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Well-orderings and finite quantifiers

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§0. **Introduction.** That the class **W** of all (non-empty) well-orderings cannot be characterized using (finite) first-order sentences is a well known result. Almost as well known is that **W** can be characterized by an infinitely long sentence involving conjunctions of countably many formulas and quantifications over countable sequences of individual variables (cf. [8]). In [4] and [5] it is shown that in order to characterize **W** in an infinitary first-order language quantifications over infinitely many individual variables are essential. The aim of this paper is to determine how much can we express, concerning well-orderings, in infinitary languages whose only non-logical constant is a binary relation symbol and which allow the conjunction/disjunction of infinitely many formulas but whose quantifiers bind single individual variables. The results in this note are obtained by an elimination of quantifiers, that is we determine a certain class of sentences, which for lack of a better name we shall call "sentences in normal form" or simply "normal sentences", such that any other sentence is equivalent (as far as well-orderings are concerned) to a disjunction of normal sentences. The method of carrying out the elimination of quantifiers is essentially an extension of the combination of the methods used by Ehrenfeucht [2] and Mostowski/Tarski [6] for the finite language.

§1. **The language $\mathbf{L}_{\alpha\omega}$.** Var_α is the set of individual variables of $\mathbf{L}_{\alpha\omega}$ and $\mathrm{Var}_\alpha=\{v_\mu\colon \mu<\alpha\}$. The atomic formulas of $\mathbf{L}_{\alpha\omega}$ are the expressions of the form: $x\doteq y$ and $x<y$ where x and y are individual variables. The set of formulas of $\mathbf{L}_{\alpha\omega}$ is the least set S which includes all the atomic formulas and such that:

(a) if $\theta\in S$, then the negation of θ, $\neg\theta$, is also a member of S,

(b) if $X=\{\theta_i\colon i\in I\}\subseteqq S$ and $|X|<\alpha$, then both the conjunction of X, $\bigwedge X$ (also written $\bigwedge_{i\in I}\theta_i$) and the disjunction of X, $\bigvee X$ (or $\bigvee_{i\in I}\theta_i$) are also members of S,

(c) if $\theta\in S$ and $x\in \mathrm{Var}_\alpha$, then both the universal quantification of θ, $\forall x\theta$ and the existential quantification of θ, $\exists x\theta$, are members of S.

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The finitary propositional connectives: $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined in the usual way in terms of $\bigwedge$, $\bigvee$ and $\neg$.

Proceeding as in the finitary languages we then could (and we shall assume that we have done so) define the following standard notions:

$\mathrm{FV}(\theta)$ the set of variables occurring free in θ,

$\mathrm{VS}(\theta)$ the set of variables occurring in θ,

$\mathrm{SF}(\theta)$ the set of subformulas of θ.

By a sentence of $\mathbf{L}_{\alpha\omega}$ we understand a formula θ of $\mathbf{L}_{\alpha\omega}$ such that θ has no free variables (i. e. that $\mathrm{FV}(\theta)=0$). An important characteristic of the languages $\mathbf{L}_{\alpha\omega}$ is that $|\mathrm{FV}(\phi)\sim\mathrm{FV}(\theta)|<\omega$ whenever $\phi\in\mathrm{SF}(\theta)$ and $\theta\in\mathbf{L}_{\alpha\omega}$; in particular a subformula of a sentence of $\mathbf{L}_{\alpha\omega}$ has finitely many variables occurring free. The *rank of the formula* θ, $rk(\theta)$, is that ordinal such that if θ is atomic, then $rk(\theta)=0$ while $rk(\neg\phi)=rk(\forall x\phi)=rk(\exists x\phi)=rk(\phi)+1$, $rk(\bigwedge X)=rk(\bigvee X)=\bigcup_{\theta\in X}(rk(\theta)+1)$. By the *quantifier degree* of a formula θ, $qd(\theta)$, we understand that ordinal such that (i) if θ is atomic, then $qd(\theta)=0$ (ii) $qd(\neg\phi)=qd(\phi)$ (iii) $qd(\bigwedge X)=qd(\bigvee X)=\bigcup_{\theta\in X}qd(\theta)$ and (iv) $qd(\forall x\phi)=qd(\exists x\phi)=qd(\phi)+1$.

If we make the assumption (and from now on we shall do so):

ASSUMPTION 1: α is a regular cardinal

then for all $\theta\in\mathbf{L}_{\alpha\omega}$ we have that $rk(\theta)<\alpha$, $qd(\theta)<\alpha$ and $|\mathrm{SF}(\theta)|<\alpha$.

In this paper the relational systems that we shall consider will be of the type $\mathfrak{A}=\langle A, R\rangle$ where $A\neq 0$ and $R\subseteq A^2$. We assume that it is known what it means for a sequence $s\in A^\alpha$ (i. e. $\mathrm{Dom}(s)=\alpha=\{\mu:\mu<\alpha\}$ and $\mathrm{Rng}(s)\subseteq A$) to satisfy the formula θ of $\mathbf{L}_{\alpha\omega}$ in the relational system $\mathfrak{A}=\langle A, R\rangle$; we shall express the condition by "$(\mathfrak{A}, s)\models\theta$". "$\mathfrak{A}\models\theta$" means that for all $s\in A^\alpha$, $(\mathfrak{A}, s)\models\theta$ If $s\in A^\Delta$ where $\{\mu: v_\mu\in\mathrm{FV}(\theta)\}\subseteq\Delta\subseteq\alpha$, then we shall define $(\mathfrak{A}, s)\models\theta$ to mean that for some (or equivalently: for all) $s^*\in A^\alpha$ such that $s\subseteq s^*$, $(\mathfrak{A}, s^*)\models\theta$. If $\mathbf{K}$ is a class of relational systems and θ a sentence, then "$\mathbf{K}$ is a model of θ", in symbols: $\mathbf{K}\models\theta$, just in case that for all $\mathfrak{A}\in\mathbf{K}$, $\mathfrak{A}\models\theta$. Conversely if Γ is a set (or class of sentences from different $\mathbf{L}_{\alpha\omega}$), then $\mathrm{Mod}(\Gamma)=\{\mathfrak{A}:\text{ for all }\theta\in\Gamma,\ \mathfrak{A}\models\theta\}$.

§2. **The language-class $\mathbf{Q}_\alpha$.** Let $\mathbf{RC}$ be the class of all infinite regular cardinals. Then let $\mathbf{L}$ be $\bigcup_{\alpha\in\mathbf{RC}}\mathbf{L}_{\alpha\omega}$. $\mathbf{L}$ will be called a language-class (we prefer to restrict the name "language" to a set). Since every well-ordered relational system can be characterized (up to isomorphism) by a sentence of $\mathbf{L}$, the class Σ of sentences of $\mathbf{L}$ which "state" that every non-empty definable subset of a linear ordering has a first element has the property that its models are precisely the well-ordered systems. As we shall see the success of Σ in characterizing $\mathbf{W}$ depends strongly on the fact that the quantifier degrees of the sentences in Σ are unbounded. Thus the

DEFINITION. $\mathbf{Q}_\alpha = \{\theta : \theta \in \mathbf{L}_{\alpha\omega} \text{ and } qd(\theta) < \alpha\}$.

