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A proof of cut-elimination theorem in simple type-theory

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In [4], G. Takeuti conjectured that the cut-elimination theorem would hold in his system GLC as well as in LK. Many attempts to prove it constructively have not yet succeeded. On the other hand, W. Tait [3] proved the cut-elimination theorem for the second order predicate logic by a non-constructive method. In this paper, we shall prove the cut-elimination theorem in simple type-theory also by a non-constructive method. Our proof will be formalizable in Zermelo's set theory, which contains neither the axiom of replacement nor the axiom of choice[1]. The author wishes to express his thanks to Professor T. Nishimura, Mr. K. Namba and Mr. T. Uesu for their kind advice and assistance.

§ 1. Complexes

The system of simple type-theory we shall use is Schütte's system in [2][2]. We shall use the notations in [2].

Let V be a semi-valuation[3]. We shall define V-complexes of type τ by induction on types.

1.1. A V-complex of type 0 is a pair $[e^0, 0]$, where e^0 is an expression of type 0.

1.2. A V-complex of type 1 is a pair $[A, p]$, where A is a well-formed formula and p is t or f satisfying the following conditions.

1.2.1. If A is t in the semi-valuation V, then $p=t$.

1.2.2. If A is f in V, then $p=f$.

1.3. Suppose that the V-complexes of type $\tau_1, \cdots, \tau_{n-1}$ and τ_n are already defined. Let $\mathfrak{C}\tau_1, \cdots, \mathfrak{C}\tau_n$ be the sets of all the V-complexes of type $\tau_1, \cdots, \tau_n$

1) Cf. Appendix 2.

2) For the sake of brevity, constants (except function constants) are omitted, since they can be identified with free variables.

3) Our proof remains valid, if the term "semi-valuation" is replaced by "partial valuation" throughout this paper. But we use only the conditions 6.1.1.–6.1.7. in [2] but do not use 6.2.1.–6.2.7. in [2].

respectively. Then a V-complex of type $\tau=(\tau_1, \cdots, \tau_n)$ is a pair $[e^\tau, p]$, where e^τ is an expression of type τ and p is a subset of $\mathfrak{C}\tau_1 \times \cdots \times \mathfrak{C}\tau_n$ satisfying the following conditions. For any V-complexes $C_1=[e_1^{\tau_1}, p_1], \cdots, C_n=[e_n^{\tau_n}, p_n]$ of type $\tau_1, \cdots, \tau_n$ respectively,

1.3.1. if the wff $(e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^\tau)$ is t in V, then $\langle C_1, \cdots, C_n\rangle \in p$, and

1.3.2. if the wff $(e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^\tau)$ is f in V, then $\langle C_1, \cdots, C_n\rangle \notin p$.

1.4. In a V-complex $[e^\tau, p]$, e^τ is called the first part of this complex and p is called the second part of it.

1.5. For any expression e^τ of type τ, there exists a p such that $[e^\tau, p]$ is a V-complex of type τ. For if $\tau=0$, we may set $p=0$, if $\tau=1$, we may take t or f as p according as the wff e^1 is t in V or not, and if $\tau=(\tau_1, \cdots, \tau_n)$, we may set $p=\{\langle C_1^{\tau_1}, \cdots, C_n^{\tau_n}\rangle \mid$ the wff $(e_1^{\tau_1}, \cdots, e_n^{\tau_n} \in e^\tau)$ is t in V, where $e_i^{\tau_i}$ is the first part of $C_i^{\tau_i} (i=1, \cdots, n)\}$.

§ 2. Correspondences

2.1. By a V-correspondence we mean a function which maps each free variable a^τ to a V-complex of type τ. In this and next paragraphs we simply say "complex" or "correspondence" instead of "V-complex" or "V-correspondence" respectively, since a semi-valuation V is fixed in these paragraphs. Henceforth Φ, Ψ, Φ', Ψ' etc. denote correspondences. $\Phi_1(a^\tau)$, $\Phi_2(a^\tau)$ denote the first or the second part of $\Phi(a^\tau)$ respectively.

2.2.1. If $\Phi(b)=\Psi(b)$ for all free variables b except a, we write $\Phi \underset{a}{\sim} \Psi$.

2.2.2. Let $a_1^{\tau_1}, \cdots, a_n^{\tau_n}$ be distinct free variables and $C_1^{\tau_1}, \cdots, C_n^{\tau_n}$ be complexes of type $\tau_1, \cdots, \tau_n$ respectively.

$$\Phi\begin{pmatrix} C_1^{\tau_1} \cdots C_n^{\tau_n} \\ a_1^{\tau_1} \cdots a_n^{\tau_n} \end{pmatrix}$$

denotes the correspondence Ψ defined by

$$\Psi(a_i^{\tau_i})=C_i^{\tau_i} \qquad (i=1, \cdots, n)$$
$$\Psi(b^\tau)=\Phi(b^\tau) \qquad (b^\tau \neq a_1^{\tau_1}, \cdots, a_n^{\tau_n}).$$

