charpit system of an intermediate integral of the first order for any initial curve ([7]).

A set of four relations

$$x'=x, \quad y'=y, \quad z'=h(x, y, z, q), \quad p'=k(x, y, z, q) \tag{1.7}$$

between x, y, z, p, q and x', y', z', p', q' is called an Imschenetsky transformation if it satisfies

$$\frac{\partial h}{\partial q} \neq 0, \quad \frac{\partial(h, k)}{\partial(z, q)} \neq 0 \tag{1.8}$$

([5]). It gives the following transformation between the two equations from

$$\frac{\partial h}{\partial q}s+\frac{\partial h}{\partial z}p+\frac{\partial h}{\partial x}-k=0 \tag{1.9}$$

to

$$\frac{\partial h}{\partial q}s'-\frac{\partial k}{\partial q}q'-\frac{dk}{dy}\frac{\partial h}{\partial q}+\frac{dh}{dy}\frac{\partial k}{\partial q}=0, \tag{1.10}$$

where we replace x, y, z, q in (1.10) by

$$x=x', \quad y=y', \quad z=h'(x', y', z', p'), \quad q=k'(x', y', z', p'),$$

solving (1.7) with respect to x, y, z, q. Take a solution $z=\phi(x, y)$ of (1.9). Then the surface $z'=h(x', y', \phi(x', y'), \phi_y(x', y'))$ gives a solution of (1.10) and $\partial z'/\partial x'=k(x', y', \phi, \phi_y)$. Conversely take a solution $z'=\phi'(x', y')$ of (1.10). Then the surface $z=h'(x, y, \phi'(x, y), \phi'_{x'}(x, y))$ gives a solution of (1.9) and $\partial z/\partial y=k'(x, y, \phi', \phi'_{x'})$. These two transformations are the inverse of each other.

The original equation (1.9) and the transformed one (1.10) are linear in

p and q' respectively. An equation of the form

$$s+M(x, y, z, q)p+N(x, y, z, q)=0 \tag{1.11}$$

can be the original equation of an Imschenetsky transformation if and only if the coefficients M and N satisfy

$$\frac{\partial M}{\partial x}-N\frac{\partial M}{\partial q}-\frac{\partial N}{\partial z}+M\frac{\partial N}{\partial q}\neq 0\,. \tag{1.12}$$

In this case, equation (1.11) is said to be of Imschenetsky type ([8]). The vanishing of the left-hand member of (1.12) is a necessary and sufficient condition that equation (1.11) be solved by Monge's method of integration ([7]).

Let a, b, c be functions of x, y, and h_0 be $\partial a/\partial x+ab-c$. Then, if $h_0\neq 0$, the set of four relations

$$x'=x,\quad y'=y,\quad z'=q+az,\quad h_0 z=p'+bz'$$

defines an Imschenetsky transformation called a Laplace transformation, which transforms an equation of Laplace linear form

$$s+a(x, y)p+b(x, y)q+c(x, y)z=0 \tag{1.13}$$

to an equation of the same type

$$s'+\left(a-\frac{\partial}{\partial y}\log h_0\right)p'+bq'+\left(c-\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}-b\frac{\partial}{\partial y}\log h_0\right)z'=0\,.$$

The h_0 is called the first invariant of equation (1.13). It is equal to the left-hand member of (1.12) if we set $M=a$ and $N=bq+cz$. The $(n+1)$-th invariant h_n of equation (1.13) is defined inductively by

$$h_n=\frac{\partial a_n}{\partial x}-\frac{\partial b}{\partial y}+h_{n-1}\,,\qquad a_n=a_{n-1}-\frac{\partial}{\partial y}\log h_{n-1}\,,\qquad a_0=a$$

under the condition that $h_{n-1}\neq 0$. The h_n is proved to be the first invariant of the n-times transformed equation by the Laplace transformation. Hence, Laplace linear equation (1.13) is reduced by n-times applications of the Laplace transformation to an equation solved by Monge's method of integration if and only if its $(n+1)$-th invariant h_n vanishes. This theorem is due to Laplace.

Let us apply the method of integration by integrable systems to equation (1.11) of Imschenetsky type. Then we shall obtain the following:

THEOREM 1. *Suppose that the Imschenetsky transformation* (1.7) *transforms equation* (1.11) *to an equation* $s'+f'=0$. *Then the original equation* (1.11) *is solved by integrable systems of order* n *if and only if the transformed equation* $s'+f'=0$ *is solved by integrable systems of order* $n-1$. *Here,* n *is a positive integer.*

This theorem was proved for $n=1$ in [7], and for $n=2$ in [8]. In both the cases the transformed equation $s'+f'=0$ was assumed to be linear in p'.

Combining Theorem 1 with Laplace's theorem, we see that Laplace linear equation (1.13) is solved by integrable systems of order n if and only if its $(n+1)$-th invariant h_n vanishes. This result was obtained in [8] by prolonging the system (1.6) for Laplace linear equation (1.13) to a complete system.

The reduced equation (1.10) by the Imschenetsky transformation (1.7) is not linear in p' in general. An equation of Imschenetsky type was said in [7], [8] to be of Laplace type if one of its reduced equations is linear in p'. In this case, every reduced equation is linear in p'. Here, we shall call such an equation of Imschenetsky type an equation of L_1-type, and define an equation of L_n-type inductively as an equation of Imschenetsky type one of whose reduced equations is of L_{n-1}-type ($n \geqq 2$). In this case, every reduced equation is of L_{n-1}-type. The left-hand member of (1.12) is called the first invariant of equation (1.11) denoted by H_0 ([8]). We shall define in § 3 the $(n+1)$-th invariant H_n ($n \geqq 1$) of (1.11) generalizing the h_n of (1.13). To give a condition that equation (1.11) be of L_n-type, we shall define also in § 3 the l-invariants l_n ($n \geqq 1$). The H_n and l_n ($n \geqq 1$) are rational functions of $q_2, \cdots, q_{n+1}$ whose coefficients are functions of x, y, z, q. If equation (1.11) takes on the form (1.13) we have $H_n = h_n$ ($n \geqq 0$) and $l_n = 0$ ($n \geqq 1$). The following theorem will be proved:

THEOREM 2. *Suppose that* $n \geqq 1$. *Then equation* (1.11) *is of* L_n*-type if and only if* $H_0 \neq 0, \cdots, H_{n-1} \neq 0$ *and* $l_1 = \cdots = l_n = 0$. *Suppose that equation* (1.11) *is of* L_n*-type. Then, it is solved by integrable systems of order* n *if and only if its* $(n+1)$*-th invariant* H_n *vanishes identically.*

An equation of Imschenetsky type is of L_1-type if one of its reduced equations is solved by Monge's method of integration ([7]). In this case, every reduced equation is solved by Monge's method. Hence, by Theorems 1 and 2, equation (1.11) can be reduced by n-times applications of the Imschenetsky transformation to an equation solved by Monge's method of integration if and only if $H_0 \neq 0, \cdots, H_{n-1} \neq 0$, $H_n = l_1 = \cdots = l_n = 0$. In order that $l_n = 0$, M and N should satisfy a system of partial differential equations with independent variables x, y, z, q. The system $l_1 = l_2 = \cdots = l_n = \cdots = 0$ is generated by $l_1 = l_2 = l_3 = 0$; if M and N satisfy $l_1 = l_2 = l_3 = 0$, then they satisfy $l_n = 0$ for every $n \geqq 1$. This is a result from the following theorem due to J. Clairin [5]:

An equation of L_3-type is transformed by a contact transformation either to a Laplace linear equation or to a Moutard equation of the form

$$s + e^z p + \frac{\partial}{\partial y}(b e^{-z}) + c = 0\,,$$

where b, c are functions of x, y.

There exists an example of an equation of L_2-type which is not linear in q ([8]). It can be transformed neither to a Laplace linear equation nor to a Moutard equation by any contact transformation.

We are always in the category of infinite differentiability. The same arguments as above can be made along the other characteristic $dy=dz-pdx=dq+fdx=0$ of equation (1.1). In this case, the Cauchy problem should be considered in the space of $x, y, z, p_1, q, p_2, \cdots, p_n$, where $p_i=\partial^i z/\partial x^i$.

REMARK 1.1. Consider in general a system Σ of Pfaffian equations $\theta_1=\cdots=\theta_t=0$. Then an integral vector field ξ of Σ is called a characteristic of Σ if at every point we have $d\theta_i(\xi, \eta)=0$, $1\leqq i\leqq t$ for all integral vectors η of Σ. In this case $c\xi$ is a characteristic of Σ for any function c. The characteristic ξ of Σ is characterized by the property that Σ is left invariant by the one-parameter group of transformations generated by ξ. The system of Pfaffian equations for defining the characteristics of Σ is called the characteristic system of Σ, which is proved to be completely integrable due to E. Cartan ([1], [2], [3, p. 101]). See Goursat's memoir ([6], pp. 6-7) and E. Cartan's one ([4], pp. 50-60, pp. 78-87). Suppose that Σ is generated by (1.2) and $dq_n+f_{n-1}dx-udy=0$. Then its characteristic system is generated by (1.5) and

$$\text{(1.14)} \qquad \frac{\partial u}{\partial p}dy=\left\{\frac{du}{dx}-\sum_{i=1}^{n} f_{i-1}\frac{\partial u}{\partial q_i}+\frac{\partial f}{\partial q}u+\left(G_{n-1}-f\frac{\partial}{\partial p}\right)f_{n-1}\right\}dy=0\,.$$

These equations (1.5) and (1.14) are derived respectively from

$$dx\wedge dp+dy\wedge dq\equiv dx\wedge(dp+fdy)\equiv 0 \qquad \bmod(\Sigma)$$

and $df_{n-1}\wedge dx-du\wedge dy\equiv 0 \bmod(\Sigma)$. Hence, our system Σ has its non-trivial characteristic if and only if u is a solution of (1.6), and in this case it is given by (1.5).

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§ 2. Imschenetsky transformation.

