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COMPLEX STRUCTURES, TOTALLY REAL AND TOTALLY GEODESIC SUBMANIFOLDS OF COMPACT 3-SYMMETRIC SPACES, AND AFFINE SYMMETRIC SPACES

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Abstract. We construct invariant complex structures of a compact 3-symmetric space by means of the canonical almost complex structure of the underlying manifold and some involutions of a Lie group. Moreover, by making use of graded Lie algebras and some invariant structures of affine symmetric spaces, we classify half dimensional, totally real and totally geodesic submanifolds of a compact 3-symmetric space with respect to each invariant complex structure.

1. Introduction. It was Gray [5] who classified 3-symmetric spaces (see also Wolf and Gray [20]), proving that each 3-symmetric space admits an invariant almost complex structure called the *canonical almost complex structure*, which is not integrable in general. According to [5], it is known that for a compact Riemannian 3-symmetric space G/H with a compact simple Lie group G, the dimension of the center $Z(H)$ of H is either 0, 1 or 2 (see also [20, Theorem 3.3]), and if the dimension of $Z(H)$ is not zero, then H is a centralizer of a toral subgroup of G. Therefore, it follows from Wang [19] that there exists a G-invariant complex structure I on G/H if $\dim Z(H) \neq 0$. Moreover, invariant (almost) complex structures on G/H had been investigated by Borel and Hirzebruch [2] (see also Nishiyama [12] and Wolf and Gray [20]). In the present paper, first we describe invariant complex structures on a compact Riemannian 3-symmetric space G/H with $\dim Z(H) \neq 0$, by means of the canonical almost complex structure and some involutive automorphisms of G (Section 3).

Half-dimensional totally real and totally geodesic submanifolds of Hermitian symmetric spaces are (non-Hermitian) symmetric R-spaces. Takeuchi [16] described those submanifolds by using graded Lie algebras of the first kind. In our previous papers [17, 18], we classified half dimensional, totally real and totally geodesic submanifolds of naturally reductive, compact Riemannian 3-symmetric spaces with respect to the canonical almost complex structures. In particular, when $\dim Z(H) \neq 0$, these submanifolds are obtained from graded Lie algebras of the second kind. More precisely, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ a noncompact dual of $\mathfrak{g}$ with a Cartan involution τ and the corresponding Cartan decomposition $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}$. For

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any graded triple $(\mathfrak{g}^*, Z, \tau)$ associated with a gradation of the second kind, we put

$$\sigma := \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}Z\right), \quad H := G^{\sigma},$$

where G^{σ} denotes the set of fixed points of σ in G. It is obvious that σ is an inner automorphism of order 3 on G. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$, and N the K-orbit in G/H at the origin $o := \{H\} \in G/H$. Then N is a half dimensional, totally real and totally geodesic submanifold of a compact Riemannian 3-symmetric space G/H with respect to the canonical almost complex structure. Conversely, any such submanifold is obtained in this manner. On the other hand, a symmetric pair of type K_ε was introduced by Oshima and Sekiguchi [14], and subsequently Kaneyuki [9] proved that any symmetric pair of type K_ε is obtained from graded Lie algebras of the first kind or of the second kind.

The main purpose of the present paper is to classify the 'real forms' of a compact Riemannian 3-symmetric space G/H, i.e., half dimensional, totally real and totally geodesic submanifolds of G/H with respect to each G-invariant complex structure, by making use of symmetric pairs of type K_ε and their invariant geometric structures (Section 5).

The organization of this paper is as follows:

In Section 2, we recall several notions and facts regarding 3-symmetric spaces and graded Lie algebras used throughout the paper.

In Section 3, we describe invariant complex structures of a compact Riemannian 3-symmetric space G/H with $\dim Z(H) \neq 0$ in terms of the canonical almost complex structures and involutions on G/H (see Proposition 3.3 and Corollary 3.4).

In Section 4, we prove that any half dimensional, totally real and totally geodesic submanifold of G/H with respect to each G-invariant complex structure is also totally real with respect to the canonical almost complex structure (Proposition 4.5).

In Section 5, by making use of symmetric pairs of type K_ε and their noncompactly causal structure (cf. Hilgert and Ólafsson [7]), we describe each invariant complex structure I and classify every real form with respect to I of G/H with $\dim Z(H) \neq 0$ (Proposition 5.5, Theorem 5.6).

2. Preliminaries. 2.1. Riemannian 3-symmetric spaces. In this subsection we recall relevant notions and results on compact Riemannian 3-symmetric spaces. Let G be a Lie group and H a compact subgroup of G, and let $\langle\,,\rangle$ be a G-invariant Riemannian metric on G/H. A Riemannian homogeneous space $(G/H\langle\,,\rangle)$ is called a *Riemannian* 3-*symmetric space* if it is not isometric to a Riemannian symmetric space and there exists an automorphism σ of order 3 on G satisfying the following:

(i) ${G^{\sigma}}_0 \subset H \subset G^{\sigma}$, where G^{σ} is the set of fixed points of σ and ${G^{\sigma}}_0$ the identity component of G^{σ}, and

(ii) the transformation of G/H induced by σ is an isometry.

We note that, except for the condition that $(G/H, \langle\,,\rangle)$ is not isometric to a Riemannian symmetric space, the definition of Riemannian 3-symmetric spaces in this paper is equivalent to that in [5] (see Proposition 5.1 and Theorem 5.4 of [5]).

In this paper, for each automorphism φ of G, we denote the differential of φ at $e \in G$ by the same symbol as φ.

Let $(G/H, \langle\ ,\ \rangle, \sigma)$ be a Riemannian 3-symmetric space with an automorphism σ of order 3 on G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively, and let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be an $\mathrm{Ad}(H)$- and σ-invariant decomposition of $\mathfrak{g}$. We note that $\mathfrak{h}$ coincides with the set $\mathfrak{g}^\sigma$ of fixed points of σ. Under the canonical identification of $\mathfrak{m}$ with the tangent space $T_o(G/H)$ of G/H at $o = \{H\}$, we define an isometry J of $(\mathfrak{m}, \langle\ ,\ \rangle)$ by

$$\sigma = -\frac{1}{2}\mathrm{Id} + \frac{\sqrt{3}}{2}J\,, \quad \mathrm{Id} = \text{the identity map of } \mathfrak{m}\,. \tag{2.1}$$

It is known that J induces a G-invariant almost complex structure on G/H, which is denoted by the same symbol as J. We call J the *canonical almost complex structure* (see [5]).

LEMMA 2.1 ([5]). *For $X, Y \in \mathfrak{m}$, we have*

$$[JX, JY]_{\mathfrak{h}} = [X, Y]_{\mathfrak{h}}\,, \quad [JX, Y]_{\mathfrak{m}} = -J[X, Y]_{\mathfrak{m}}\,.$$

Next, we describe an inner automorphism of order 3 on a compact simple Lie algebra. Let $\mathfrak{g}$ be a compact simple Lie algebra and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}_c$ and $\mathfrak{t}_c$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ be the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$, and let $\Pi(\mathfrak{g}_c, \mathfrak{t}_c) = \{\alpha_1, \dots, \alpha_n\}$ be a fundamental root system of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ for some lexicographic ordering. We define $H_j \in \mathfrak{t}_c$, $j = 1, \dots, n$, by

$$\alpha_i(H_j) = \delta_{ij}\,. \tag{2.2}$$

Then each inner automorphism of order 3 on $\mathfrak{g}$ is given by the following lemma (cf. Wolf and Gray [20] and Helgason [6]).

LEMMA 2.2. *Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$, and σ an inner automorphism of order 3 on G. Let $\delta = \sum_{p=1}^n m_p\alpha_p$ denote the highest root of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$. Then σ is conjugate to $\mathrm{Ad}(g_i)$ for an element g_i of G which has one of the following forms*:

(1) $g_0 = \exp\{(2\pi\sqrt{-1}/3)H_i\}$ $(m_i = 3)$,
(2) $g_1 = \exp\{(2\pi\sqrt{-1}/3)H_i\}$ $(m_i = 2)$,
(3) $g_2 = \exp\{(2\pi\sqrt{-1}/3)(H_j + H_k)\}$ $(m_j = m_k = 1)$,
(4) $g_3 = \exp\{(2\pi\sqrt{-1}/3)H_i\}$ $(m_i = 1)$.

REMARK 2.3. (1) In the case (4), we see that the pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$ is (Hermitian) symmetric.

(2) Let $\mathfrak{z}(\mathfrak{g}^\sigma)$ be the center of $\mathfrak{g}^\sigma$. If $\sigma = \mathrm{Ad}(g_k)$ for $k = 0, 1, 2$, then the dimension of $\mathfrak{z}(\mathfrak{g}^\sigma)$ is equal to k.

2.2. Graded Lie algebras. In this subsection we recall several notions and results on graded Lie algebras. Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\boldsymbol{R}$. Let τ be a Cartan involution of $\mathfrak{g}^*$ and

$$\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}\,, \quad \tau|_{\mathfrak{k}} = \mathrm{Id}_{\mathfrak{k}}\,, \quad \tau|_{\mathfrak{p}} = -\mathrm{Id}_{\mathfrak{p}}\,, \tag{2.3}$$

the Cartan decomposition of $\mathfrak{g}^*$ corresponding to τ. Take a gradation of the ν-th kind on $\mathfrak{g}^*$:

$$\begin{aligned}&\mathfrak{g}^* = \mathfrak{g}^*_{-\nu} + \cdots + \mathfrak{g}^*_0 + \cdots + \mathfrak{g}^*_\nu\,, \quad \mathfrak{g}^*_1 \neq \{0\}\,, \quad \mathfrak{g}^*_\nu \neq \{0\}\,,\\ &[\mathfrak{g}^*_p, \mathfrak{g}^*_q] \subset \mathfrak{g}^*_{p+q}\,, \quad \tau(\mathfrak{g}^*_p) = \mathfrak{g}^*_{-p}\,, \quad -\nu \leq p, q \leq \nu\,.\end{aligned} \tag{2.4}$$

It is known that there exists a unique element $Z \in \mathfrak{p} \cap \mathfrak{g}^*_0$, called the characteristic element of the gradation (2.4), such that

$$\mathfrak{g}^*_p = \{X \in \mathfrak{g}^*; [Z, X] = pX\}\,, \quad -\nu \leq p \leq \nu\,. \tag{2.5}$$

A triple $(\mathfrak{g}^*, Z, \tau)$ is called a *graded triple*. Let

$$\mathfrak{g}^* = \sum_{i=-\nu}^{\nu} \mathfrak{g}^*_i\,, \quad \bar{\mathfrak{g}}^* = \sum_{i=-\bar{\nu}}^{\bar{\nu}} \bar{\mathfrak{g}}^*_i$$

be two graded Lie algebras. These gradations are said to be *isomorphic* if $\nu = \bar{\nu}$ and there exists an isomorphism $\phi : \mathfrak{g}^* \to \bar{\mathfrak{g}}^*$ such that $\phi(\mathfrak{g}^*_i) = \bar{\mathfrak{g}}^*_i$ for $-\nu \leq i \leq \nu$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let Δ denote the set of restricted roots of $\mathfrak{g}^*$ with respect to $\mathfrak{a}$. We denote by $\Pi = \{\lambda_1, \ldots, \lambda_l\}$ a fundamental root system of Δ with respect to a lexicographic ordering of $\mathfrak{a}$. We call subsets $\{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ of Π a *partition* of Π if $\Pi_1 \neq \emptyset$, $\Pi_m \neq \emptyset$ and

$$\Pi = \Pi_0 \cup \Pi_1 \cup \cdots \cup \Pi_m \ \ \text{(disjoint union)}\,.$$

Let Π and $\bar{\Pi}$ be fundamental root systems of noncompact semisimple Lie algebras $\mathfrak{g}^*$ and $\bar{\mathfrak{g}}^*$, respectively. Partitions $\{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ of Π and $\{\bar{\Pi}_0, \bar{\Pi}_1, \ldots, \bar{\Pi}_n\}$ of $\bar{\Pi}$ are said to be *equivalent* if there exists an isomorphism ϕ from the Dynkin diagram of Π to that of $\bar{\Pi}$ such that $m = n$ and $\phi(\Pi_i) = \bar{\Pi}_i$, $i = 0, 1, \ldots, m$.

Let $\{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ be a partition of $\Pi = \{\lambda_1, \ldots, \lambda_l\}$. We define a map $h_\Pi : \Delta \to \boldsymbol{Z}$ by

$$h_\Pi(\lambda) := \sum_{\lambda_i \in \Pi_1} k_i + 2 \sum_{\lambda_i \in \Pi_2} k_i + \cdots + m \sum_{\lambda_i \in \Pi_m} k_i\,, \quad \lambda = \sum_{i=1}^{l} k_i \lambda_i \in \Delta\,. \tag{2.6}$$

Then there exists a unique $Z \in \mathfrak{a}$ such that $\lambda(Z) = h_\Pi(\lambda)$ for all $\lambda \in \Delta$. Since $h_\Pi(\lambda) \in \boldsymbol{Z}$ for all $\lambda \in \Delta$, there exists a gradation whose characteristic element equals Z, which is called *the gradation defined by a partition* $\{\Pi_0, \Pi_1, \ldots, \Pi_m\}$ of Π. Moreover, Kaneyuki and Asano [10] proved the following theorem.

THEOREM 2.4 ([10]). *Let $\mathfrak{g}^*$ be a noncompact semisimple Lie algebra over $\boldsymbol{R}$ and Π a fundamental root system of $\mathfrak{g}^*$. Then the correspondence*

$$\{\Pi_0, \Pi_1, \ldots, \Pi_m\} \mapsto \textit{the gradation defined by } \{\Pi_0, \Pi_1, \ldots, \Pi_m\}$$

induces a bijection between the set of equivalence classes of partitions of Π and the set of isomorphism classes of gradations of $\mathfrak{g}^*$.

Define $h_i \in \mathfrak{a}$, $i = 1, 2, \ldots, l$, by

$$\lambda_j(h_i) = \delta_{ij}\,, \tag{2.7}$$

and denote the highest root of Δ by

$$\delta_{\mathfrak{a}} := \sum_{i=1}^{l} n_i \lambda_i \,. \tag{2.8}$$

According to Faraut, Kaneyuki, Korányi, Lu and Roos [4, pp. 115, Proposition I.2.7], the following Proposition holds.

PROPOSITION 2.5 ([4]). *Let $Z \in \mathfrak{a}$ be a characteristic element of a graded Lie algebra of the second kind defined by a partition of Π. Then*

$$Z = h_i \,, \quad \text{or } h_j + h_k \,,$$

with $n_i = 2$ and $n_j = n_k = 1$.

Finally, we clarify the relation between H_i and h_j. Let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$ containing $\mathfrak{a}$. We denote by $\mathfrak{g}_c$ and $\mathfrak{t}_c$ the complexifications of $\mathfrak{g}^*$ and $\mathfrak{t}^*$, respectively. Suppose that $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ and Δ have compatible orderings.

