J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On the rank and curvature of non-singular complex hypersurfaces in a complex projective space*

By Katsumi NOMIZU

(Received July 23, 1968)

Let M be a non-singular connected complex hypersurface in the complex projective space $P^{n+1}(C)$ with Fubini-Study metric of constant holomorphic sectional curvature 1. In [2] it was shown that the rank of the second fundamental form A of M at a point x of M is determined by the curvature tensor of M at x. Thus the rank of A is intrinsic at each point and is simply called the rank of M.

In the present paper we shall obtain the following results:

THEOREM 1. *If M is compact and if the rank of M is $\leqq n-1$ at every point, then M is imbedded as a projective hyperplane in $P^{n+1}(C)$.*

THEOREM 2. *Let $n \geqq 3$. If M is compact and if the sectional curvature of M with respect to the induced Kählerian metric is $\geqq \frac{1}{4}$ for every tangent 2-plane, then M is imbedded as a projective hyperplane.*

1. Preliminaries. We recall the terminology and a few results from [1] and [2]. Let M be a complex hypersurface in $P^{n+1}(C)$. Let J denote the complex structures of $P^{n+1}(C)$ and M, and let g denote the Fubini-Study metric of holomorphic sectional curvature 1 in $P^{n+1}(C)$ as well as the Kählerian metric induced on M. For each point x_0 of M, choose a field of unit normals ξ defined on a neighborhood U of x_0.

Denoting by $\tilde{\nabla}$ and ∇ the Kählerian connections of $P^{n+1}(C)$ and M, we have the basic formulas (cf. [1])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi + k(X, Y)J\xi$$

$$\tilde{\nabla}_X \xi = -AX + s(X)J\xi ,$$

where X and Y are vector fields tangent to M, h and k are bilinear symmetric forms, s is a 1-form, and A is a tensor field of type (1, 1), called the second fundamental form. Moreover, we have $h(X, Y) = g(AX, Y)$, $k(X, Y) = g(JAX, Y)$, and $AJ = -JA$. The Gauss equation expresses the curvature ten-

* Work supported by an NSF Grant GP-7610.

sor R of M as follows:

$$R(X, Y)=\tilde{R}(X, Y)+D(X, Y),$$

where $\tilde{R}$ is the curvature tensor of $P^{n+1}(C)$ given by

$$\tilde{R}(X, Y)=\frac{1}{4}\{X\wedge Y+JX\wedge JY+2g(X, JY)J\}$$

and D is a tensor of type (1, 3) defined by

$$D(X, Y)=AX\wedge AY+JAX\wedge JAY.$$

In these formulas, $X\wedge Y$, where $X, Y\in T_x(M)$, denotes the skew-symmetric endomorphism of the tangent space $T_x(M)$ defined by

$$(X\wedge Y)(Z)=g(Y, Z)X-g(X, Z)Y, \qquad Z\in T_x(M).$$

In [2] it was shown that the kernel of A at $x\in M$ is equal to $\{X\in T_x(M);\ (R-\tilde{R})(X, Y)=0$ for all $Y\in T_x(M)\}$. Thus the rank of A at x is intrinsic; we call it the rank of M at the point x.

Suppose that, at every point of an open subset W, the rank of M is equal to a constant, say, $2r$, where r is a positive integer. Then we get a distribution T^0 of dimension $2n-2r$ which assigns to each $x\in W$ the kernel of A. In the arguments leading to Proposition 1 in [2], it is shown that T^0 is involutive and invariant by the complex structure J so that any of its maximal integral manifolds is a complex submanifold which is, in fact, totally geodesic in $P^{n+1}(C)$. This means that there exist a projective subspace P^{n-r} in $P^{n+1}(C)$ and an open subset U of $P^{n+1}(C)$ such that $U\cap P^{n-r}\subset M$. We shall make use of this fact in the following section.

2. Proof of Theorem 1. If the rank of M is zero everywhere, then M is totally geodesic and hence is a projective hyperplane. Assume that the rank of M has a maximum, say, $2r>0$, at some point x_0 of M. Then the rank of A is identically equal to $2r$ in a neighborhood W of x_0. As we stated in section 1, there exist a projective subspace P^{n-r} in $P^{n+1}(C)$ and an open subset U in $P^{n+1}(C)$ such that $U\cap P^{n-r}\subset M$. We shall now show that the entire subspace P^{n-r} is contained in M[1]. Let $(z_0, z_1, \cdots, z_{n+1})$ be a projective coordinate system in $P^{n+1}(C)$ such that the subspace P^{n-r} is given by $z_0=z_1=\cdots=z_r=0$. Since M is algebraic by a well-known theorem of Chow, it is the zero set of a certain homogeneous polynomial $f(z_0, z_1, \cdots, z_{n+1})$. Denoting the coordinates of the point x_0 by $(a_0, a_1, \cdots, a_{n+1})$, we may assume that

$$U=\{(z_0, z_1, \cdots, z_{n+1});\ |z_k-a_k|<d\}, \qquad d>0.$$

1) We thank H. Hironaka for conversations which have led to the following reasoning.

The condition $U \cap P^{n-r} \subset M$ implies that $f(0, 0, \cdots, 0, z_{r+1}, \cdots, z_{n+1}) = 0$ for all $(z_{r+1}, \cdots, z_{n+1})$ such that $|z_k - a_k| < d$, $r+1 \leqq k \leqq n+1$. It follows that $f(0, 0, \cdots, 0, z_{r+1}, \cdots, z_{n+1}) = 0$ for all $z_{r+1}, \cdots, z_{n+1}$. Thus P^{n-r} is contained in M.

Now by the proposition below we may conclude that M is a projective hyperplane; but then the rank of M is identically 0. This contradiction coming from the assumption that the rank of M is not 0 at some point, we have proved Theorem 1.