In this section we shall prove Theorem 1 stated in the introduction. By the definition, the first invariant H_0 of equation (1.11) is given by

$$\text{(2.1)} \qquad H_0=X_1M-Z_1N\,,$$

where X_1, Z_1 are the operators defined by

$$X_1=\frac{\partial}{\partial x}-N\frac{\partial}{\partial q}, \qquad Z_1=\frac{\partial}{\partial z}-M\frac{\partial}{\partial q}.$$

PROPOSITION 2.1. *Equation* (1.11) *can be the original equation of an Imschenetsky transformation* (1.7) *if and only if* $H_0\neq 0$. *In this case we have*

$$H_0=-\frac{\partial(h,k)}{\partial(z,q)}\left(\frac{\partial h}{\partial q}\right)^{-2}. \tag{2.2}$$

PROOF. Suppose that equation (1.11) is the original equation (1.9) of the Imschenetsky transformation (1.7). Then we have

$$Z_1h=\frac{\partial h}{\partial z}-M\frac{\partial h}{\partial q}=0, \qquad X_1h=\frac{\partial h}{\partial x}-N\frac{\partial h}{\partial q}=k, \tag{2.3}$$

since the coefficients M, N of the original equation are given by $\frac{\partial h}{\partial z}\Big/\frac{\partial h}{\partial q}$ and $\left(\frac{\partial h}{\partial q}\right)^{-1}\left(\frac{\partial h}{\partial x}-k\right)$ respectively. By the definition of Imschenetsky transformation, h and k satisfy (1.8). Hence, by (2.1) and (2.3), we have

$$\begin{aligned}
H_0&=X_1M-Z_1N=\left(\frac{\partial h}{\partial q}\right)^{-1}(X_1M-Z_1N)\frac{\partial h}{\partial q}=\left(\frac{\partial h}{\partial q}\right)^{-1}[Z_1, X_1]h\\
&=\left(\frac{\partial h}{\partial q}\right)^{-1}([Z_1, X_1]+X_1Z_1)h=\left(\frac{\partial h}{\partial q}\right)^{-1}Z_1X_1h=\left(\frac{\partial h}{\partial q}\right)^{-1}Z_1k\\
&=\left(\frac{\partial h}{\partial q}\right)^{-1}\left(\frac{\partial k}{\partial z}-M\frac{\partial k}{\partial q}\right)=-\left(\frac{\partial h}{\partial q}\right)^{-2}\left(\frac{\partial h}{\partial z}\frac{\partial k}{\partial q}-\frac{\partial h}{\partial q}\frac{\partial k}{\partial z}\right)\\
&=-\left(\frac{\partial h}{\partial q}\right)^{-2}\frac{\partial(h,k)}{\partial(z,q)}\neq 0.
\end{aligned}$$

Conversely, suppose that M and N satisfy $H_0\neq 0$. Take a solution h of $Z_1h=0$ satisfying $\partial h/\partial q\neq 0$, and define k by $X_1h=k$. Then they satisfy the identity (2.2). Hence, we have $\partial(h,k)/\partial(z,q)\neq 0$ by the assumption that $H_0\neq 0$. Therefore, these h and k define an Imschenetsky transformation which can be applied to equation (1.11).

PROPOSITION 2.2. *Suppose that equation* (1.11) *is of Imschenetsky type and reduced to* $s'+f'=0$ *by* (1.7), *and that* $s^*+f^*=0$ *is the reduced equation of* (1.11) *by another Imschenetsky transformation. Then from the former we obtain the latter changing* x', y', z' *to* x^*, y^*, z^*, *where* $x^*=x'$, $y^*=y'$, $z^*=\lambda(x',y',z')$.

PROOF. Suppose that h^* and k^* define the Imschenetsky transformation which reduces (1.11) to $s^*+f^*=0$. Then h^* is a solution of $Z_1h^*=0$. Since h is a solution of $Z_1h=0$ satisfying $\partial h/\partial q\neq 0$, h^* is expressed in the form $h^*=\lambda(x,y,h)$, where $\partial\lambda/\partial h\neq 0$. The k^* is given by

$$k^*=X_1h^*=X_1\lambda=\frac{\partial\lambda}{\partial x}+\frac{\partial\lambda}{\partial h}X_1h=\frac{\partial\lambda}{\partial x}+\frac{\partial\lambda}{\partial h}k.$$

Change x', y', z' to x^*, y^*, z^*, where $x^*=x', y^*=y', z^*=\lambda(x', y', z')$. Then we have

$$s^*+f^*=s^*-\Big(\frac{\partial h^*}{\partial q}\Big)^{-1}\Big(\frac{\partial k^*}{\partial q}q^*+\frac{dk^*}{dy}\frac{\partial h^*}{\partial q}-\frac{dh^*}{dy}\frac{\partial k^*}{\partial q}\Big)$$
$$=\frac{\partial\lambda}{\partial z'}\Big\{s'-\Big(\frac{\partial h}{\partial q}\Big)^{-1}\Big(\frac{\partial k}{\partial q}q'+\frac{dk}{dy}\frac{\partial h}{\partial q}-\frac{dh}{dy}\frac{\partial k}{\partial q}\Big)\Big\}$$
$$=\frac{\partial\lambda}{\partial z'}(s'+f')\,.$$

LEMMA 2.1. *Let f be $M(x, y, z, q)p+N(x, y, z, q)$ and f_i $(i\geqq 0)$ be the function defined by* (1.3). *Then the f_i takes on the form A_ip+B_i for each $i\geqq 0$. Here, A_i and B_i are the functions of $x, y, z, q_1, \cdots, q_{i+1}$ defined by $A_0=M$, $B_0=N$ and*

$$A_i=G_iA_{i-1}-MA_{i-1}\,,\quad B_i=G_iB_{i-1}-NA_{i-1}\,,\qquad i\geqq 1\,. \tag{2.4}$$

PROOF. Since $A_0=M$ and $B_0=N$, we get $f_0=f=Mp+N=A_0p+B_0$. Suppose that $f_{i-1}=A_{i-1}p+B_{i-1}$. Then, by (2.4), we have

$$f_i=\Big(G_i-f\frac{\partial}{\partial p}\Big)f_{i-1}=\Big\{G_i-(Mp+N)\frac{\partial}{\partial p}\Big\}(A_{i-1}p+B_{i-1})$$
$$=(G_iA_{i-1}-MA_{i-1})p+G_iB_{i-1}-NA_{i-1}=A_ip+B_i\,.$$

PROPOSITION 2.3. *Let n be a positive integer. Then, equation* (1.11) *is solved by integrable systems of order n if and only if the system of two linear equations*

$$Z_nu+\frac{\partial M}{\partial q}u+M_n=0\,,\qquad X_nu+\frac{\partial N}{\partial q}u+N_n=0 \tag{2.5}$$

with independent variables $x, y, z, q_1, \cdots, q_n$ has the rank greater than zero, where Z_n, X_n are the operators defined by

$$Z_n=\frac{\partial}{\partial z}-\sum_{j=1}^{n}A_{j-1}\frac{\partial}{\partial q_j}\,,\quad X_n=\frac{\partial}{\partial x}-\sum_{j=1}^{n}B_{j-1}\frac{\partial}{\partial q_j}\,, \tag{2.6}$$

and M_n, N_n are the functions of $x, y, z, q_1, \cdots, q_n$ defined by

$$M_n=G_{n-1}A_{n-1}-MA_{n-1}\,,\qquad N_n=G_{n-1}B_{n-1}-NA_{n-1}\,. \tag{2.7}$$

PROOF. Replace f in (1.6) by $Mp+N$. Then, by Lemma 2.1, the second equation of (1.6) is written in the form

$$p\Big(Z_nu+\frac{\partial M}{\partial q}u+M_n\Big)+X_nu+\frac{\partial N}{\partial q}u+N_n=0\,.$$

Since the functions A_i, B_i, $\partial M/\partial q$, $\partial N/\partial q$, M_n, N_n do not involve p, the compatibility condition between the two equations in (1.6) is given by the first equation of (2.5). Hence, the system (1.6) is equivalent to the system consisting of (2.5) and $\partial u/\partial p=0$. Each of the two equations of (2.5) and every equation produced by them as a compatibility condition are compatible with $\partial u/\partial p=0$. Thus the two systems (1.6) and (2.5) have the same rank, since the number of independent variables of the second system is diminished from that of the first system by one.

LEMMA 2.2. *Let X_i, Z_i, G_i be the operators defined by* (2.6) *and* (1.4). *Then we have the following identities:*

$$[Z_{i+1}, G_i]=-MZ_i+(G_iA_i)\frac{\partial}{\partial q_{i+1}}, \qquad i\geqq 1, \tag{2.8}$$

$$[X_{i+1}, G_i]=-NZ_i+(G_iB_i)\frac{\partial}{\partial q_{i+1}}, \qquad i\geqq 1. \tag{2.9}$$

PROOF. By the definition (2.4) of A_j, we have

$$\begin{aligned}[Z_{i+1}, G_i]&=\Big[\frac{\partial}{\partial z}-\sum_{j=0}^{i}A_j\frac{\partial}{\partial q_{j+1}}, \frac{\partial}{\partial y}+q\frac{\partial}{\partial z}+\sum_{j=1}^{i}q_{j+1}\frac{\partial}{\partial q_j}\Big]\\ &=-A_0\frac{\partial}{\partial z}-\sum_{j=1}^{i}(A_j-G_jA_{j-1})\frac{\partial}{\partial q_j}+(G_iA_i)\frac{\partial}{\partial q_{i+1}}\\ &=-M\frac{\partial}{\partial z}+\sum_{j=1}^{i}MA_{j-1}\frac{\partial}{\partial q_j}+(G_iA_i)\frac{\partial}{\partial q_{i+1}}\\ &=-MZ_i+(G_iA_i)\frac{\partial}{\partial q_{i+1}}.\end{aligned}$$

Similarly we can prove (2.9) by the definition (2.4) of B_j.

LEMMA 2.3. *Suppose that $n\geqq 2$, and change the independent variables $x, y, z, q_1, \cdots, q_n$ to $x', y', z', p', q_1', \cdots, q_{n-1}'$, where*

$$\begin{cases} x'=x,\ y'=y,\ z'=h,\ p'=k,\ q'=G_1h=\dfrac{dh}{dy}+\dfrac{\partial h}{\partial q}q_2, \\ q_i'=G_i\cdots G_1h=G_{i-1}(G_{i-1}\cdots G_1h)+\dfrac{\partial h}{\partial q}q_{i+1}, \qquad 1<i<n. \end{cases} \tag{2.10}$$

Here G_i $(1\leqq i<n)$ is the operator defined by (1.4), *and h, k are functions of x, y, z, q satisfying* (1.8) *and* (2.3). *Then, we have*

$$Z_n=H_0\frac{\partial h}{\partial q}\frac{\partial}{\partial p'}, \tag{2.11}$$

$$X_n=\frac{d}{dx'}-\sum_{i=1}^{n-1}f'_{i-1}\frac{\partial}{\partial q_i'}+(X_1k)\frac{\partial}{\partial p'}. \tag{2.12}$$

Here, f_i' $(0\leqq i<n-1)$ is the function defined by

$$\begin{cases} f_i'=\Big(G_i'-f'\dfrac{\partial}{\partial p'}\Big)f'_{i-1},\quad G_i'=\dfrac{d}{dy'}+\sum_{j=1}^{i}q'_{j+1}\dfrac{\partial}{\partial q'_j}, \quad i\geqq 1, \\ f_0'=f'=-\dfrac{dk}{dy}-\Big(\dfrac{\partial h}{\partial q}\Big)^{-1}\Big(q'-\dfrac{dh}{dy}\Big)\dfrac{\partial k}{\partial q}. \end{cases} \tag{2.13}$$

PROOF. By (2.8) and (2.3), we have

$$Z_nq_i'=Z_{i+1}(G_i\cdots G_1)h=(G_i-M)Z_i(G_{i-1}\cdots G_1)h= \tag{2.14}$$
$$\cdots=(G_i-M)\cdots(G_1-M)Z_1h=0$$

for each i $(1\leqq i<n)$, since $(G_{j-1}\cdots G_1)h$ is a function of $x, y, z, q_1, \cdots, q_j$. By (2.3) and (2.2), we get

$$Z_1k=\frac{\partial k}{\partial z}-M\frac{\partial k}{\partial q}=\frac{\partial k}{\partial z}-\Big(\frac{\partial h}{\partial z}\Big/\frac{\partial k}{\partial q}\Big)\frac{\partial k}{\partial q} \tag{2.15}$$
$$=-\Big(\frac{\partial h}{\partial q}\Big)^{-1}\frac{\partial(h,k)}{\partial(z,q)}=H_0\frac{\partial h}{\partial q}.$$