2.3. We shall extend a correspondence Φ to $\tilde{\Phi}$ which maps each expression e^τ to a complex of type τ.

2.3.1. First we define the first part $\tilde{\Phi}_1(e)$ of $\tilde{\Phi}(e)$. Let $a_1, \cdots, a_n$ be all the free variables contained in an expression $e=e(a_1, \cdots, a_n)$ and let $\Phi_1(a_i)=e_i$ $(i=1, \cdots, n)$. Then we set

$$\tilde{\Phi}_1(e)=e(e_1, \cdots, e_n).$$

2.3.2. Thus, $\tilde{\Phi}(e)$ will be defined when the second part $\tilde{\Phi}_2(e)$ is well-defined so that $\tilde{\Phi}(e)=[\tilde{\Phi}_1(e), \tilde{\Phi}_2(e)]$ is indeed a complex. The definition proceeds by the induction on the number of stages to construct e. Suppose that $\tilde{\Phi}_2(d)$ and therefore $\tilde{\Phi}(d)$ be well-defined for any expression d which was constructed in an earlier stage than that of e. Then we define $\tilde{\Phi}_2(e)$ by cases and prove that $[\tilde{\Phi}_1(e), \tilde{\Phi}_2(e)]$ is a complex.

Case 1. e is a free variable a.

Then we set $\tilde{\Phi}_2(e)=\Phi_2(a)$.

Case 2. e is of the form $\varphi(d_1, \cdots, d_n)$, where φ is a function constant. We set $\tilde{\Phi}_2(e)=0$.

Case 3. e is of the form $(d_1, \cdots, d_n \in d)$.

$$\tilde{\Phi}_2(e) \overset{\text{def}}{=} \begin{cases} t, \text{ if } \langle\tilde{\Phi}(d_1), \cdots, \tilde{\Phi}(d_n)\rangle \in \tilde{\Phi}_2(d), \\ f, \text{ otherwise.} \end{cases}$$

Case 4. e is of the form $\neg A$.

$$\tilde{\Phi}_2(e) \overset{\text{def}}{=} \begin{cases} t, \text{ if } \tilde{\Phi}_2(A)=f, \\ f, \text{ otherwise.} \end{cases}$$

Case 5. e is of the form $A \vee B$.

$$\tilde{\Phi}_2(e) \overset{\text{def}}{=} \begin{cases} t, \text{ if } \tilde{\Phi}_2(A)=t \text{ or } \tilde{\Phi}_2(B)=t, \\ f, \text{ otherwise.} \end{cases}$$

Case 6. e is of the form $\exists x^\tau A(x^\tau)$.

$$\tilde{\Phi}_2(e) \overset{\text{def}}{=} \begin{cases} t, \text{ if there exists a correspondence } \Psi \\ \quad \text{such that } \Psi \underset{a^\tau}{\sim} \Phi \text{ and } \tilde{\Psi}_2(A(a^\tau))=t. \\ f, \text{ otherwise,} \end{cases}$$

where a^τ is the first free variable of type τ (in a fixed enumeration) which does not occur in e.

Case 7. e is of the form $\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n})$.

$$\tilde{\Phi}_2(e) \overset{\text{def}}{=} \left\{ \langle C_1^{\tau_1}, \cdots, C_n^{\tau_n}\rangle \mid C_i^{\tau_i} \in \mathfrak{C}_{\tau_i} \text{ and} \right.$$

$$\left. \widetilde{\Phi\begin{pmatrix} C_1^{\tau_1} \cdots C_n^{\tau_n} \\ a_1^{\tau_1} \cdots a_n^{\tau_n} \end{pmatrix}}_2 (A(a_1^{\tau_1}, \cdots, a_n^{\tau_n}))=t \right\},$$

where $a_i^{\tau_i}$ is the first free variable of type τ_i (in a fixed enumeration) which does not occur in e and differs from $a_1^{\tau_1}, \cdots, a_{i-1}^{\tau_{i-1}}$ $(i=1, \cdots, n)$.

2.3.3. Next we prove that $\tilde{\Phi}(e)=[\tilde{\Phi}_1(e), \tilde{\Phi}_2(e)]$ defined above is a complex. Similarly to 2.3.2 the proof proceeds by the induction on the number of stages to construct e.

Case 1. e is a free variable a.

$$\tilde{\Phi}(e)=[\tilde{\Phi}_1(a),\ \tilde{\Phi}_2(a)]=[\Phi_1(a),\ \Phi_2(a)]=\Phi(a).$$

Hence $\tilde{\Phi}(e)$ is a complex.

Case 2. e is of the form $\varphi(d_1, \cdots, d_n)$, where φ is a function constant.

It is clear that $\tilde{\Phi}(e)$ is a complex, since e is of type 0.

Case 3. e is of the form $(d_1, \cdots, d_n \in d)$.

Let $\tilde{\Phi}(d_i)=C_i=[d'_i, p_i]$ $(i=1, \cdots, n)$ and $\tilde{\Phi}(d)=C=[d', p]$.

Then $\tilde{\Phi}_1(e)$ is $(d'_1, \cdots, d'_n \in d')$.

Suppose that $\tilde{\Phi}_1(e)$ is t in V. Since $\tilde{\Phi}(d)=[d', p]$ is a complex by the induction hypothesis,

$$\langle C_1, \cdots, C_n\rangle \in p,$$

by 1.3.1. I.e.

$$\langle \tilde{\Phi}(d_1), \cdots, \tilde{\Phi}(d_n)\rangle \in \tilde{\Phi}_2(d).$$

Hence by the definition, $\tilde{\Phi}_2(e)=t$. Similarly if $\tilde{\Phi}_1(e)$ is f in V, then $\tilde{\Phi}_2(e)=f$. So $\tilde{\Phi}(e)$ is a complex by 1.2.1 and 1.2.2.