LEMMA 2.6. *Let λ_i be any root in Π.*

(1) *If there exists a unique $\alpha_j \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$ such that $\alpha_j|_{\mathfrak{a}} = \lambda_i$, then $h_i = H_j$.*

(2) *If there exist two fundamental roots $\alpha_j, \alpha_k \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$ such that $\alpha_j|_{\mathfrak{a}} = \alpha_k|_{\mathfrak{a}} = \lambda_i$, then $h_i = H_j + H_k$.*

PROOF. (1) From the classification of the Satake diagrams (cf. Araki [1] and [6]) for $\alpha_p \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$, $p \neq j$, it follows that $\alpha_p|_{\mathfrak{a}} = 0$ or $\alpha_p|_{\mathfrak{a}} = \lambda_q$ for some q, $q \neq i$. Thus we have

$$\alpha_p(h_i) = \alpha_p|_{\mathfrak{a}}(h_i) = 0 \,, \quad \alpha_j(h_i) = \lambda_i(h_i) = 1 \,,$$

which implies that $h_i = H_j$.

(2) Similarly as above, for $\alpha_p \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$, $p \neq j, k$, it follows that $\alpha_p|_{\mathfrak{a}} = 0$ or $\alpha_p|_{\mathfrak{a}} = \lambda_q$ for some q, $q \neq i$. Therefore

$$\alpha_p(h_i) = \alpha_p|_{\mathfrak{a}}(h_i) = 0 \,, \quad \alpha_m(h_i) = \alpha_m|_{\mathfrak{a}}(h_i) = \lambda_i(h_i) = 1 \,, \quad m = j, k \,,$$

which implies that $h_i = H_j + H_k$. □

3. Invariant complex structures and J. In this section we use the same notation as in Section 2.1. Let $(G/H, \langle\, , \rangle, \sigma)$ be a compact, simply connected Riemannian 3-symmetric space such that G is a compact simple Lie group, σ is inner and the dimension of the center $Z(H)$ of H is not zero. In this case, H is a centralizer of a toral subgroup of G and so H is connected. Moreover, it is known that $(G/H, \langle\, , \rangle, \sigma)$ admits a G-invariant complex structure (cf. Wang [19]). In the remaining part of this paper we assume that a compact Riemannian 3-symmetric space $(G/H, \langle\, , \rangle, \sigma)$ is of inner type such that G is a compact simple Lie group, H is a centralizer of a toral subgroup of G and $\langle\, , \rangle$ is induced from a biinvariant metric on G.

In this section we construct invariant complex structures on a 3-symmetric space $(G/H, \langle\, , \rangle, \sigma)$ by means of J and some involutive automorphisms of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Since σ is inner, there exists a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{h}$. Let $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ be the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$ and $\mathfrak{g}^\alpha$ the root space for $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$. We take the Weyl basis $\{E_\alpha \in \mathfrak{g}^\alpha; \alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)\}$ of $\mathfrak{g}_c$ so that

$$A_\alpha := E_\alpha - E_{-\alpha}\,, \quad B_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha}) \in \mathfrak{g}\,, \quad B(E_\alpha, E_{-\alpha}) = 1\,,$$

where B denotes the Killing form of $\mathfrak{g}$. The following lemma is obvious.

LEMMA 3.1. *For $T \in \mathfrak{t}_c$ and $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, we have*

$$\mathrm{Ad}(\exp T)(E_\alpha) = e^{\alpha(T)} E_\alpha\,.$$

Since $\mathfrak{t} \subset \mathfrak{h}$, there is a subset Δ_0 of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ such that

$$\mathfrak{h}_c = \mathfrak{t}_c + \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha\,. \tag{3.1}$$

Let I denote a G-invariant complex structure of $(G/H, \langle\ ,\ \rangle, \sigma)$. Then we have $\mathfrak{m}_c = \mathfrak{m}_+ \oplus \mathfrak{m}_-$ (direct sum), where $\mathfrak{m}_\pm$ denote the $\pm\sqrt{-1}$-eigenspaces of I, respectively. Set $\mathfrak{a}^+ := \mathfrak{h}_c + \mathfrak{m}_+$. Since I is G-invariant, it follows that $m_\pm$ are $\mathrm{ad}(\mathfrak{h})$-invariant, and furthermore $\mathfrak{a}^+$ is a Lie subalgebra of $\mathfrak{g}_c$ (cf. Borel and Hirzebruch [2] and Nishiyama [12]). Since $\mathfrak{t}_c \subset \mathfrak{h}_c \subset \mathfrak{a}^+$, there exists a subset Δ^+ of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ such that

$$\mathfrak{a}^+ = \mathfrak{h}_c + \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha\,. \tag{3.2}$$

Moreover, since I is integrable, it follows that

$$\begin{aligned} &\Delta(\mathfrak{g}_c, \mathfrak{t}_c) = \Delta_0 \cup \Delta^+ \cup (-\Delta^+) \ \ \text{(disjoint union)}\,, \\ &\alpha \in \Delta_0 \cup \Delta^+\,, \quad \beta \in \Delta^+\,, \quad \alpha + \beta \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c) \Rightarrow \alpha + \beta \in \Delta^+\,, \end{aligned} \tag{3.3}$$

(cf. [12]), and hence by [2] (see also [12, Theorem 1]), there exists a fundamental root system $\tilde{\Pi} = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}$ of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ such that $\Pi(\mathfrak{h}) := \tilde{\Pi} \cap \Delta_0$ is a fundamental root system of Δ_0 and

$$\Delta^+ = [\tilde{\Pi}]^+ - [\Pi(\mathfrak{h})]^+\,. \tag{3.4}$$

Here, $[\tilde{\Pi}]^+$ and $[\Pi(\mathfrak{h})]^+$ denote the sets of positive roots generated by $\tilde{\Pi}$ and $\Pi(\mathfrak{h})$, respectively. Note that

$$I|_{\mathfrak{g}^\alpha} = \sqrt{-1}\,\mathrm{Id}_{\mathfrak{g}^\alpha} \ \ (\alpha \in \Delta^+)\,, \quad I|_{\mathfrak{g}^\beta} = -\sqrt{-1}\,\mathrm{Id}_{\mathfrak{g}^\beta} \ \ (\beta \in -\Delta^+)\,. \tag{3.5}$$

Conversely, if there exists a subset Δ^+ of Δ satisfying (3.3), then the linear automorphism I of $\mathfrak{m} = \sum_{\alpha \in \Delta^+}(\boldsymbol{R}A_\alpha + \boldsymbol{R}B_\alpha)$ given by (3.5) induces a G-invariant complex structure on G/H (cf. [12]).

Since $\sigma^3 = \mathrm{Id}$, it follows from Lemma 3.1 that for $\alpha \in \Delta^+$ and a primitive cubic root of unity $\xi = e^{2\pi\sqrt{-1}/3}$, we have $\sigma(E_\alpha) = \xi E_\alpha$ or $\sigma(E_\alpha) = \xi^2 E_\alpha$. Define subsets $\Delta_1{}^+, \Delta_2{}^+$ of Δ^+ by

$$\Delta^+{}_i := \{\alpha \in \Delta^+; \sigma(E_\alpha) = \xi^i E_\alpha\}\,, \quad i = 1, 2\,. \tag{3.6}$$

Let $\mathfrak{z}(\mathfrak{h})$ be the center of $\mathfrak{h}$. Then, by Remark 2.3 (2), the dimension of $\mathfrak{z}(\mathfrak{h})$ is 1 or 2. Moreover

LEMMA 3.2. (1) *If* $\dim \mathfrak{z}(\mathfrak{h}) = 1$, *then there exists* $\tilde{\alpha}_{i_0} \in \tilde{\Pi}$ *such that* $\tilde{\Pi} - \Pi(\mathfrak{h}) = \{\tilde{\alpha}_{i_0}\}$, $m_{i_0} = 2$ *and*

$$\sigma = \mathrm{Ad}\left(\exp \varepsilon \frac{2\pi}{3} \sqrt{-1} \tilde{H}_{i_0}\right).$$

(2) *If* $\dim \mathfrak{z}(\mathfrak{h}) = 2$, *then there exist* $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2} \in \tilde{\Pi}$ *such that* $\tilde{\Pi} - \Pi(\mathfrak{h}) = \{\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}\}$, $m_{i_1} = m_{i_2} = 1$, *and*

$$\sigma = \mathrm{Ad}\left(\exp \varepsilon \frac{2\pi}{3} \sqrt{-1} (\tilde{H}_{i_1} + \tilde{H}_{i_2})\right).$$

Here $\tilde{\alpha}_i(\tilde{H}_j) = \delta_{ij}$, $\varepsilon = 1$ *or* -1 *and we denote the highest root of* $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ *with respect to* $\tilde{\Pi}$ *by* $\sum_i m_i \tilde{\alpha}_i$.

PROOF. (1) Set $\mathfrak{z}(\mathfrak{h}) = \boldsymbol{R}\sqrt{-1}Z$. Since $\mathfrak{h}$ is the centralizer of $\mathfrak{z}(\mathfrak{h})$ in $\mathfrak{g}$, it follows from (3.1) that

$$\tilde{\alpha}_i(Z) = 0, \quad \tilde{\alpha}_j(Z) \neq 0, \quad \tilde{\alpha}_i \in \Pi(\mathfrak{h}), \quad \tilde{\alpha}_j \in \tilde{\Pi} - \Pi(\mathfrak{h}). \tag{3.7}$$

Then there exists a unique $\tilde{\alpha}_{i_0} \in \tilde{\Pi}$ such that $\tilde{\Pi} - \Pi(\mathfrak{h}) = \{\tilde{\alpha}_{i_0}\}$. Indeed, if there are $\tilde{\alpha}_i$, $\tilde{\alpha}_j \in \tilde{\Pi} - \Pi(\mathfrak{h})$, $i \neq j$, then we obtain $\tilde{\alpha}_k(\tilde{H}_i) = \tilde{\alpha}_k(\tilde{H}_j) = 0$ for any $\tilde{\alpha}_k \in \Pi(\mathfrak{h})$, and hence, by (3.1), $\sqrt{-1}\tilde{H}_i$ and $\sqrt{-1}\tilde{H}_j$ are in $\mathfrak{z}(\mathfrak{h})$. This contradicts the assumption.

Now, we may put $Z = \tilde{H}_{i_0}$ and $\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)c\tilde{H}_{i_0}\})$ for some $c \in \boldsymbol{Z}$. Moreover, since $\sigma(E_{\tilde{\alpha}_{i_0}}) = \xi E_{\tilde{\alpha}_{i_0}}$ or $\xi^2 E_{\tilde{\alpha}_{i_0}}$ by Lemma 3.1, we can put $c = 1$ or -1. From the classification of root systems of simple Lie algebras, it is easy to see that if $m_{i_0} \geq 3$, then there exists $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ such that

$$\alpha = \sum_j n_j \tilde{\alpha}_j, \quad n_{i_0} = 3. \tag{3.8}$$

By (3.4) and the fact that $\sigma = \mathrm{Ad}(\exp\{\pm(2\pi\sqrt{-1}/3)\tilde{H}_{i_0}\})$, we obtain $\sigma(E_\alpha) = E_\alpha$, i.e., $E_\alpha \in \mathfrak{h}_c$. However, we have $\alpha(Z) \neq 0$ from (3.8), and so $E_\alpha \notin \mathfrak{h}_c$. Hence $m_{i_0} \leq 2$. In the case where $m_{i_0} = 1$, it is known that $(G/H, \langle\, ,\rangle, \sigma)$ is isometric to a Hermitian symmetric space (cf. [20, Theorem 3.3]). Consequently, we obtain $m_{i_0} = 2$.

(2) Next, we assume that $\dim \mathfrak{z}(\mathfrak{h}) = 2$. By a similar argument as above, we can see that there exist $\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2} \in \tilde{\Pi}$ such that

$$\tilde{\Pi} - \Pi(\mathfrak{h}) = \{\tilde{\alpha}_{i_1}, \tilde{\alpha}_{i_2}\}.$$

In this case, we have $\mathfrak{z}(\mathfrak{h}) = \boldsymbol{R}\sqrt{-1}\tilde{H}_{i_1} + \boldsymbol{R}\sqrt{-1}\tilde{H}_{i_2}$ and

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}(a\tilde{H}_{i_1} + b\tilde{H}_{i_2})\right), \quad a, b \in \boldsymbol{Z}.$$

Since $\sigma(E_{\tilde{\alpha}_{i_1}}) = \xi^k E_{\tilde{\alpha}_{i_1}}$ and $\sigma(E_{\tilde{\alpha}_{i_2}}) = \xi^l E_{\tilde{\alpha}_{i_2}}$, k, $l = 1$ or 2, it follows that a, $b \not\equiv 0$ (mod 3). Hence we may assume that $(a, b) = \pm(1, 1)$ or $\pm(1, -1)$. From the classification of root systems, we can take a root $\alpha = \sum_k n_k \tilde{\alpha}_k$ so that $n_{i_1} = n_{i_2} = 1$. If $(a, b) = \pm(1, -1)$, then $\sigma(E_\alpha) = E_\alpha$, and this contradicts the fact that $\alpha \in \Delta^+$. Therefore we have

$$\sigma = \mathrm{Ad}\left(\exp \pm \frac{2\pi}{3}\sqrt{-1}(\tilde{H}_{i_1} + \tilde{H}_{i_2}) \right). \tag{3.9}$$

Finally, we show that $m_{i_1} = m_{i_2} = 1$. Suppose that $m_{i_1} + m_{i_2} \geq 3$. Then, from the classification of root systems, it is easy to see that there exists $\alpha = \sum_j n_j \tilde{\alpha}_j \in \Delta^+$ such that $n_{i_1} + n_{i_2} = 3$. By (3.9), it is easy to see that $\sigma(E_\alpha) = E_\alpha$, which is a contradiction. Consequently, we have $m_{i_1} = m_{i_2} = 1$. □

Since $\mathfrak{g}^\sigma = \mathfrak{g}^{\sigma^{-1}} (= \mathfrak{h})$, we may assume that

$$\sigma = \begin{cases} \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)\tilde{H}_{i_0}\}) & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 1\,, \\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(\tilde{H}_{i_1} + \tilde{H}_{i_2})\}) & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 2\,. \end{cases} \tag{3.10}$$

Considering (3.6) and (3.10) together with Lemma 3.2, we obtain

$$\begin{aligned} \Delta^+{}_1 &= \begin{cases} \{\beta = \sum_j n_j \tilde{\alpha}_j;\ n_{i_0} = 1\} & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 1\,, \\ \{\beta = \sum_j n_j \tilde{\alpha}_j;\ (n_{i_1}, n_{i_2}) = (1,0) \text{ or } (0,1)\} & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 2\,, \end{cases} \\ \Delta^+{}_2 &= \begin{cases} \{\beta = \sum_j n_j \tilde{\alpha}_j;\ n_{i_0} = 2\} & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 1\,, \\ \{\beta = \sum_j n_j \tilde{\alpha}_j;\ n_{i_1} = n_{i_2} = 1\} & \text{if } \dim \mathfrak{z}(\mathfrak{h}) = 2\,. \end{cases} \end{aligned} \tag{3.11}$$

Now, we shall describe each G-invariant complex structure of G/H in terms of the canonical almost complex structure J of $(G/H, \langle\ ,\ \rangle, \sigma)$.