Hence, we obtain

$$Z_n=(Z_nx')\frac{\partial}{\partial x'}+(Z_ny')\frac{\partial}{\partial y'}+(Z_nz')\frac{\partial}{\partial z'}+(Z_np')\frac{\partial}{\partial p'}+\sum_{i=1}^{n-1}(Z_nq_i')\frac{\partial}{\partial q_i'}$$
$$=H_0\frac{\partial h}{\partial q}\frac{\partial}{\partial p'}.$$

By (2.9), we have

$$X_nq_i'=X_{i+1}(G_i\cdots G_1)h=(G_iX_i-NZ_i)(G_{i-1}\cdots G_1)h$$
$$=G_iX_i(G_{i-1}\cdots G_1)h=\cdots=(G_i\cdots G_1)X_1h=(G_i\cdots G_1)k$$

for each $i(1\leqq i<n)$. Here

$$G_i=(G_ix')\frac{\partial}{\partial x'}+(G_iy')\frac{\partial}{\partial y'}+(G_iz')\frac{\partial}{\partial z'}+(G_ip')\frac{\partial}{\partial p'}+\sum_{j=1}^{n-1}(G_iq_j')\frac{\partial}{\partial q_j'} \tag{2.16}$$
$$=\frac{\partial}{\partial y'}+q'\frac{\partial}{\partial z'}-f'\frac{\partial}{\partial p'}+\sum_{j=1}^{i-1}q_{j+1}'\frac{\partial}{\partial q_j'}+\sum_{j=i}^{n-1}(G_iq_j')\frac{\partial}{\partial q_j'}$$
$$=G_{i-1}'-f'\frac{\partial}{\partial p'}+\sum_{j=i}^{n-1}(G_iq_j')\frac{\partial}{\partial q_j'},\qquad i\geqq 1,$$

since we have

$$G_1k=\frac{dk}{dy}+q_2\frac{\partial k}{\partial q}=\frac{dk}{dy}+\Big(\frac{\partial h}{\partial q}\Big)^{-1}\Big(q'-\frac{dh}{dy}\Big)\frac{\partial k}{\partial q}=-f'.$$

Hence, we obtain

$$X_nq_i'=(G_i\cdots G_1)k=-(G_i\cdots G_2)f' \tag{2.17}$$
$$=-(G_i\cdots G_3)\Big(G_1'-f'\frac{\partial}{\partial p'}\Big)f'=-(G_i\cdots G_3)f_1'=\cdots=-f_{i-1}'$$

for each $i(1\leqq i<n)$, since f_j' is a function of $x', y', z', q_1', \cdots, q_{j+1}'$. By this identity and $X_1h=k$, we have (2.12).

LEMMA 2.4. *Suppose that $n\geqq 2$, and change the independent variables $x, y, z, q_1, \cdots, q_n$ and the unknown function u in the system* (2.5) *to $x', y', z', p', q_1', \cdots,$*

q'_{n-1} *and* u' *by* (2.10) *and*

$$u' = G_{n-1}(G_{n-1} \cdots G_1 h) + \frac{\partial h}{\partial q} u = G_{n-1} q'_{n-1} + \frac{\partial h}{\partial q} u, \tag{2.18}$$

where G_i $(1 \leqq i < n)$ *is the operator defined by* (1.14), *and* h, k *are functions of* x, y, z, q *satisfying* (1.8) *and* (2.3). *Then the first and the second equations of* (2.5) *are transformed to*

$$H_0 \frac{\partial u'}{\partial p'} = 0 \tag{2.19}$$

and

$$\begin{aligned}\left(\frac{\partial h}{\partial q}\right)^{-1} \left\{ \frac{du'}{dx'} - \sum_{i=1}^{n-1} f'_{i-1} \frac{\partial u'}{\partial q'_i} + \frac{\partial f'}{\partial q'} u' + \left(G'_{n-2} - f' \frac{\partial}{\partial p'}\right) f'_{n-2} \right\}& \\ + \left(\frac{\partial h}{\partial q}\right)^{-1} (X_1 k) \frac{\partial u'}{\partial p'} = 0&\end{aligned} \tag{2.20}$$

respectively, where f'_i $(i \geqq 0)$, f' *are the functions defined by* (2.13).

PROOF. Since

$$\frac{\partial f'}{\partial q'} = -\frac{\partial k}{\partial q} \Big/ \frac{\partial h}{\partial q},$$

we have

$$\left\{\begin{aligned} Z_1 \frac{\partial h}{\partial q} &= \frac{\partial}{\partial q}(Z_1 h) + \frac{\partial M}{\partial q}\frac{\partial h}{\partial q} = \frac{\partial M}{\partial q}\frac{\partial h}{\partial q}, \\ X_1 \frac{\partial h}{\partial q} &= \frac{\partial}{\partial q}(X_1 h) + \frac{\partial N}{\partial q}\frac{\partial h}{\partial q} = \frac{\partial k}{\partial q} + \frac{\partial N}{\partial q}\frac{\partial h}{\partial q} \\ &= \frac{\partial h}{\partial q}\left(\frac{\partial k}{\partial q} \Big/ \frac{\partial h}{\partial q} + \frac{\partial N}{\partial q}\right) = \frac{\partial h}{\partial q}\left(\frac{\partial N}{\partial q} - \frac{\partial f'}{\partial q'}\right), \end{aligned}\right. \tag{2.21}$$

and

$$Z_1\left(\frac{\partial h}{\partial q}\right)^{-1} = -\frac{\partial M}{\partial q}\left(\frac{\partial h}{\partial q}\right)^{-1}, \quad X_1\left(\frac{\partial h}{\partial q}\right)^{-1} = -\left(\frac{\partial N}{\partial q} - \frac{\partial f'}{\partial q'}\right)\left(\frac{\partial h}{\partial q}\right)^{-1}. \tag{2.22}$$

By (2.8) and (2.14), we have

$$\begin{aligned} Z_n G_{n-1} q'_{n-1} &= \left(G_{n-1} Z_n - M Z_{n-1} + G_{n-1} A_{n-1} \frac{\partial}{\partial q_n}\right) q'_{n-1} \\ &= -M Z_{n-1} q'_{n-1} + G_{n-1} A_{n-1} \frac{\partial q'_{n-1}}{\partial q_n} \\ &= -M\left(Z_n + A_{n-1} \frac{\partial}{\partial q_n}\right) q'_{n-1} + G_{n-1} A_{n-1} \frac{\partial h}{\partial q} \\ &= (-M A_{n-1} + G_{n-1} A_{n-1}) \frac{\partial h}{\partial q} \\ &= M_n \frac{\partial h}{\partial q}, \qquad n \geqq 2, \end{aligned} \tag{2.23}$$

since

$$Z_{n-1}=Z_n+A_{n-1}\frac{\partial}{\partial q_n},\quad \frac{\partial q'_{n-1}}{\partial q_n}=\frac{\partial h}{\partial q}.$$

By (2.9), (2.17) and (2.16), we have

$$\begin{aligned}(2.24)\qquad X_nG_{n-1}q'_{n-1}&=\Big(G_{n-1}X_n-NZ_{n-1}+G_{n-1}B_{n-1}\frac{\partial}{\partial q_n}\Big)q'_{n-1}\\
&=-G_{n-1}f'_{n-2}-NA_{n-1}\frac{\partial h}{\partial q}+G_{n-1}B_{n-1}\frac{\partial h}{\partial q}\\
&=-\Big(G'_{n-2}-f'\frac{\partial}{\partial p'}+G_{n-1}q'_{n-1}\frac{\partial}{\partial q'_{n-1}}\Big)f'_{n-2}+(G_{n-1}B_{n-1}-NA_{n-1})\frac{\partial h}{\partial q}\\
&=-\Big(G'_{n-2}-f'\frac{\partial}{\partial p'}\Big)f'_{n-2}-G_{n-1}q'_{n-1}\frac{\partial f'}{\partial q'}+N_n\frac{\partial h}{\partial q},\qquad n\geqq 2,\end{aligned}$$

since

$$Z_{n-1}q'_{n-1}=\Big(Z_n+A_{n-1}\frac{\partial}{\partial q_n}\Big)q'_{n-1}=A_{n-1}\frac{\partial h}{\partial q},\quad \frac{\partial f'_{n-2}}{\partial q'_{n-1}}=\frac{\partial f'}{\partial q'}.$$

Hence, by (2.22), (2.23) and (2.24), we obtain

$$\begin{aligned}Z_nu+\frac{\partial h}{\partial q}u+M_n&=Z_n\Big\{\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})\Big\}\\
&+\frac{\partial M}{\partial q}\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})+M_n=-\frac{\partial M}{\partial q}\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})\\
&+\Big(\frac{\partial h}{\partial q}\Big)^{-1}\Big(Z_nu'-M_n\frac{\partial h}{\partial q}\Big)+\frac{\partial M}{\partial q}\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})+M_n\\
&=\Big(\frac{\partial h}{\partial q}\Big)^{-1}Z_nu'\end{aligned}$$

and

$$\begin{aligned}&X_nu+\frac{\partial N}{\partial q}u+N_n\\
&=X_n\Big\{\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})\Big\}+\frac{\partial N}{\partial q}\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})+N_n\\
&=-\Big(\frac{\partial N}{\partial q}-\frac{\partial f'}{\partial q'}\Big)\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})\\
&\quad+\Big(\frac{\partial h}{\partial q}\Big)^{-1}\Big\{X_nu'+\Big(G'_{n-2}-f'\frac{\partial}{\partial p'}\Big)f'_{n-2}+G_{n-1}q'_{n-1}\frac{\partial f'}{\partial q'}-N_n\frac{\partial h}{\partial q}\Big\}\\
&\quad+\frac{\partial N}{\partial q}\Big(\frac{\partial h}{\partial q}\Big)^{-1}(u'-G_{n-1}q'_{n-1})+N_n\\
&=\Big(\frac{\partial h}{\partial q}\Big)^{-1}\Big\{X_nu'+\Big(G'_{n-2}-f'\frac{\partial}{\partial p'}\Big)f'_{n-2}+\frac{\partial f'}{\partial q'}u'\Big\}.\end{aligned}$$

Hence, by Lemma 2.3, the first and the second equations of (2.5) are transformed to (2.19) and (2.20) respectively.

REMARK 2.1. Suppose that $z=\phi(x, y)$ is a solution of (1.11) of Imschenetsky type, and that $z'=\phi'(x', y')$ is the transformed surface by (1.7). Then $q'_i=\partial^i\phi'/\partial y'^i$ ($i\geqq 1$) is given by (2.10), where we replace q_j by $\partial^j\phi/\partial y^j$.

PROOF OF THEOREM 1. In [7], this theorem was proved for an equation of L_1-type in the case where $n=1$. Also it was shown that equation (1.11) of Imschenetsky type is of L_1-type if the reduced equation is solved by integrable systems of order 0. In [8], it was proved that equation (1.11) of Imschenetsky type is of L_1-type if it is solved by integrable systems of the first order. Hence, we have Theorem 1 for $n=1$. The f' in (2.13) gives the reduced equation $s'+f'=0$ of (1.11) by (1.7). Let f'_i ($0\leqq i\leqq n-2$) be the function defined by (2.13), and n be greater than one. Then, by Lemma 2.4, the system (2.5) and the system of (2.19) and (2.20) have the same rank. The reduced equation is solved by integrable systems of order $n-1$ if and only if the system of two linear equations

$$\frac{\partial u'}{\partial p'}=0\,,$$

$$\frac{du'}{dx'}-\sum_{i=1}^{n-1}f'_{i-1}\frac{\partial u'}{\partial q'_i}+\frac{\partial f'}{\partial q'}u'+\Big(G'_{n-2}-f'\frac{\partial}{\partial p'}\Big)f'_{n-2}=0$$

has the rank greater than zero. This system is equivalent to the system of (2.19) and (2.20). By Proposition 2.3, the original equation (1.11) is solved by integrable systems of order n if and only if the system (2.5) has the rank greater than zero. Hence, we have Theorem 1 for $n\geqq 2$, and hence for every $n\geqq 1$.