Case 4. e is of the form $\neg A$.

Let $\tilde{\Phi}_1(A)=B$. Then $\tilde{\Phi}_1(e)=\neg B$.

Therefore if $\tilde{\Phi}_1(e)$ is t in V, B is f in V by 6.1.1 in [2]. So $\tilde{\Phi}_2(A)=f$ by the induction hypothesis and 1.2.2, and hence $\tilde{\Phi}_2(e)=t$ by the definition.

Similarly if $\tilde{\Phi}_1(e)$ is f in V, then $\tilde{\Phi}_2(e)=f$. Thus $\tilde{\Phi}(e)$ is a complex.

Case 5. e is of the form $A \vee B$.

Similar to the case 4.

Case 6. e is of the form $\exists x^\tau A(x^\tau, a_1, \cdots, a_n)$, where $a_1, \cdots, a_n$ are all the free variables occurring in e.

Let $\Phi(a_i)=[d_i, p_i]$ $(i=1, \cdots, n)$. Then $\tilde{\Phi}_1(e)$ is $\exists x^\tau A(x^\tau, d_1, \cdots, d_n)$. Suppose that $\tilde{\Phi}_1(e)$ is t in V. Then by 6.1.5 in [2] there exists an expression d^τ such that $A(d^\tau, d_1, \cdots, d_n)$ is t in V. By 1.5 there exists a p such that $C^\tau=[d^\tau, p]$ is a complex. We set $\Psi=\Phi\binom{C^\tau}{a^\tau}$, where a^τ is the free variable mentioned in 2.3.2 case 6. Then by the induction hypothesis,

$$[\tilde{\Psi}_1(A(a^\tau, a_1, \cdots, a_n)),\ \tilde{\Psi}_2(A(a^\tau, a_1, \cdots, a_n))]$$

is a complex. But $\tilde{\Psi}_1(A(a^\tau, a_1, \cdots, a_n))$ is $A(d^\tau, d_1, \cdots, d_n)$. Since it is t in V, $\tilde{\Psi}_2(A(a, a_1, \cdots, a_n))=t$. Therefore $\tilde{\Phi}_2(\exists x^\tau A(x^\tau, a_1, \cdots, a_n))=t$ by $\Psi \underset{a^\tau}{\sim} \Phi$. Next suppose that $\tilde{\Phi}_1(e)$ is f in V. Let Ψ be an arbitrary correspondence such that $\Psi \underset{a^\tau}{\sim} \Phi$ and let $\Psi(a^\tau)=[d^\tau, p]$. Since $\exists x^\tau A(x^\tau, d_1, \cdots, d_n)$ is f in V, $A(d^\tau, d_1, \cdots, d_n)$ is also f in V by 6.1.6 in [2]. But $A(d^\tau, d_1, \cdots, d_n)$ is $\tilde{\Psi}_1(A(a^\tau, a_1, \cdots, a_n))$. Therefore by the induction hypothesis $\tilde{\Psi}_2(A(a^\tau, a_1, \cdots, a_n))=f$. So there is no Ψ such that $\Psi \underset{a^\tau}{\sim} \Phi$ and $\tilde{\Psi}_2(A(a^\tau, a_1, \cdots, a_n))=t$. Hence $\tilde{\Phi}_2(\exists x^\tau A(x^\tau, a_1, \cdots, a_n))=f$. Thus $\tilde{\Phi}(e)$ is a complex by 1.2.1 and 1.2.2.

Case 7. e is of the form $\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}, b_1, \cdots, b_m)$, where $b_1, \cdots, b_m$ are all the free variables occurring in e. Let $\Phi(b_j) = [d_j, p_j]$ ($j = 1, \cdots, m$) and let $C_i^{\tau_i} = [c_i^{\tau_i}, q_i]$ be an arbitrary complex of type τ_i ($i = 1, \cdots, n$). Then $\tilde{\Phi}_1(e)$ is $\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}, d_1, \cdots, d_m)$. Now suppose that the wff $(c_1^{\tau_1}, \cdots, c_n^{\tau_n} \in \lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}, d_1, \cdots, d_m))$ is t in V. Then the wff $A(c_1^{\tau_1}, \cdots, c_n^{\tau_n}, d_1, \cdots, d_m)$ is also t in V by 6.1.7 in [2]. Let $a_i^{\tau_i} (i = 1, \cdots, n)$ be as in 2.3.2 case 7, and let

$$\Psi = \Phi \begin{pmatrix} C_1^{\tau_1} \cdots C_n^{\tau_n} \\ a_1^{\tau_1} \cdots a_n^{\tau_n} \end{pmatrix}.$$

Then $\tilde{\Psi}_1(A(a_1^{\tau_1}, \cdots, a_n^{\tau_n}, b_1, \cdots, b_m))$ is $A(c_1^{\tau_1}, \cdots, c_n^{\tau_n}, d_1, \cdots, d_m)$.

