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Pseudo-uniform reducibility

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1. Introduction

In [1] we showed:

THEOREM A. *If every recursive set is representable in a theory* (T) *then* (T) *is undecidable.*

THEOREM B. *If every recursive set is definable in* (T) *and if* (T) *is consistent, then the set* T_0 *of Gödel numbers of the provable sentences of* (T) *is recursively inseparable from the set* R_0 *of Gödel numbers of the refutable sentences of* (T).

The above propositions combine notions of recursive function theory with those of mathematical logic—i.e. with the concept of a "first order theory". In this note we obtain generalizations of these propositions which are purely recursive function theoretic in nature. We also show that the conclusions of Theorems A and B hold under still weaker hypotheses.

2. Pseudo-uniform reducibility

The word "number" shall mean natural number. We use "A", "B", "α", "β" for sets of natural numbers. A set A is (many-one) reducible to α if there is a recursive function $g(\psi)$ (called a (many-one) reduction of A to α) such that $A=g^{-1}(\alpha)$—i.e. for each number i, $i\in A\leftrightarrow g(i)\in\alpha$. Consider now a collection Σ of recursively enumerable sets. The collection Σ is *uniformly reducible* to α (as defined in [2]) if there is a recursive function $g(x,y)$ (called a uniform reduction of Σ to α) such that for every i for which $\omega_i\in\Sigma$, the function $g(i,y)$ (as a function of the one variable y) is a reduction of ω_i to Σ.[2)] Thus, if Σ is uniformly reducible to α, then not only is every element of Σ reducible to α, but given any such element ω_i (in the sense of given its index i) we can *effectively* find a reduction of it to α.

It is trivial to verify that if some non-recursive set is reducible to α,

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2) By ω_i, we mean the set of all numbers x satisfying the condition $(\exists y)\ T_1(i,x,y)$ where $T_1(z,x,y)$ is the predicate of Kleene's enumeration theorem [3].

then α is non-recursive. Hence, it follows that if every recursively enumerable set is reducible to α, then α is non-recursive (since there exists a recursively enumerable set which is not recursive). This fact is well known. Suppose that every *recursive* set is reducible to α; does it follow that α is non-recursive? Clearly not, for if α is any non-empty set whose complement is also non-empty, then every recursive set A is reducible to α (just take an element a_1 of α and an element a_2 of $\tilde{\alpha}$ and define $g(x)=a_1$ if $x \in A$; $g(x)=a_2$ if $x \notin A$). Since A is recursive, $g(x)$ is a recursive function, and clearly a reduction of A to α. Suppose that the collection of all recursive sets is *uniformly* reducible to α; does it follow that α is non-recursive? In [**2**] we showed that this hypothesis implies not only that α is non-recursive, but that the complement of α is productive. Thus, to establish the non-recursivity of a set α, the hypothesis that all recursive sets be reducible to α is too weak, and the hypothesis of uniform reducibility is stronger than necessary. We now consider a notion which is of intermediate strength.

We shall say that Σ is *pseudo-uniformly reducible* to α if there is a recursive function $g(x, y)$ (called a pseudo-uniform reduction of Σ to α) such that for every set $A \in \Sigma$, there is a number a such that $g(a, y)$ (as a function of the one variable y) is a reduction of A to α. We note that this definition (unlike that of uniform reducibility) does *not* require that such a number a be an index of the set A, nor that there be a recursive function $\varphi(x)$ which assigns to any index of A such a number a. If there were such a recursive function $\varphi(x)$, then Σ would indeed be *uniformly* reducible to α under the function $g(\varphi(x), y)$. We shall soon see that a sufficient condition for α to be non-recursive is that the collection of all recursive sets be pseudo-uniformly reducible to α. And in light of our next proposition, we feel that this fact constitutes the mathematical essence of Theorem A.

The notion of pseudo-uniform reducibility arises naturally in connection with metamathematics in the following way. Suppose we have a theory (T) with standard formalizations (cf. [**4**]). Let $F_1, F_2, \cdots, F_n, \cdots$ be an effective enumeration of all the formulas with exactly one free variable; let Δ_i be the numeral designating the natural number i; let g be an effective Gödel numbering of all closed sentences; let T be the set of all provable (closed) sentences and R the set of all (closed) sentences whose negation is provable; let T_0, R_0 respectively be the set of Gödel numbers of the provable, refutable sentences of (T); let $\varphi(i, j)$ be the Gödel number of $F_i(\Delta_j)$. Under the usual requirements of "effectiveness" of the Gödel numbering and of the sequence $\Delta_0, \Delta_1, \Delta_2, \cdots, \Delta_i$, the function $\varphi(x, y)$ is (general) recursive.