One of the principal results of the paper is to show that $\mathbf{Q}_\alpha$, as far as well-orderings are concerned, does not allow us to express any more than $\mathbf{L}_{\alpha\omega}$, that is every sentence of $\mathbf{Q}_\alpha$ is equivalent to a sentence of $\mathbf{L}_{\alpha\omega}$ (cf. Theorem 5.16, p. 487). However, even though $\mathbf{Q}_\alpha$ is no stronger than $\mathbf{L}_{\alpha\omega}$ (for well-orderings; in general $\mathbf{Q}_\alpha$ is much stronger than $\mathbf{L}_{\alpha\omega}$) there are certain advantages to working with $\mathbf{Q}_\alpha$, the most important being that for $\mathbf{Q}_\alpha$ the distributivity law holds; i.e. if θ is a formula of $\mathbf{Q}_\alpha$ which is a disjunction of conjunctions (conjunction of disjunctions), then θ is logically equivalent to a formula θ^* of $\mathbf{Q}_\alpha$ which is a conjunction of disjunctions (disjunction of conjunctions).

§ 3. Certain classes of relational systems.

W has already been defined as the class of all non-empty well-orderings, i.e. $\mathfrak{A} = \langle A, R\rangle \in \mathbf{W}$ just in case that $A \neq 0$ and R well-orders A. We shall let $\mathbf{T}_\alpha$ be the $\mathbf{L}_{\alpha\omega}$-theory of well-orderings, that is

(3.1) $\quad \mathbf{T}_\alpha = \{\theta : \theta \text{ is a sentence of } \mathbf{L}_{\alpha\omega} \text{ and } \mathbf{W} \models \theta\}$.

The following classes of relational systems and sentences are natural classes to consider:

(3.2) $\quad \mathbf{T}_\alpha^{\mathbf{Q}} = \{\theta : \theta \text{ is a sentence of } \mathbf{Q}_\alpha \text{ and } \mathbf{W} \models \theta\}$,

(3.3) $\quad \mathbf{W}_\alpha = \mathrm{Mod}\,(\mathbf{T}_\alpha)$,

(3.4) $\quad \mathbf{W}_\alpha^{\mathbf{Q}} = \mathrm{Mod}\,(\mathbf{T}_\alpha^{\mathbf{Q}})$.

It is clear that $\mathbf{W} \subseteqq \mathbf{W}_\alpha^{\mathbf{Q}} \subseteqq \mathbf{W}_\alpha$. It will be shown in Theorem 5.19, p. 24 that $\mathbf{W}_\alpha^{\mathbf{Q}} = \mathbf{W}_\alpha \neq \mathbf{W}$. Just as in the case of the finitary first-order language we shall also consider the class of linear orderings in which every non-empty definable subset has a first element. For that purpose let $\mathbf{D}_\alpha$ be the set of all sentences of $\mathbf{L}_{\alpha\omega}$ of the form

$$\forall v_0 \forall v_1 (v_0 < v_1 \vee v_0 \doteq v_1 \vee v_1 < v_0)$$
$$\wedge \forall x_0 \cdots \forall x_{k-1} (\exists x_k \theta \rightarrow \exists x_k (\theta \wedge \forall z(\theta[x_k/z] \rightarrow z \doteq x_k \vee x_k < z)))$$

where θ is a formula of $\mathbf{L}_{\alpha\omega}$ with finitely many free variables such that $\mathrm{FV}(\theta) \subseteqq \{x_i : i \leqq k\}$, $z \notin \mathrm{FV}(\theta)$ and $\theta[x_k/z]$ is the formula $\exists x_k(x_k \doteq z \wedge \theta)$. Then we let

(3.5) $\quad \mathbf{M}_\alpha = \mathrm{Mod}\,(\mathbf{D}_\alpha)$,

(3.6) $\quad \mathbf{M}_\alpha^{\mathbf{Q}} = \mathrm{Mod}\,(\mathbf{D}_\alpha^{\mathbf{Q}})$,

where $\mathbf{D}_\alpha^{\mathbf{Q}}$ is like $\mathbf{D}_\alpha$ except that the formulas should be from $\mathbf{Q}_\alpha$ instead of $\mathbf{L}_{\alpha\omega}$. From the obvious properties of well-orderings we immediately obtain the following inclusions:

$$\mathbf{W} \subseteqq \mathbf{W}_\alpha^{\mathbf{Q}} \subseteqq \mathbf{M}_\alpha^{\mathbf{Q}} \subseteqq \mathbf{M}_\alpha, \qquad \mathbf{W} \subseteqq \mathbf{W}_\alpha \subseteqq \mathbf{M}_\alpha .$$

We shall eventually show that $\mathbf{W}_\alpha = \mathbf{M}_\alpha = \mathbf{M}_\alpha^{\mathbf{Q}} = \mathbf{W}_\alpha^{\mathbf{Q}}$ (cf. Theorem 5.19, p. 488).

§ 4. The sentences in normal form. In order to define the normal sentences we need to give certain formulas of $\mathbf{L}_{\alpha\omega}$ related to the notion of a limit (or derived) point in a linear ordering. In the definitions that follow it is assumed that μ, ξ and λ are ordinals and that λ is a non-zero limit ordinal (if in addition μ, ξ and λ are ordinals strictly smaller that α then it is easy to verify that the formulas defined are indeed formulas of $\mathbf{L}_{\alpha\omega}$).

4.1 DEFINITION. Lim_μ *(read: v_0 is a μ-limit point) is the formula such that*

(i) $\mathrm{Lim}_0 = v_0 \doteq v_0$,

(ii) $\mathrm{Lim}_{\xi+1} = \mathrm{Lim}_\xi \wedge \exists v_1(v_1 < v_0 \wedge \mathrm{Lim}_\xi [v_0/v_1])$
$\wedge \forall v_1(v_1 < v_0 \wedge \mathrm{Lim}_\xi [v_0/v_1] \rightarrow \exists v_2(v_1 < v_2 \wedge v_2 < v_0 \wedge \mathrm{Lim}_\xi [v_0/v_2]))$,

(iii) $\mathrm{Lim}_\lambda = \bigwedge_{\xi<\lambda} \mathrm{Lim}_\xi$.

4.2 DEFINITION. Las_μ *(read: v_0 is the last μ-limit point) is the formula*

$$\mathrm{Lim}_\mu \wedge \neg\exists v_1(v_0 < v_1 \wedge \mathrm{Lim}_\mu [v_0/v_1]) .$$

4.3 DEFINITION. End_μ^{-1} *(read: there are no μ-limit points) is the sentence*

$$\neg\exists v_0 \, \mathrm{Lim}_\mu .$$

4.4 DEFINITION. End_μ^0 *(read: the μ-limit points are unbounded) is the sentence:*

$$\exists v_0 \, \mathrm{Lim}_\mu \wedge \forall v_0(\mathrm{Lim}_\mu \rightarrow \exists v_1(v_0 < v_1 \wedge \mathrm{Lim}_\mu [v_0/v_1])) .$$

4.5 DEFINITION. End_μ^{n+1} *(where n is a natural number and is read: the μ-end number is $n+1$) is the sentence:*

$$\exists^{n+1} v_0 \, \mathrm{Lim}_\mu \vee \exists v_0(\mathrm{Las}_{\mu+1} \wedge \exists^n v_1(v_0 < v_1 \wedge \mathrm{Lim}_\mu [v_0/v_1])) ,$$

where $\exists^k x\theta$ is the formula that "expresses" the condition that there are exactly k things x such that θ.