So by the induction hypothesis,

$$\tilde{\Psi}_2(A(a_1^{\tau_1}, \cdots, a_n^{\tau_n}, b_1, \cdots, b_m)) = t.$$

Hence $\langle C_1^{\tau_1}, \cdots, C_n^{\tau_n} \rangle \in \tilde{\Phi}_2(e)$ by definition. Next suppose that the wff

$$(c_1^{\tau_1}, \cdots, c_n^{\tau_n} \in \lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}, d_1, \cdots, d_m))$$

is f in V. Then $A(c_1^{\tau_1}, \cdots, c_n^{\tau_n}, d_1, \cdots, d_m)$ is also f in V. So by the induction hypothesis

$$\tilde{\Psi}_2(A(a_1^{\tau_1}, \cdots, a_n^{\tau_n}, b_1, \cdots, b_m)) = f.$$

Hence $\langle C_1^{\tau_1}, \cdots, C_n^{\tau_n} \rangle \notin \tilde{\Phi}_2(e)$ by the definition. Accordingly $[\tilde{\Phi}_1(e), \tilde{\Phi}_2(e)]$ is a complex by 1.3.1 and 1.3.2.

§ 3. Preliminary results

The following 3.1 (lemma) is easily seen by the induction on the number of stages to construct e.

3.1. Lemma. *Let $e(a_1^{\tau_1}, \cdots, a_n^{\tau_n})$ be an expression which does not contain free variables other than $a_1^{\tau_1}, \cdots, a_n^{\tau_n}$, and let $b_1^{\tau_1}, \cdots, b_n^{\tau_n}$ be distinct free variables. If Φ, Ψ are correspondences such that $\Phi(a_i^{\tau_i}) = \Psi(b_i^{\tau_i}) (i = 1, \cdots, n)$, then $\tilde{\Phi}(e(a_1^{\tau_1}, \cdots, a_n^{\tau_n})) = \tilde{\Psi}(e(b_1^{\tau_1}, \cdots, b_n^{\tau_n}))$.*

3.2. Corollary. *The value $\tilde{\Phi}(e)$ depends only on the values of Φ for the free variables occurring in e; i.e. if $\Phi(a) = \Psi(a)$ for every free variable a occurring in e, then $\tilde{\Phi}(e) = \tilde{\Psi}(e)$.*

3.3. Corollary.

3.3.1.

$$\tilde{\Phi}_2(\exists x^\tau A(x^\tau)) = \begin{cases} t, & \text{if there exists a correspondence } \Psi \\ & \text{such that } \Psi \underset{a^\tau}{\sim} \Phi \text{ and } \tilde{\Psi}_2(A(a^\tau)) = t, \\ f, & \text{otherwise,} \end{cases}$$

where a^τ is an arbitrary free variable of type τ which does not occur in $\exists x^\tau A(x^\tau)$.

3.3.2.

$$\tilde{\Phi}_2(\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}))$$

$$= \left\{ \langle C_1^{\tau_1}, \cdots, C_n^{\tau_n} \rangle \middle| C_i^{\tau_i} \in \mathfrak{C}_{\tau_i} \text{ and } \Phi\binom{\overbrace{C_1^{\tau_1} \cdots C_n^{\tau_n}}}{a_1^{\tau_1} \cdots a_n^{\tau_n}}_{?}(A(a_1^{\tau_1}, \cdots, a_n^{\tau_n})) = t \right\},$$

where $a_1^{\tau_1}, \cdots, a_n^{\tau_n}$ *are arbitrary distinct free variables which do not occur in* $\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n})$.

3.4. LEMMA. *Let* $e(a_1, \cdots, a_m)$ *be an expression*[4], *let* $e_i (i = 1, \cdots, m)$ *be an expression with the same type as* a_i, *let* Φ *be a correspondence and let*

$$\Psi = \Phi\binom{\tilde{\Phi}(e_1), \cdots, \tilde{\Phi}(e_m)}{a_1, \quad \cdots, a_m}.$$

Then $\tilde{\Phi}(e(e_1, \cdots, e_m)) = \tilde{\Psi}(e(a_1, \cdots, a_m))$.

PROOF. It is clear that

$$\tilde{\Phi}_1(e(e_1, \cdots, e_m)) = \tilde{\Psi}_1(e(a_1, \cdots, a_m)).$$

We prove

$$\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = \tilde{\Psi}_2(e(a_1, \cdots, a_m)),$$

by the induction on the number of stages to construct $e(a_1, \cdots, a_m)$.

Case 1 (i). $e(a_1, \cdots, a_m)$ is a_i.

Then $e(e_1, \cdots, e_m)$ is e_i.

Hence $\tilde{\Phi}(e(e_1, \cdots, e_m)) = \tilde{\Phi}(e_i) = \Psi(a_i) = \tilde{\Psi}(e(a_1, \cdots, a_m))$.

Case 1 (ii). $e(a_1, \cdots, a_m)$ is b ($b \neq a_1, \cdots, a_m$).

Then $e(e_1, \cdots, e_m)$ is b.

Hence $\tilde{\Phi}(e(e_1, \cdots, e_m)) = \Phi(b) = \Psi(b) = \tilde{\Psi}(e(a_1, \cdots, a_m))$.

Case 2. $e(a_1, \cdots, a_m)$ is of the form $\varphi(d_1, \cdots, d_k)$, where φ is a function constant.

In this case, both $e(a_1, \cdots, a_m)$ and $e(e_1, \cdots, e_m)$ are of type 0.

Hence $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = \tilde{\Psi}_2(e(a_1, \cdots, a_m)) = 0$.