PROPOSITION 3.3. (1) *For any G-invariant complex structure I of G/H, define a mapping $\varphi : \mathfrak{g} \to \mathfrak{g}$ by $\varphi|_{\mathfrak{h}} := \mathrm{Id}_{\mathfrak{h}}$, $\varphi|_{\mathfrak{m}} := I \circ J$. Then φ is an involutive automorphism of $\mathfrak{g}$.*

(2) *Conversely, let φ be an involutive automorphism of G such that $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$, $\varphi \neq \mathrm{Id}$. Then $I := -\varphi|_{\mathfrak{m}} \circ J$ is a complex structure of $\mathfrak{m}$ and induces a G-invariant complex structure of G/H.*

PROOF. (1) Since I is a G-invariant complex structure, there exists a subset Δ^+ of Δ satisfying (3.3). As above, we take $\tilde{\Pi}$, $\Pi(\mathfrak{h})$ and $\Delta^+{}_i$, $i = 1, 2$, for Δ^+. Let β_i and γ_i, $i = 1, 2$, be elements in $\Delta^+{}_i$. Then it follows from Lemma 3.2 and (3.11) that

$$\beta_1 + \gamma_2\,, \quad \beta_2 + \gamma_2 \notin \Delta(\mathfrak{g}_c, \mathfrak{t}_c)\,. \tag{3.12}$$

Moreover, from (2.1) and (3.6) we have

$$J(E_{\pm\beta_1}) = \pm\sqrt{-1}E_{\pm\beta_1}\,, \quad J(E_{\pm\beta_2}) = \mp\sqrt{-1}E_{\pm\beta_2}\,. \tag{3.13}$$

Therefore, by (3.5) and (3.13), we get

$$\varphi(E_{\beta_1}) = -E_{\beta_1}\,, \quad \varphi(E_{\beta_2}) = E_{\beta_2}\,, \quad \varphi(E_{-\beta_1}) = -E_{-\beta_1}\,, \quad \varphi(E_{-\beta_2}) = E_{-\beta_2}\,. \tag{3.14}$$

In particular, we have $I \circ J = J \circ I$ and $\varphi^2 = \mathrm{Id}$.

Next, we shall show that $\varphi \in \mathrm{Aut}(\mathfrak{g})$. For $X, Y \in \mathfrak{h}$ we obtain

$$\varphi[X, Y] = [X, Y] = [\varphi(X), \varphi(Y)]\,,$$

because $[X, Y] \in \mathfrak{h}$ and $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$. Since $\beta_1 + \gamma_1 \in \Delta^{+}{}_2$ if $\beta_1 + \gamma_1 \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, it follows from (3.14) that

$$\varphi[E_{\beta_1}, E_{\gamma_1}] = [E_{\beta_1}, E_{\gamma_1}] = [\varphi(E_{\beta_1}), \varphi(E_{\gamma_1})] .$$

Similarly, we obtain

$$\begin{aligned} \varphi[E_{\beta_1}, E_{-\gamma_1}] &= [E_{\beta_1}, E_{-\gamma_1}] = [\varphi(E_{\beta_1}), \varphi(E_{-\gamma_1})] , \\ \varphi[E_{\beta_2}, E_{-\gamma_1}] &= -[E_{\beta_2}, E_{-\gamma_1}] = [\varphi(E_{\beta_2}), \varphi(E_{-\gamma_1})] , \\ \varphi[E_{\beta_1}, E_{-\gamma_2}] &= -[E_{\beta_1}, E_{-\gamma_2}] = [\varphi(E_{\beta_1}), \varphi(E_{-\gamma_2})] , \end{aligned}$$

and hence $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$ for any X, $Y \in \mathfrak{m}$. Furthermore, since I and J are G-invariant, it is obvious that

$$I \circ J \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ I \circ J , \quad X \in \mathfrak{h} ,$$

which implies that $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$ for $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$. We have thus proved (1).

(2) Let φ be an involutive automorphism of G such that $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$ and $\varphi \neq \mathrm{Id}$. Since $\mathfrak{h}$ contains a maximal abelian subalgebra of $\mathfrak{g}$, an involution φ is inner. Therefore, since J is G-invariant, φ is inner and H is connected, it follows that $\varphi \circ J = J \circ \varphi$ and $I := -\varphi|_{\mathfrak{m}} \circ J$ is a complex structure of $\mathfrak{m}$ such that I is $\mathrm{Ad}(H)$-invariant. Hence I induces a G-invariant almost complex structure of G/H, denoted by the same symbol I. Now, we see that the Nijenhuis tensor

$$S_I(x, y) = [Ix, Iy] - [x, y] - I[x, Iy] - I[Ix, y] ,$$

x, y being vector fields of G/H, is identically zero. To prove this, it only has to show that $S_I = 0$ at o, since S_I is a tensor and I is G-invariant. Let $\pi : G \to G/H$ be the canonical projection and W an open subset in $\mathfrak{m}$ such that $0 \in W$ and the mapping

$$\pi \circ \exp : W \to \pi(\exp W)$$

is diffeomorphic. For $X \in \mathfrak{m}$, we denote by X_* the vector field on $\pi(\exp W)$ defined by

$$(X_*)_{\pi(\exp x)} := (d \exp x)_*(X) .$$

According to Nomizu [13], the Levi-Civita connection ∇ of $(G/H, \langle\ ,\ \rangle, \sigma)$ at o is given by

$$(3.15) \qquad (\nabla_{X_*} Y_*)_o = \frac{1}{2}[X, Y]_{\mathfrak{m}} , \quad X, Y \in \mathfrak{m} ,$$

and therefore we get

$$(3.16) \qquad [X_*, Y_*]_o = [X, Y]_{\mathfrak{m}} .$$

By the definition of X_* and G-invariance of I and J, it follows that

$$(3.17) \qquad I(X_*) = (IX)_* , \quad J(X_*) = (JX)_* .$$

By making use of (3.16) and (3.17), for $X, Y \in \mathfrak{m}$ we have

$$
\begin{aligned}
S_I(X_*, Y_*)_o &= [I(X_*), I(Y_*)]_o - [X_*, Y_*]_o - I([X_*, I(Y_*)]_o) - I([I(X_*), Y_*]_o) \\
&= [(IX)_*, (IY)_*]_o - [X_*, Y_*]_o - I([X_*, (IY)_*]_o) - I([(IX)_*, Y_*]_o) \\
&= [IX, IY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - I([X, IY]_{\mathfrak{m}}) - I([IX, Y]_{\mathfrak{m}}) \\
&= \varphi([JX, JY]_{\mathfrak{m}}) - [X, Y]_{\mathfrak{m}} - \varphi(J[X, \varphi(JY)]_{\mathfrak{m}}) \\
&\quad - \varphi(J[\varphi(JX), Y]_{\mathfrak{m}})\,.
\end{aligned} \tag{3.18}
$$

Applying Lemma 2.1 and the commutativity of φ and J to (3.18), we obtain

$$
\begin{aligned}
S_I(X_*, Y_*)_o &= -\varphi([X, Y]_{\mathfrak{m}}) - [X, Y]_{\mathfrak{m}} - \varphi([X, \varphi(Y)]_{\mathfrak{m}}) - \varphi([\varphi(X), Y]_{\mathfrak{m}}) \\
&= -\varphi([X, Y]_{\mathfrak{m}}) - [X, Y]_{\mathfrak{m}} - [\varphi(X), Y]_{\mathfrak{m}} - [X, \varphi(Y)]_{\mathfrak{m}}\,.
\end{aligned} \tag{3.19}
$$

Let $\mathfrak{m}(\varphi, \pm 1)$ be the ± 1-eigenspaces of $\varphi|_{\mathfrak{m}}$. If $X, Y \in \mathfrak{m}(\varphi, -1)$, then $[X, Y]_{\mathfrak{m}} \in \mathfrak{m}(\varphi, 1)$ and it follows from (3.19) that

$$S_I(X_*, Y_*)_o = -\{[X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}}\} = 0\,.$$

Similarly, if $X \in \mathfrak{m}(\varphi, -1)$, $Y \in \mathfrak{m}(\varphi, 1)$, then

$$S_I(X_*, Y_*)_o = -\{-[X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}}\} = 0\,.$$

Finally, we suppose that $X, Y \in \mathfrak{m}(\varphi, 1)$. Then we have

$$S_I(X_*, Y_*)_o = -\{[X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}} + [X, Y]_{\mathfrak{m}}\} = -4[X, Y]_{\mathfrak{m}}\,.$$

To complete the proof of (2), we show that $[X, Y]_{\mathfrak{m}} = 0$ for any $X, Y \in \mathfrak{m}(\varphi, 1)$. Noting Lemma 2.2 and Remark 2.3, we may assume that there exist a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contained in $\mathfrak{h}$, the root system $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ and a fundamental root system $\Pi(\mathfrak{g}_c, \mathfrak{t}_c) = \{\alpha_1, \dots, \alpha_n\}$ such that

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}Z\right),$$

where $\sqrt{-1}Z \in \mathfrak{t}$ is one of the following forms:

(i) $\sqrt{-1}Z = \sqrt{-1}H_{j_1}$ $(m_{j_1} = 2)$, (ii) $\sqrt{-1}Z = \sqrt{-1}(H_{j_2} + H_{j_3})$ $(m_{j_2} = m_{j_3} = 1)$.

Here $H_i \in \mathfrak{t}_c$, $1 \le i \le n$, is given by (2.2) and $\delta = \sum_{p=1}^{n} m_p \alpha_p$ is the highest root of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ with respect to $\Pi(\mathfrak{g}_c, \mathfrak{t}_c)$. Since φ is an involution of inner type, we can put $\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}T)$ for some $\sqrt{-1}T \in \mathfrak{t}$. In the case (i), for $\alpha_j \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$, $j \neq j_1$, we have $E_{\alpha_j} \in \mathfrak{h}_c$ and so

$$\varphi(E_{\alpha_j}) = E_{\alpha_j}\,, \quad j \neq j_1\,.$$

Therefore, it follows from Lemma 3.1 that

$$T = aH_{j_1} + \sum_{j \neq j_1} a_j H_j\,, \quad a_j \equiv 0 \ (\mathrm{mod}\ 2)\,,$$

and hence $\varphi = \mathrm{Ad}(\exp a\pi\sqrt{-1}H_{j_1})$. Moreover, since φ is a nonidentical involution, it follows from Lemma 3.1 that

$$\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}H_{j_1})\,. \tag{3.20}$$

By Lemma 3.1 and (3.20), it is easy to see that $\mathfrak{m}(\varphi, 1)$ is spanned by the following vectors:

$$A_\alpha = E_\alpha - E_{-\alpha}\,, \quad B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})\,, \quad \alpha = \sum_{j=1}^{n} n_j\alpha_j \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)\,, \quad n_{j_1} = 2\,.$$

Now, set

$$\Delta(\varphi, 1) := \left\{\alpha = \sum_{j=1}^{n} n_j\alpha_j \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)\ ;\ n_{j_1} = \pm 2\right\}.$$

If $\alpha, \beta \in \Delta(\varphi, 1)$ and $\alpha + \beta \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, then $\alpha + \beta$ must be of the form $\alpha + \beta = \sum_j k_j\alpha_j$, $k_{j_1} = 0$, since $m_{j_1} = 2$. Hence we have $[E_\alpha, E_\beta] \in \mathfrak{h}_c$ and

$$[\mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, 1)]_{\mathfrak{m}} = \{0\}\,.$$

In the case (ii), by a similar argument as above, the involution φ has the following form:

$$\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}(aH_{j_2} + bH_{j_3}))\,, \quad a, b \in \boldsymbol{Z}\,, \tag{3.21}$$

and so we may assume that

$$(a, b) = (1, 0)\,, \quad (0, 1) \quad \text{or} \quad (1, 1)\,.$$

Then $\mathfrak{m}(\varphi, 1)$ is spanned by the following vectors:

- $A_\alpha,\ B_\alpha$, where $\alpha = \sum_{j=1}^{n} k_j\alpha_j \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, $k_{j_2} = 0$, $k_{j_3} = 1$, if $(a, b) = (1, 0)$,
- $A_\alpha,\ B_\alpha$, where $\alpha = \sum_{j=1}^{n} k_j\alpha_j \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, $k_{j_2} = 1$, $k_{j_3} = 0$, if $(a, b) = (0, 1)$,
- $A_\alpha,\ B_\alpha$, where $\alpha = \sum_{j=1}^{n} k_j\alpha_j \in \Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, $k_{j_2} = k_{j_3} = 1$, if $(a, b) = (1, 1)$.

Since $m_{j_2} = m_{j_3} = 1$, we can easily check that $[\mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, 1)]_{\mathfrak{m}} = \{0\}$ for each case, and this completes the proof of the proposition.

From the proof of Proposition 3.3, we have the following

COROLLARY 3.4. *Let φ be an involutive automorphism of G such that $\varphi \neq \mathrm{Id}$ and $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$, and let $\Pi(\mathfrak{g}_c, \mathfrak{t}_c) = \{\alpha_1, \ldots, \alpha_n\}$, H_i and m_i, $1 \le i \le n$, be as in Section* 2.1.

(1) *Suppose that* $\dim \mathfrak{z}(\mathfrak{h}) = 1$ *and* $\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_{j_1}\})$ *for some* $\alpha_{j_1} \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$ *with* $m_{j_1} = 2$. *Then*

$$\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}H_{j_1})\,.$$

(2) *Suppose that* $\dim \mathfrak{z}(\mathfrak{h}) = 2$ *and* $\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_{j_2} + H_{j_3})\})$ *for some* $\alpha_{j_2},\ \alpha_{j_3} \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$ *with* $m_{j_2} = m_{j_3} = 1$. *Then*

$$\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}H_{j_2})\,, \quad \mathrm{Ad}(\exp \pi\sqrt{-1}H_{j_3}) \quad \textit{or} \quad \mathrm{Ad}(\exp \pi\sqrt{-1}(H_{j_2} + H_{j_3}))\,.$$

REMARK 3.5. A Riemannian 3-symmetric space $(G/H, \langle\ ,\ \rangle, \sigma)$ is not Kählerian for any G-invariant complex structure I on G/H. Indeed, for $X, Y \in \mathfrak{m}$, it follows from (3.15) and Lemma 2.1 together with Proposition 3.3 that

$$\begin{aligned}(\nabla_{X_*} I)_o(Y_*) &= \frac{1}{2}\{[X, IY]_{\mathfrak{m}} - I([X, Y]_{\mathfrak{m}})\} \\ &= \frac{1}{2}\{[X, -\varphi(JY)]_{\mathfrak{m}} + \varphi J([X, Y]_{\mathfrak{m}})\} \\ &= \frac{1}{2}\varphi J([\varphi(X) + X, Y]_{\mathfrak{m}}),\end{aligned}$$

where φ is the involution of $\mathfrak{g}$ such that $I = -\varphi J$. If $\nabla I = 0$, then it follows that

$$[\mathfrak{m}(\varphi, 1), \mathfrak{m}]_{\mathfrak{m}} = \{0\}. \tag{3.22}$$

Therefore, since $\langle\ ,\ \rangle$ is biinvariant, we obtain

$$\langle[\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, -1)], \mathfrak{m}(\varphi, 1)\rangle = \langle[\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, 1)], \mathfrak{m}(\varphi, -1)\rangle = \{0\},$$

which implies that

$$[\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, -1)]_{\mathfrak{m}} = \{0\}. \tag{3.23}$$

Since $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant decomposition, we have

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{3.24}$$

Moreover, since $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$, we obtain

$$[\mathfrak{h}, \mathfrak{m}(\varphi, \pm 1)] \subset \mathfrak{m}(\varphi, \pm 1). \tag{3.25}$$

Put $\mathfrak{l} := \mathfrak{m}(\varphi, -1) + [\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, -1)]$. Then it follows from (3.22), (3.23) and (3.25) that $\mathfrak{l}$ is an ideal of $\mathfrak{g}$, which is a contradiction. Consequently, $(G/H, \langle\ ,\ \rangle, \sigma)$ is not Kählerian.