A formula $F(x)$ is said to *represent* the set of all numbers n for which $F(\Delta_n) \in T$. We pointed out in [**2**] that if a set A is representable in (T), then

A is (many-one) reducible to T_0. We now note the following stronger fact:

PROPOSITION 1. *If each element of a collection Σ is representable in (T), then the collection Σ is pseudo-uniformly reducible to T_0.*

PROOF. For each element A of Σ there is, by hypothesis, a formula $F_i(x)$ which represents A in (T). Then for every number $j, j \in A \leftrightarrow F_i(\Delta_j) \in T \leftrightarrow \varphi(i, j) \in T_0$. Thus $\varphi(x, y)$ is a pseudo-uniform reduction of Σ to T_0.

We now show

THEOREM 1. *If the collection of all recursive sets is pseudo-uniformly reducible to α, then α is not recursive.*

We actually show Theorem 1 in the following stronger form.

THEOREM 1'. *Each of the following conditions implies the next.*

(a) *The collection of recursive sets is pseudo-uniformly reducible to α.*

(b) *There is a recursive function $g(x)$ such that for every recursive set A, there is a number i such that $i \in A \leftrightarrow g(i) \in \alpha$.*

(c) *α is not recursive.*

PROOF. Suppose (a); let $f(x, y)$ be such a uniform reduction. Define $g(x) = f(x, x)$. Then $g(x)$ is recursive. Let A be any recursive set. By hypothesis there is a number i such that for every number $y, i \in A \leftrightarrow f(i, y) \in \alpha$. Setting $y = i, i \in A \leftrightarrow f(i, i) \in \alpha \cdots$ i. e. $i \in A \leftrightarrow g(i) \in \alpha$. Thus (a)$\Rightarrow$(b).

Suppose (b). We must show that α is not recursive. Suppose it were. Then $\tilde{\alpha}$ would be recursive. Then $g^{-1}(\alpha)$ is recursive [$g^{-1}(\tilde{\alpha}) = df$ the set of all i such that $g(i) \in \tilde{\alpha}$]. Then there is a number i such that $i \in g^{-1}(\tilde{\alpha}) \leftrightarrow g(i) \in \alpha$. But $i \in g^{-1}(\tilde{\alpha}) \leftrightarrow g(i) \in \tilde{\alpha}$. Hence $g(i) \in \tilde{\alpha} \leftrightarrow g(i) \in \alpha$, which is impossible.

In view of Proposition 1, Theorem 1 is indeed a generalization of Theorem A.

We also note that the statement (b)$\Rightarrow$(c) of Theorem 1' is a stronger statement than Theorem 1, and implies the following stronger form of Theorem A (by setting $g(i) = \varphi(i, i)$).

THEOREM A'. *If for every recursive set A, there is a number i such that $i \in A \leftrightarrow F_i(\Delta_i) \in T$, then T_0 is non-recursive.*

The hypothesis of Theorem A' is obviously weaker than that of Theorem A, for the latter says that for any recursive set A there is a number i such that for *every* j (whether equal to i or not), $j \in A \leftrightarrow F_i(\Delta_j) \in T$.

3. Pseudo-uniform reducibility of ordered pairs

Let A, B, α, β be number sets. A recursive function $f(x)$ is a (many-one) *reduction* of the ordered pair (A, B) to the ordered pair (α, β) (as defined in [2]) if $f(x)$ is simultaneously a reduction of A to α and of B to β.—i. e. for every number i: (1) $i \in A \leftrightarrow f(i) \in \alpha$; (2) $i \in B \leftrightarrow f(i) \in \beta$.

Consider now a collection Σ of ordered pairs of number sets. We shall say that Σ is *pseudo-uniformly reducible* to a pair (α, β) if there is a recursive function $f(x, y)$ (which we will call a pseudo-uniform reduction of Σ to (α, β)) such that for every pair (A, B) in Σ, there is a number i such that $f(i, y)$ (as a function of the one variable y) is a reduction of (A, B) to (α, β).[3]

The obvious analogue of Proposition 1 is

PROPOSITION 2. *Let S be a collection of sets and let Σ be the collection of all ordered pairs $(A, \tilde{A})$ such that $A \in S$. Then if every element of S is definable in (T), and if (T) is consistent, then Σ is pseudo-uniformly reducible to the pair (T_0, R_0).*

PROOF. As in the proof of Proposition 1, let $\varphi(i, j)$ be the Gödel number of $F_i(\Delta_j)$. Let $A \in S$. Then for some number i, $F_i(x)$ defines A in (T). Thus for all j, $j \in A \Rightarrow F_i(\Delta_j) \in T$ and $j \in \tilde{A} \Rightarrow F_i(\Delta_j) \in R$. Since (T) is consistent, then $j \in A \leftrightarrow F_i(\Delta_j) \in T$, and $j \in \tilde{A} \leftrightarrow F_i(\Delta_j) \in R$. [For $F_i(\Delta_j) \in T \Rightarrow F_i(\Delta_j) \notin R \Rightarrow j \notin \tilde{A} \Rightarrow j \in A$. Similarly $F_i(\Delta_j) \in R \Rightarrow j \in \tilde{A}$.] Thus $j \in A \leftrightarrow \varphi(i, j) \in T_0$ and $j \in \tilde{A} \leftrightarrow \varphi(i, j) \in R_0$. Hence $\varphi(i, y)$ is a reduction of $(A, \tilde{A})$ to (T_0, R_0)