Case 3. $e(a_1, \cdots, a_m)$ is of the form

$$(d_1(a_1, \cdots, a_m), \cdots, d_k(a_1, \cdots, a_m) \in d(a_1, \cdots, a_m)).$$

Then $e(e_1, \cdots, e_m)$ is

$$(d_1(e_1, \cdots, e_m), \cdots, d_k(e_1, \cdots, e_m) \in d(e_1, \cdots, e_m)).$$

Suppose that $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = t$. Then by the definition,

$$\langle \tilde{\Phi}(d_1(e_1, \cdots, e_m)), \cdots, \tilde{\Phi}(d_k(e_1, \cdots, e_m)) \rangle \in \tilde{\Phi}_2(d(e_1, \cdots, e_m)).$$

But $\tilde{\Phi}(d_i(e_1, \cdots, e_m)) = \tilde{\Psi}(d_i(a_1, \cdots, a_m))$, $(i = 1, \cdots, k)$ and $\tilde{\Phi}_2(d(e_1, \cdots, e_m)) = \tilde{\Psi}_2(d(a_1, \cdots, a_m))$, by the induction hypothesis.

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4) e may contain free variables other than $a_1, \cdots, a_m$.

Hence

$$\langle \tilde{\Psi}(d_1(a_1, \cdots, a_m)), \cdots, \tilde{\Psi}(d_k(a_1, \cdots, a_m))\rangle \in \tilde{\Psi}_2(d(a_1, \cdots, a_m)).$$

Accordingly $\tilde{\Psi}_2(e(a_1, \cdots, a_m)) = t$. Similarly if $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = f$, then $\tilde{\Psi}_2(e(a_1, \cdots, a_m)) = f$.

Thus $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = \tilde{\Psi}_2(e(a_1, \cdots, a_m))$.

Case 4, 5. $e(a_1, \cdots, a_m)$ is of the form $\neg A$ or $A \vee B$.

The proposition is clear by the definition and the induction hypothesis.

Case 6. $e(a_1, \cdots, a_m)$ is of the form $\exists x^\tau A(x^\tau, a_1, \cdots, a_m)$.

Then $e(e_1, \cdots, e_m)$ is $\exists x^\tau A(x^\tau, e_1, \cdots, e_m)$. Let a^τ be a free variable of type τ, which is different from $a_1, \cdots, a_m$ and contained neither in $e(a_1, \cdots, a_m)$ nor in $e_1, \cdots, e_m$. Now suppose that $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = t$.

Then there exists a correspondence Φ' such that

$$\Phi' \underset{a^\tau}{\sim} \Phi \text{ and } \tilde{\Phi}'_2(A(a^\tau, e_1, \cdots, e_m)) = t^{5)}.$$

Since e_i does not contain a^τ, $\tilde{\Phi}'(e_i) = \tilde{\Phi}(e_i)$ by 3.2.

Let Ψ' be $\Psi\begin{pmatrix}\Phi'(a^\tau)\\ a^\tau\end{pmatrix}$.

Then

$$\begin{aligned}\Psi' &= \Phi\begin{pmatrix}\tilde{\Phi}(e_1), & \cdots, & \tilde{\Phi}(e_m), & \tilde{\Phi}'(a^\tau)\\ a_1, & \cdots, & a_m, & a^\tau\end{pmatrix}\\ &= \Phi'\begin{pmatrix}\tilde{\Phi}(e_1), & \cdots, & \tilde{\Phi}(e_m)\\ a_1, & \cdots, & a_m\end{pmatrix}\\ &= \Phi'\begin{pmatrix}\tilde{\Phi}'(e_1), & \cdots, & \tilde{\Phi}'(e_m)\\ a_1, & \cdots, & a_m\end{pmatrix}.\end{aligned}$$

So by the induction hypothesis

$$\tilde{\Psi}'_2(A(a^\tau, a_1, \cdots, a_m)) = t.$$

Since $\Psi' \underset{a^\tau}{\sim} \Psi$, $\tilde{\Psi}_2(\exists x^\tau A(x^\tau, a_1, \cdots, a_m)) = t^{5)}$.

Conversely, if $\tilde{\Psi}_2(e(a_1, \cdots, a_m)) = t$, then

$$\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = t.$$

Hence $\tilde{\Phi}_2(e(e_1, \cdots, e_m)) = \tilde{\Psi}_2(e(a_1, \cdots, a_m))$.

Case 7. $e(a_1, \cdots, a_m)$ is of the form

$$\lambda x_1^{\tau_1} \cdots x_n^{\tau_n} A(x_1^{\tau_1}, \cdots, x_n^{\tau_n}, a_1, \cdots, a_m).$$

Let $b_1^{\tau_1}, \cdots, b_n^{\tau_n}$ be new variables which are different from each other and from $a_1, \cdots, a_m$ and contained neither in $e(a_1, \cdots, a_m)$ nor in $e_i (i = 1, \cdots, m)$.

Now suppose that $\langle C_1^{\tau_1}, \cdots, C_n^{\tau_n}\rangle \in \tilde{\Phi}_2(e(e_1, \cdots, e_m))$.

5) Cf. 3.3.1.