4.6 DEFINITION. *If θ is a formula of $\mathbf{L}_{\alpha\omega}$ and ϕ is a formula of $\mathbf{L}_{\alpha\omega}$ such that $v_0 \in \mathrm{FV}(\phi)$, then $(\theta)^\phi$ is the formula (of $\mathbf{L}_{\alpha\omega}$) obtained by relativizing the quantifiers in θ to ϕ (i.e. replacing $\forall x \cdots$ by $\forall x(\phi[v_0/x] \rightarrow \cdots$ and correspondingly for $\exists x$).*

4.7 DEFINITION. Den_0^∞ *is any sentence of $\mathbf{L}_{\omega,\omega}$ such that for any system $\mathfrak{A} = \langle A, R\rangle$, $\mathfrak{A} \models \mathrm{Den}_0^\infty$ just in case that either $\mathfrak{A} \cong \langle \omega, \in_\omega\rangle$ or that for some $a \in A$, $\langle \omega, \in_\omega\rangle$ is isomorphic to the subsystem of $\mathfrak{A}$ determined by the set $\{b: \langle b, a\rangle \in R\}$ (in other words that $\langle \omega, \in_\omega\rangle$ is either isomorphic to $\mathfrak{A}$ or isomorphic to an initial segment of $\mathfrak{A}$).*

4.8 DEFINITION. $\mathrm{Den}_\mu^\infty = (\mathrm{Den}_0^\infty) \mathrm{Lim}_\mu$.

4.9 DEFINITION. Den_{μ}^{n+1} *(read: the μ-derived number is $n+1$, $n<\omega$) is the sentence:*

$$\exists^{n+1}v_0\,\mathrm{Lim}_{\mu}\,.$$

For typographical reasons we shall let Den_{μ}^{0} be same sentence as End_{μ}^{-1}.

Given a linearly ordered system $\mathfrak{A}=\langle A, R\rangle$ we let (i) $\mathrm{Lm}_{\mu}(\mathfrak{A})=\{b: (\mathfrak{A}, \langle b\rangle)\models \mathrm{Lim}_{\mu}\}$, (ii) $\mathrm{Ls}_{\mu}(\mathfrak{A})$ be the unique element $b\in A$ such that $(\mathfrak{A}, \langle b\rangle)\models \mathrm{Las}_{\mu}$ if such a unique element exists, otherwise $\mathrm{Ls}_{\mu}(\mathfrak{A})$ is left undefined, (iii) $\mathrm{Ed}_{\mu}(\mathfrak{A})$ be the unique $s\in\{-1\}\cup\omega$ such that $\mathfrak{A}\models\mathrm{End}_{\mu}^{s}$ if such a unique s exists, otherwise it is left undefined and (iv) correspondingly for $\mathrm{Dn}_{\mu}(\mathfrak{A})$. Finally if ρ is an ordinal, then we let ρ be the relational system $\langle\rho, \in_{\rho}\rangle$.

We shall make use of the following properties of ordinals and since they can be established without much difficulty we shall omit the proofs.

4.10 PROPOSITIONS. *If ρ, σ and μ are ordinals then:*

(.1) $\mathrm{Dn}_{\mu}(\rho)=0$ *if and only if* $\rho\leqq\omega^{\mu}$,

(.2) $\mathrm{Dn}_{\mu}(\rho)=0$ *and for all* $\xi<\mu$, $\mathrm{Ed}_{\xi}(\rho)=0$ *if and only if* $\rho=\omega^{\mu}$,

(.3) $\mathrm{Dn}_{\mu}(\rho)=n+1$ *if and only if for some* ξ, $0<\xi<\omega^{\mu}$, $\rho=\omega^{\mu}\cdot(n+1)+\xi$,

(.4) $\mathrm{Dn}_{\mu}(\rho)=\infty$ *if and only if* $\omega^{\mu+1}\leqq\rho$,

(.5) $\mathrm{Ed}_{\mu}(\rho)=0$ *if and only if for all* $\xi\leqq\mu$, $\mathrm{Ed}_{\xi}(\rho)=0$,

(.6) *if* $\mathrm{Ed}_{\mu}(\rho)=0$ *then* $\mathrm{Dn}_{\mu}(\rho)=\infty$,

(.7) *if* $\rho=\omega^{\mu}\cdot\eta_0+\delta$, $\sigma=\omega^{\mu}\cdot\eta_1+\delta$ *where* $\eta_0, \eta_1>0$ *and* $\delta<\omega^{\mu}$ *then for all* $\xi<\mu$, $\mathrm{Dn}_{\xi}(\rho)=\mathrm{Dn}_{\xi}(\sigma)$ *and* $\mathrm{Ed}_{\xi}(\rho)=\mathrm{Ed}_{\xi}(\sigma)$,

(.8) *if* ρ, $\sigma<\omega^{\mu}$ *and for all* $\xi<\mu$, $\mathrm{Ed}_{\xi}(\rho)=\mathrm{Ed}_{\xi}(\sigma)$, *then* $\rho=\sigma$,

(.9) *if* $\mathrm{Dn}_{\mu}(\rho)=\mathrm{Dn}_{\mu}(\sigma)\neq\infty$ *and for all* $\xi\leqq\mu$, $\mathrm{Ed}_{\xi}(\rho)=\mathrm{Ed}_{\xi}(\sigma)$, *then* $\rho=\sigma$.

It is clear from the above propositions that not all possible combinations of end-numbers and derived numbers are taken by well-ordered systems. This leads to the following

4.12 DEFINITION. *If f and g are functions such that $f\in(\omega\cup\{\infty\})^{\mu}$, $g\in(\omega\cup\{-1\})^{\mu}$, then (f, g) is a "μ-consistent pair" just in case that for some non-zero ordinal ρ, the sentence $\mathrm{Cp}((f, g))$, where*

$$\mathrm{Cp}((f, g))=\bigwedge\nolimits_{\xi<\mu}(\mathrm{Den}_{\xi}^{f(\xi)}\wedge\mathrm{End}_{\xi}^{g(\xi)})$$

is satisfied in ρ.

4.13 DEFINITION. *The set $\mathbf{NS}_{\mu}$ of sentences in normal form is the set of sentences of $\mathbf{L}_{\alpha\omega}$ $(\mu<\alpha)$ such that*

$$\mathbf{NS}_{\mu}=\{\mathrm{Cp}(x): \textit{for some } \xi<\mu,\ x \textit{ is a } \xi\textit{-consistent pair}\}\,.$$

In order to determine the cardinality $|\mathbf{NS}_{\mu}|$ of $\mathbf{NS}_{\mu}$ we need the following:

4.14 PROPOSITION. *To every ordinal ρ there corresponds at least one ordinal $\xi<\omega^{\mu+1}$ such that for all $\eta<\mu$,*

$$\mathrm{Dn}_{\eta}(\rho)=\mathrm{Dn}_{\eta}(\xi)\quad\textit{and}\quad\mathrm{Ed}_{\eta}(\rho)=\mathrm{Ed}_{\eta}(\xi)\,.$$

PROOF. Follows from propositions 4.10.