§ 3. Invariants H_n and l_n.

In this section we shall define the invariants H_n and l_n ($n\geqq 1$) of equation (1.11) and show some of their properties. Let us define the $(n+1)$-th invariant H_n ($n\geqq 1$) of (1.11) from H_0 inductively by

$$\text{(3.1)}\qquad H_n=H_{n-1}+X_{n+1}C_n+N(\partial M/\partial q)-G_1(\partial N/\partial q')\,,\qquad n\geqq 1\,,$$

under the condition that $H_0\neq 0, \cdots, H_{n-1}\neq 0$, where C_n is a function of x, y, z, $q_1, \cdots, q_{n+1}$ defined inductively by

$$\text{(3.2)}\qquad C_n=C_{n-1}-G_n\log H_{n-1}\ (n\geqq 1)\,,\qquad C_0=M\,.$$

Then, H_n and C_n ($n\geqq 1$) are rational functions of $q_2, \cdots, q_{n+1}$ whose coefficients are functions of x, y, z, q. Under the condition that $H_0\neq 0, \cdots, H_{k-1}\neq 0$, we define the operators Y_{ki} ($k\geqq 0$, $i\geqq 1$) by $Y_{0i}=-Z_i$ and

$$Y_{ki}=\sum_{j=1}^{i} A_{k,j-1}\frac{\partial}{\partial q_j} \qquad (k\geqq 1),$$

where A_{kj} is a function of $x, y, z, q_1, \cdots, q_{j+1}$ defined inductively with respect to k by

(3.3) $$H_kA_{k+1,j}=X_{j+1}A_{kj}+Y_{k,j+1}B_j-\pi_kA_{kj}\ (k\geqq 0), \qquad A_{0j}=A_j$$

for each $j\geqq 0$. Here, we put

$$\pi_k=\partial N/\partial q\ (k\geqq 1), \qquad \pi_0=0.$$

The A_{kj} $(k, j\geqq 1)$ is a rational function of $q_2, \cdots, q_{j+1}$ whose coefficients are functions of x, y, z, q. By the definition, we have

(3.4) $$H_kY_{k+1,i}=[X_i, Y_{ki}]-\pi_kY_{ki}\ (k\geqq 0), \qquad Y_{0i}=-Z_i,\ i\geqq 1.$$

We define the n-th l-invariant l_n of equation (1.11) by

(3.5) $$(-1)^n l_n=Y_{n-1,n+1}A_{nn}-Y_{n,n+1}A_{n-1,n}-e_{n-1}A_{nn}$$

under the condition that $H_0\neq 0, \cdots, H_{n-1}\neq 0$, where we put

$$e_j=(-1)^j\partial\log H_{j-1}/\partial q_j\ (j\geqq 1), \qquad e_0=-\partial M/\partial q.$$

The l_n is a rational function of $q_2, \cdots, q_{n+1}$ whose coefficients are functions of x, y, z, q. By the definition (3.5), $(-1)^n l_n$ is the coefficient of $\partial/\partial q_{n+1}$ in the operator $[Y_{n-1,n+1}, Y_{n,n+1}]-e_{n-1}Y_{n,n+1}$.

LEMMA 3.1. *Suppose that* $k\geqq 0$, *and that* $H_0\neq 0, \cdots, H_{k-1}\neq 0$, $l_1=\cdots=l_{k-1}=0$. *Then we have the following identities:*

(i) $A_{sj}=0\ (0\leqq j<s-1)$, $A_{s,s-1}=(-1)^{s-1}$, $1\leqq s\leqq k$;

(ii) $Y_{s+1,s+1}H_s+X_se_s-\pi_se_s-Y_{s1}\pi_{s+1}=0$, $0\leqq s\leqq k-1$;

(iii) $[Y_{si}, Y_{ti}]-e_sY_{ti}=0$, $0\leqq s\leqq k-2$, $s<t\leqq k$, $i\geqq 1$;

(iv) $Y_{sk}H_{k-1}=0$, $0\leqq s\leqq k-2$;

(v) $A_{si}=(G_i-C_s)A_{s,i-1}-A_{s-1,i-1}$, $0\leqq s\leqq k$, $i\geqq 1$;

(vi) $[Y_{s,i+1}, G_i]=-Y_{s-1,i}-C_sY_{si}-G_iA_{si}\dfrac{\partial}{\partial q_{i+1}}$, $0\leqq s\leqq k$, $i\geqq 1$;

(vii) $A_{ss}=(-1)^s\sum_{j=1}^{s} C_j$, $0\leqq s\leqq k$;

(viii) $(-1)^sH_s=X_{s+1}A_{ss}+Y_{s,s+1}B_s-\pi_sA_{ss}$, $0\leqq s\leqq k$;

(ix) $l_s=Y_{s-1,s+1}C_s-G_{s-1}e_{s-1}+C_{s-1}e_{s-1}+e_{s-2}$, $1\leqq s\leqq k$.

Here, we put $Y_{-1,i}=0\ (i\geqq 1)$, $A_{-1,j}=0\ (j\geqq 0)$, $e_{-1}=0$.

PROOF. For $k=0$, we have (v), (vi) and (viii) by (2.4), (2.8) and (2.1) respectively. Since $A_{00}=A_0=M$, we get (vii). Hence, Lemma 3.1 is valid for $k=0$. Suppose that $H_0\neq 0, \cdots, H_{k-1}\neq 0$, $l_1=\cdots=l_{k-1}=0$, and that the identities (i)-(ix) are valid. Then, by (i), Y_{ki} does not involve $\partial/\partial q_j$ for any $j<k$, and its coefficient of $\partial/\partial q_k$ is $(-1)^{k-1}$ if $i\geqq k$. By (v), (vii) and (3.2), we have

$$(3.6)\qquad \frac{\partial A_{sk}}{\partial q_{k+1}}=\frac{\partial A_{s,k-1}}{\partial q_k}=\cdots=\frac{\partial A_{ss}}{\partial q_{s+1}}=(-1)^s\frac{\partial C_s}{\partial q_{s+1}}=-e_s\,,\qquad 0\leqq s\leqq k\,.$$

In addition to the above assumption, let us suppose that $H_k\neq 0$ and $l_k=0$, and prove successively the identities (i)-(ix) for $k+1$.

(i). The coefficient of $\partial/\partial q_{j+1}$ in the right-hand side of (3.4) vanishes for any $j<k-1$, since $Y_{ki}B_j=0$ for such j. For $j=k-1$, we have

$$X_kA_{k,k-1}+Y_{kk}B_{k-1}-\pi_kA_{k,k-1}=0\,,$$

since

$$A_{k,k-1}=(-1)^{k-1}\,,\qquad Y_{kk}B_{k-1}=(-1)^{k-1}\,\partial B_{k-1}/\partial q_k=(-1)^{k-1}\pi_k\,.$$

Hence we have $A_{k+1,j}=0$ for any $j\leqq k-1$. For $j=k$, we have

$$A_{k+1,k}=H_k^{-1}(X_{k+1}A_{kk}+Y_{k,k+1}B_k-\pi_kA_{kk})=(-1)^k$$

by (viii). Therefore, $Y_{k+1,i}=0$ for $i\leqq k$, and the coefficient of $\partial/\partial q_{k+1}$ in $Y_{k+1,i}$ is $(-1)^k$ for $i\geqq k+1$.

(ii). Suppose that $s=0$. Then, by (2.1), we have

$$\begin{aligned}\frac{\partial}{\partial q}H_0&=\frac{\partial}{\partial q}(X_1M)-\frac{\partial}{\partial q}(Z_1N)\\&=\Big(X_1\frac{\partial}{\partial q}-\frac{\partial N}{\partial q}\frac{\partial}{\partial q}\Big)M-\Big(Z_1\frac{\partial}{\partial q}-\frac{\partial M}{\partial q}\frac{\partial}{\partial q}\Big)N\\&=-X_1e_0-Z_1\pi_1\,.\end{aligned}$$

Suppose that $1\leqq s\leqq k$. Then, by (3.1), we have

$$\begin{aligned}\frac{\partial}{\partial q_{s+1}}H_s&=\frac{\partial}{\partial q_{s+1}}H_{s-1}+\frac{\partial}{\partial q_{s+1}}(X_{s+1}C_s)+\frac{\partial}{\partial q_{s+1}}\Big(N\frac{\partial M}{\partial q}\Big)-\frac{\partial}{\partial q_{s+1}}\Big(G_1\frac{\partial N}{\partial q}\Big)\\&=\Big(X_{s+1}\frac{\partial}{\partial q_{s+1}}-\frac{\partial B_s}{\partial q_{s+1}}\frac{\partial}{\partial q_{s+1}}\Big)C_s-\Big(G_1\frac{\partial}{\partial q_{s+1}}+\frac{\partial}{\partial q_s}\Big)\frac{\partial N}{\partial q}\\&=(-1)^{s-1}(X_{s+1}e_s-\pi_se_s-Y_{s1}\pi_{s+1})\,,\end{aligned}$$

since H_{s-1}, $N\dfrac{\partial M}{\partial q}$, $\dfrac{\partial N}{\partial q}$ do not involve q_{s+1} and

$$\frac{\partial B_s}{\partial q_{s+1}}=\pi_{s+1}=\pi_s\,,\qquad \frac{\partial C_s}{\partial q_{s+1}}=(-1)^{s-1}e_s\,,$$

$$Y_{11}\pi_2=\frac{\partial}{\partial q_1}\frac{\partial N}{\partial q}\,,\quad Y_{s1}\pi_{s+1}=0=(-1)^{s-1}\frac{\partial}{\partial q_s}\frac{\partial N}{\partial q}\quad(1<s\leqq k)\,.$$

Hence, we obtain (ii) for $0\leqq s\leqq k$ by

$$Y_{s+1,s+1}H_s=(-1)^s\frac{\partial}{\partial q_{s+1}}H_s\,,\qquad 0\leqq s\leqq k\,.$$

(iii) and (iv). First, we shall prove (iii) for $s=k-1$ and $t=k$;