We set $\Phi'=\Phi\binom{C_1^{\tau_1}\cdots C_n^{\tau_n}}{b_1^{\tau_1}\cdots b_n^{\tau_n}}$ and $\Psi'=\Psi\binom{C_1^{\tau_1}\cdots C_n^{\tau_n}}{b_1^{\tau_1}\cdots b_n^{\tau_n}}$.
Then

$$\tilde{\Phi}'_2(A(b_1^{\tau_1},\cdots,b_n^{\tau_n},e_1,\cdots,e_m))=t^{6)}.$$

And $\tilde{\Phi}'(e_i)=\tilde{\Phi}(e_i)$ $(i=1,\cdots,m)$, since e_i does not contain $b_1,\cdots,b_n$.
Accordingly,

$$\Psi'=\Phi'\binom{\tilde{\Phi}(e_1),\cdots,\tilde{\Phi}(e_m)}{a_1,\cdots,a_m}$$

$$=\Phi'\binom{\tilde{\Phi}'(e_1),\cdots,\tilde{\Phi}'(e_m)}{a_1,\cdots,a_m}.$$

Hence by the induction hypothesis

$$\tilde{\Psi}'_2(A(b_1^{\tau_1},\cdots,b_n^{\tau_n},a_1,\cdots,a_m))=t.$$

and hence

$$\langle C_1^{\tau_1},\cdots,C_n^{\tau_n}\rangle\in\tilde{\Psi}_2(\lambda x_1^{\tau_1}\cdots x_n^{\tau_n}A(x_1^{\tau_1},\cdots,x_n^{\tau_n},a_1,\cdots,a_m))$$
$$=\tilde{\Psi}_2(e(a_1,\cdots,a_m)).$$

Conversely if $\langle C_1^{\tau_1},\cdots,C_n^{\tau_n}\rangle\in\tilde{\Psi}_2(e(a_1,\cdots,a_m))$, then $\langle C_1^{\tau},\cdots,C_n^{\tau_n}\rangle\in\tilde{\Phi}_2(e(e_1,\cdots,e_m))$.

Thus $\tilde{\Phi}_2(e(e_1,\cdots,e_m))=\tilde{\Psi}_2(e(a_1,\cdots,a_m))$.

The proof of 3.4 is now completed.

3.5. COROLLARY.

3.5.1. *If* $\tilde{\Phi}_2(A(e))=t$, *then* $\tilde{\Phi}_2(\exists xA(x))=t$.

3.5.2. $\tilde{\Phi}_2((e_1,\cdots,e_n\in\lambda x_1\cdots x_nA(x,\cdots,x_n)))=\tilde{\Phi}_2(A(e_1,\cdots,e_n))$.

PROOF. Suppose that $\tilde{\Phi}_2(A(e))=t$.

Let Ψ be $\Phi\binom{\tilde{\Phi}(e)}{a}$, where a is not contained in $\exists xA(x)$. Then by the lemma 3.4 $\tilde{\Psi}_2(A(a))=t$. Since $\Psi\underset{a}{\sim}\Phi$, $\tilde{\Phi}_2(\exists xA(x))=t$ by the definition.

Next suppose that $\tilde{\Phi}_2((e_1,\cdots,e_n\in\lambda x_1\cdots x_nA(x_1,\cdots,x_n)))=t$. Let p be

$$\tilde{\Phi}_2(\lambda x_1\cdots x_nA(x_1,\cdots,x_n))$$
$$=\left\{\langle C_1,\cdots,C_n\rangle\middle|\widetilde{\Phi\binom{C_1\cdots C_n}{a_1\cdots a_n}}_2(A(a_1,\cdots,a_n))=t\right\}.$$

where $a_1,\cdots,a_n$ are not contained in $\lambda x_1\cdots x_nA(x_1,\cdots,x_n)$. Then $\langle\tilde{\Phi}(e_1),\cdots,\tilde{\Phi}(e_n)\rangle\in p$. That is,

$$\widetilde{\Phi\binom{\tilde{\Phi}(e_1),\cdots,\tilde{\Phi}(e_n)}{a_1,\cdots,a_n}}_2(A(a_1,\cdots,a_n))=t.$$

6) Cf. 3.3.2.

Hence by the lemma 3.4, $\tilde{\Phi}_2(A(e_1, \cdots, e_n))=t$.

Similarly if $\tilde{\Phi}_2((e_1, \cdots, e_n \in \lambda x_1 \cdots x_n A(x_1, \cdots, x_n)))=f$, then $\tilde{\Phi}_2(A(e_1, \cdots, e_n))=f$.

3.6.1. Let F be a wff. Positive parts (p. p.'s) and negative parts (n. p.'s) of F are called explicit parts (e. p.'s) of F.

3.6.2. Let $F[A]$ be a wff with an e. p. A in just one indicated place. Moreover let B be an e. p. of F. If A is a subexpression of B in F (i. e. all the symbols in A are those of B), we say B includes A in F. If A, B have no symbol in common in F, we say A, B are disjoint in F.

3.7. LEMMA. *Let F be a wff. If $\tilde{\Phi}_2(F)=f$, then $\tilde{\Phi}_2(A)=f$ for all p. p. A of F and $\tilde{\Phi}_2(A)=t$ for all n. p. A of F.*

PROOF. If A is F itself, the proposition is clear. If $\neg B$ is a p. p. of F and $\tilde{\Phi}_2(\neg B)=f$, then B is a n. p. of F and $\tilde{\Phi}_2(B)=t$. If $\neg B$ is a n. p. of F and $\tilde{\Phi}_2(\neg B)=t$, then B is a p. p. of F and $\tilde{\Phi}_2(B)=f$. If $B \vee C$ is a p. p. of F and $\tilde{\Phi}_2(B \vee C)=f$, then both B and C are p. p.'s of F and $\tilde{\Phi}_2(B)=\tilde{\Phi}_2(C)=f$. So by the definition of p. p.'s and n. p.'s, the proof is complete.

