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On the unit group of an absolutely cyclic number field of degree five

Dedicated to Professor Iyanaga on his 60th birthday

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1. Let K be a Galois extension of odd degree n over the rational number field $\boldsymbol{Q}$. Then K is totally real and the group of units of K has $(n-1)$ generators mod ± 1. Let $\boldsymbol{H}$ be the group of totally positive units of K. Then $\boldsymbol{H}$ has also $(n-1)$ generators, and it is known that in case $n=3$ these generators can be taken to conjugate to each other (cf. Hasse [1]). We shall show in this paper that the same is true for $n=5$.

In the following let K be a cyclic field of degree 5 over $\boldsymbol{Q}$, σ a generator of the Galois group $G(K/\boldsymbol{Q})$ and $\boldsymbol{H}$ the group of totally positive units of K. For $\xi \in K$, $\xi^{(i)}$ means $\sigma^{i-1}(\xi) \in K$ $(i=1, 2, 3, 4, 5)$. Then the points

$$\boldsymbol{P}(\xi)=(\log \xi^{(1)}, \log \xi^{(2)}, \log \xi^{(3)}, \log \xi^{(4)}, \log \xi^{(5)}) \in \boldsymbol{R}^5$$

for $\xi \in \boldsymbol{H}$ form a lattice $\boldsymbol{L}$ lying in the hyperplane $\pi : x_1+x_2+x_3+x_4+x_5=0$. Obviously the five points $\boldsymbol{P}(\xi^{(1)}, \cdots, \boldsymbol{P}(\xi^{(5)})$ lie at the same distance from the origin O of $\boldsymbol{R}^5$.

Let $\eta(\neq 1)$ be a unit in $\boldsymbol{H}$ such that $\boldsymbol{P}(\eta) \in \boldsymbol{L}$ lies nearest to O. Then our main result is that $\boldsymbol{H}$ is generated by any four of $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta^{(4)}, \eta^{(5)}$, or geometrically expressed, $\boldsymbol{L}$ is generated by $\boldsymbol{P}(\eta^{(1)}), \cdots, \boldsymbol{P}(\eta^{(5)})$.

We shall namely prove the following theorem.

THEOREM. *Let K be an absolutely cyclic field of degree* 5, *and $\boldsymbol{H}$ the group of totally positive units of K. Then $\boldsymbol{H}$ is generated by $\eta \in \boldsymbol{H}$ and its conjugates, where η is an element* $(\neq 1)$ *of $\boldsymbol{H}$ such that*

$$\sum_{i=1}^{5} (\log \eta^{(i)})^2 \leqq \sum_{i=1}^{5} (\log \xi^{(i)})^2$$

holds for any element $\xi \in \boldsymbol{H}$ $(\xi \neq 1)$.

2. We shall first prove the following general proposition. Let $\boldsymbol{M}$ be an n-dimensional lattice in $\boldsymbol{R}^n$, which is generated by n vectors $\overrightarrow{OQ}_1, \overrightarrow{OQ}_2, \cdots, \overrightarrow{OQ}_n$. Let d_i be the length of $\overrightarrow{OQ}_i$ $(i=1, 2, \cdots, n)$.

(A) For any point $X \in \boldsymbol{R}^n$, there exists a point Y of $\boldsymbol{M}$, such that the distance

$$\overline{XY} \leqq \frac{1}{2}\left(\sum_{i=1}^{n} d_i^2\right)^{1/2}.$$

Here we can replace the sign $\leqq$ by $<$ except the case: $\overrightarrow{OQ_i} \perp \overrightarrow{OQ_j}$; for any $i \neq j$.

PROOF. We shall prove it by induction on the dimension n.

1) If $n=1$ the assertion is trivial.

2) For $n \geqq 2$ let $\boldsymbol{N}$ be the sublattice of $\boldsymbol{M}$ generated by $\overrightarrow{OQ_1}, \cdots, \overrightarrow{OQ_{n-1}}$. Then $\boldsymbol{M}=\boldsymbol{Z} \cdot \overrightarrow{OQ_n}+\boldsymbol{N}$, and each $i\overrightarrow{OQ_n}+\boldsymbol{N}$ forms an $n-1$ dimensional lattice in the hyperplane π_i, where $\pi_i /\!/ \pi_j$ $i \neq j$. For any given point $X \in \boldsymbol{R}^n$ we can choose a suitable i and a point $Z \in \pi_i$ such that $\overrightarrow{XZ} \perp \pi_i$ and $\overline{XZ} \leqq \frac{d_n}{2}$. We can replace $\leqq$ by $<$, if $\overrightarrow{OQ_n}$ is not orthogonal to π_i. With respect to the point $Z \in \pi_i$, and the lattice $\boldsymbol{N}$, we can apply the assumption of the induction. Hence there exists a point Y of $\boldsymbol{N}$ such that $\overline{YZ} \leqq \frac{1}{2}\left(\sum_{i=1}^{n-1} d_i^2\right)^{\frac{1}{2}}$. Then we have $\overline{XY}^2=\overline{XZ}^2+\overline{YZ}^2 \leqq \frac{1}{4} \sum_{i=1}^{n} d_i^2$ and we can replace $\leqq$ by $<$ except $\overrightarrow{OQ_i} \perp \overrightarrow{OQ_j}$ for any $i \neq j$. Q. E. D.

3. Now we proceed to the proof of the theorem. With the same notations as in the introduction, let $\tilde{\boldsymbol{L}}$ denote the lattice in π generated by $P(\eta^{(1)}), \cdots, P(\eta^{(5)})$. Our aim is to prove $\tilde{\boldsymbol{L}}=\boldsymbol{L}$. Now it is known that, l being an odd prime, any cyclic field of degree l over $\boldsymbol{Q}$ has the property that any $l-1$ among $\xi^{(i)}$ $i=1,2, \cdots, l$ forms a system of independent units in K for any non rational unit ξ in K (cf. Hilbert [2] § 55). This implies obviously $\dim \tilde{\boldsymbol{L}}=4$. Take $\tilde{\boldsymbol{L}}$ as the lattice $\boldsymbol{M}$ in Proposition (A). Then $Q_i=P(\eta^{(i)})$ $(i=1,2,3,4)$ generate $\tilde{\boldsymbol{L}}$ and $d_1=\cdots=d_4=\left(\sum_{i=1}^{5}(\log \eta^{(i)})^2\right)^{1/2}$. Moreover, for some $i \neq j$ $\overrightarrow{OQ_i}$ is not orthogonal to $\overrightarrow{OQ_j}$. Hence from Proposition (A) follows the proposition:

(B) For any point X of π there exists a point Y of $\tilde{\boldsymbol{L}}$ such that the distance $\overline{XY}<d$, where $d^2=\sum_{i=1}^{5}\left(\log \eta^{(i)}\right)^2$.

It is rather a routine reasoning to deduce our theorem from Proposition (B).

REMARK. Whether the similar result holds for a prime p $(\neq 3, \neq 5)$ or not is an open problem.

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References

[1] H. Hasse, Arithmetische Bestimmung von Grundeinheit in zyklischen kubischen und biquadratischen Zahlkörpern, Abh. Deutsch. Akad. Wiss. Berlin, 1948, Nr. 2 (1950).

[2] D. Hilbert, Die Theorie der algebraischen Zahlkörper, Gesamm. Abhand. Band 1, Springer, Berlin, 1932, 63-68.