(3.7) $$[Y_{k-1,i},\,Y_{ki}]-e_{k-1}Y_{ki}=0\,,\qquad i\geqq 1\,.$$

Let E_j denote the coefficient of $\partial/\partial q_j$ in the left-hand side of (3.7). Then E_j vanishes for any $j<k$, since Y_{ki} does not involve $\partial/\partial q_j$ and $Y_{kj}A_{k-1,j-1}=0$ for such j. For $j=k$, we have

$$\begin{aligned}E_k&=Y_{k-1,k}A_{k,k-1}-Y_{kk}A_{k-1,k-1}-e_{k-1}A_{k,k-1}\\&=Y_{k-1,k}(-1)^{k-1}-(-1)^{k-1}(\partial/\partial q_k)A_{k-1,k-1}-e_{k-1}(-1)^{k-1}=0\end{aligned}$$

by (3.6). For $j=k+1$, we have $E_{k+1}=(-1)^k l_k$ by the definition (3.5) of l_k. For $j\geqq k$, by (v) and (vi), we have

$$\begin{aligned}E_{j+1}&=Y_{k-1,j+1}A_{kj}-Y_{k,j+1}A_{k-1,j}-e_{k-1}A_{kj}\\&=Y_{k-1,j+1}\{(G_j-C_k)A_{k,j-1}-A_{k-1,j-1}\}\\&\quad-Y_{k,j+1}\{(G_j-C_{k-1})A_{k-1,j-1}-A_{k-2,j-1}\}-e_{k-1}A_{kj}\\&=(G_jY_{k-1,j}-Y_{k-2,j}-C_{k-1}Y_{k-1,j})A_{k,j-1}\\&\quad-(Y_{k-1,k+1}C_k)A_{k,j-1}-C_kY_{k-1,j}A_{k,j-1}-Y_{k-1,j}A_{k-1,j-1}\\&\quad-(G_jY_{kj}-Y_{k-1,j}-C_kY_{kj})A_{k-1,j-1}+(Y_{kk}C_{k-1})A_{k-1,j-1}\\&\quad+C_{k-1}Y_{kj}A_{k-1,j-1}+Y_{kj}A_{k-2,j-1}\\&\quad-e_{k-1}\{(G_j-C_k)A_{k,j-1}-A_{k-1,j-1}\}\\&=(G_j-C_k-C_{k-1})(Y_{ki}A_{k-1,j-1}-Y_{k-1,j}A_{k,j-1}-e_{k-1}A_{k,j-1})\\&\quad-(Y_{k-1,k+1}C_k-G_{k-1}e_{k-1}+C_{k-1}e_{k-1})A_{k,j-1}\\&\quad-(Y_{k-2,j}A_{k,j-1}-Y_{kj}A_{k-2,j-1})\,,\end{aligned}$$

since

$$Y_{kk}C_{k-1}=(-1)^{k-1}\,\partial C_{k-1}/\partial q_k=-e_{k-1}\,.$$

Here, we have

$$Y_{k-2,j}A_{k,j-1}-Y_{kj}A_{k-2,j-1}=e_{k-2}A_{k,j-1}$$

by the identity

$$[Y_{k-2,i},\,Y_{ki}]=e_{k-2}Y_{ki}$$

which we obtain from (iii), taking $s=k-2$ and $t=k$. Hence, by (ix), we have

(3.8) $$E_{j+1}=(G_j-C_k-C_{k-1})E_j-l_kA_{k,j-1}\,,\qquad j\geqq k\,.$$

Since we assumed that $l_k=0$, by (3.8) we have $E_j=0$ for all $j>0$. Thus (3.7) has been proved. Secondly, we shall prove the identity

$$[Y_{si}, Y_{k+1,i}]-e_s Y_{k+1,i}=-(Y_{si}\log H_k)Y_{k+1,i}, \qquad i\geqq 1 \tag{3.9}$$

for each s $(0\leqq s\leqq k-1)$. From (iii) we obtain

$$[Y_{si}, Y_{ki}]=e_s Y_{ki}, \qquad 0\leqq s<k-1,$$

taking $t=k$. Hence, by (3.4) and the Jacobi identity, we have

$$\begin{aligned}
[Y_{si}, Y_{k+1,i}]-e_s Y_{k+1,i} &= [Y_{si}, H_k^{-1}([X_i, Y_{ki}]-\pi_k Y_{ki})]-e_s Y_{k+1,i}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}+H_k^{-1}[Y_{si}, ([X_i, Y_{ki}]-\pi_k Y_{ki})]-e_s Y_{k+1,i}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}-e_s Y_{k+1,i}+H_k^{-1}\{-(Y_{si}\pi_k)Y_{ki}\\
&\quad -\pi_k[Y_{si}, Y_{ki}]+[Y_{si}, [X_i, Y_{ki}]]\}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}-e_s Y_{k+1,i}+H_k^{-1}\{-(Y_{si}\pi_k+\pi_k e_s)Y_{ki}\\
&\quad +[X_i, [Y_{si}, Y_{ki}]]-[[X_i, Y_{si}], Y_{ki}]\}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}-e_s Y_{k+1,i}+H_k^{-1}\{-(Y_{si}\pi_k+\pi_k e_s)Y_{ki}\\
&\quad +[X_i, e_s Y_{ki}]-[H_s Y_{s+1,i}+\pi_s Y_{si}, Y_{ki}]\}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}-e_s Y_{k+1,i}+H_k^{-1}\{-(Y_{si}\pi_k+\pi_k e_s)Y_{ki}\\
&\quad +(X_i e_s)Y_{ki}+e_s[X_i, Y_{ki}]+(Y_{ki}H_s)Y_{s+1,i}-H_s[Y_{s+1,i}, Y_{ki}]\\
&\quad +(Y_{ki}\pi_s)Y_{si}-\pi_s[Y_{si}, Y_{ki}]\}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}-e_s Y_{k+1,i}+H_k^{-1}\{-(Y_{si}\pi_k+\pi_k e_s)Y_{ki}+(X_i e_s)Y_{ki}\\
&\quad +e_s(H_k Y_{k+1,i}+\pi_k Y_{ki})+(Y_{ki}H_s)Y_{s+1,i}-H_s[Y_{s+1,i}, Y_{ki}]-\pi_s e_s Y_{ki}\}\\
&=-(Y_{si}\log H_k)Y_{k+1,i}+H_k^{-1}\{(X_i e_s-Y_{si}\pi_{s+1}-\pi_s e_s)Y_{ki}\\
&\quad +(Y_{ki}H_s)Y_{k+1,i}-H_s[Y_{s+1,i}, Y_{ki}]\},
\end{aligned}$$

since $Y_{ki}\pi_s=0$ and $Y_{si}\pi_k=Y_{si}\pi_{s+1}$. Here, suppose that $s=k-1$. Then we have

$$(Y_{ki}H_s)Y_{s+1,i}-H_s[Y_{s+1,i}, Y_{ki}]=(Y_{s+1,i}H_s)Y_{ki},$$

since $[Y_{s+1,i}, Y_{ki}]=[Y_{ki}, Y_{ki}]=0$. Suppose that $0\leqq s<k-1$. Then, by (iii) or (3.7), we have

$$[Y_{s+1,i}, Y_{ki}]=e_{s+1}Y_{ki},$$

and

$$\begin{aligned}
(Y_{ki}H_s)Y_{s+1,i}-H_s[Y_{s+1,i}, Y_{ki}] &= -H_s e_{s+1}Y_{ki}\\
&=(-1)^s\Big(H_s\frac{\partial}{\partial q_{s+1}}\log H_s\Big)Y_{ki}=(Y_{s+1,i}H_s)Y_{ki},
\end{aligned}$$

since $Y_{ki}H_s=0$. Hence, for any s $(0\leqq s\leqq k-1)$, we obtain

$$(X_i e_s-Y_{si}\pi_{s+1}-\pi_s e_s)Y_{ki}+(Y_{ki}H_s)Y_{k+1,i}-H_s[Y_{s+1,i}, Y_{ki}]=0$$

by (ii). Thus we have the identity (3.9) for each s $(0\leqq s\leqq k-1)$. The coefficient of $\partial/\partial q_{k+1}$ in the left-hand side of (3.9) vanishes, since we have

$$\begin{aligned} &Y_{s,k+1}A_{k+1,k}-Y_{k+1,k+1}A_{sk}-e_sA_{k+1,k} \\ &\qquad = Y_{s,k+1}(-1)^k-(-1)^k\frac{\partial}{\partial q_{k+1}}A_{sk}-e_s(-1)^k=0 \end{aligned}$$

by (3.6). The coefficient of $\partial/\partial q_{k+1}$ in $Y_{k+1,i}$ is $(-1)^k$ if $i\geqq k+1$. Hence, we have $Y_{si}\log H_k=0$ $(i\geqq k+1)$ and

$$[Y_{si},\ Y_{k+1,i}]-e_sY_{k+1,i}=0\,,\qquad 0\leqq s\leqq k-1,\ i\geqq 1\,.$$

Thus we obtain (iii) and (iv) for $k+1$.

(v). Let us prove the identity

$$\begin{aligned} (3.10)\qquad &X_{i+1}A_{ki}+Y_{k,i+1}B_i-\pi_kA_{ki} \\ &\qquad =(G_i-C_k)(X_iA_{k,i-1}+Y_{ki}B_{i-1}-\pi_kA_{k,i-1})-H_kA_{k,i-1},\quad i\geqq 1\,. \end{aligned}$$

By (v), (2.4), (2.9) and (vi), we have

$$\begin{aligned} &X_{i+1}A_{ki}+Y_{k,i+1}B_i-\pi_kA_{ki} \\ &\quad =X_{i+1}(G_iA_{k,i-1}-C_kA_{k,i-1}-A_{k-1,i-1})+Y_{k,i+1}(G_iB_{i-1}-NA_{i-1})-\pi_kA_{ki} \\ &\quad =(G_iX_i-NZ_i)A_{k,i-1}-(X_{i+1}C_k)A_{k,i-1}-C_kX_iA_{k,i-1}-X_iA_{k,i-1} \\ &\qquad +(G_iY_{ki}-Y_{k-1,i}-C_kY_{ki})B_{i-1}-(Y_{k1}N)A_{i-1}-NY_{ki}A_{i-1} \\ &\qquad -\pi_k(G_iA_{k,i-1}-C_kA_{k,i-1}-A_{k-1,i-1}) \\ &\quad =(G_i-C_k)(X_iA_{k,i-1}+Y_{ki}B_{i-1}-\pi_kA_{k,i-1}) \\ &\qquad -(X_{k+1}C_k+N\,\partial M/\partial q-G_i\pi_k)A_{k,i-1} \\ &\qquad -(X_iA_{k-1,i-1}+Y_{k-1,i}B_{i-1}-\pi_kA_{k-1,i-1}) \\ &\qquad -N(Z_iA_{k,i-1}+Y_{ki}A_{i-1}-(\partial M/\partial q)A_{k,i-1})-(Y_{k1}N)A_{i-1}\,. \end{aligned}$$

Here, suppose that $k=0$. Then we have (3.10), since

$$\begin{aligned} &(X_{k+1}C_k+N\,\partial M/\partial q-G_i\pi_k)A_{k,i-1}+N(Z_iA_{k,i-1}+Y_{ki}A_{i-1}-(\partial M/\partial q)A_{k,i-1}) \\ &\qquad +(Y_{k1}N)A_{i-1}=(X_1M-Z_1N)A_{i-1}=H_0A_{i-1}=H_kA_{k,i-1}\,. \end{aligned}$$

Suppose that $k\geqq 1$. Then we have

$$Z_iA_{k,i-1}+Y_{ki}A_{i-1}-(\partial M/\partial q)A_{k,i-1}=0\,,$$

since we obtain

$$[Y_{0i},\ Y_{ki}]=e_0Y_{ki}$$

from (iii) or (3.9). If $k=1$, then we have $\pi_k=Y_{k1}N$ and $\pi_{k-1}=\pi_0=0$, and if $k>1$, then we have $\pi_k=\pi_{k-1}$ and $Y_{k1}N=0$. Hence, for any $k\geqq 1$, we obtain

$$X_i A_{k-1,i-1} + Y_{k-1,i} B_{i-1} - \pi_k A_{k-1,i-1} + (Y_{k1} N) A_{i-1}$$
$$= X_i A_{k-1,i-1} + Y_{k-1,i} B_{i-1} + \pi_{k-1} A_{k-1,i-1} = H_{k-1} A_{k,i-1}$$

by (3.3). Therefore, for $k \geqq 1$, we have (3.10) by (3.1). Thus, for any $k \geqq 0$, we obtain the identity (3.10). Since we assumed that $H_k \neq 0$, by (3.2) and (3.3) we have

$$A_{k+1,i} = (G_i - C_{k+1}) A_{k+1,i-1} - A_{k,i-1}, \qquad i \geqq 1, \tag{3.11}$$

multiplying both sides of (3.10) by H_k^{-1}.