4. Totally real and totally geodesic submanifolds. In this section we shall investigate a relationship between half dimensional, totally real and totally geodesic submanifolds of $(G/H, \langle\ ,\ \rangle, \sigma)$ with respect to I and those with respect to J. Let ∇ be the Levi-Civita connection on $(G/H, \langle\ ,\ \rangle, \sigma)$ and I a G-invariant complex structure on G/H. For vector fields X, Y of G/H, we set

$$\tilde{\nabla}_X Y := \nabla_X Y + I((\nabla_X I)(Y)). \tag{4.1}$$

Then $\tilde{\nabla}$ is an affine connection on G/H, since I and $(\nabla I)(X, Y) := (\nabla_X I)(Y)$ are tensor fields on G/H. Let N be a half dimensional, totally real and totally geodesic submanifold of $(G/H, \langle\ ,\ \rangle, \sigma)$ with respect to I.

LEMMA 4.1. *N is also totally geodesic with respect to $\tilde{\nabla}$.*

PROOF. First, note that $(G/H, \langle\ ,\ \rangle, I)$ is an almost Hermitian manifold, since J and φ preserve $\langle\ ,\ \rangle$. Let X, Y be vector fields of G/H which are tangent to N. Because N is totally geodesic, a vector field $\nabla_X Y$ is tangent to N. Moreover, by the assumption on N and the fact that $(G/H, \langle\ ,\ \rangle, I)$ is an almost Hermitian manifold, it follows that $\nabla_X(IY)$ is perpendicular to N, and hence $I(\nabla_X(IY))$ is tangent to N. Since

$$\tilde{\nabla}_X Y = \nabla_X Y + I(\nabla_X(IY) - I(\nabla_X Y)) = 2\nabla_X Y + I(\nabla_X(IY)),$$

it follows that $\tilde{\nabla}_X Y$ is tangent to N. $\square$

Let $\tilde{T}$ be the torsion tensor of $\tilde{\nabla}$. For $X \in \mathfrak{m}$, let X_* be as in the proof of Proposition 3.3. Then

$$\tilde{T}(X_*, Y_*) = \tilde{\nabla}_{X_*} Y_* - \tilde{\nabla}_{Y_*} X_* - [X_*, Y_*] = [X_*, Y_*] + I(\nabla_{X_*}(IY)_* - \nabla_{Y_*}(IX)_*)\,,$$

and hence by (3.15) and (3.16) we have

$$\tilde{T}(X, Y) = [X, Y]_{\mathfrak{m}} + \frac{1}{2}(I[X, IY]_{\mathfrak{m}} - I[Y, IX]_{\mathfrak{m}})\,, \quad X, Y \in \mathfrak{m}\,. \tag{4.2}$$

Since I is G-invariant, we may assume that $o \in N$. Put $U = T_o N (\subset \mathfrak{m})$. Then it follows from Lemma 4.1 that $\tilde{T}(U, U) \subset U$. Therefore, by (4.2), we obtain

$$[X, Y]_{\mathfrak{m}} + \frac{1}{2}(I[X, IY]_{\mathfrak{m}} - I[Y, IX]_{\mathfrak{m}}) \in U\,, \quad X, Y \in U\,. \tag{4.3}$$

On the other hand, the integrability of I implies (see (3.18))

$$[IX, IY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - I[X, IY]_{\mathfrak{m}} - I[IX, Y]_{\mathfrak{m}} = 0\,. \tag{4.4}$$

Hence, by (4.3) and (4.4), we obtain

$$[X, Y]_{\mathfrak{m}} + [IX, IY]_{\mathfrak{m}} \in U, \quad X, Y \in U\,. \tag{4.5}$$

Let φ be an involutive automorphism of G such that $I = -\varphi|_{\mathfrak{m}} \circ J$ and let $\mathfrak{m}(\varphi, \pm 1)$ be the eigenspaces of $\varphi|_{\mathfrak{m}}$ with eigenvalues ± 1 as in Section 3. Then we have a decomposition $\mathfrak{m} = \mathfrak{m}(\varphi, 1) + \mathfrak{m}(\varphi, -1)$ of $\mathfrak{m}$. It follows from the proof of Proposition 3.3 that

$$[\mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, 1)]_{\mathfrak{m}} = \{0\}\,. \tag{4.6}$$

For $X \in \mathfrak{m}$, let X_+ (resp. X_-) be the $\mathfrak{m}(\varphi, 1)$-component (resp. $\mathfrak{m}(\varphi, -1)$-component) of X.

LEMMA 4.2. *For $X, Y \in U$, we have*

$$[X_+, Y_-]_{\mathfrak{m}} \in U.$$

PROOF. Let X and Y be in U. Since φ and J are commutative, it follows from Lemma 2.1 that

$$[IX, IY]_{\mathfrak{m}} = -[\varphi(X), \varphi(Y)]_{\mathfrak{m}}\,. \tag{4.7}$$

Using (4.5) and (4.7), we obtain

$$\begin{aligned} [X, Y]_{\mathfrak{m}} + [IX, IY]_{\mathfrak{m}} &= [X_+ + X_-, Y_+ + Y_-]_{\mathfrak{m}} - [X_+ - X_-, Y_+ - Y_-]_{\mathfrak{m}} \\ &= 2([X_+, Y_-]_{\mathfrak{m}} + [X_-, Y_+]_{\mathfrak{m}}) \in U\,. \end{aligned} \tag{4.8}$$

By (4.5) and the fact that $\langle\,,\,\rangle$ is induced from the Killing form of $\mathfrak{g}$, it follows that

$$0 = \langle [X, Y]_{\mathfrak{m}} + [IX, IY]_{\mathfrak{m}}, IX\rangle = \langle [IX, X]_{\mathfrak{m}}, Y\rangle\,,$$

and hence $[IX, X]_{\mathfrak{m}} \in IU$, i.e.,

$$I[IX, X]_{\mathfrak{m}} \in U\,, \quad X \in U\,. \tag{4.9}$$

Then, by Lemma 2.1 and (4.9), it follows that

$$\begin{aligned} I[IX, X]_{\mathfrak{m}} &= -\varphi J[-J\varphi(X), X]_{\mathfrak{m}} = \varphi[\varphi(X), X]_{\mathfrak{m}} \\ &= [X, \varphi(X)]_{\mathfrak{m}} = [X_+ + X_-, X_+ - X_-]_{\mathfrak{m}} \\ &= 2[X_-, X_+]_{\mathfrak{m}} \in U , \quad X \in U . \end{aligned} \tag{4.10}$$

Therefore, by replacing X in (4.10) with $X + Y$, we obtain

$$[X_+, X_-]_{\mathfrak{m}} + [Y_+, Y_-]_{\mathfrak{m}} + [X_+, Y_-]_{\mathfrak{m}} + [Y_+, X_-]_{\mathfrak{m}} \in U ,$$

and hence, by (4.10), we have

$$[X_+, Y_-]_{\mathfrak{m}} + [Y_+, X_-]_{\mathfrak{m}} \in U . \tag{4.11}$$

Finally, it follows from (4.8) and (4.11) that $[X_+, Y_-]_{\mathfrak{m}} \in U$ for $X, Y \in U$. □

Next, we consider $[X_-, Y_-]_{\mathfrak{m}}$. For $X, Y, Z \in U$, we have

$$\begin{aligned} \langle [X_-, Y_-]_{\mathfrak{m}}, IZ \rangle &= -\langle [X_-, IZ]_{\mathfrak{m}}, Y_- \rangle = \langle [X_-, JZ_+ - JZ_-]_{\mathfrak{m}}, Y_- \rangle \\ &= -\langle J[X_-, Z_+]_{\mathfrak{m}}, Y_- \rangle + \langle J[X_-, Z_-]_{\mathfrak{m}}, Y_- \rangle . \end{aligned}$$

Since $\varphi J = J\varphi$ and $[X_-, Z_-]_{\mathfrak{m}} \in \mathfrak{m}(\varphi, 1)$, we obtain $J(\mathfrak{m}(\varphi, \pm 1)) = \mathfrak{m}(\varphi, \pm 1)$ and $\langle J[X_-, Z_-]_{\mathfrak{m}}, Y_- \rangle = 0$. Moreover, since $[X_-, Z_+]_{\mathfrak{m}} \in \mathfrak{m}(\varphi, -1)$, we obtain

$$\begin{aligned} -\langle J[X_-, Z_+]_{\mathfrak{m}}, Y_- \rangle + \langle J[X_-, Z_-]_{\mathfrak{m}}, Y_- \rangle &= -\langle J[X_-, Z_+]_{\mathfrak{m}}, Y_- \rangle \\ &= -\langle J[X_-, Z_+]_{\mathfrak{m}}, Y \rangle = \langle [X_-, Z_+]_{\mathfrak{m}}, JY \rangle . \end{aligned}$$

Therefore, by Lemma 4.2, we get

$$\begin{aligned} \langle [X_-, Y_-]_{\mathfrak{m}}, IZ \rangle &= \langle [X_-, Z_+]_{\mathfrak{m}}, JY \rangle = \langle [X_-, Z_+]_{\mathfrak{m}}, -\varphi IY \rangle \\ &= \langle -\varphi([X_-, Z_+]_{\mathfrak{m}}), IY \rangle = \langle [X_-, Z_+]_{\mathfrak{m}}, IY \rangle = 0 , \end{aligned}$$

which implies that

$$[X_-, Y_-]_{\mathfrak{m}} \in U , \quad X, Y \in U . \tag{4.12}$$

From (4.6), (4.12) and Lemma 4.2, we obtain the following lemma.

LEMMA 4.3. $[U, U]_{\mathfrak{m}} \subset U$.

Put $\mathfrak{b} := U + [U, U]$ $(= U + [U, U]_{\mathfrak{h}})$. Since N is a totally geodesic submanifold of a naturally reductive homogeneous space $(G/H, \langle\,,\,\rangle)$, the subspace U is curvature invariant. Therefore, by Proposition 3.4 [Chapter X, 11], we have for $X, Y, Z \in U$

$$\frac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}} - [[X, Y]_{\mathfrak{h}}, Z] \in U . \tag{4.13}$$

It follows from Lemma 4.3 and (4.13) that $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$. In particular, N is an orbit of a Lie subgroup with Lie algebra $\mathfrak{b}$ of G.

Next, we consider $\varphi(U)$. By (4.6) we have

$$\varphi([X, Y]_{\mathfrak{m}}) = -[X_+, Y_-]_{\mathfrak{m}} - [X_-, Y_+]_{\mathfrak{m}} + [X_-, Y_-]_{\mathfrak{m}} ,$$

and hence by (4.12) and Lemma 4.2,

$$\varphi([U, U]_{\mathfrak{m}}) \subset U . \tag{4.14}$$

Since $\mathfrak{m} = U \oplus IU$, it follows that

$$[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}} = [U + IU, U + IU]_{\mathfrak{m}} = [U, U]_{\mathfrak{m}} + [IU, IU]_{\mathfrak{m}} + [IU, U]_{\mathfrak{m}} .$$

For $X, Y, Z \in U$, Lemma 4.3 implies that

$$\langle [IX, Y]_{\mathfrak{m}}, Z\rangle = \langle [Y, Z]_{\mathfrak{m}}, IX\rangle = 0$$

and hence

$$[IU, U]_{\mathfrak{m}} \subset IU = U^{\perp} . \tag{4.15}$$

Moreover, by (4.7) and (4.14), we obtain

$$[IX, IY]_{\mathfrak{m}} = -[\varphi(X), \varphi(Y)]_{\mathfrak{m}} = -\varphi([X, Y]_{\mathfrak{m}}) \in U , \quad X, Y \in U . \tag{4.16}$$

Therefore we have a decomposition

$$[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}} = ([U, U]_{\mathfrak{m}} + [IU, IU]_{\mathfrak{m}}) \oplus [IU, U]_{\mathfrak{m}} . \tag{4.17}$$

LEMMA 4.4. $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}$.