Since the case when $\alpha=\omega$, $\mathbf{L}_{\alpha\omega}$ is (isomorphic) to the usual finitary first order language we shall from now on make the following

ASSUMPTION 2. α is a regular cardinal $>\omega$.

From the assumption 2, proposition 4.14 and the fact that $\omega^{\mu}<\alpha$ whenever $\mu<\alpha$ we immediately obtain.

4.15 PROPOSITION.

(.1) *If* $\mu<\alpha$, *then* $|\mathbf{NS}_{\mu}|<\alpha$,

(.2) *if* $\mu<\alpha$ *and* $X\subseteqq \mathbf{NS}_{\mu}$, *then* $\bigvee X\in \mathbf{L}_{\alpha\omega}$.

In the case of the finitary language $\mathbf{L}_{\omega\omega}$ every finite ordinal is definable and not surprisingly the corresponding result is true in $\mathbf{L}_{\alpha\omega}$. That is to every ordinal $\mu<\alpha$ there corresponds a sentence Ord_{μ} of $\mathbf{L}_{\alpha\omega}$ such that for every relational system $\mathfrak{A}$, $\mathfrak{A}\models \mathrm{Ord}_{\mu}$ if and only if $\mathfrak{A}\cong\mu$ (cf. [7]). Using the sentences Ord_{μ} we can then show that if $\mathfrak{A}=\langle A, R\rangle\in\mathbf{M}_{\alpha}$ then if $|\mathfrak{A}|$ $(=|A|)$ $<\alpha$, $\mathfrak{A}$ is a well-ordering while if $|\mathfrak{A}|\geqq\alpha$, then α is either isomorphic to $\mathfrak{A}$ or else it is isomorphic to an initial segment of $\mathfrak{A}$. Moreover this is true not only for $\mathfrak{A}$ but for every interval $\mathfrak{B}$ of $\mathfrak{A}$ (cf. definition 4.17 below).

4.16 DEFINITION. *If* $\mathfrak{A}=\langle A, R\rangle$, $a\in A$ *and* $a'\in A$, *then*

(i) $[*,\infty)^{\mathfrak{A}}=A$,

(ii) $[*,a)^{\mathfrak{A}}=\{b:\langle b,a\rangle\in R\}$,

(iii) $[a,a')^{\mathfrak{A}}=\{b: b=a \text{ or } (\langle a,b\rangle\in R \text{ and } \langle b,a'\rangle\in R)\}$,

(iv) $[a,\infty)^{\mathfrak{A}}=\{b: b=a \text{ or } \langle a,b\rangle\in R\}$,

(v) $[x,y)^{\mathfrak{A}}=$ *the subsystem of* $\mathfrak{A}$ *determined by the set* $[x,y)^{\mathfrak{A}}$ *(provided it is not empty and that* $\{x,y\}\subseteqq A\cup\{*,\infty\}$*).*

4.17 DEFINITION. *The set on intervals of* $\mathfrak{A}$, $\mathrm{Int}(\mathfrak{A})$, *is the set*

$$\{[x,y)^{\mathfrak{A}}: \{x,y\}\subseteqq A\cup\{*,\infty\}\}.$$

4.18 THEOREM. *If* $\mathfrak{A}\in\mathbf{M}_{\alpha}$, *then*

(i) $\mathrm{Int}(\mathfrak{A})\subseteqq\mathbf{M}_{\alpha}$,

(ii) *if* $|\mathfrak{A}|<\alpha$, *then* $\mathfrak{A}$ *is a well-ordering,*

(iii) *if* $|\mathfrak{A}|>\alpha$, *then either* $\mathfrak{A}\cong\alpha$ *or else* $\mathfrak{A}$ *contains an initial segment isomorphic to* α.

PROOF. (i) Let $\mathfrak{A}\in\mathbf{M}_{\alpha}$ and $\mathfrak{B}\in\mathrm{Int}(\mathfrak{A})$. To show that $\mathfrak{B}\in\mathbf{M}_{\alpha}$ we must show (roughly speaking) that every non-empty definable subset of $\mathfrak{B}$ has a first element; but because of the relation of $\mathfrak{B}$ to $\mathfrak{A}$ a definable subset of $\mathfrak{B}$ is a definable subset of $\mathfrak{A}$ and hence (i) follows. (ii) and (iii) have essentially been proved by the remarks prior to 4.16.

4.19 THEOREM. *If* $\mathfrak{A}\in\mathbf{W}_{\alpha}$, *then to every* $\mu<\alpha$, *there corresponds at least one ordinal* ρ *such that for every* $\xi<\mu$, ρ *and* $\mathfrak{A}$ *satisfy exactly the same sentences of the form* End_{ξ}^{k} *and* Den_{ξ}^{s}.

PROOF. According to 4.14 and 3.1 $\varphi=\bigvee_{\theta\in\mathbf{NS}_{\mu}}\theta$ is a sentence of $\mathbf{T}_{\alpha}$. Hence

$\mathfrak{A} \models \varphi$, whenever $\mathfrak{A} \in \mathbf{W}_\alpha$. Thus for some $\theta \in \mathbf{NS}_\mu$, $\mathfrak{A} \models \theta$. Therefore from 4.12 and 4.13 it follows that there exists an ordinal ρ with the required properties.

The properties mentioned in 4.18 and 4.19 are the main tools used in the elimination of quantifiers. Thus the following definition:

4.20 DEFINITION. *$\mathbf{K}_\alpha$ is the class of linearly ordered systems $\mathfrak{A}$ having a first element such that for every $\mathfrak{B} \in \mathrm{Int}(\mathfrak{A})$:*

(a) *if $|\mathfrak{B}| < \alpha$, then $\mathfrak{B}$ is a well-ordering,*

(b) *if $|\mathfrak{B}| \geqq \alpha$, then either $\alpha \cong \mathfrak{B}$ or α is isomorphic to an initial segment of $\mathfrak{B}$ (we shall denote this condition by $\alpha \leqq \mathfrak{B}$),*

(c) *to every $\mu < \alpha$ there corresponds an ordinal ρ such that for every $\xi < \mu$, $\mathfrak{B}$ and ρ satisfy the same sentences of the form End_ξ^k and Den_ξ^k.*

Most of our work will now be done with the class $\mathbf{K}_\alpha$ and eventually we shall show that

$$\mathbf{K}_\alpha = \mathbf{W}_\alpha^{\mathbf{Q}} = \mathbf{W}_\alpha = \mathbf{M}_\alpha = \mathbf{M}_\alpha^{\mathbf{Q}} \neq \mathbf{W}\,.$$

§5. The elimination of quantifiers.