3.8. LEMMA. *Let $F[A]$ be a wff with an e. p. A in just one indicated place and let $\tilde{\Phi}_2(F[A])=t$, Moreover suppose that for each e. p. B of $F[A]$ which is disjoint with A, the following conditions are satisfied.*

3.8.1. *If B is a p. p. of $F[A]$, then $\tilde{\Phi}_2(B)=f$, and*

3.8.2. *if B is a n. p. of $F[A]$, then $\tilde{\Phi}_2(B)=t$.*

Then for each e. p. C of $F[A]$ which includes A, the following conditions are satisfied.

3.8.3. *If C is a p. p. of $F[A]$, then $\tilde{\Phi}_2(C)=t$, and*

3.8.4. *if C is a n. p. of $F[A]$, then $\tilde{\Phi}_2(C)=f$.*

PROOF. If C is $F[A]$ itself, the proposition is clear. If $\neg C$ is a p. p. of $F[A]$ and $\tilde{\Phi}_2(\neg C)=t$ and C includes A, then C is a n. p. and $\tilde{\Phi}_2(C)=f$. If $\neg C$ is a n. p. of $F[A]$ and $\tilde{\Phi}_2(\neg C)=f$ and C includes A, then C is a p. p. and $\tilde{\Phi}_2(C)=t$. Next suppose that $C \vee D$ is a p. p. of $F[A]$ and $\tilde{\Phi}_2(C \vee D)=t$ and C includes A. Then C, D are p. p.'s of $F[A]$ and D is disjoint with A. Hence by 3.8.1 $\tilde{\Phi}_2(D)=f$. Therefore $\tilde{\Phi}_2(C)$ must be t since $\tilde{\Phi}_2(C \vee D)=t$. Similarly if $D \vee C$ is a p. p. of $F[A]$ and $\tilde{\Phi}_2(D \vee C)=t$ and C includes A, then C is a p. p. and $\tilde{\Phi}_2(C)=t$. It completes the proof of 3.8.

§ 4. Cut-elimination theorem

4.1. LEMMA. *If F is derivable, then $\tilde{\Phi}_2(F)=t$ for any V-correspondence Φ. (V is an arbitrary semi-valuation.)*

PROOF. We shall prove this proposition by the induction of the derivability order of F. (See 4.1 in [2]).

Case 1. F is an axiom, i. e. F is $F[P_+, P_-]$, where P is a prime wff.

Suppose that $\tilde{\Phi}_2(F)=f$. Then by the lemma 3.7, $\tilde{\Phi}_2(P_+)=f$ and $\tilde{\Phi}_2(P_-)=t$. This is a contradiction, since P_+ and P_- are the same wff. Hence $\tilde{\Phi}_2(F[P_+, P_-])=t$.

Case 2. (S1.) $F[A_-], F[B_-]\to F[A\vee B_-]$. Suppose that $\tilde{\Phi}_2(F[A_-])=t$, $\tilde{\Phi}_2(F[B_-])=t$ and $\tilde{\Phi}_2(F[A\vee B_-])=f$ and we shall lead a contradiction. From $\tilde{\Phi}_2(F[A\vee B_-])=f$ we have $\tilde{\Phi}_2(A\vee B)=t$ by the lemma 3.7. Hence $\tilde{\Phi}_2(A)=t$ or $\tilde{\Phi}_2(B)=t$.

Subcase (i). $\tilde{\Phi}_2(A)=t$.

Consider any e. p. C of $F[A_-]$ which is disjoint with A. The C in $F[A\vee B_-]$ which is in the corresponding place is an e. p. of $F[A\vee B_-]$. Hence $\tilde{\Phi}_2(C)=f$ or t according as C is a p. p. or n. p. of $F[A\vee B_-]$ by the lemma 3.7. Accordingly the conditions of the lemma 3.8 are fulfilled. Therefore $\tilde{\Phi}_2(A)=f$ since A is a n. p. of $F[A_-]$ and includes A. This contradicts the hypothesis.

Subcase (ii). $\tilde{\Phi}_2(B)=t$.

Similar to the subcase (i).

Case 3. (S2.) $F[A(a^\tau)_-]\to F[\exists x^\tau A(x^\tau)_-]$, where a^τ does not occur in the conclusion. Suppose that $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_-])=f$. Then by the lemma 3.7 $\tilde{\Phi}_2(\exists x^\tau A(x^\tau))=t$. Hence there exists a V-correspondence Ψ such that $\Psi \underset{a^\tau}{\sim} \Phi$ and $\Psi(A(a^\tau))=t$ (cf. 3.3.1). But by the induction hypothesis $\tilde{\Psi}_2(F[A(a^\tau)_-])=t$. Moreover if C is an e. p. of $F[A(a^\tau)_-]$ which is disjoint with $A(a^\tau)_-$, then by the assumption and the lemma 3.7, $\tilde{\Psi}_2(C)=\tilde{\Phi}_2(C)=f$ or t according as C is a p. p. or n. p., for C does not contain a^τ and $\Psi \underset{a^\tau}{\sim} \Phi$. Therefore by the lemma 3.8, $\Psi(A(a^\tau))=f$. This is a contradiction. Accordingly $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_-])=t$.