PROOF. Let V be the orthogonal complement of $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}$ in $\mathfrak{m}$. Since

$$\langle [\mathfrak{m}, V]_{\mathfrak{m}}, \mathfrak{m}\rangle = -\langle [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}, V\rangle = \{0\} ,$$

we have

$$V = \{X \in \mathfrak{m} \, ; [X, \mathfrak{m}]_{\mathfrak{m}} = \{0\}\} . \tag{4.18}$$

Then $V = (V \cap \mathfrak{m}(\varphi, 1)) \oplus (V \cap \mathfrak{m}(\varphi, -1))$, because $\varphi(V) = V$. By (4.18), a subspace $[V \cap \mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, -1)]$ is contained in $\mathfrak{h}$. However, $[V \cap \mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, -1)]$ is contained in the (-1)-eigenspace of φ, and so $[V \cap \mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, -1)] \subset \mathfrak{m}(\varphi, -1) \subset \mathfrak{m}$. Hence we have

$$[V \cap \mathfrak{m}(\varphi, 1), \mathfrak{m}(\varphi, -1)] = \{0\} , \tag{4.19}$$

and similarly

$$[V \cap \mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, 1)] = \{0\} . \tag{4.20}$$

Now, consider a canonical decomposition $\mathfrak{g} = (\mathfrak{h} + \mathfrak{m}(\varphi, 1)) \oplus \mathfrak{m}(\varphi, -1)$ corresponding to an orthogonal symmetric Lie algebra $(\mathfrak{g}, \varphi)$. Since $\mathfrak{g}$ is simple and $\mathfrak{m}(\varphi, -1) \oplus [\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, -1)]$ is an ideal of $\mathfrak{g}$, we have

$$[\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, -1)] = \mathfrak{h} + \mathfrak{m}(\varphi, 1) . \tag{4.21}$$

If $V \cap \mathfrak{m}(\varphi, 1) \neq \{0\}$, then it follows from (4.19) and (4.21) that $[V \cap \mathfrak{m}(\varphi, 1), \mathfrak{g}] = \{0\}$, which contradicts the fact that $\mathfrak{g}$ is simple. Therefore we may suppose that $V \subset \mathfrak{m}(\varphi, -1)$. By (3.24) and (4.18) together with the Jacobi identity, we obtain

$$[\mathfrak{h}, V] \subset V . \tag{4.22}$$

Noting (4.22) together with (4.20), we obtain $[\mathfrak{h}+\mathfrak{m}(\varphi, 1), V] \subset V$. Since $\mathfrak{g} = (\mathfrak{h}+\mathfrak{m}(\varphi, 1)) \oplus \mathfrak{m}(\varphi, -1)$ is a canonical decomposition and the isotropy representation of an irreducible symmetric space is irreducible, we have $V = \{0\}$ or $\mathfrak{m}(\varphi, -1)$. If $V = \mathfrak{m}(\varphi, -1)$, then it follows from (4.20) and (4.21) that $[\mathfrak{m}(\varphi, -1), \mathfrak{m}(\varphi, 1)] = \{0\}$ and $[\mathfrak{g}, \mathfrak{m}(\varphi, 1)] = \{0\}$, which means that $\mathfrak{m}(\varphi, 1) = \{0\}$. However, in this case, a pair $(\mathfrak{g}, \mathfrak{h})$ is symmetric corresponding to $(\mathfrak{g}, \varphi)$ because $\mathfrak{m} = \mathfrak{m}(\varphi, -1)$. Consequently, we have $V = \{0\}$, and hence $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}}$. □

Combining (4.15), (4.16), (4.17) and Lemma 4.3 together with Lemma 4.4, we obtain

$$U = [U, U]_{\mathfrak{m}} + [IU, IU]_{\mathfrak{m}}, \quad IU = [IU, U]_{\mathfrak{m}}.$$

Then it follows from Lemma 2.1, Proposition 3.3 and Lemma 4.3 that

$$\varphi([IU, IU]_{\mathfrak{m}}) = [I^2 JU, I^2 JU]_{\mathfrak{m}} = [JU, JU]_{\mathfrak{m}} = -[U, U]_{\mathfrak{m}} \subset U.$$

Therefore, by (4.14), we obtain

$$\varphi(U) = \varphi([U, U]_{\mathfrak{m}}) + \varphi([IU, IU]_{\mathfrak{m}}) \subset U. \tag{4.23}$$

PROPOSITION 4.5. *Let I be a G-invariant complex structure of $(G/H, \langle\,,\,\rangle, \sigma)$ and N a half dimensional, totally real and totally geodesic submanifold of $(G/H, \langle\,,\,\rangle, \sigma)$ with respect to I. Then N is also totally real with respect to J.*

PROOF. As before, we may assume that $o \in N$, and put $U = T_o N \subset \mathfrak{m}$. By the assumption, we have an orthogonal decomposition $\mathfrak{m} = U \oplus IU$. Then it follows from Proposition 3.3 and (4.23) that

$$JU = -I \circ \varphi(U) = IU.$$

Hence $\mathfrak{m} = U \oplus JU$ is an orthogonal decomposition of $\mathfrak{m}$. As stated under Lemma 4.3, N is an orbit of a Lie subgroup of G, and J is G-invariant. Hence we get an orthogonal decomposition

$$T_x(G/H) = T_x N \oplus J(T_x N), \quad x \in N,$$

and the proposition is proved.

REMARK 4.6. According to [17], each Riemannian 3-symmetric space $(G/H, \langle\,,\,\rangle, \sigma)$ and its half dimensional, totally real and totally geodesic submanifold of $(G/H, \langle\,,\,\rangle, \sigma)$ with respect to J are equivalent to one of those constructed from graded Lie algebras of the second kind as follows: Let $\mathfrak{g}^*$ be a noncompact simple Lie algebra over $\boldsymbol{R}$ and τ a Cartan involution of $\mathfrak{g}^*$. Then we have the Cartan decomposition $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}$ as in Section 2.2. Take a graded triple $(\mathfrak{g}^*, Z, \tau)$ associated with a gradation

$$\mathfrak{g}^* = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_{-1} + \mathfrak{g}^*_0 + \mathfrak{g}^*_1 + \mathfrak{g}^*_2, \quad \mathfrak{g}^*_1 \neq \{0\}, \quad \mathfrak{g}^*_2 \neq \{0\}$$

of the second kind on $\mathfrak{g}^*$. Define an inner automorphism σ of order 3 on the compact dual $\mathfrak{g} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ of $\mathfrak{g}^*$ by

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}Z\right),$$

and put $\mathfrak{h} = \mathfrak{g}^\sigma$, the set of fixed points of σ. Let G be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$. Let H and K be the analytic subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively. Then the dimension of the center $Z(H)$ of H is nonzero, and $N = K \cdot o$ is a half dimensional, totally real and totally geodesic submanifold of $(G/H, \langle\,,\rangle, \sigma)$ with respect to J. We call $((G/H, \langle\,,\rangle, \sigma), N)$ a TRG-*pair corresponding to a graded triple* $(\mathfrak{g}^*, Z, \tau)$.

5. Involutions and graded Lie algebras. In this section we shall investigate G-invariant complex structures and half dimensional, totally real and totally geodesic submanifolds of $(G/H, \langle\,,\rangle, \sigma)$ with respect to those complex structures by making use of some affine symmetric pairs associated with graded Lie algebras.

Let $(\mathfrak{l}, \theta)$ be a symmetric pair of type K_ε (see Oshima and Sekiguchi [14] for the definition of symmetric pairs of type K_ε). Then $(\mathfrak{l}, \theta)$ is either a *symmetric pair of type* K_εI or a *symmetric pair of type* K_εII, which were introduced by Kaneyuki [9]. More precisely, Kaneyuki [9] proved that for a symmetric pair $(\mathfrak{l}, \theta)$ of type K_ε there exists a graded Lie algebra:

$$\mathfrak{l} = \mathfrak{l}_{-\nu} + \cdots + \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1 + \cdots + \mathfrak{l}_\nu\,, \quad \mathfrak{l}_1 \neq \{0\}\,, \quad \mathfrak{l}_\nu \neq \{0\}$$

of the ν-th kind, $\nu = 1, 2$, with the characteristic element Z and a grade-reversing Cartan involution τ such that

$$\theta = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)\tau\,, \tag{5.1}$$

which commutes with τ. A symmetric pair $(\mathfrak{l}, \theta)$ is called a symmetric pair of type K_εI if $\nu = 1$. Furthermore, $(\mathfrak{l}, \theta)$ is called a symmetric pair of type K_εII if $\nu = 2$ and $(\mathfrak{l}, \theta)$ is not isomorphic to a symmetric pair of type K_εI (see [9] for details).

Let $((G/H, \langle\,,\rangle, \sigma), N)$ be a TRG-pair corresponding to a graded triple $(\mathfrak{g}^*, Z, \tau)$ of the second kind. Let $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition corresponding to τ, and $\mathfrak{g} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ the compact dual of $\mathfrak{g}^*$. Then by Remark 4.6 we obtain

$$\sigma = \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}Z\right), \quad \mathfrak{h} = \mathfrak{g}^\sigma\,, \quad N = K \cdot o\,.$$

Suppose that $(\mathfrak{g}^*, Z, \tau)$ is a graded triple associated with a gradation

$$\mathfrak{g}^* = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_{-1} + \mathfrak{g}^*_0 + \mathfrak{g}^*_1 + \mathfrak{g}^*_2\,, \quad \mathfrak{g}^*_1 \neq \{0\}\,, \quad \mathfrak{g}^*_2 \neq \{0\}$$

of the second kind on a simple Lie algebra $\mathfrak{g}^*$. Since τ is a grade-reversing Cartan involution, we have

$$\mathfrak{g}^*_p + \mathfrak{g}^*_{-p} = \mathfrak{k} \cap (\mathfrak{g}^*_p + \mathfrak{g}^*_{-p}) \oplus \mathfrak{p} \cap (\mathfrak{g}^*_p + \mathfrak{g}^*_{-p})\,, \quad p = 0, 1, 2\,. \tag{5.2}$$

Let θ be an involution on $\mathfrak{g}^*$ given by (5.1). It is easy to see that the set $\mathfrak{k}_\varepsilon$ of fixed points of θ is given by

$$\mathfrak{k}_\varepsilon = (\mathfrak{k} \cap (\mathfrak{g}^*_{-2} + \mathfrak{g}^*_0 + \mathfrak{g}^*_2)) \oplus (\mathfrak{p} \cap (\mathfrak{g}^*_{-1} + \mathfrak{g}^*_1))\,, \tag{5.3}$$

and so $\mathfrak{g}^*$ is decomposed into $\mathfrak{g}^* = \mathfrak{k}_\varepsilon + \mathfrak{p}_\varepsilon$. Here

$$\mathfrak{p}_\varepsilon = (\mathfrak{k} \cap (\mathfrak{g}^*_{-1} + \mathfrak{g}^*_1)) \oplus (\mathfrak{p} \cap (\mathfrak{g}^*_{-2} + \mathfrak{g}^*_0 + \mathfrak{g}^*_2))\,. \tag{5.4}$$

Let $\theta^a := \theta\tau$ be the associated involution of θ (cf. Hilgert and Ólafsson [7], and [15]). Then we have

$$\theta^a = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)\,, \tag{5.5}$$

and an orthogonal decomposition $\mathfrak{g}^* = \mathfrak{k}_\varepsilon{}^a \oplus \mathfrak{p}_\varepsilon{}^a$ of $\mathfrak{g}^*$, where

$$\mathfrak{k}_\varepsilon{}^a := (\mathfrak{g}^*)^{\theta^a} = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_0 + \mathfrak{g}^*_2\,, \quad \mathfrak{p}_\varepsilon{}^a := \mathfrak{g}^*_{-1} + \mathfrak{g}^*_1\,.$$

Set

$$\mathfrak{k}^{\mathrm{ad}} := (\mathfrak{k} \cap \mathfrak{k}_\varepsilon{}^a) \oplus \sqrt{-1}(\mathfrak{p} \cap \mathfrak{k}_\varepsilon{}^a)\,. \tag{5.6}$$

Since

$$\mathrm{Ad}(\exp t\sqrt{-1}Z)(X_p) = e^{t\sqrt{-1}p}X_p\,, \quad X_p \in \mathfrak{g}^*_p\,, \quad t \in \boldsymbol{R}\,, \tag{5.7}$$

it follows that

$$\mathfrak{g}^{\theta^a} = \mathfrak{k}^{\mathrm{ad}}\,, \tag{5.8}$$

so $(\mathfrak{g}, \mathfrak{k}^{\mathrm{ad}})$ is a symmetric pair of compact type.

Let $\mathfrak{t}^*$ be a Cartan subalgebra of $\mathfrak{g}^*$ containing the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Denote the complexifications of $\mathfrak{g}^*$ and $\mathfrak{t}^*$ by $\mathfrak{g}_c$ and $\mathfrak{t}_c$, respectively. Let $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$, Δ, $\Pi(\mathfrak{g}_c, \mathfrak{t}_c) = \{\alpha_1, \dots, \alpha_n\}$, $\Pi = \{\lambda_1, \dots, \lambda_l\}$ and H_i, $1 \le i \le n$, be as in Section 2. Suppose that $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ and Δ have compatible orderings.

LEMMA 5.1. *Let $\mathfrak{z}(\mathfrak{g}^*_0)$ be the center of $\mathfrak{g}^*_0$.*

(1) *If $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = \{0\}$, then $\dim \mathfrak{z}(\mathfrak{g}^*_0) = \dim \mathfrak{z}(\mathfrak{h}) = 1$.*

(2) *If $\dim \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = 1$, then $\dim \mathfrak{z}(\mathfrak{g}^*_0) = \dim \mathfrak{z}(\mathfrak{h}) = 2$, and $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \subset \mathfrak{k} \cap \mathfrak{k}_\varepsilon$ or $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \subset \mathfrak{p} \cap \mathfrak{p}_\varepsilon$.*

PROOF. First of all, we note that (5.7) implies that

$$\mathfrak{h} = \mathfrak{k} \cap \mathfrak{g}^*_0 \oplus \sqrt{-1}(\mathfrak{p} \cap \mathfrak{g}^*_0)\,, \tag{5.9}$$

$Z \in \mathfrak{z}(\mathfrak{g}^*_0)$ and $\sqrt{-1}Z \in \mathfrak{z}(\mathfrak{h})$. In particular, we have $\dim \mathfrak{z}(\mathfrak{g}^*_0) = \dim \mathfrak{z}(\mathfrak{h})$.

From Theorem 3.2, Theorem 3.3 and Theorem 4.3 of [8], we see that

$$\dim \mathfrak{z}(\mathfrak{g}^*_0) - \dim \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = 1\,,$$

and $\dim \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = 0$ or 1. Since $\mathfrak{k}_\varepsilon{}^a = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_0 + \mathfrak{g}^*_2$ and τ is a grade-reversing Cartan involution, it follows that $\mathfrak{k}_\varepsilon{}^a$ is τ-stable, which implies that $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a)$ is also τ-stable. Hence, we obtain

$$\mathfrak{k}_\varepsilon{}^a = \mathfrak{k} \cap \mathfrak{k}_\varepsilon{}^a + \mathfrak{p} \cap \mathfrak{k}_\varepsilon{}^a = \mathfrak{k} \cap \mathfrak{k}_\varepsilon + \mathfrak{p} \cap \mathfrak{p}_\varepsilon\,, \quad \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = \mathfrak{k} \cap \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) + \mathfrak{p} \cap \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a)\,.$$

Therefore, if $\dim \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = 1$, then it follows that $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \subset \mathfrak{k} \cap \mathfrak{k}_\varepsilon$ or $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \subset \mathfrak{p} \cap \mathfrak{p}_\varepsilon$. □

LEMMA 5.2. *Let φ be an involution on $\mathfrak{g}$ such that $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$ and $\varphi \neq \mathrm{Id}$.*

(1) *If $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = \{0\}$, then $\varphi = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)$.*

(2) *If* $\dim \mathfrak{z}({\mathfrak{k}_\varepsilon}^a) = 1$ *and* $\mathfrak{z}({\mathfrak{k}_\varepsilon}^a) \subset \mathfrak{k} \cap \mathfrak{k}_\varepsilon$, *then there exists* $\sqrt{-1}Z_0 \in \mathfrak{k} \cap \mathfrak{k}_\varepsilon$ *such that* $\mathfrak{z}({\mathfrak{k}_\varepsilon}^a) = \boldsymbol{R}\sqrt{-1}Z_0$ *and that the mapping* $\mathrm{ad}\sqrt{-1}Z_0 : {\mathfrak{p}_\varepsilon}^a \to {\mathfrak{p}_\varepsilon}^a$ *satisfies* $\big(\mathrm{ad}\sqrt{-1}Z_0\big)^2 = -\mathrm{Id}$. *In this case, the involution* φ *coincides with either*

$$\mathrm{Ad}(\exp \pi\sqrt{-1}Z) \quad \textit{or} \quad \mathrm{Ad}\bigg(\exp \frac{\pi}{2}\sqrt{-1}(Z \pm Z_0)\bigg).$$

(3) *If* $\dim \mathfrak{z}({\mathfrak{k}_\varepsilon}^a) = 1$ *and* $\mathfrak{z}({\mathfrak{k}_\varepsilon}^a) \subset \mathfrak{p} \cap \mathfrak{p}_\varepsilon$, *then there exists* $X^0 \in \mathfrak{p} \cap \mathfrak{p}_\varepsilon$ *such that* $\mathfrak{z}({\mathfrak{k}_\varepsilon}^a) = \boldsymbol{R}X^0$ *and that the eigenvalues of* $\mathrm{ad}X^0 : {\mathfrak{p}_\varepsilon}^a \to {\mathfrak{p}_\varepsilon}^a$ *are* ± 1. *In this case, the involution* φ *coincides with either*

$$\mathrm{Ad}(\exp \pi\sqrt{-1}Z) \quad \textit{or} \quad \mathrm{Ad}\bigg(\exp \frac{\pi}{2}\sqrt{-1}(Z \pm X^0)\bigg).$$

PROOF. Note that it follows from (2.5) that $\varphi_0 := \mathrm{Ad}(\exp \pi\sqrt{-1}Z)$ is an involution on $\mathfrak{g}$ satisfying $\varphi_0|_{\mathfrak{h}} = \mathrm{Id}$ and $\varphi_0 \neq \mathrm{Id}$.