PROPOSITION. *Let M be a compact complex hypersurface in $P^{n+1}(C)$. If M contains a certain projective subspace P^{n-r}, where $2r \leqq n-1$, then M is a projective hyperplane.*

The following proof is an extension of the argument for Theorem 6 in [2], which will now be contained in our Theorem 1. We write the homogeneous polynomial f defining M in the form

$$f(z_0, z_1, \cdots, z_{n+1}) = F(z_{r+1}, \cdots, z_{n+1}) + \sum_{k=0}^{r} z_k f_k(z_{r+1}, \cdots, z_{n+1})$$

$$+ \sum_{k_0 + \cdots + k_r \geqq 2} z_0^{k_0} z_1^{k_1} \cdots z_r^{k_r} f_{k_0 k_1 \cdots k_r}(z_{r+1}, \cdots, z_{n+1})$$

where F, f_k and $f_{k_0 k_1 \cdots k_r}$ are homogeneous polynomials in the variables $z_{r+1}, \cdots, z_{n+1}$. Since $P^{n-r} \subset M$, we have $f(0, \cdots, 0, z_{r+1}, \cdots, z_{n+1}) = 0$ for all $z_{r+1}, \cdots, z_{n+1}$. Thus F is identically 0. Consequently, we get

$$\partial f / \partial z_j = f_j + \sum_{k_0 + \cdots + k_r \geqq 2} k_j z_0^{k_0} \cdots z_j^{k_j - 1} \cdots z_r^{k_r} f_{k_0 \cdots k_r}, \qquad 0 \leqq j \leqq r,$$

and

$$\partial f / \partial z_m = \sum_{j=0}^{r} z_j \partial f_j / \partial z_m + \sum_{k_0 + \cdots + k_r \geqq 2} z_0^{k_0} \cdots z_r^{k_r} \partial f_{k_0 \cdots k_r} / \partial z_m, \quad r+1 \leqq m \leqq n+1.$$

At $(0, \cdots, 0, z_{r+1}, \cdots, z_{n+1}) \in P^{n-r} \subset M$, we have

$$\partial f / \partial z_j = f_j, \ 0 \leqq j \leqq r, \quad \text{and} \quad \partial f / \partial z_m = 0, \ r+1 \leqq m \leqq n+1.$$

We consider $r+1$ homogeneous polynomials f_j, $0 \leqq j \leqq r$. If $r+1 \leqq n-r$ (that is, $2r \leqq n-1$ as we are assuming), it follows from the dimension theorem for intersections of varieties (cf. the main theorem of § 5, [3]) that, unless f_j's are constants, there is a non-trivial common solution $(b_{r+1}, \cdots, b_{n+1})$ of the system of equations $f_j = 0$, $0 \leqq j \leqq r$. Then the point $(0, \cdots, 0, b_{r+1}, \cdots, b_{n+1})$ of $P^{n+1}(C)$ lies in M and, at that point, all partial derivatives $\partial f / \partial z^k$, $0 \leqq k \leqq n+1$, are 0. This contradicts the premise that f defines our non-singular hypersurface M. We conclude that f_j's are constants and f is of degree 1, that is, M is a projective hyperplane.

3. PROOF OF THEOREM 2. At any point x of M, there is an orthonormal basis in $T_x(M)$ of the form $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ such that

$$Ae_i = \lambda_i e_i \quad \text{and} \quad A(Je_i) = -\lambda_i(Je_i), \qquad 1 \leqq i \leqq n,$$

where we may assume $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$ (see Lemma 1 of [**1**]). By Corollary 1 of [**1**], the sectional curvature for the plane spanned by e_j and Je_k, $j \neq k$, is equal to $\frac{1}{4} - \lambda_j\lambda_k$. Since all sectional curvatures are $\geqq \frac{1}{4}$, we have $-\lambda_j\lambda_k \geqq 0$. Thus, if $\lambda_1 > 0$, then $\lambda_2 = \cdots = \lambda_n = 0$ and the rank of M is at most 2 at x. If $\lambda_1 = 0$, then, of course, A is 0 at x. Thus our assumption on sectional curvature implies that the rank of M is at most 2 everywhere. Since $n \geqq 3$ by assumption, Theorem 1 can be applied to conclude that M is a projective hyperplane.

4. Remark. For a compact connected Kählerian manifold M of complex dimension n, consider the following conditions:

a) M admits a holomorphic, isometric imbedding into $P^{n+1}(C)$;

b) all sectional curvatures of M are $\geqq \frac{1}{4}$;

c) the scalar curvature of M is constant.

Our Theorem 2 says that, for $n \geqq 3$, conditions a) and b) imply that M is holomorphically isometric to $P^n(C)$. A result due to Berger and Goldberg-Bishop [**4**] implies that b) and c) give rise to the same conclusion. Finally, Kobayashi [**5**] proved, by using the main theorem in [**1**], that a) and c) imply that M is either $P^n(C)$ or a complex quadric, provided that $n \geqq 2$.

Brown University

Bibliography

[1] B. Smyth, Differential geometry of complex hypersurfaces, Ann. of Math., **85** (1967), 246-266.

[2] K. Nomizu and B. Smyth, Differential geometry of complex hypersurfaces II, J. Math. Soc. Japan, **20** (1968), 498-521.

[3] P. Samuel, Méthodes d'algèbre abstraite en géométrie algébrique, second edition, Springer-Verlag, Berlin, 1967.

[4] I. Goldberg and R. L. Bishop, On the topology of positively curved Kähler manifolds II, Tôhoku Math. J., **17** (1965), 310-318.

[5] S. Kobayashi, Hypersurfaces of complex projective space with constant scalar curvature, J. Differential Geometry, **1** (1967), 369-370.