(vi). By (3.11), we have

$$[Y_{k+1,i+1}, G_i] = \left[\sum_{j=0}^{i} A_{k+1,j} \frac{\partial}{\partial q_{j+1}}, \frac{d}{dy} + \sum_{j=1}^{i} q_{j+1} \frac{\partial}{\partial q_j}\right] \tag{3.12}$$
$$= A_{k+1,0} \frac{\partial}{\partial z} + \sum_{j=0}^{i} (A_{k+1,j} - G_j A_{k+1,j-1}) \frac{\partial}{\partial q_j} - G_i A_{k+1,i} \frac{\partial}{\partial q_{i+1}}$$
$$= A_{k+1,0} \frac{\partial}{\partial z} - \sum_{j=0}^{i} (C_{k+1} A_{k+1,j-1} + A_{k,j-1}) \frac{\partial}{\partial q_j} - G_i A_{k+1,i} \frac{\partial}{\partial q_{i+1}}$$
$$= -Y_{ki} - C_{k+1} Y_{k+1,i} - G_i A_{k+1,i} \frac{\partial}{\partial q_{i+1}}, \qquad i \geqq 0.$$

(vii). Take $i = k+1$ in (3.11). Then we have

$$A_{k+1,k+1} = (G_{k+1} - C_{k+1}) A_{k+1,k} - A_{kk}$$
$$= -(G_{k+1} - C_{k+1})(-1)^k - (-1)^k \sum_{j=1}^{k} C_j = (-1)^{k+1} \sum_{j=1}^{k} C_j.$$

(viii). By (2.4) and (3.12), we have

$$Y_{k+1,k+2} B_{k+1} = Y_{k+1,k+2}(G_{k+1} B_k - N A_k)$$
$$= (G_{k+1} Y_{k+1,k+1} - Y_{k,k+1} - C_{k+1} Y_{k+1,k+1}) B_k$$
$$-(Y_{k+1,1} N) A_k - N Y_{k+1,k+1} A_k$$
$$= (-1)^k \left(G_1 \frac{\partial N}{\partial q} - C_{k+1} \pi_{k+1} - N \frac{\partial M}{\partial q}\right) - Y_{k,k+1} B_k - (Y_{k+1,1} N) A_k,$$

since

$$Y_{k+1,k+1} B_k = (-1)^k \frac{\partial N}{\partial q} = (-1)^k \pi_{k+1}, \qquad Y_{k+1,k+1} A_k = (-)^k \frac{\partial M}{\partial q}.$$

Hence we obtain

$$X_{k+2} A_{k+1,k+1} + Y_{k+1,k+2} B_{k+1} - \pi_{k+1} A_{k+1,k+1}$$
$$= (-1)^{k+1} \left(X_{k+2} C_{k+1} + N \frac{\partial M}{\partial q} - G_1 \frac{\partial N}{\partial q}\right)$$
$$-(X_{k+1} A_{kk} + Y_{k,k+1} B_k - \pi_{k+1} A_{kk}) - (Y_{k+1,1} N) A_k$$

by

(3.13) $$A_{k+1,k+1}=(-1)^{k+1}C_{k+1}-A_{kk}\,,\qquad k\geqq 0\,.$$

Here, suppose that $k=0$. Then we have

$$X_{k+1}A_{kk}+Y_{k,k+1}B_k-\pi_{k+1}A_{kk}+(Y_{k+1,1}N)A_k=X_1M-Z_1N=H_0\,.$$

Suppose that $k\geqq 1$. Then we have

$$X_{k+1}A_{kk}+Y_{k,k+1}B_k-\pi_{k+1}A_{kk}+(Y_{k+1,1}N)A_k=(-1)^kH_k$$

by (viii), since $\pi_{k+1}=\pi_k$ and $Y_{k+1,k}N=0$. Hence, for any $k\geqq 0$, we obtain the identity (viii) for $k+1$ by (3.1).

(ix). By (3.6) and (3.11), we have

$$\begin{aligned}
&Y_{k+1,k+2}A_{k,k+1}+e_kA_{k+1,k+1}\\
&\quad=\Big\{(-1)^k\frac{\partial}{\partial q_{k+1}}+A_{k+1,k+1}\frac{\partial}{\partial q_{k+2}}\Big\}A_{k,k+1}+e_kA_{k+1,k+1}\\
&\quad=(-1)^k\frac{\partial}{\partial q_{k+1}}(G_{k+1}A_{kk}-C_kA_{kk}-A_{k-1,k})\\
&\quad=(-1)^k\Big(G_{k+1}\frac{\partial}{\partial q_{k+1}}+\frac{\partial}{\partial q_k}\Big)A_{kk}+e_kA_{kk}+(-1)^kC_ke_k+(-1)^ke_{k-1}\\
&\quad=(-1)^{k+1}G_ke_k-Y_{k,k+1}A_{kk}+(-1)^k(C_ke_k+e_{k-1})\,.
\end{aligned}$$

Hence, by (3.5) and (3.13), we obtain

$$\begin{aligned}
l_{k+1}&=(-1)^{k+1}(Y_{k,k+2}A_{k+1,k+1}-Y_{k+1,k+2}A_{k,k+1}-e_kA_{k+1,k+1})\\
&=Y_{k,k+2}C_{k+1}-G_ke_k+C_ke_k+e_{k-1}\,.
\end{aligned}$$

REMARK 3.1. Under the same condition that is assumed in Lemma 3.1, the operators $[X_i,\,Y_{ki}]-\pi_kY_{ki}$ and $[Y_{k-1,i},\,Y_{ki}]-e_{k-1}Y_{ki}$ $(i\geqq 1)$ do not involve $\partial/\partial q_j$ for any $j\leqq k$, as it was shown in the proof of (i) and of (iii), (iv) respectively.

By the identity (viii) in Lemma 3.1 we have the following:

COROLLARY 3.1. *Under the same condition that is assumed in Lemma* 3.1, *the coefficient of* $\partial/\partial q_{k+1}$ *in the operator* $[X_i,\,Y_{ki}]-\pi_kY_{ki}$ $(i\geqq k+1)$ *is* $(-1)^kH_k$.

PROPOSITION 3.1. *Suppose that* $k\geqq 2$, *and that* $H_0\neq 0,\cdots,H_{k-1}\neq 0$, $l_1=0$, $\cdots,l_{k-1}=0$. *Then we have*

(3.14) $$H_{k-2}l_k=-Y_{k-2,k+1}H_k\,.$$

PROOF. For $i\geqq 1$, let us prove the identity

(3.15) $$\begin{aligned}&[Y_{k-2,i},\,[X_i,\,Y_{ki}]-\pi_kY_{ki}]\\ &\qquad=e_{k-2}([X_i,\,Y_{ki}]-\pi_kY_{ki})-H_{k-2}([Y_{k-1,i},\,Y_{ki}]-e_{k-1}Y_{ki})\,.\end{aligned}$$

By (ii), (iii) in Lemma 3.1 and the Jacobi identity, we have

$$
\begin{aligned}
&[Y_{k-2,i}, [X_i, Y_{ki}]-\pi_k Y_{ki}]\\
&\quad=[Y_{k-2,i}, [X_i, Y_{ki}]]-(Y_{k-2,i}\pi_k)Y_{ki}-\pi_k[Y_{k-2,i}, Y_{ki}]\\
&\quad=[X_i, [Y_{k-2,i}, Y_{ki}]]-[[X_i, Y_{k-2,i}], Y_{ki}]-(Y_{k-2,i}\pi_k)Y_{ki}-\pi_k e_{k-2}Y_{ki}\\
&\quad=[X_i, e_{k-2}Y_{ki}]-[H_{k-2}Y_{k-1,i}+\pi_{k-2}Y_{k-2,i}, Y_{ki}]\\
&\qquad-(Y_{k-2,i}\pi_k)Y_{ki}-\pi_k e_{k-2}Y_{ki}\\
&\quad=(X_i e_{k-2})Y_{ki}+e_{k-2}[X_i, Y_{ki}]+(Y_{ki}H_{k-2})Y_{k-1,i}-H_{k-2}[Y_{k-1,i}, Y_{ki}]\\
&\qquad+(Y_{ki}\pi_{k-2})Y_{k-2,i}-\pi_{k-2}[Y_{k-2,i}, Y_{ki}]-(Y_{k-2,i}\pi_k)Y_{ki}-\pi_k e_{k-2}Y_{ki}\\
&\quad=(X_i e_{k-2}-\pi_{k-2}e_{k-2}-Y_{k-2,i}\pi_k)Y_{ki}+e_{k-2}[X_i, Y_{ki}]\\
&\qquad-H_{k-2}[Y_{k-1,i}, Y_{ki}]-\pi_k e_{k-2}Y_{ki}\\
&\quad=-Y_{k-1,k-1}H_{k-2}Y_{ki}+e_{k-2}[X_i, Y_{ki}]-H_{k-2}[Y_{k-1,i}, Y_{ki}]-\pi_k e_{k-2}Y_{ki}\\
&\quad=e_{k-2}([X_i, Y_{ki}]-\pi_k Y_{ki})-H_{k-2}([Y_{k-1,i}, Y_{ki}]-e_{k-1}Y_{ki}),
\end{aligned}
$$

since $Y_{ki}H_{k-2}=0$, $Y_{ki}\pi_{k-2}=0$ and $\pi_k=\pi_{k-1}$. Thus we have (3.15). By the definition, $(-1)^k l_k$ is the coefficient of $\partial/\partial q_{k+1}$ in the operator $[Y_{k-1,k+1}, Y_{k,k+1}]-e_{k-1}Y_{k,k+1}$. By Corollary 3.1, $(-1)^k H_k$ is the coefficient of $\partial/\partial q_{k+1}$ in the operator $[X_{k+1}, Y_{k,k+1}]-\pi_k Y_{k,k+1}$. Hence, comparing the coefficient of $\partial/\partial q_{k+1}$ in the left-hand side of (3.15) for $i=k+1$ with that of $\partial/\partial q_{k+1}$ in the right-hand side, we have

$$(-1)^k Y_{k-2,k+1}H_k-(-1)^k H_k\frac{\partial A_{k-2,k}}{\partial q_{k+1}}=e_{k-2}(-1)^k H_k-(-1)^k H_{k-2}l_k .$$

Since $\partial A_{k-2,k}/\partial q_{k+1}=-e_{k-2}$, we obtain (3.14).