(1) By Lemma 5.1, we have $\dim \mathfrak{z}(\mathfrak{g}_0^*) = \dim \mathfrak{z}(\mathfrak{h}) = 1$. Then (1) of the lemma follows from Corollary 3.4.

(2) By Lemma 5.1, we have $\dim \mathfrak{z}(\mathfrak{h}) = 2$. Let $\mathfrak{p}^{\mathrm{ad}}$ denote the orthogonal complement of $\mathfrak{k}^{\mathrm{ad}}$ in $\mathfrak{g}$. Then it follows from (5.6) that

$$\mathfrak{p}^{\mathrm{ad}} = (\mathfrak{k} \cap {\mathfrak{p}_\varepsilon}^a) \oplus \sqrt{-1}(\mathfrak{p} \cap {\mathfrak{p}_\varepsilon}^a) = (\mathfrak{k} \cap (\mathfrak{g}_{-1}^* + \mathfrak{g}_1^*)) \oplus \sqrt{-1}(\mathfrak{p} \cap (\mathfrak{g}_{-1}^* + \mathfrak{g}_1^*)).$$

In this case, $\mathfrak{k}^{\mathrm{ad}}$ has 1-dimensional center $\mathfrak{z}(\mathfrak{k}^{\mathrm{ad}})$, and hence $(\mathfrak{g}, \mathfrak{k}^{\mathrm{ad}})$ is a Hermitian symmetric pair of compact type. Moreover, there exists $\sqrt{-1}Z_0 \in \mathfrak{k} \cap \mathfrak{k}^{\mathrm{ad}}$ such that $\mathfrak{z}(\mathfrak{k}^{\mathrm{ad}}) = \boldsymbol{R}\sqrt{-1}Z_0$ and $\mathrm{ad}\sqrt{-1}Z_0 : \mathfrak{p}^{\mathrm{ad}} \to \mathfrak{p}^{\mathrm{ad}}$ satisfies $\big(\mathrm{ad}\sqrt{-1}Z_0\big)^2 = -\mathrm{Id}$ on $\mathfrak{p}^{\mathrm{ad}}$. Since

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \tag{5.10}$$

it follows from (5.6) that

$$\begin{aligned} &\mathrm{ad}\sqrt{-1}Z_0(\mathfrak{k} \cap (\mathfrak{g}_1^* + \mathfrak{g}_{-1}^*)) \subset \mathfrak{k} \cap (\mathfrak{g}_1^* + \mathfrak{g}_{-1}^*), \\ &\mathrm{ad}\sqrt{-1}Z_0(\mathfrak{p} \cap (\mathfrak{g}_1^* + \mathfrak{g}_{-1}^*)) \subset \mathfrak{p} \cap (\mathfrak{g}_1^* + \mathfrak{g}_{-1}^*). \end{aligned}$$

Therefore, $\mathrm{ad}\sqrt{-1}Z_0({\mathfrak{p}_\varepsilon}^a) \subset {\mathfrak{p}_\varepsilon}^a$ and $(\mathrm{ad}\sqrt{-1}Z_0)^2 = -\mathrm{Id}$ on ${\mathfrak{p}_\varepsilon}^a$, since $(\mathrm{ad}\sqrt{-1}Z_0)^2 = -\mathrm{Id}$ on $\mathfrak{p}^{\mathrm{ad}}$. Then $\mathrm{Ad}(\exp \pi\sqrt{-1}Z_0) = -\mathrm{Id}$ on $\mathfrak{p}^{\mathrm{ad}}$, and the set of fixed points of $\mathrm{Ad}(\exp \pi\sqrt{-1}Z_0)$ in $\mathfrak{g}$ coincides with $\mathfrak{k}^{\mathrm{ad}}$. Hence it follows from (5.5) and (5.8) that

$$\mathrm{Ad}(\exp \pi\sqrt{-1}Z) = \mathrm{Ad}(\exp \pi\sqrt{-1}Z_0). \tag{5.11}$$

From (5.11), the automorphisms

$$\nu_\pm := \mathrm{Ad}\bigg(\exp \frac{\pi}{2}\sqrt{-1}(Z \pm Z_0)\bigg) \tag{5.12}$$

of $\mathfrak{g}$ are involutive and $\nu_\pm|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$. Moreover, since $\sqrt{-1}Z_0$ is in $\mathfrak{z}(\mathfrak{k}^{\mathrm{ad}})$ and Z is the characteristic element, it follows that $\mathrm{Ad}(\exp\{(\pi\sqrt{-1}/2)Z\}) \neq \mathrm{Id}$ and $\mathrm{Ad}(\exp\{(\pi\sqrt{-1}/2)Z_0\}) = \mathrm{Id}$ on $\mathfrak{k}^{\mathrm{ad}}$, and hence $\nu_\pm \neq \mathrm{Id}$. By Corollary 3.4, we can see that $\mathrm{Ad}(\exp \pi\sqrt{-1}Z)$ and $\nu_\pm$ are the only involutions satisfying the assumption.

(3) The same argument as above implies that there exists $\sqrt{-1}X^0 \in \sqrt{-1}\mathfrak{p} \cap \mathfrak{k}^{\mathrm{ad}}$ such that $\mathfrak{z}(\mathfrak{k}^{\mathrm{ad}}) = \boldsymbol{R}\sqrt{-1}X^0$ and the mapping $\mathrm{ad}\sqrt{-1}X^0 : \mathfrak{p}^{\mathrm{ad}} \to \mathfrak{p}^{\mathrm{ad}}$ satisfies $(\mathrm{ad}\sqrt{-1}X^0)^2 = -\mathrm{Id}$. Therefore, it follows that $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = \boldsymbol{R}X^0$ and $(\mathrm{ad}X^0)^2 = \mathrm{Id}$ on $\mathfrak{p}_\varepsilon{}^a$, and thus the eigenvalues of $\mathrm{ad}X^0 : \mathfrak{p}_\varepsilon{}^a \to \mathfrak{p}_\varepsilon{}^a$ are ± 1. In this case, we have $\mathrm{Ad}(\exp \pi\sqrt{-1}X^0)|_{\mathfrak{p}^{\mathrm{ad}}} = -\mathrm{Id}_{\mathfrak{p}^{\mathrm{ad}}}$ and

$$\mathrm{Ad}(\exp \pi\sqrt{-1}Z) = \mathrm{Ad}(\exp \pi\sqrt{-1}X^0)\,.$$

Therefore the automorphisms

$$\varphi_\pm := \mathrm{Ad}\Big(\exp \frac{\pi}{2}\sqrt{-1}(Z \pm X^0)\Big) \tag{5.13}$$

satisfy $\varphi_\pm|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$ and $\varphi_\pm \neq \mathrm{Id}$. It follows from Corollary 3.4 that φ is one of

$$\mathrm{Ad}(\exp \pi\sqrt{-1}Z) \quad \text{and} \quad \varphi_\pm\,,$$

and so (3) is obtained. □

Let φ_0 be the involution of $\mathfrak{g}$ given by

$$\varphi_0 := \mathrm{Ad}(\exp \pi\sqrt{-1}Z)\,, \tag{5.14}$$

and let $\nu_\pm$ and $\varphi_\pm$ be as in (5.12) and (5.13), respectively.

LEMMA 5.3. *We have*

$$\varphi_0(\mathfrak{k}) = \mathfrak{k}\,, \quad \varphi_\pm(\mathfrak{k}) = \mathfrak{k}\,, \quad \nu_\pm(\mathfrak{k}) \neq \mathfrak{k}\,.$$

PROOF. Since $Z, X^0 \in \mathfrak{p}$, it follows that $\tau(Z) = -Z$ and $\tau(X^0) = -X^0$. Then

$$\tau\varphi_0\tau^{-1} = \mathrm{Ad}(\exp \pi\sqrt{-1}\tau(Z)) = \mathrm{Ad}(\exp -\pi\sqrt{-1}Z) = \varphi_0{}^{-1} = \varphi_0\,,$$

which implies that $\tau\varphi_0 = \varphi_0\tau$. Hence we have $\varphi_0(\mathfrak{k}) = \mathfrak{k}$, and similarly $\varphi_\pm(\mathfrak{k}) = \mathfrak{k}$. On the other hand, since $\sqrt{-1}Z_0 \in \mathfrak{k}$, it follows that

$$\tau\nu_+\tau^{-1} = \mathrm{Ad}\Big(\exp \frac{\pi}{2}\sqrt{-1}(-Z + Z_0)\Big) = \nu_-{}^{-1} = \nu_-\,,$$

and hence $\tau\nu_\pm \neq \nu_\pm\tau$, which implies that $\nu_\pm(\mathfrak{k}) \neq \mathfrak{k}$. □

As stated in Section 3, we assume that $(G/H, \langle\,,\rangle, \sigma)$ is a compact Riemannian 3-symmetric space of inner type such that G is simple, H is a centralizer of a toral subgroup of G and $\langle\,,\rangle$ is induced from a biinvariant metric on G. Let $((G/H, \langle\,,\rangle, \sigma), I, N)$ denote a triplet of a Riemannian 3-symmetric space $(G/H, \langle\,,\rangle, \sigma)$, a G-invariant complex structure I of $(G/H, \langle\,,\rangle, \sigma)$ and a half dimensional, totally real and totally geodesic submanifold N with respect to I. We call $((G/H, \langle\,,\rangle, \sigma), I, N)$ a *TRG-triple*. Moreover, we call two TRG-triples $((G/H, \langle\,,\rangle, \sigma), I, N)$ and $((\bar{G}/\bar{H}, \langle\,,\rangle, \bar{\sigma}), \bar{I}, \bar{N})$ are *equivalent* if there exists an isometry $f : (G/H, \langle\,,\rangle) \to (\bar{G}/\bar{H}, \langle\,,\rangle)$ such that $f_* \circ I = \bar{I} \circ f_*$ and $f(N) = \bar{N}$.

REMARK 5.4. Let $(G/H, \langle\,,\rangle, \sigma)$ be a Riemannian 3-symmetric space such that $\dim Z(H) = 2$. Then we may assume that $\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_j + H_k)\})$ for some

$\alpha_j, \alpha_k \in \Pi(\mathfrak{g}_c, \mathfrak{t}_c)$ with $m_j = m_k = 1$. From Corollary 3.4, each G-invariant complex structure I corresponds to one of the following involutions:

$$\mathrm{Ad}(\exp \pi\sqrt{-1}(H_j + H_k))\,, \quad \mathrm{Ad}(\exp \pi\sqrt{-1}H_j)\,, \quad \mathrm{Ad}(\exp \pi\sqrt{-1}H_k)\,.$$

Let I_j (resp. I_k) be the G-invariant complex structure corresponding to $\mathrm{Ad}(\exp \pi\sqrt{-1}H_j)$ (resp. $\mathrm{Ad}(\exp \pi\sqrt{-1}H_k)$). Let $((G/H, \langle\,,\rangle, \sigma), I_j, N)$ be a TRG-triple. Suppose that the TRG-pair $((G/H, \langle\,,\rangle, \sigma), N)$ corresponds to a graded triple $(\mathfrak{g}^*, Z = H_j + H_k, \tau)$ associated with a gradation of the second kind. Then it follows from Lemma 5.2 that I_j corresponds to one of $\varphi_\pm$ and $\nu_\pm$. Moreover, since $\nu_\pm|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$, $\nu_\pm(\mathfrak{m}) = \mathfrak{m}$, $N = K \cdot o$ and N is totally real with respect to J, it follows from Lemma 5.3 that

$$\nu_\pm \circ J(T_oN) = \nu_\pm \circ J(\mathfrak{k} \cap \mathfrak{m}) = \nu_\pm(\sqrt{-1}\mathfrak{p} \cap \mathfrak{m}) \neq \sqrt{-1}\mathfrak{p} \cap \mathfrak{m} = (T_oN)^\perp\,,$$

and hence I_j does not correspond to $\nu_\pm$.

Let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}^*$ and $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}^*$ corresponding to a Cartan involution τ, as before. Let $(\mathfrak{g}^*, \mu)$ be a symmetric pair such that $\mu\tau = \tau\mu$ and $\mathfrak{g}^* = \mathfrak{g}^{*\mu} + \mathfrak{q}$ the μ-invariant decomposition of $\mathfrak{g}^*$. A symmetric pair $(\mathfrak{g}^*, \mu)$ is called a *noncompactly causal* if there exists a $\tilde{G}^\mu$-invariant, regular, closed and convex cone C in $\mathfrak{q}$ such that $C^o \cap \mathfrak{p} \neq \emptyset$ (see [7]). Here, C^o denotes the interior of C.

The following proposition explains a relation between TRG-triples and symmetric pairs of type K_ε.

PROPOSITION 5.5. *Each TRG-triple is equivalent to* $((G/H, \langle\,,\rangle, \sigma), I, N)$ *such that* $((G/H, \langle\,,\rangle, \sigma), N)$ *is a TRG-pair corresponding to a graded triple* $(\mathfrak{g}^*, Z, \tau)$ *associated with a gradation*

$$\mathfrak{g}^* = \mathfrak{g}^*_{-2} + \mathfrak{g}^*_{-1} + \mathfrak{g}^*_0 + \mathfrak{g}^*_1 + \mathfrak{g}^*_2$$

of the second kind, which is defined by a partition $\{\Pi_0, \Pi_1\}$ *of* Π. *Moreover, the following holds.*

(1) *If* $(\mathfrak{g}^*, \theta)$ *is of type* K_εII, *then* I *corresponds to the involution* $\varphi_0 = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)$.