We now show

THEOREM 2. *Let Σ_R be the collection of all complementary pairs of recursive sets and let α, β be disjoint. Then if Σ_R is pseudo-uniformly reducible to (α, β), then (α, β) is recursively inseparable.*[4]

We in fact shall show the stronger fact:

THEOREM 2′. *Each of the following conditions implies the next:*

(a) *Σ_R is pseudo-uniformly reducible to (α, β) [α, β are disjoint].*

(b) *There is a recursive function $g(x)$ such that for each pair $(A, \tilde{A}) \in \Sigma$, there is a number i such that $i \in A \leftrightarrow g(i) \in \alpha$ and $i \in \tilde{A} \leftrightarrow g(i) \in \beta$.*

(c) *The pair $(g^{-1}(\alpha), g^{-1}(\beta))$ is recursively inseparable.*

(d) *The pair (α, β) is recursively inseparable—in fact, the subset $gg^{-1}\alpha$ of α is recursively inseparable from the subset $gg^{-1}\beta$ of β.*

PROOF. (1) (a)$\Rightarrow$(b). Let $f(x, y)$ be a pseudo-uniform reduction of Σ_R to (α, β). As in the proof of Theorem 1′, let $g(x)$ be the recursive function $f(x, x)$. Let $(A, \tilde{A}) \in \Sigma$ and let i be such that $f(i, y)$ is a reduction of $(A, \tilde{A})$ to (α, β). Since $f(i, y)$ is a reduction of A to α, then (by the argument in the proof of Theorem 1′) $i \in A \leftrightarrow g(i) \in \alpha$. Similarly, since $f(i, y)$ is a reduction of $\tilde{A}$ to β, then $i \in \tilde{A} \leftrightarrow g(i) \in \beta$.

(2) (b)$\Rightarrow$(c). Suppose $g(x)$ is as in (b). Suppose $(g^{-1}(\alpha), g^{-1}(\beta))$ were

3) Again, this notion is midway in strength between the notions: (1) every element of Σ is reducible to (α, β); (2) Σ is uniformly reducible to (α, β), as defined in [2]. The latter says that given indices i, j of A, B where $(A, B) \in \Sigma$, we can *effectively* find a number i such that $f(i, y)$ is a reduction of (A, B) to (α, β).

4) A pair is called recursively inseparable if there is no recursive superset of one disjoint from the other.

recursively separable. Then there is a recursive superset A of $g^{-1}(\beta)$ disjoint from $g^{-1}(\alpha)$. Hence, $g^{-1}(\beta) \subseteqq A$; $g^{-1}(\alpha) \subseteqq \tilde{A}$. By the hypothesis of (b), there is an i such that $i \in A \leftrightarrow g(i) \in \alpha$ and $i \in \tilde{A} \leftrightarrow g(i) \in \beta$. Hence, $i \in A \Rightarrow g(i) \in \alpha \Rightarrow i \in g^{-1}(\alpha) \Rightarrow i \in \tilde{A}$, and $i \in \tilde{A} \Rightarrow g(i) \in \beta \Rightarrow i \in g^{-1}(\beta) \Rightarrow i \in A$.

Thus $i \in A \leftrightarrow i \in \tilde{A}$, which is impossible. Hence $g^{-1}(\alpha), g^{-1}(\beta)$ are recursively inseparable.

(3) (c)$\Rightarrow$(d). We have shown in [2] (p. 62, Proposition 4, Ch. II) that if (A_1, A_2) is recursively inseparable and if (A_1, A_2) is reducible to (B_1, B_2) (or even if there is a recursive function which maps A_1 into B_1 and A_2 into B_2) then (B_1, B_2) is in turn recursively inseparable. But clearly g maps $g^{-1}(\alpha)$ into $gg^{-1}\alpha$ and $g^{-1}(\beta)$ into $gg^{-1}\beta$.

Theorem 2 and Proposition 2 clearly imply Theorem B. But again, the statement (b)$\Rightarrow$(d) of Theorem 2′ is stronger than Theorem 2, and implies the following stronger form of Theorem B.

THEOREM B′. *A sufficient condition for the nucleii (T_0, R_0) of a consistent theory (T) to be recursively inseparable is that for every recursive set A there exists a number i such that $i \in A \leftrightarrow F_i(\Delta_i) \in T$ and $i \in \tilde{A} \leftrightarrow F_i(\Delta_i) \in R$.*

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