5.1 DEFINITION (Ehrenfeucht/Fraisse, cf. [3]). *If $\mathfrak{A} = \langle A, R\rangle$, $\mathfrak{B} = \langle B, S\rangle$, $x \in A^n$, $y \in B^m$ and $n, m < \omega$, then*

(.1) *$(\mathfrak{A}, x) \equiv_0 (\mathfrak{B}, y)$ if and only if $n = m$ and $\{\langle x_i, y_i\rangle : i < n\}$ is an isomorphism from $\mathfrak{A}$ restricted to $\{x_i : i < n\}$ into $\mathfrak{B}$,*

(.2) *$(\mathfrak{A}, x) \equiv_{\xi+1} (\mathfrak{B}, y)$ if and only if for all $a \in A$ there exists a $b \in B$ such that $(\mathfrak{A}, x^\frown\langle a\rangle) \equiv_\xi (\mathfrak{B}, y^\frown\langle b\rangle)$ and for all $b \in B$ there exists an $a \in A$ such that $(\mathfrak{A}, x^\frown\langle a\rangle) \equiv_\xi (\mathfrak{B}, y^\frown\langle b\rangle)$ (where if s and t are sequences then $s^\frown t$ is the concatenation of s and t),*

(.3) *if $0 < \lambda = \cup\lambda$, then $(\mathfrak{A}, x) \equiv_\lambda (\mathfrak{B}, y)$ if and only if for all $\xi < \lambda$ $(\mathfrak{A}, x) \equiv_\xi (\mathfrak{B}, y)$,*

(.4) *$\mathfrak{A} \equiv_\mu \mathfrak{B}$ if and only if $(\mathfrak{A}, 0) \equiv_\mu (\mathfrak{B}, 0)$.*

The following theorem can be proved by the methods in [2] and thus its proof is omitted.

5.2 THEOREM. *If $\mu < \alpha$, $\mathfrak{A} = \langle A, R\rangle$, $\mathfrak{B} = \langle B, S\rangle$, $x \in A^n$ and $y \in B^n$, then the following two conditions are equivalent:*

(1) *$(\mathfrak{A}, x) \equiv_\mu (\mathfrak{B}, y)$,*

(2) *to every formula $\theta \in \mathbf{Q}_\alpha$ such that $\mathrm{FV}(\theta) \subseteqq \{v_i : i < n\}$ and $qd(\theta) \leqq \mu$, $(\mathfrak{A}, x) \models \theta$ if and only if $(\mathfrak{B}, y) \models \theta$.*

An immediate corollary of the above is that if $\mathfrak{A} \equiv_\alpha \mathfrak{B}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathbf{Q}_\alpha$ (and hence a fortiori $\mathbf{L}_{\alpha\omega}$)-elementarily equivalent. It is not difficult to give examples of relational systems which are $\mathbf{L}_{\alpha\omega}$ elementarily equivalent but not $\mathbf{Q}_\alpha$-elementarily equivalent. However we shall prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are restricted to be members of $\mathbf{K}_\alpha$ then $\mathbf{Q}_\alpha$-elementary equivalence coincides with $\mathbf{L}_{\alpha\omega}$-elementary equivalence.

The following properties of the members of $\mathbf{K}_\alpha$ are required.

5.4 PROPOSITION. *If* $\mathfrak{A} \in \mathbf{K}_\alpha$, *then* $\mathrm{Int}(\mathfrak{A}) \subseteqq \mathbf{K}_\alpha$ *and*

(.1) *if* $\mathrm{Dn}_\mu(\mathfrak{A}) = 0$ *and* $\mu < \alpha$, *then for some* $\rho \leqq \omega^\mu$, $\mathfrak{A} \cong \rho$,

(.2) *if* $\mathrm{Dn}_\mu(\mathfrak{A}) = n+1$ *and* $\mu < \alpha$, *then for some* $\rho < \omega^\mu$, $\mathfrak{A} \cong \omega^\mu \cdot (n+1) + \rho$,

(.3) *if* $\mathrm{Dn}_\mu(\mathfrak{A}) = \infty$ *where* $\mu < \alpha$, *then* $\omega^{\mu+1} \leqq \mathfrak{A}$.

PROOF. Recall that if $\mathfrak{A} \in \mathbf{K}_\alpha$, then for every interval $\mathfrak{B}$ of $\mathfrak{A}$, either $\alpha \leqq \mathfrak{B}$ or else $\mathfrak{B}$ is a well-ordering. Apply then 4.10.

Combining the above with 4.10.9 we then obtain

5.5 PROPOSITION. *If* $\mathfrak{A}$, $\mathfrak{B} \in \mathbf{K}_\alpha$, $\mu < \alpha$, $\mathrm{Dn}_\mu(\mathfrak{A}) = \mathrm{Dn}_\mu(\mathfrak{B}) \neq \infty$ *and* $\mathrm{Ed}_\xi(\mathfrak{A}) = \mathrm{Ed}_\xi(\mathfrak{B})$ *whenever* $\xi \leqq \mu$, *then* $\mathfrak{A} \cong \mathfrak{B}$.

5.6 PROPOSITION. *If* $\mathfrak{A} = \langle A, R \rangle \in \mathbf{K}_\alpha$, $\mathfrak{B} = \langle B, S \rangle \in \mathrm{Int}(\mathfrak{A})$ *and* $\mu < \alpha$, *then*

(.1) *if* $b \in \mathrm{Lm}_\mu(\mathfrak{A})$, $b \in B$ *and for some* $a \in B$, $\langle a, b \rangle \in R$, *then* $b \in \mathrm{Lm}_\mu(\mathfrak{B})$,

(.2) *if* $\mathrm{Ed}_\mu(\mathfrak{B}) = 0$, *then for all* $b \in B$ *and all* $\xi \leqq \mu$, $\mathrm{Ed}([b, \infty)^{\mathfrak{B}}) = 0$,

(.3) *if* $\mathrm{Lm}_{\mu+1}(\mathfrak{B}) = 0$ *and* $\mathrm{Ed}_\mu(\mathfrak{B}) = n+1$, *then* $\mathrm{Dn}_\mu(\mathfrak{B}) = n+1$,

(.4) *if* $\mathrm{Ls}_{\mu+1}(\mathfrak{B}) = b \in B$, $c \in B$ *and* $\langle c, b \rangle \in R$, *then for all* $\xi \leqq \mu$, $\mathrm{Ed}_\xi(\mathfrak{B}) = \mathrm{Ed}([c, \infty)^{\mathfrak{B}})$,

(.5) *if* $\{b, c\} \subseteqq \mathrm{Lm}_\mu(\mathfrak{A})$, $\langle b, c \rangle \in R$ *and there does not exist an* $a \in \mathrm{Lm}_\mu(\mathfrak{B})$ *such that* $\langle b, a \rangle \in R$ *and* $\langle a, c \rangle \in R$, *then* $[b, c)^{\mathfrak{B}} \cong \omega^\mu$.

PROOF. (.1) follows from the condition that all members of $\mathbf{K}_\alpha$ are linearly ordered. (.2) assume $\mathrm{Ed}_\mu(\mathfrak{B}) = 0$. From the definition of End_μ^0 (cf. 4.4) and (.1) it follows that $\mathrm{Ed}_\mu([b, \infty)^{\mathfrak{B}}) = 0$. From condition (c) of 4.20 it follows that from $\mathrm{Ed}_\mu([b, \infty)^{\mathfrak{B}}) = 0$ we obtain that for all $\xi \leqq \mu$, $\mathrm{Ed}_\xi([b, \infty)^{\mathfrak{B}}) = 0$; thus (.2). Part (.3) is immediate from the definitions (cf. 4.5, 4.1 and 4.9). To prove (.4) we first note that the condition $b = \mathrm{Ls}_{\mu+1}(\mathfrak{B}) \in B$ tells us that b is a $\mu+1$-limit point and the last such (in $\mathfrak{B}$). The case $\mathrm{Ed}_\mu(\mathfrak{B}) = 0$ is taken care by (.2). On the other hand we cannot have that $\mathrm{Ed}_\mu(\mathfrak{B}) = -1$ since b is a μ-limit point. Using (.1) we see that $\mathrm{Ed}_\mu([c, \infty)^{\mathfrak{B}}) \neq -1$. The proof of (.4) is completed by considering the definition of End_μ^{n+1}. For the proof of (.5) it suffices to remark that from conditions (a), (b) of the definition of $\mathbf{K}_\alpha$ (cf. 4.20) $[b, c)^{\mathfrak{B}}$ must be a well-ordering and then it is immediate from the properties of ordinals that $[b, c)^{\mathfrak{B}} \cong \omega^\mu$.