(2) *If* $(\mathfrak{g}^*, \theta)$ *is of type* K_εI, *then* I *corresponds to one of the following involutions*:

$$\varphi_0 = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)\,, \quad \varphi_\pm = \mathrm{Ad}\left(\exp \frac{\pi}{2}\sqrt{-1}(Z \pm X^0)\right),$$

where X^0 *denotes the cone-generating element* (*in the sense of* [7]) *in* $\mathfrak{p} \cap \mathfrak{p}_\varepsilon$.

PROOF. First of all, we determine the possibilities of $(\mathfrak{g}^*, Z, \tau)$. From Proposition 4.5 and Remark 4.6, for any $((G/H, \langle\,,\rangle, \sigma), I, N)$ we may assume that the TRG-pair $((G/H, \langle\,,\rangle, \sigma), N)$ corresponds to a graded triple $(\mathfrak{g}^*, Z, \tau)$ associated with a simple graded Lie algebra $\mathfrak{g}^* = \sum_{p=-2}^{2} \mathfrak{g}^*{}_p$. In particular,

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}Z\right), \quad N = K \cdot o\,. \tag{5.15}$$

Since σ is of order 3, it follows from [18, Proposition 5.1] that there exists an element w in the Weyl group of $(\mathfrak{g}^*, \mathfrak{a})$ such that

$$\mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}w(Z)\right) = \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}h\right),$$

where $h \in \mathfrak{a}$ has one of the following forms:

$$h_i \ (n_i = 1, 2, 3)\,, \quad h_j + h_k \ (n_j = n_k = 1)\,,$$

denoting by $\delta_{\mathfrak{a}} = \sum_p n_p\lambda_p$ the highest root of Δ defined in (2.8). However, if $h = h_i$, $n_i = 1$, then $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair, and if $h = h_i$, $n_i = 3$, then $\mathfrak{z}(\mathfrak{h}) = \{0\}$ (see [18, Lemma 5.3]), which contradict the assumption on $(G/H, \langle\,,\rangle, \sigma)$. Therefore, there exists an inner automorphism ν of $\mathfrak{k}$ such that

$$\nu\sigma\nu^{-1} = \mathrm{Ad}\left(\exp\frac{2\pi}{3}\sqrt{-1}h\right), \quad h = h_i \ (n_i = 2) \ \text{ or } \ h_j + h_k \ (n_j = n_k = 1)\,. \tag{5.16}$$

Since $\nu \in \mathrm{Int}(\mathfrak{k})$ and $N = K \cdot o$, we have $\nu(N) = N$. Hence $((G/H, \langle\,,\rangle, \sigma), I, N)$ is equivalent to $((G/\tilde{H}, \langle\,,\rangle, \nu\sigma\nu^{-1}), \tilde{I}, N)$, where $\tilde{H} := G^{\nu\sigma\nu^{-1}}$ and $\tilde{I} := \nu I\nu^{-1}$. Also, by (5.16) a TRG-pair $((G/\tilde{H}, \langle\,,\rangle, \nu\sigma\nu^{-1}), N)$ corresponds to a graded triple $(\mathfrak{g}^*, h, \tau)$, which is defined by a partition $\{\Pi_0, \Pi_1\}$ of Π such that

$$\Pi_1 = \{\lambda_i\} \quad \text{or} \quad \{\lambda_j, \lambda_k\}\,, \quad (n_i = 2,\ n_j = n_k = 1)\,. \tag{5.17}$$

Therefore, we may suppose that $((G/H, \langle\,,\rangle, \sigma), I, N)$ is a TRG-triple such that the TRG-pair $((G/H, \langle\,,\rangle, \sigma), N)$ corresponds to a graded triple $(\mathfrak{g}^*, Z, \tau)$ associated with $\mathfrak{g}^* = \sum_{p=-2}^{2} \mathfrak{g}^*{}_p$ such that

$$Z = h_i \ \text{ or } \ h_j + h_k\,, \quad n_i = 2\,, \ \ n_j = n_k = 1 \tag{5.18}$$

for some i, j, k. If $Z = h_i$ for some i, $1 \leq i \leq l$, with $n_i = 2$, then for $X \in \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \cap (\mathfrak{p} \cap \mathfrak{p}_\varepsilon)$, it is obvious that $X \in \mathfrak{a}$, since $\mathfrak{a} \subset \mathfrak{g}_0^* \subset \mathfrak{k}_\varepsilon{}^a$. Let X_λ be a vector in $\mathfrak{g}^{*\lambda}$, which denotes the root space for $\lambda \in \Delta$ of $\mathfrak{g}^*$. Then it follows from the definition (2.7) of $h_p \in \mathfrak{a}$ that $X_{\lambda_p} \in \mathfrak{g}_0^*$ for $p \neq i$ and so $\lambda_p(X) = 0$, $p \neq i$, which implies that $X = ch_i = cZ$ for some $c \in \boldsymbol{R}$. Moreover, since $[X, \mathfrak{g}_{\pm 2}^*] = \{0\}$, it follows that $c = 0$. Thus we obtain

$$\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \cap (\mathfrak{p} \cap \mathfrak{p}_\varepsilon) = \{0\}\,, \tag{5.19}$$

if $Z = h_i$ with $n_i = 2$.

Now, we prove (2) of the proposition. Suppose that $(\mathfrak{g}^*, \theta)$ is of type K_εI. Then there exists a graded triple $(\mathfrak{g}^*, Z', \tau)$ associated with a gradation of the first kind on $\mathfrak{g}^*$ such that

$$\theta = \mathrm{Ad}(\exp \pi\sqrt{-1}Z')\tau\,,$$

and hence $(\mathfrak{g}^*, \theta)$ is a noncompactly causal symmetric pair (see Theorem 3.1 of [9]. Also, see Proposition 3.2.1 and Theorem 3.2.4 of [7]). Therefore, since Z' is a cone-generating element, it follows from [7, Proposition 3.1.11] that

$$\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \cap (\mathfrak{p} \cap \mathfrak{p}_\varepsilon) = \boldsymbol{R}Z'(\neq \{0\})\,,$$

so by (5.19) there exist j, k, $1 \leq j, k \leq l$, with $n_j = n_k = 1$ such that $Z = h_j + h_k$. Conversely, we assume that $Z = h_j + h_k$ for some j, k with $n_j = n_k = 1$. Put $X^0 := h_j - h_k$. Since $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{p}_\varepsilon$ and

$$\mathfrak{g}^*_p = \sum \{\mathfrak{g}^{*\lambda};\ \lambda(h_j) + \lambda(h_k) = p\,, \quad \lambda \in \Delta\}, \quad p = \pm 1, \pm 2\,,$$
$$[h_j, \mathfrak{g}^*_0] = [h_k, \mathfrak{g}^*_0] = \{0\}\,,$$

it follows that

$$X^0 \in \mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) \cap (\mathfrak{p} \cap \mathfrak{p}_\varepsilon)\,, \quad \mathrm{Spec}(\mathrm{ad} X^0 : \mathfrak{p}_\varepsilon{}^a \to \mathfrak{p}_\varepsilon{}^a) = \{\pm 1\}\,. \tag{5.20}$$

Hence X^0 is a cone-generating element of $(\mathfrak{g}^*, \theta)$, and Lemma 5.1 implies that $\mathfrak{z}(\mathfrak{k}_\varepsilon{}^a) = \boldsymbol{R} X^0 \subset \mathfrak{p} \cap \mathfrak{p}_\varepsilon$ and

$$\mathrm{Ad}(\exp \pi \sqrt{-1} Z) = \mathrm{Ad}(\exp \pi \sqrt{-1} X^0)\,,$$

since $\varphi_\pm$ is an involution. Therefore, $\theta = \mathrm{Ad}(\exp \pi \sqrt{-1} X^0)\tau$ and thus $(\mathfrak{g}^*, \theta)$ is of type K_εI. In this case, it follows from Lemma 5.2 that I corresponds to one of automorphisms φ_0 and $\varphi_\pm$. Then by Lemma 5.3 and Proposition 3.3, we obtain $I(\mathfrak{k} \cap \mathfrak{m}) = -J(\mathfrak{k} \cap \mathfrak{m})$ and thus $N = K \cdot o$ is totally real with respect to each I.

Finally, we prove (1) of the proposition. Suppose that $(\mathfrak{g}^*, \theta)$ is of type K_εII. By the above argument it follows that $Z = h_i$ for some i with $n_i = 2$. In this case, it follows from (5.19), Lemma 5.2 and Remark 5.4 that I corresponds to φ_0. □

Finally, we prove the following theorem which classifies TRG-triples.

THEOREM 5.6. *Under the same notation as in Proposition* 5.5, *each TRG-triple is equivalent to one of* $((G/H, \langle\ ,\ \rangle, \sigma), I, N = K \cdot o)$ *listed in the following Table* 1 *and Table* 2.

PROOF. Let $((G/H, \langle\ ,\ \rangle, \sigma), I, N = K \cdot o)$ be a TRG-triple. As before, let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of G, H and K, respectively. Moreover, let K_0 be the Lie subgroup of K satisfying $N = K \cdot o = K/K_0$, and $\mathfrak{k}_0$ the Lie algebra of K_0. Then it follows from (5.9) that $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{h} = \mathfrak{k} \cap \mathfrak{g}^*_0$, which is a maximal compact Lie subalgebra of $\mathfrak{g}^*_0$.

First of all, suppose that $\mathfrak{g} = \mathfrak{e}_6$. Then the possibilities of $\mathfrak{g}^*$ are

$$\mathfrak{e}_{6(6)}\,, \quad \mathfrak{e}_{6(2)}\,, \quad \mathfrak{e}_{6(-14)} \quad \text{and} \quad \mathfrak{e}_{6(-26)}\,.$$

If $\mathfrak{g}^* = \mathfrak{e}_{6(2)}$, then $\mathfrak{k} = \mathfrak{su}(6) \oplus \mathfrak{su}(2)$, and the Satake diagram of $\mathfrak{g}^*$ and the Dynkin diagram of Π are given in Figure 1, where

$$\lambda_1 = \alpha_2|_\mathfrak{a}\,, \quad \lambda_2 = \alpha_4|_\mathfrak{a}\,, \quad \lambda_3 = \alpha_3|_\mathfrak{a} = \alpha_5|_\mathfrak{a}\,, \quad \lambda_4 = \alpha_1|_\mathfrak{a} = \alpha_6|_\mathfrak{a}\,. \tag{5.21}$$

It is known that the highest roots δ of $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ and $\delta_\mathfrak{a}$ of Δ are given respectively by

$$\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\,, \quad \delta_\mathfrak{a} = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4\,.$$

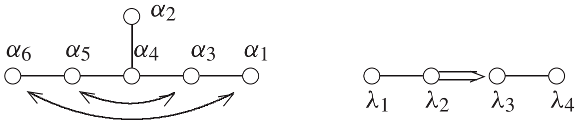

FIGURE 1. The Satake diagram of $\mathfrak{e}_{6(2)}$.

By the proof of Proposition 5.5, a symmetric pair $(\mathfrak{g}^*, \theta)$ is of type K_εII, and a partition $\{\Pi_0, \Pi_1\}$ of Π is given by $\Pi_1 = \{\lambda_1\}$ or $\{\lambda_4\}$. Then it follows from [8, Theorem 3.3] that

$$\mathfrak{g}_0^* \cong \begin{cases} \mathfrak{su}(3,3) \oplus \boldsymbol{R} & \text{if } \Pi_1 = \{\lambda_1\}\,, \\ \mathfrak{so}(5,3) \oplus \boldsymbol{R} \oplus \sqrt{-1}\boldsymbol{R} & \text{if } \Pi_1 = \{\lambda_4\}\,, \end{cases}$$

which implies that

$$\mathfrak{k}_0 \cong \begin{cases} \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(3)) & \text{if } \Pi_1 = \{\lambda_1\}\,, \\ \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \sqrt{-1}\boldsymbol{R} & \text{if } \Pi_1 = \{\lambda_4\}\,. \end{cases}$$

Moreover, it follows from (5.21) and Lemma 2.6 that $Z = h_1 = H_2$ or $Z = h_4 = H_1 + H_6$, and [6, Theorem 5.15] implies that $\mathfrak{h}$ has the following form:

$$\mathfrak{h} \cong \begin{cases} \mathfrak{a}_5 \oplus \boldsymbol{R} & \text{if } \sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_2\})\,, \\ \mathfrak{d}_4 \oplus \boldsymbol{R}^2 & \text{if } \sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1 + H_6)\})\,. \end{cases}$$

Consequently, it follows from [5, Table V] and Proposition 5.5 that each TRG-triple is equivalent to one of the following:

$$((\{E_6/\boldsymbol{Z}_3\}/\{[(SU(6)/\boldsymbol{Z}_3) \times T^1]/\boldsymbol{Z}_2\},\ \langle\,,\rangle,\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_2\})),\ I_0,\ K\cdot o)\,,$$
$$((\{E_6/\boldsymbol{Z}_3\}/\{(SO(8) \times SO(2) \times SO(2))/\boldsymbol{Z}_2\},\ \langle\,,\rangle,$$
$$\mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1 + H_6)\})),\ I_0,\ K\cdot o)\,,$$

where I_0 denotes the G-invariant complex structure on G/H corresponding to $\varphi_0 = \mathrm{Ad}(\exp \pi\sqrt{-1}Z)$, and K is the analytic subgroup of G with Lie algebra $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$. In particular, it follows that

$$(\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \mathfrak{k}_0) \cong \begin{cases} (\mathfrak{e}_6,\ \mathfrak{su}(6) \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{su}(6) \oplus \mathfrak{su}(2),\ \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(3))) & \text{if } \Pi_1 = \{\lambda_1\}\,, \\ (\mathfrak{e}_6,\ \mathfrak{so}(8) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2),\ \mathfrak{su}(6) \oplus \mathfrak{su}(2)\,, & \\ \qquad \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \sqrt{-1}\boldsymbol{R}) & \text{if } \Pi_1 = \{\lambda_4\}\,. \end{cases}$$

If $\mathfrak{g}^* = \mathfrak{e}_{6(-14)}$, then $\mathfrak{k} = \mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R}$, and the Satake diagram of $\mathfrak{g}^*$ and the Dynkin diagram of Π are given in Figure 2, where

$$\lambda_1 = \alpha_2|_{\mathfrak{a}}\,, \quad \lambda_2 = \alpha_1|_{\mathfrak{a}} = \alpha_6|_{\mathfrak{a}}\,.$$

The highest root $\delta_{\mathfrak{a}}$ of Δ is $\delta_{\mathfrak{a}} = 2\lambda_1 + 2\lambda_2$. Then a symmetric pair $(\mathfrak{g}^*, \theta)$ is of type K_εII, and a partition $\{\Pi_0, \Pi_1\}$ of Π is given by $\Pi_1 = \{\lambda_1\}$ or $\{\lambda_2\}$. As above, it follows from [8, Theorem 3.3] that

$$\mathfrak{k}_0 \cong \begin{cases} \mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1)) & \text{if } \Pi_1 = \{\lambda_1\}\,, \\ \mathfrak{so}(7) \oplus \sqrt{-1}\boldsymbol{R} & \text{if } \Pi_1 = \{\lambda_2\}\,. \end{cases}$$

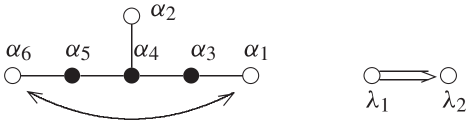

FIGURE 2. The Satake diagram of $\mathfrak{e}_{6(-14)}$.