Case 4. (S3.) $F[\exists x^\tau A(x^\tau)_+]\vee A(e^\tau)\to F[\exists x^\tau A(x^\tau)_+]$. Suppose that $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_+]\vee A(e^\tau))=t$ and $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_+])=f$. Then $\tilde{\Phi}_2(A(e^\tau))$ must be t. Accordingly by 3.5.1 $\tilde{\Phi}_2(\exists x^\tau A(x^\tau))=t$. But $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_+])=f$. Therefore $\tilde{\Phi}_2(\exists x^\tau A(x^\tau))=f$ by the lemma 3.7. This is a contradiction. Hence $\tilde{\Phi}_2(F[\exists x^\tau A(x^\tau)_+])=t$.

Case 5. (S4 a, b.) $F[A(e_1, \cdots, e_n)_\pm]\to F[(e_1, \cdots, e_n\in\lambda x_1\cdots x_n A(x_1, \cdots, x_n))_\pm]$. Suppose that $\tilde{\Phi}_2(F[A(e_1, \cdots, e_n)_\pm])=t$. Then we have $\tilde{\Phi}_2(F[(e_1, \cdots, e_n\in\lambda x_1\cdots x_n A(x_1, \cdots, x_n))_\pm])$ by a similar argument as in Case 2, using 3.5.2, 3.7 and 3.8.

Case 6. (S5.) $F\vee\exists x^1 \urcorner(x^1\vee \urcorner x^1)\to F$. Suppose that $\tilde{\Phi}_2(F\vee\exists x^1\urcorner(x^1\vee\urcorner x^1))=t$. Clearly $\tilde{\Phi}_2(\exists x^1\urcorner(x^1\vee\urcorner x^1))=f$. Hence $\tilde{\Phi}_2(F)$ must be t. This completes the proof of 4.1.

4.2. THEOREM. *If F is derivable, then it is strictly derivable.*

PROOF. If F is not strictly derivable, by 6.7 in [2] there exists a semi-valuation V in which F is f. Let Φ be a V-correspondence such that $\Phi(a)$ is of the form $[a, p_a]$ for each free variable a[7]. Then for every expression e,

7) Such a correspondence exists by 1.5.

$\tilde{\Phi}(e)$ is of the form $[e, q_e]$. In particular, $\tilde{\Phi}(F)$ is of the form $[F, q_F]$. Since F is f in V, q_F must be f. (c.f. 1.2.2.) That is, $\Phi_2(F)=f$. Therefore F cannot be derivable by the lemma 4.1. q.e.d.

Appendix 1.

Our method can be directly applied to GLC or other modified systems of simple type theory.

Appendix 2.

Every expressions, types, semi-valuations, V-complexes, the set of all the V-complexes of type τ, etc. are regarded as sets in Zermelo's set theory $\boldsymbol{Z}$ by a certain formalization, while a V-correspondence Φ cannot be regarded as a set in $\boldsymbol{Z}$. But for a given expression e, the value $\Phi(e)$ depends on only a finite number of the values of Φ by 3.2. So our proof goes also when definition of V-correspondence is changed so that the domains of them are finite sets of free variables. After this modification, a V-correspondence can be regarded as a set in $\boldsymbol{Z}$, and hence the formalization of our proof in $\boldsymbol{Z}$ can be easily established. I think that the fact is very important by the following reason.

We denote the axiom system of natural number theory with or without the induction by $\tilde{\Gamma}_a$ or Γ_a respectively. The proof of cut-elimination theorem in GLC is not formalizable in the system $\tilde{\Gamma}_a$ in GLC. (It is known that this system is weaker than $[\boldsymbol{Z}]$[8]).

In fact, it is that the following sequents are provable in GLC;

$$\tilde{\Gamma}_a \rightarrow Cons_{\mathrm{LK}}(\Gamma_a)$$

$$\tilde{\Gamma}_a \rightarrow Cons_{\mathrm{LK}}(\Gamma_a) \wedge CE \supset Cons_{\mathrm{GLC}}(\Gamma_a)$$

$$\tilde{\Gamma}_a \rightarrow Cons_{\mathrm{GLC}}(\Gamma_a) \supset Cons_{\mathrm{GLC}}(\tilde{\Gamma}_a)^{9)}.$$

where $Cons_{\mathrm{LK}}(\Gamma_a)$ etc. denote the arithmetical statement which asserts that Γ_a is consistent in LK etc. and CE denotes the statement which asserts the cut-elimination theorem in GLC. Hence if

$$\tilde{\Gamma}_a \rightarrow CE$$

were provable in GLC, we would have

$$\tilde{\Gamma}_a \rightarrow \mathrm{Cons}_{\mathrm{GLC}}(\tilde{\Gamma}_a)$$

in GLC, which is impossible by Gödel's theorem. From this argument it seems

8) Cf. [7].
9) Cf. [4] 9.28.

likely that the proof of cut-elimination theorem cannot be essentially reduced to one which is based on a weaker standpoint (in particular, the finite standpoint) than $\boldsymbol{Z}$.

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