The following proposition requires a little more work.

5.7 PROPOSITION. *If* $\mathfrak{A} = \langle A, R \rangle$, $\mathfrak{B} = \langle B, S \rangle$ *are members of* $\mathbf{K}_\alpha$ $\mu < \alpha$, $a = \mathrm{Ls}_{\mu+1}(\mathfrak{A}) \in A$, $b = \mathrm{Ls}_{\mu+1}(\mathfrak{B}) \in B$, *and for all* $\xi \leqq \mu+1$, $\mathrm{Ed}_\xi(\mathfrak{A}) = \mathrm{Ed}_\xi(\mathfrak{B})$, *then* $[a, \infty)^{\mathfrak{A}} \cong [b, \infty)^{\mathfrak{B}}$.

PROOF. Assume the antecedent. Since $a = \mathrm{Ls}_{\mu+1}(\mathfrak{A})$ and $b = \mathrm{Ls}_{\mu+1}(\mathfrak{B})$ it follows that $\mathrm{Dn}_{\mu+1}([a, \infty)^{\mathfrak{A}}) = \mathrm{Dn}_{\mu+1}([b, \infty)^{\mathfrak{B}}) = 0$. Thus for some ordinals ρ_1 and ρ_2 smaller than or equal to $\omega^{\mu+2}$ we have that $[a, \infty)^{\mathfrak{A}} \cong \rho_1$ and $[b, \infty)^{\mathfrak{B}} \cong \rho_2$. Let $\omega^{\eta_0} \cdot n_0 + \cdots + \omega^{\eta_k} \cdot n_k$ and $\omega^{\delta_0} \cdot m_0 + \cdots + \omega^{\delta_s} \cdot m_s$ be the Cantor normal forms of ρ_1 and ρ_2 respectively (cf. [1]). From the assumption that for all $\xi \leqq \mu+1$, $\mathrm{Ed}_\xi(\mathfrak{A}) = \mathrm{Ed}_\xi(\mathfrak{B})$ we obtain that $k = s$ and that for all $i \leqq k$, $\eta_i = \delta_i$ and $m_i = n_i$;

in other words that $\rho_1=\rho_2$.

5.8 DEFINITION. *If* $\mathfrak{A}=\langle A, R\rangle$, $\mathfrak{B}=\langle B, S\rangle$, *then*

(.1) $\mathfrak{A}\doteq_\mu\mathfrak{B}$, *just in case that for all* $\xi\leqq\mu$, $\mathfrak{A}$ *and* $\mathfrak{B}$ *satisfy exactly the same sentences of the forms* End_ξ^ξ *and* Den_ξ^k,

(.2) *if* $x\in A^n$ *and* $y\in B^n$ *and for all* $i<n-1$ $(\langle x_i, x_{i+1}\rangle\in R$ *and* $\langle y_i, y_{i+1}\rangle\in S)$ *then* $(\mathfrak{A}, x)\doteq_\mu(\mathfrak{B}, y)$ *just in case that*

(a) $[*, x_0)^{\mathfrak{A}}\doteq_\mu[*, y_0)^{\mathfrak{B}}$,

(b) $[x_{n-1}, \infty)^{\mathfrak{A}}\doteq_\mu[y_{n-1}, \infty)^{\mathfrak{B}}$,

(c) *for all* $i<n-1$, $[x_i, x_{i+1})^{\mathfrak{A}}\doteq_\mu[y_i, y_{i+1})^{\mathfrak{B}}$.

All the results obtained so far were preparatory lemmas to the following theorem:

5.9 THEOREM. *If* $\mathfrak{A}=\langle A, R\rangle$, $\mathfrak{B}=\langle B, S\rangle$ *are members of* $\mathbf{K}_\alpha$, $\mu<\alpha$ *and* $\mathfrak{A}\doteq_{\mu+1}\mathfrak{B}$, *then for all* $a\in A$, *there exists a* $b\in B$ *such that* $[*, a)^{\mathfrak{A}}\doteq_\mu[*, b)^{\mathfrak{B}}$ *and* $[a, \infty)^{\mathfrak{A}}\doteq_\mu[b, \infty)^{\mathfrak{B}}$.

PROOF. Assume the antecedent. The proof is divided into 9 cases.

CASE 1. *For some* $\xi\leqq\mu+1$, $\mathrm{Dn}_\xi(\mathfrak{A})\neq\infty$.

Result is then obtained by applying 5.5, page 484.

Because of case 1 it suffices to prove the theorem under the further condition:

CONDITION 1.

$$\mathrm{Dn}_{\mu+1}(\mathfrak{A})=\infty\,.$$

In view of 5.4 it follows then that:

(a) $$\omega^{\mu+2}\leqq\mathfrak{A}\quad\text{and}\quad\omega^{\mu+2}\leqq\mathfrak{B}\,.$$

CASE 2. $\mathrm{Ed}_{\mu+1}(\mathfrak{B})=-1$.

Under condition 1 this case does not arise.

CASE 3. $\mathrm{Ed}_{\mu+1}(\mathfrak{A})=0$.

Let $\rho<\omega^{\mu+2}$ be an ordinal such that $[*, a)^{\mathfrak{A}}\doteq_\mu\rho$ (cf. 4.20(c) and 4.14). Choose then for b the element of B such that $[*, b)^{\mathfrak{B}}\cong\rho$ (such an element exists because of (a)). The proof is then completed by applying 5.6.

CASE 4. $\mathrm{Ed}_{\mu+1}(\mathfrak{A})=n+1$ *and* $\mathrm{Lm}_{\mu+2}(\mathfrak{A})=0$.

This case cannot arise under condition 1.

CASE 5. $\mathrm{Ed}_{\mu+1}(\mathfrak{A})=n+1$ *and* $\mathrm{Lm}_{\mu+2}(\mathfrak{B})=0$.

This case cannot arise under condition 1.

Because of cases 1-5 it suffices to prove the theorem under the further:

CONDITION 2.

$$\mathrm{Ed}_{\mu+1}(\mathfrak{A})=n+1\,,$$
$$\mathrm{Ls}_{\mu+2}(\mathfrak{A})=a_0\in A\quad\text{and}\quad\mathrm{Ls}_{\mu+2}(\mathfrak{B})=b_0\in B\,.$$

CASE 6. $\langle a_0, a\rangle\in R$ *or* $a_0=a$.

Consider $[a_0, \infty)^{\mathfrak{A}}$ and $[b_0, \infty)^{\mathfrak{B}}$. Applying 5.7 we obtain that $[a_0, \infty)^{\mathfrak{A}} \cong [b_0, \infty)^{\mathfrak{B}}$. Choose then for b the element corresponding, under the isomorphism, to a.