REMARK 3.2. A contact transformation leaves the type of equation $s+f=0$ invariant for any f if and only if it is a composition of the following three transformations:

$$x^*=x, \qquad y^*=y, \qquad z^*=\lambda(x, y, z), \tag{3.16}$$

$$x^*=\phi(x), \qquad y^*=\psi(y), \qquad z^*=z, \tag{3.17}$$

and $x^*=y$, $y^*=x$, $z^*=z$. The last transformation changes one of the two characteristics of $s+f=0$ to the other. Under the transformations (3.16) and (3.17), the transformed equation $s^*+f^*=0$ satisfies

$$\partial^2 f/\partial p^2=\frac{\partial\lambda}{\partial z}(\partial^2 f^*/\partial p^{*2}), \qquad \partial^2 f/\partial q^2=\frac{\partial\lambda}{\partial z}(\partial^2 f^*/\partial q^{*2})$$

and

$$\partial^2 f/\partial p^2=\Big(\frac{d\phi}{dx}\Big)^{-1}\frac{d\psi}{dy}(\partial^2 f^*/\partial p^{*2}), \quad \partial^2 f/\partial q^2=\frac{d\phi}{dx}\Big(\frac{d\psi}{dy}\Big)^{-1}\partial^2 f^*/\partial q^{*2}$$

respectively. Hence the type (1.11) of the equation is left invariant by the transformations (3.16) and (3.17). By these transformations, $q_1, \cdots, q_i, \cdots$ are changed to $q_1^*, \cdots, q_i^*, \cdots$, where $q_{i+1}^* = G_i q_i^*$ $(i \geqq 1)$, $q_1^* = G_0 \lambda$ and $(d\psi/dy) q_{i+1}^* = G_i q_i^*$ $(i \geqq 1)$, $(d\psi/dy) q_1^* = q_1$ respectively. Let H_k^* and l_k^* be the $(k+1)$-th invariant and k-th l-invariant of the transformed equation $s^* + f^* = 0$ respectively. Then, under the transformation (3.17), we have

$$H_k = \frac{d\phi}{dx}\frac{d\psi}{dy} H_k^*, \qquad l_{k+1} = \left(\frac{d\psi}{dy}\right)^{1-k} l_{k+1}^*, \qquad k \geqq 0.$$

Suppose that $H_0 \neq 0, \cdots, H_{k-1} \neq 0$ and $l_1 = \cdots = l_k = 0$. Then we have $H_k = H_k^*$ under the transformation (3.16). In addition to this condition, suppose that $H_k \neq 0$. Then we have $l_{k+1} = (\partial\lambda/\partial z) l_{k+1}^*$.

§ 4. Invariants H_n' and l_n' of the reduced equation.

In this section we shall prove Theorem 2 stated in the introduction. By the definition, an equation of Imschenetsky type is of L_1-type if and only if the reduced equation is linear in p', and is of L_n-type if and only if the reduced equation is of L_{n-1}-type $(n \geqq 2)$.

PROPOSITION 4.1. *Suppose that equation* (1.11) *is of Imschenetsky type. Then it is of L_1-type if and only if $l_1 = 0$.*

PROOF. Suppose that the Imschenetsky transformation (1.7) can be applied to (1.11). Then the function f' in (2.13) gives the reduced equation $s' + f' = 0$. Let us define the function f_j' $(j \geqq 0)$ of $x', y', z', p', q_1', \cdots, q_{j+1}'$ by (2.3), and change the independent variables $x, y, z, q_1, \cdots, q_i$ to $x', y', z', p', q_1', \cdots, q_{i-1}'$ by (2.10). Then, by (2.11) and (2.12) we have

$$Y_{1i} = H_0^{-1}[Z_i, X_i] = \frac{\partial h}{\partial q}\Big(\frac{\partial}{\partial z'} - \sum_{j=1}^{i-1} \frac{\partial f_{j-1}'}{\partial p'}\frac{\partial}{\partial q_j'}\Big) + \frac{\partial k}{\partial q}\frac{\partial}{\partial p'}, \quad i \geqq 2, \tag{4.1}$$

since we get

$$Z_1 X_1 k - X_1\Big(H_0 \frac{\partial h}{\partial q}\Big) = (H_0 Y_{11} - X_1 Z_1) k - X_1 Z_1 k = H_0 \frac{\partial k}{\partial q}$$

by (2.15). Hence by (2.11) we obtain

$$[Z_i, Y_{1i}] - \frac{\partial M}{\partial q} Y_{1i} = -H_0\Big(\frac{\partial h}{\partial q}\Big)^2 \sum_{j=1}^{i-1} \frac{\partial^2 f_{j-1}'}{\partial p'^2}\frac{\partial}{\partial q_j'}, \qquad i \geqq 2, \tag{4.2}$$

since we have

$$Z_1 \frac{\partial h}{\partial q} = \frac{\partial M}{\partial q}\frac{\partial h}{\partial q},$$

$$Z_1 \frac{\partial k}{\partial q} - \frac{\partial}{\partial q}\Big(H_0 \frac{\partial h}{\partial q}\Big) = \Big(\frac{\partial}{\partial q} Z_1 + \frac{\partial M}{\partial q}\frac{\partial}{\partial q}\Big) k - \frac{\partial}{\partial q} Z_1 k = \frac{\partial M}{\partial q}\frac{\partial k}{\partial q}$$

by (2.21) and (2.15). By the definition, l_1 is the coefficient of $\partial/\partial q_2$ in the left-hand side of (4.2) for $i = 2$. Hence we have

$$l_1 = -H_0 \frac{\partial h}{\partial q} (\partial^2 f'/\partial p'^2),$$

since

$$\frac{\partial}{\partial q_k} = \frac{\partial h}{\partial q} \frac{\partial}{\partial q'_{k-1}} + \sum_{j=k}^{i-1} \frac{\partial q'_j}{\partial q_k} \frac{\partial}{\partial q'_j}, \qquad k \geqq 2. \tag{4.3}$$

Therefore, the reduced equation $s' + f' = 0$ is linear in p' if and only if $l_1 = 0$.

Suppose that equation (1.11) is of L_1-type. Then the reduced equation takes on the form

$$s' + M'(x', y', z', q')p' + N'(x', y', z', q') = 0. \tag{4.4}$$

For this equation, let us define the invariants H'_k, l'_k and the operators X'_i, Z'_i, Y'_{ji} as we defined them for the equation (1.11). Suppose that $i \geqq 2$, and that x', y', z', p', $q'_1, \cdots, q'_{i-1}$ are functions of $x, y, z, q_1, \cdots, q_i$ defined by (2.10). Then, by (2.11), (2.12) and (4.1), we have

$$X_i = X'_{i-1} + p'Z'_{i-1} + (X_1 k)\frac{\partial}{\partial p'}, \quad Y_{1i} = \frac{\partial h}{\partial q} Z'_{i-1} + \frac{\partial k}{\partial q} \frac{\partial}{\partial p'}. \tag{4.5}$$

PROPOSITION 4.2. *Suppose that equation* (1.11) *is of L_1-type. Then we have* $H_1 = H'_0$.

PROOF. By (4.5) and (2.21), we have

$$\begin{aligned} [X_i, Y_{1i}] - \frac{\partial N}{\partial q} Y_{1i} &= \Big(X_1 \frac{\partial h}{\partial q}\Big) Z'_{i-1} + \Big(X_1 \frac{\partial k}{\partial q}\Big) \frac{\partial}{\partial p'} \\ &\quad + \frac{\partial h}{\partial q} [X'_{i-1}, Z'_{i-1}] - \frac{\partial k}{\partial q} Z'_{i-1} - (Y_{1i} X_1 k) \frac{\partial}{\partial p'} - \frac{\partial N}{\partial q} Y_{1i} \\ &= -\frac{\partial h}{\partial q} [Z'_{i-1}, X'_{i-1}], \qquad i \geqq 2, \end{aligned} \tag{4.6}$$

since

$$X_1 \frac{\partial k}{\partial q} - Y_{1i} X_1 k = \Big(\frac{\partial}{\partial q} X_1 + \frac{\partial N}{\partial q} \frac{\partial}{\partial q}\Big) k - \frac{\partial}{\partial q} X_1 k = \frac{\partial N}{\partial q} \frac{\partial k}{\partial q}.$$

By Corollary 3.1, $-H_1$ is the coefficient of $\partial/\partial q_2$ in $[X_2, Y_{12}] - \pi_1 Y_{12}$, and H'_0 is the coefficient of $\partial/\partial q'$ in $[Z'_1, X'_1]$ by the definition. Hence, taking account of (4.3), we have $H_1 = H'_0$ by (4.6).

PROPOSITION 4.3. *Suppose that equation* (1.11) *is of L_1-type, and that* $H_1 \neq 0$. *Then we have*

$$H_2 = H'_1 - p' l'_1, \qquad l_2 = -\frac{\partial h}{\partial q} l'_1.$$

PROOF. Multiplying each side of (4.6) by $H_1^{-1} = (H'_0)^{-1}$, we have

$$Y_{2i} = -\frac{\partial h}{\partial q} Y'_{1,i-1}, \qquad i \geqq 2. \tag{4.7}$$

By this identity and (4.5), we obtain

$$[X_i,\ Y_{2i}]-\frac{\partial N}{\partial q}Y_{2i}=-\Big(X_1\frac{\partial h}{\partial q}\Big)Y'_{1,i-1} \tag{4.8}$$

$$-\frac{\partial h}{\partial q}([X'_{i-1},\ Y'_{1,i-1}]+p'[Z'_{i-1},\ Y'_{1,i-1}])+\frac{\partial N}{\partial q}\frac{\partial h}{\partial q}Y'_{1,i-1}$$

$$=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\ Y'_{1,i-1}]-\frac{\partial N'}{\partial q'}Y'_{1,i-1}\Big)$$

$$-\frac{\partial h}{\partial q}p'\Big([Z'_{i-1},\ Y'_{1,i-1}]-\frac{\partial M'}{\partial q'}Y'_{1,i-1}\Big),\qquad i\geqq 2,$$

since we have

$$X_1\frac{\partial h}{\partial q}-\frac{\partial N}{\partial q}\frac{\partial h}{\partial q}=-\frac{\partial h}{\partial q}\frac{\partial f'}{\partial q'}=-\frac{\partial h}{\partial q}\Big(\frac{\partial M'}{\partial q'}p'+\frac{\partial N'}{\partial q'}\Big)$$

by (2.21). Hence, comparing the coefficient of $\partial/\partial q_3$ in the left-hand side of the first identity of (4.8) for $i=3$ with that of $\partial/\partial q'_2$ in the right-hand side of the second identity, we have $H_2=H'_1-p'l'_1$. By Proposition 3.1, we obtain

$$l_2=H_0^{-1}Z_3H_2=\frac{\partial h}{\partial q}\frac{\partial}{\partial p'}(H'_1-p'l'_1)=-\frac{\partial h}{\partial q}l'_1.$$

LEMMA 4.1. *Suppose that* $k\geqq 2$, $H_0\neq 0,\ \cdots,\ H_k\neq 0$, $H'_0\neq 0,\ \cdots,\ H'_{k-1}\neq 0$, $l_1=\cdots=l_k=l'_1=\cdots=l'_{k-1}=0$, $H_k=H'_{k-1}$, *and that for* $i\geqq 2$ *we have*

$$[X_i,\ Y_{ki}]-\frac{\partial N}{\partial q}Y_{ki}=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\ Y'_{k-1,i-1}]-\frac{\partial N'}{\partial q'}Y'_{k-1,i-1}\Big). \tag{4.9}$$

Then we have

$$[X_i,\ Y_{k+1,i}]-\frac{\partial N}{\partial q}Y_{k+1,i}=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\ Y'_{k,i-1}]-\frac{\partial N'}{\partial q'}Y'_{k,i-1}\Big),\quad i\geqq 2$$

and

$$H_{k+1}=H'_k,\qquad l_{k+1}=-\frac{\partial h}{\partial q}l'_k.$$

PROOF. Multiplying both sides of (4.9) by $H_k^{-1}=(H'_{k-1})^{-1}$, we have

$$Y_{k+1,i}=-\frac{\partial h}{\partial q}Y'_{k,i-1},\qquad i\geqq 2. \tag{4.10}$$

By the identity (iii) in Lemma 3.1, we have

$$[Z'_{i-1},\ Y'_{k,i-1}]-\frac{\partial M'}{\partial q'}Y'_{k,i-1}=0,$$

since we can take $s=0$ and $t=k$ by the assumption that $k\geqq 2$. Hence, by (4.5) and (4.10), we obtain

$$(4.11)\qquad [X_i,\, Y_{k+1,i}]-\frac{\partial N}{\partial q}Y_{k+1,i}$$
$$=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\, Y'_{k,i-1}]-\frac{\partial N'}{\partial q'}Y'_{k,i-1}\Big)$$
$$-\frac{\partial h}{\partial q}p'\Big([Z'_{i-1},\, Y'_{k,i-1}]-\frac{\partial M'}{\partial q'}Y'_{k,i-1}\Big)$$
$$=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\, Y'_{k,i-1}]-\frac{\partial N'}{\partial q'}Y'_{k,i-1}\Big),\qquad i\geqq 2.$$

By Corollary 3.1, $(-1)^{k+1}H_{k+1}$ is the coefficient of $\partial/\partial q_{k+2}$ in $[X_{k+2},\, Y_{k+1,k+2}]-\pi_{k+1}Y_{k+1,k+2}$, and $(-1)^kH'_k$ is the coefficient of $\partial/\partial q'_{k+1}$ in $[X'_{k+1},\, Y'_{k,k+1}]-\pi'_kY'_{k,k+1}$. Hence, by (4.11), we have $H_{k+1}=H'_k$. By Proposition 3.1, we obtain

$$l_{k+1}=-H_{k-1}^{-1}Y_{k+1,k+2}H_{k+1}=(H'_{k-2})^{-1}\frac{\partial h}{\partial q}Y'_{k,k+1}H'_k=-\frac{\partial h}{\partial q}l'_k.$$

PROPOSITION 4.4. *Suppose that $k\geqq 3$, and that the coefficients M and N of equation* (1.11) *satisfy $H_0\neq 0,\cdots,H_{k-1}\neq 0$, $l_1=\cdots=l_{k-1}=0$. Then we have*

$$(4.12)\qquad H_k=H'_{k-1},\qquad l_k=-\frac{\partial h}{\partial q}l'_{k-1}.$$

PROOF. Suppose that $k=3$. Then, by Proposition 4.3, we have $l'_1=-\Big(\frac{\partial h}{\partial q}\Big)^{-1}l_2=0$ and $H'_1=H_2\neq 0$. Hence, by the identity (iii) in Lemma 3.1, we obtain

$$[Z'_{i-1},\, Y'_{1,i-1}]-\frac{\partial M'}{\partial q'}Y'_{1,i-1}=0$$

and

$$[X_i,\, Y_{2i}]-\frac{\partial N}{\partial q}Y_{2i}=-\frac{\partial h}{\partial q}\Big([X'_{i-1},\, Y'_{1,i-1}]-\frac{\partial N'}{\partial q'}Y'_{1,i-1}\Big)$$

by (4.8). Therefore, we have (4.12) and (4.9) for $k=3$ by Lemma 4.1. Suppose that $H_0\neq 0,\cdots,H_k\neq 0$, $l_1=\cdots=l_k=0$, and that the identities (4.12) and (4.9) are valid for each of $3,\cdots,k$. Then we have $H'_0=H_1,\cdots,H'_{k-1}=H_k$ and $l'_1=\cdots=l'_{k-1}=0$. Hence, by Lemma 4.1, we have (4.2) and (4.9) for $k+1$.

Let us prove the first part of Theorem 2 stated in the introduction.

THEOREM 4.1. *Equation* (1.11) *of Imschenetsky type is of L_n-type if and only if $H_0\neq 0,\cdots,H_{n-1}\neq 0$ and $l_1=\cdots=l_n=0$.*

PROOF. By Proposition 4.1, this theorem is valid for $n=1$. Since an equation of L_1-type is of Imschenetsky type and linear in p, an equation of L_2-type is of L_1-type. Hence, an equation of L_n-type is of L_{n-1}-type. Suppose that Theorem 4.1 is valid for $n=k$. Then the reduced equation $s'+f'=0$ of (1.11) is of L_k-type if and only if $H'_0\neq 0,\cdots,H'_{k-1}\neq 0$, and $l'_1=\cdots=l'_k$

$=0$. Suppose that equation (1.11) is of L_{k+1}-type. Then it is of L_k-type, and we have $H_0\neq 0, \cdots, H_{k-1}\neq 0$, $l_1=\cdots=l_k=0$. Since the reduced equation is of L_k-type, we have $H'_0\neq 0, \cdots, H'_{k-1}\neq 0$ and $l'_1=\cdots=l'_k=0$. Hence, by Proposition 4.3 and Proposition 4.4, we obtain $l_{k+1}=-(\partial h/\partial q)l'_k=0$ and $H_k=H'_{k-1}\neq 0$. Conversely, suppose that $H_0\neq 0, \cdots, H_k\neq 0$ and $l_1=\cdots=l_{k+1}=0$. Then, by Proposition 4.3 and Proposition 4.4, we have $l'_k=-(\partial h/\partial q)^{-1}l_k=0$ and $H'_{k-1}=H_k\neq 0$. Hence, the reduced equation (4.4) is of L_k-type, and the original equation (1.11) is of L_{k+1}-type.

Let us prove the second part of Theorem 2 stated in the introduction.

THEOREM 4.2. *Suppose that equation* (1.11) *is of L_n-type. Then it is solved by integrable systems of order n if and only if $H_n=0$.*

PROOF. In [7] it was proved that an equation of L_1-type is solved by integrable systems of the first order if and only if it is reduced by the associated Imschenetsky transformation to an equation whose first invariant H'_0 vanishes. By Proposition 4.2, H'_0 vanishes if and only if $H_1=0$. Hence Theorem 4.2 is valid for $n=1$. Suppose that Theorem 4.2 is valid for $n=k\geqq 1$, and equation (1.11) is of L_{k+1}-type. Then the reduced equation (4.4) is solved by integrable systems of order k if and only if $H'_k=0$. By Theorem 4.1, we have $H_0\neq 0, \cdots, H_k\neq 0$ and $l_1=\cdots=l_{k+1}=0$. Hence, by Propositions 4.3 and 4.4, we have $H'_k=H_{k+1}$. By Theorem 1, equation (1.11) of L_{k+1}-type is solved by integrable systems of order $k+1$ if and only if the reduced equation (4.4) is solved by integrable systems of order k. Hence, equation (1.11) of L_{k+1}-type is solved by integrable systems of order $k+1$ if and only if $H_{k+1}=0$.

REMARK 4.1. Suppose that $H_0\neq 0, \cdots, H_{k-1}\neq 0$, $l_1=\cdots=l_k=0$ and $H_k=0$. Then, by (3.10), we have

$$[X_i, Y_{ki}]-\pi_k Y_{ki}=0\,, \qquad i\geqq 1\,.$$

For any $n\geqq k$, the system (2.5) is prolonged to a complete system consisting of (2.5) and

$$Y_{sn}u+e_s u-(A_{sn}+e_s q_{n+1})=0\,, \qquad 1\leqq s\leqq k$$

by adding the compatibility conditions. Here, $A_{sn}+e_s q_{n+1}$ is a function of $x, y, z, q_1, \cdots, q_n$, since A_{sn} is linear with respect to q_{n+1}, and $\partial A_{sn}/\partial q_{n+1}=-e_s$. Hence the rank of the system (2.5) is $n-k+1$. Therefore, if equation (1.11) is of L_k-type and if it is solved by integrable systems of order k, then it is solved by integrable systems of order n for every $n\geqq k$. This argument gives another proof of Theorem 4.2. Suppose that equation (1.11) is not of Imschenetsky type; $H_0=0$. Then equation (1.11) is solved by integrable systems of order n for every $n\geqq 0$.

REMARK 4.2. In [8], for equation (1.11) of Imschenetsky type, its invariant H_{1i} $(1 \leqq i \leqq 4)$ was defined by

$$H_{11}=\frac{\partial L_1}{\partial q}, \qquad H_{13}=\frac{dL_1}{dy}-ML_1-Z_1M,$$

$$H_{12}=\frac{\partial L_2}{\partial q}, \qquad H_{14}=\frac{dL_2}{dy}-NL_1-Z_1N-2H_0,$$

where

$$L_1=Z_1\log H_0+\frac{\partial M}{\partial q}, \qquad L_2=X_1\log H_0+\frac{\partial N}{\partial q}.$$

By these H_{1i}, the invariants l_1 and H_1 are expressed in the form

$$l_1=H_{11}q_2+H_{13}, \qquad H_1=-H_{12}q_2-H_{14}.$$

Hence, Proposition 4.1 gives the same condition $H_{11}=H_{13}=0$ as obtained in [8] for equation (1.11) of Imschenetsky type to be of L_1-type. In [8], Proposition 4.2 was expressed in the identities

$$A=-H_{12}/\frac{\partial h}{\partial q}, \qquad B=-H_{14}+\frac{dh}{dy}H_{12}/\frac{\partial h}{\partial q},$$

where A and B are the coefficients of H_0' which is linear in q'; $H_0'=Aq'+B$. Also, such identities that are equivalent to those in Proposition 4.3 were obtained in [8].

EXAMPLE 4.1 (Moutard equation). Equation of the form

$$s^*+\frac{\partial}{\partial x}(\alpha e^{z^*})+\frac{\partial}{\partial y}(\beta e^{-z^*})+\gamma=0$$

is called a Moutard equation, where α, β, γ are functions of x, y. Suppose that $\alpha\neq 0$. Then this equation is transformed to

$$s+pe^z+\frac{\partial}{\partial y}(be^{-z})+c=0 \tag{4.13}$$

by changing the dependent variable z^* to z, where $z=z^*+\log\alpha$, and

$$b=\alpha\beta, \qquad c=\gamma-\frac{\partial^2\log\alpha}{\partial x\partial y}.$$

For the equation (4.13), we have

$$H_0=-be^{-z}q+\frac{\partial b}{\partial y}e^{-z}-b.$$

Suppose that $b\neq 0$. Then, the Imschenetsky transformation

$$x'=x, \qquad y'=y, \qquad z'=\log\left(q+e^z-\frac{\partial\log b}{\partial y}\right),$$

$$p'=be^{-z}-b'e^{-z'}, \qquad b'=b+\frac{\partial^2\log b}{\partial x\partial y}+c$$

reduces (4.13) to a Moutard equation

$$s'+p'e^{z'}+\frac{\partial}{\partial y}(b'e^{-z'})+c'=0\,,$$

where

$$c'=b'-b\,.$$

Conversely, the equation (4.13) is reduced from a Moutard equation

$$s''+\frac{\partial}{\partial x}\{(c-b)e^{-z''}\}-q''e^{z''}-c=0$$

by the Imschenetsky transformation

$$x''=x,\quad y''=y,\quad z''=\log(p+be^{-z}),\quad q''=(b-c)e^{-z''}-e^{z}\,.$$

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