Because of case 6 we make the further condition:

CONDITION 3.

$$\langle a, a_0 \rangle \in R .$$

It follows then from the conditions and 5.6.4 that:

$$\text{(b)} \qquad \text{if } \xi \leqq \mu+1, \text{ then } \mathrm{Ed}_\xi(\mathfrak{A}) = \mathrm{Ed}_\xi([a, \infty)^{\mathfrak{A}}) = \mathrm{Ed}_\xi(\mathfrak{B}) .$$

CASE 7. *For some* $\xi \leqq \mu+1$, $\mathrm{Dn}_\xi([*, a)^{\mathfrak{A}}) \neq \infty$.

Let ρ be the least ξ such that $\mathrm{Dn}_\xi([*, a)^{\mathfrak{A}}) \neq \infty$ and let $m = \mathrm{Dn}_\xi([*, a)^{\mathfrak{A}})$. From 5.4 we obtain that for some δ, $\delta < \omega^\rho$ and $[*, a)^{\mathfrak{A}} \cong \omega^\rho \cdot m + \delta$. Because of (a) we must have that $\omega^{\mu+2} \leqq [a, \infty)^{\mathfrak{A}}$. Hence for all $\xi \leqq \mu$, $\mathrm{Dn}_\xi([a, \infty)^{\mathfrak{A}}) = \infty$. Thus if for b we choose the element of B such that $[*, b)^{\mathfrak{B}} \cong \omega^\rho \cdot m + \delta < \omega^{\mu+2}$, then we also have that $\mathrm{Dn}_\xi([b, \infty)^{\mathfrak{B}}) = \infty$ whenever $\xi \leqq \mu$. The proof of this case is completed by applying (b).

For the remaining cases we add the further condition:

CONDITION 4.

$$\mathrm{Dn}_{\mu+1}([*, a)^{\mathfrak{A}}) = \infty .$$

From the conditions we then obtain:

$$\text{(c)} \qquad \omega^{\mu+2} \leqq [*, a)^{\mathfrak{A}} \text{ and for all } \xi \leqq \mu,\ \mathrm{Dn}_\xi([*, a)^{\mathfrak{A}}) = \infty .$$

CASE 8. $\mathrm{Dn}_\mu([a, \infty)^{\mathfrak{A}}) = n$.

Because of condition 3, this case cannot arise.

CASE 9. $\mathrm{Dn}_\mu([a, \infty)^{\mathfrak{A}}) = \infty$.

Let $\rho < \omega^{\mu+2}$ be such that $\rho \doteq_\mu [*, a)^{\mathfrak{A}}$. For b we may choose the element of B such that $[*, b) \cong \rho$. Then because of (a) b must precede b_0. Thus using 5.6.4 we obtain that for all $\xi \leqq \mu$, $\mathrm{Ed}_\xi([b, \infty)^{\mathfrak{B}}) = \mathrm{Ed}_\xi(\mathfrak{B}) = \mathrm{Ed}_\xi(\mathfrak{A}) = \mathrm{Ed}([a, \infty)^{\mathfrak{A}})$. Furthermore we also have that $\mathrm{Dn}_\xi[b, \infty)^{\mathfrak{B}} = \infty$ whenever $\xi \leqq \mu$. Thus the proof is complete.

From definition 5.8 we immediately obtain the following:

5.10 PROPOSITION. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are members of* $\mathbf{K}_\alpha$ *and* λ *is a non-zero limit ordinal and if* $\mathfrak{A} \doteq_\lambda \mathfrak{B}$ *then for all* $\mu < \lambda$, $\mathfrak{A} \doteq_\mu \mathfrak{B}$.

Finally we arrive at our first result:

5.11 THEOREM. *If* $\mu < \alpha$, $\mathfrak{A} = \langle A, R \rangle \in \mathbf{K}_\alpha$, $\mathfrak{B} = \langle B, S \rangle \in \mathbf{K}_\alpha$, $x \in A^n$, $y \in B^n$ *and* $(\mathfrak{A}, x) \doteq_\mu (\mathfrak{B}, y)$, *then* $(\mathfrak{A}, x) \equiv_\mu (\mathfrak{B}, y)$.

PROOF. The brunt of the proof has already been eliminated by proving 5.10 and 5.9. All that remains to be done is a simple induction on μ, and thus it is omitted (cf. [2]).

5.12 COROLLARY. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are members of* $\mathbf{K}_\alpha$ *then the following three* *conditions are equivalent:*

(1) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* $\mathbf{Q}_\alpha$*-elementarily equivalent,*

(2) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* $\mathbf{L}_{\alpha\omega}$*-elementarily equivalent,*

(3) *for all* $\mu<\alpha$, $\mathrm{Dn}_\mu(\mathfrak{A})=\mathrm{Dn}_\mu(\mathfrak{B})$ *and* $\mathrm{Ed}_\mu(\mathfrak{A})=\mathrm{Ed}_\mu(\mathfrak{B})$.

PROOF. That (1)$\Rightarrow$(2) is immediate because every sentence of $\mathbf{L}_{\alpha\omega}$ is also a sentence of $\mathbf{Q}_\alpha$. That (2)$\Rightarrow$(3) follows because Den^s_μ and End^k_μ are sentences of $\mathbf{L}_{\alpha\omega}$. That (3)$\Rightarrow$(1) follows from 5.11 and 5.2.

From the above we then obtain:

5.13 COROLLARY. *If* $\beta\geqq\alpha$ *and* τ *is an ordered system of order-type* $\beta+\beta\cdot\omega^*$ *then* β *and* τ *are* $\mathbf{Q}_\alpha$*-elementarily equivalent.*

PROOF. It is clear that for all $\mu<\alpha$, $\mathrm{Dn}_\mu(\beta)=\infty=\mathrm{Dn}_\mu(\tau)$ and $\mathrm{Ed}_\mu(\beta)=\mathrm{Ed}_\mu(\tau)$. Furthermore τ satisfies conditions (a) and (b) of 4.20 thus $\tau\in\mathbf{K}_\alpha$. Hence, by 5.12 β and τ are elementarily equivalent.

An immediate consequence of 5.13 is

5.14 COROLLARY. $\mathbf{W}\neq\mathbf{W}^{\mathbf{Q}}_{\alpha}$.

By being more careful with our bounds it is possible to modify our proofs to obtain the following improvement of 5.13 (obtained first by Karp [**4**]).

5.15 THEOREM (Karp). *If* γ *is an ordinal such that for all* $\mu<\gamma$, $\omega^\mu<\gamma$, *then for every ordinal* $\beta\geqq\gamma$, β *and any linearly ordered system of order type* $\beta+\beta\cdot\omega^*$ *satisfy the same sentences* θ *of* $\mathbf{L}$ *such that* $qd(\theta)<\gamma$.

5.16 THEOREM. *To every sentence* θ *of* $\mathbf{Q}_\alpha$ *there corresponds a set* $X\subseteqq\mathbf{NS}_\alpha$ *such that* $\bigvee X$ *is a sentence of* $\mathbf{L}_{\alpha\omega}$ *and*

$$\mathbf{K}_\alpha\models(\theta\leftrightarrow\bigvee X)$$

PROOF. Assume that θ is a sentence of $\mathbf{Q}_\alpha$ and that $\mu=qd(\theta)$ ($<\alpha$). Then let

(i) $\mathbf{H}=\{\mathfrak{A}:\mathfrak{A}\in\mathbf{K}_\alpha$ and $\mathfrak{A}\models\theta\}$,

(ii) $X=\{\phi:\phi\in\mathbf{NS}_{\mu+1}$ and for some $\mathfrak{B}\in\mathbf{H}$, $\mathfrak{B}\models\phi\}$.

The proof of the theorem is then completed in three steps.

STEP 1. $\mathfrak{A}\in\mathbf{K}_\alpha\Rightarrow(\mathfrak{A}\models\theta\Rightarrow\mathfrak{A}\models\bigvee X)$

Assume that $\mathfrak{A}\in\mathbf{K}_\alpha$ and that $\mathfrak{A}\models\theta$. Then $\mathfrak{A}\in\mathbf{H}$, and therefore (by 4.20 (c)) $\mathfrak{A}\models\bigvee X$.

STEP 2. $\mathfrak{A}\in\mathbf{K}_\alpha\Rightarrow(\mathfrak{A}\models\bigvee X\Rightarrow\mathfrak{A}\models\theta)$.

Assume that $\mathfrak{A}\in\mathbf{K}_\alpha$ and that $\mathfrak{A}\models\vee X$. Then for some $\phi\in X\subseteqq\mathbf{NS}_{\mu+1}$ $\mathfrak{A}\models\phi$. From 4.13 and 5.8 it then follows that for some $\mathfrak{B}\in\mathbf{H}\subseteqq\mathbf{K}_\alpha$, $\mathfrak{A}\doteq_\mu\mathfrak{B}$. Since $\mathfrak{B}\in\mathbf{H}$ we have that $\mathfrak{B}\models\theta$. Because $qd(\theta)=\mu$, 5.11 and 5.1 we obtain that $\mathfrak{A}\models\theta$.

STEP 3. $|X|<\alpha$ and $\bigvee X\in\mathbf{L}_{\alpha\omega}$.

This follows from 4.15.

5.17 THEOREM. $\mathbf{W}\neq\mathbf{W}^{\mathbf{Q}}_{\alpha}=\mathbf{W}_\alpha=\mathbf{K}_\alpha$.

PROOF. 5.1 told us that $\mathbf{W} \neq \mathbf{W}^{\mathbf{Q}}_{\alpha}$. From the definitions and 4.18 and 4.19 we have that $\mathbf{W}^{\mathbf{Q}}_{\alpha} \subseteqq \mathbf{W}_{\alpha} \subseteqq \mathbf{K}_{\alpha}$. Thus in order to prove 5.17 it suffices to show that $\mathfrak{A} \in \mathbf{W}^{\mathbf{Q}}_{\alpha}$ whenever $\mathfrak{A} \in \mathbf{K}_{\alpha}$. Thus assume that

(a) $\mathfrak{A} \in \mathbf{K}_{\alpha}$

(b) θ is a sentence of $\mathbf{Q}_{\alpha}$

(c) $\mathbf{W} \models \theta$ (i. e. θ is a sentence true in all well-orderings)

(d) $\mu = qd(\theta)$.

We must show that $\mathfrak{A} \models \theta$. Since $\mathfrak{A} \in \mathbf{K}_{\alpha}$ and $\mu < \alpha$ it follows from 4.20(c) and 5.8 that there must exist an ordinal ρ such that

(i) $\mathfrak{A} \doteq_{\mu} \rho$.

Since $qd(\theta) = \mu$, it follows from (i), 5.11 and 5.2 that $\mathfrak{A} = \theta$ if and only if $\rho \models \theta$. But by (c) $\rho \models \theta$. Hence $\mathfrak{A} \models \theta$.

5.18 THEOREM. $\mathbf{K}_{\alpha} = \mathbf{M}_{\alpha}$.

PROOF. Since $\mathbf{K}_{\alpha} = \mathbf{W}_{\alpha} \subseteqq \mathbf{M}_{\alpha}$ in order to prove 5.18 it is sufficient to prove $\mathbf{M}_{\alpha} \subseteqq \mathbf{K}_{\alpha}$. In view of 4.18 it then suffices to prove that for all $\mathfrak{A} \in \mathbf{M}_{\alpha}$ and for all $\mu < \alpha$, $\mathfrak{A} \models \bigvee \mathbf{NS}_{\mu}$. Thus the problem of showing that $\mathbf{K}_{\alpha} = \mathbf{M}_{\alpha}$ reduces to showing that certain sentences of $\mathbf{L}_{\alpha\omega}$ which are true in all well-orderings are semantical consequences of the sentences $\mathbf{D}_{\alpha}$ (which state that the ordering relation must be a linear ordering in which every definable non-empty subset has a first element). The latter is a routine (but long and uninteresting) verification and thus it shall be omitted.

5.19 THEOREM. $\mathbf{W} \neq \mathbf{W}_{\alpha} = \mathbf{W}^{\mathbf{Q}}_{\alpha} = \mathbf{M}_{\alpha} = \mathbf{M}^{\mathbf{Q}}_{\alpha} = \mathbf{K}_{\alpha}$.

PROOF. $\mathbf{M}_{\alpha} \supseteqq \mathbf{M}^{\mathbf{Q}}_{\alpha} \supseteqq \mathbf{W}^{\mathbf{Q}}_{\alpha} = \mathbf{K}_{\alpha}$. But $\mathbf{M}_{\alpha} = \mathbf{K}_{\alpha}$, therefore $\mathbf{M}_{\alpha} = \mathbf{M}^{\mathbf{Q}}_{\alpha}$.

A simple consequence of our results is an extension of the results of Fraisse and Ehrenfeucht to the language $\mathbf{L}_{\alpha\omega}$.

5.20 THEOREM. *If ρ and δ are ordinals (greater than 0), then*

(.1) *if $\rho < \alpha$, then ρ and δ are $\mathbf{L}_{\alpha\omega}(\mathbf{Q}_{\alpha})$-elementarily equivalent if and only if $\rho = \delta$,*

(.2) *if $\rho \geqq \alpha$, then ρ and δ are $\mathbf{L}_{\alpha\omega}(\mathbf{Q}_{\alpha})$-elementarily equivalent if and only if there exist ordinals η_0, η_1 and ξ such that $\eta_0, \eta_1 > 0$, $\xi < \alpha$, $\rho = \alpha \cdot \eta_0 + \xi$ and $\rho = \alpha \cdot \eta_1 + \xi$.*

To conclude we give a classification of the elementary (in $\mathbf{L}_{\alpha\omega}$) types of the $\mathbf{L}_{\alpha\omega}$ theory of well-orderings.

5.21 DEFINITION. $\mathbf{ET}_{\mu} = \{\mathfrak{A} : \mathfrak{A} \in \mathbf{W}$ *and such that* $\mathfrak{A}$ *and* μ *are* $\mathbf{L}_{\alpha\omega}$*-elementarily equivalent*).

5.22 THEOREM. (.1) $\{\mathbf{ET}_{\mu} : \mu < \alpha \cdot 2\}$ *is a partition of* $\mathbf{W}_{\alpha}$.

(.2) *if $\mu < \alpha$, then* $\mathbf{ET}_{\mu} = \mathrm{Iso}(\mu)$.

(.3) *if $\mu > \alpha$, then* $\mathbf{ET}_{\mu}$ *contains non-well-ordered systems of arbitrary large cardinalities.*

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