Moreover, it follows from Lemma 2.6 that $Z = h_1 = H_2$ or $Z = h_2 = H_1 + H_6$, and as in the above case we have

$$\mathfrak{h} \cong \begin{cases} \mathfrak{a}_5 \oplus \boldsymbol{R} & \text{if } \sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_2\}) \,, \\ \mathfrak{d}_4 \oplus \boldsymbol{R}^2 & \text{if } \sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1 + H_6)\}) \,. \end{cases}$$

Hence, by [5, Table V] and Proposition 5.5, each TRG-triple is equivalent to one of the following:

$$((\{E_6/\boldsymbol{Z}_3\}/\{[(SU(6)/\boldsymbol{Z}_3) \times T^1]/\boldsymbol{Z}_2\},\ \langle\,,\rangle,\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_2\})),\ I_0,\ K \cdot o)\,,$$
$$((\{E_6/\boldsymbol{Z}_3\}/\{(SO(8) \times SO(2) \times SO(2))/\boldsymbol{Z}_2\},\ \langle\,,\rangle,$$
$$\mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1 + H_6)\})),\ I_0,\ K \cdot o)\,,$$

where K is the analytic subgroup of G with Lie algebra $\mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R}$. In particular, we obtain

$$(\mathfrak{g},\ \mathfrak{h},\ \mathfrak{k},\ \mathfrak{k}_0) \cong \begin{cases} (\mathfrak{e}_6,\ \mathfrak{su}(6) \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R}, \mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1))) \\ \qquad\qquad \text{if } \Pi_1 = \{\lambda_1\}\,, \\ (\mathfrak{e}_6,\ \mathfrak{so}(8) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2),\ \mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R}, \mathfrak{so}(7) \oplus \sqrt{-1}\boldsymbol{R}) \\ \qquad\qquad \text{if } \Pi_1 = \{\lambda_2\}\,. \end{cases}$$

If $\mathfrak{g}^* = \mathfrak{e}_{6(-26)}$, then $\mathfrak{k} = \mathfrak{f}_4$, and the Satake diagram of $\mathfrak{g}^*$ and the Dynkin diagram of Π are given in Figure 3, where

$$\lambda_1 = \alpha_1|_{\mathfrak{a}}\,, \quad \lambda_2 = \alpha_6|_{\mathfrak{a}}\,.$$

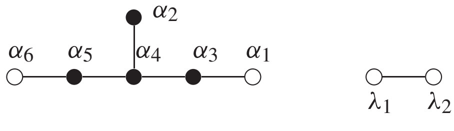

FIGURE 3. The Satake diagram of $\mathfrak{e}_{6(-26)}$.

The highest root $\delta_{\mathfrak{a}}$ of Δ is $\delta_{\mathfrak{a}} = \lambda_1 + \lambda_2$. Therefore a symmetric pair $(\mathfrak{g}^*, \theta)$ is of type K_εI, and a partition $\{\Pi_0, \Pi_1\}$ of Π is given by $\Pi_1 = \{\lambda_1, \lambda_2\}$. Then, it follows from [8, Theorem 3.3] that $\mathfrak{k}_0 \cong \mathfrak{so}(8)$. Moreover, it follows from Lemma 2.6 that $Z = H_1 + H_6$ and thus we have

$$\sigma = \mathrm{Ad}\left(\exp \frac{2\pi}{3}\sqrt{-1}(H_1 + H_6)\right), \quad \mathfrak{h} \cong \mathfrak{d}_4 \oplus \boldsymbol{R}^2\,.$$

Consequently, by [5, Table V] and Proposition 5.5, each TRG-triple is equivalent to

$$((\{E_6/\boldsymbol{Z}_3\}/\{(SO(8)\times SO(2)\times SO(2))/\boldsymbol{Z}_2\},\ \langle\,,\rangle,\\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1+H_6)\})),\ I,\ K\cdot o)\,,$$

where K is the analytic subgroup of G with Lie algebra $\mathfrak{f}_4$, and I corresponds to one of the following involutions:

$$\varphi_0=\mathrm{Ad}(\exp\pi\sqrt{-1}(H_1+H_6))\,,\quad \varphi_+=\mathrm{Ad}(\exp\pi\sqrt{-1}H_1)\,,\\ \varphi_-=\mathrm{Ad}(\exp\pi\sqrt{-1}H_6)\,.$$

In particular, we have

$$(\mathfrak{g},\ \mathfrak{h},\ \mathfrak{k},\ \mathfrak{k}_0)\cong(\mathfrak{e}_6,\ \mathfrak{so}(8)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2),\ \mathfrak{f}_4,\ \mathfrak{so}(8))\,.$$

Finally, we suppose that $\mathfrak{g}^*=\mathfrak{e}_{6(6)}$. Then $\mathfrak{k}=\mathfrak{sp}(4)$ and the Dynkin diagram of Π coincides with the Satake diagram of $\mathfrak{g}^*$. Thus $\alpha_i=\lambda_i$, $i=1,\dots,6$, and

$$\delta=\delta_{\mathfrak{a}}=\lambda_1+2\lambda_2+2\lambda_3+3\lambda_4+2\lambda_5+\lambda_6\,.$$

By virtue of the proof of Proposition 5.5, if $(\mathfrak{g}^*,\theta)$ is of type K_εII, then $\Pi_1=\{\lambda_i\}$, $i=2,3$ or 5. Similarly, if $(\mathfrak{g}^*,\theta)$ is of type K_εI, then $\Pi_1=\{\lambda_1,\lambda_6\}$. Noting that $\mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_3\})$ and $\mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_5\})$ are conjugate under $\mathrm{Aut}(\mathfrak{e}_6)$, we see from a similar argument as above that

$$\mathfrak{k}_0\cong\begin{cases}\mathfrak{so}(6) & \text{if } \Pi_1=\{\lambda_2\}\,,\\ \mathfrak{so}(5)\oplus\mathfrak{so}(2) & \text{if } \Pi_1=\{\lambda_3\}\,,\\ \mathfrak{so}(4)\oplus\mathfrak{so}(4) & \text{if } \Pi_1=\{\lambda_1,\lambda_6\}\,,\end{cases}$$

and each TRG-triple is equivalent to one of

$$((\{E_6/\boldsymbol{Z}_3\}/\{[(SU(6)/\boldsymbol{Z}_3)\times T^1]/\boldsymbol{Z}_2\},\ \langle\,,\rangle,\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_2\})),\ I_0,\ K\cdot o)\,,$$

$$((\{E_6/\boldsymbol{Z}_3\}/\{[S(U(5)\times U(1))\times SU(2)]/\boldsymbol{Z}_2\},\ \langle\,,\rangle,\\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)H_3\}))\,,I_0,\ K\cdot o)\,,$$

$$((\{E_6/\boldsymbol{Z}_3\}/\{(SO(8)\times SO(2)\times SO(2))/\boldsymbol{Z}_2\},\ \langle\,,\rangle\,,\\ \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)(H_1+H_6)\})),\ I,\ K\cdot o)\,,$$

where K is the analytic subgroup of G with Lie algebra $\mathfrak{sp}(4)$, and I corresponds to one of the following involutions:

$$\varphi_0=\mathrm{Ad}(\exp\pi\sqrt{-1}(H_1+H_6))\,,\quad \varphi_+=\mathrm{Ad}(\exp\pi\sqrt{-1}H_1)\,,\\ \varphi_-=\mathrm{Ad}(\exp\pi\sqrt{-1}H_6)\,.$$

In particular, we obtain

$$(\mathfrak{g},\mathfrak{h},\mathfrak{k},\mathfrak{k}_0)\cong\begin{cases}(\mathfrak{e}_6,\ \mathfrak{su}(6)\oplus\sqrt{-1}\boldsymbol{R},\ \mathfrak{sp}(4),\ \mathfrak{so}(6)) & \text{if } \Pi_1=\{\lambda_2\}\,,\\ (\mathfrak{e}_6,\ \mathfrak{s}(\mathfrak{u}(5)\oplus\mathfrak{u}(1))\oplus\mathfrak{su}(2),\ \mathfrak{sp}(4),\ \mathfrak{so}(5)\oplus\mathfrak{so}(2)) & \text{if } \Pi_1=\{\lambda_3\}\,,\\ (\mathfrak{e}_6,\ \mathfrak{so}(8)\oplus\mathfrak{so}(2)\oplus\mathfrak{so}(2),\ \mathfrak{sp}(4),\ \mathfrak{so}(4)\oplus\mathfrak{so}(4)) & \text{if } \Pi_1=\{\lambda_1,\lambda_6\}\,.\end{cases}$$

For other cases, we can classify TRG-triples analogously. □

REMARK 5.7. For the case where $(\mathfrak{g}^*, \theta)$ is of type K_εI, we can obtain a graded triple $(\mathfrak{g}^*, Z, \tau)$ associated with a gradation of the first kind defined by a partition of Π such that $(\mathfrak{g}^*, \theta)$ corresponds to $(\mathfrak{g}^*, Z, \tau)$ by the following way: Suppose that $((G/H, \langle\,,\rangle, \sigma), I, N)$ is a TRG-triple such that $(\mathfrak{g}^*, \theta)$ is of type K_εI. Then as in the proof of Proposition 5.5 there exists a partition $\{\Pi_0, \Pi_1\}$ of Π with $\Pi_1 = \{\lambda_j, \lambda_k\}$, $n_j = n_k = 1$ such that $(\mathfrak{g}^*, \theta)$ corresponds to a graded Lie algebra defined by $\{\Pi_0, \Pi_1\}$. Put $\lambda_0 := -\delta_{\mathfrak{a}}$ and let t_p, $0 \le p \le l,\ p \ne k$, be an element of $\mathfrak{a}$ given by $\lambda_q(t_p) = \delta_{pq},\ 0 \le q \le l,\ q \ne k$. Then, since $n_j = n_k = 1$, it is easy to see that

$$t_0 = -h_k\,, \quad t_p = h_p - n_p h_k\,, \quad p \ne 0\,. \tag{5.22}$$

Set $\hat{\Pi} := \{\lambda_p\ ;\ 0 \le p \le l,\ p \ne k\}$, which is a fundamental root system of $\mathfrak{g}^*$ with respect to $\mathfrak{a}$. Moreover, the Dynkin diagram of $\hat{\Pi}$ is the subdiagram of the extended Dynkin diagram of Π consisting of $\hat{\Pi}$. Then by (5.22) we have

$$\mathrm{Ad}(\exp \pi\sqrt{-1}t_j) = \mathrm{Ad}(\exp \pi\sqrt{-1}(h_j - h_k)) = \mathrm{Ad}(\exp \pi\sqrt{-1}(h_j + h_k))\,,$$

and hence a symmetric pair $(\mathfrak{g}^*, \theta)$ corresponds to a gradation of the first kind defined by a partition $\{\hat{\Pi}_0, \hat{\Pi}_1\}$ of $\hat{\Pi}$ given by $\hat{\Pi}_1 = \{\lambda_j\}$.

For example, if $\mathfrak{g}^* = \mathfrak{e}_{6(6)}$ and $\Pi_1 = \{\lambda_1, \lambda_6\}$, then the Dynkin diagram of $\hat{\Pi} := \{\lambda_0, \lambda_1, \dots, \lambda_5\}$ is given in Figure 4.

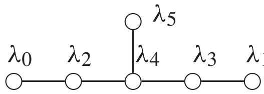

FIGURE 4. The Dynkin diagram of $\mathfrak{e}_{6(6)}$.

Therefore, the gradation defined by a partition $\{\hat{\Pi}_0, \hat{\Pi}_1 = \{\lambda_1\}\}$ of $\hat{\Pi}$ is isomorphic to that defined by a partition $\{\Pi_0, \Pi_1 = \{\lambda_1\}\}$ of Π, and it follows from [9] that $(\mathfrak{g}^*, \mathfrak{k}_\varepsilon) \cong (\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))$ with the numbering I 15.

REMARK 5.8. In Tables 1 and 2, we adopt the numbering of fundamental roots in Bourbaki [3]. Moreover, the numbering of symmetric pairs $(\mathfrak{g}^*, \theta)$ is due to Kaneyuki [9].

TABLE 1. TRG-triples with $\dim Z(H) = 1$.

TRG-triple $((G/H, \langle\,,\,\rangle, \sigma),\ I,\ N = K \cdot o = K/K_0)$.
$\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{k}_0$ are Lie algebras of G, H, K and K_0, respectively.
$\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)Z\})$, $\ I = -\varphi_0 \circ J$.
$(\mathfrak{g}^*, \theta)$ is of type K_εII corresponding to a GLA defined by a partition $\{\Pi_0, \Pi_1\}$ of Π.

$\mathfrak{g}$	$\mathfrak{h}$	Z	$(\mathfrak{g}^*,\ \mathfrak{k},\ \mathfrak{k}_0)$	$(\Pi,\ \Pi_1)$	$(\mathfrak{g}^*, \theta)$
$\mathfrak{so}(2n+1)$ $(n \geq 2)$	$\mathfrak{u}(i) \oplus \mathfrak{so}(2n-2i+1)$ $(2 \leq i \leq n)$	H_i	$(\mathfrak{so}(l, m), \mathfrak{so}(l) \oplus \mathfrak{so}(m),\ \mathfrak{so}(i) \oplus \mathfrak{so}(l-i) \oplus \mathfrak{so}(m-i))$ $(i \leq l \leq n,\ m = 2n+1-l)$	$(\mathfrak{b}_l, \{\lambda_i\})$	II 12
$\mathfrak{sp}(n)$ $(n \geq 3)$	$\mathfrak{u}(i) \oplus \mathfrak{sp}(n-i)$ $(1 \leq i \leq n-1)$	H_i	$(\mathfrak{sp}(n, \boldsymbol{R}),\ \mathfrak{u}(n),\ \mathfrak{so}(i) \oplus \mathfrak{u}(n-i))$	$(\mathfrak{c}_n, \{\lambda_i\})$	II 13
	$\mathfrak{u}(2i) \oplus \mathfrak{sp}(n-2i)$ $(2 \leq 2i \leq n-1)$	H_{2i}	$(\mathfrak{sp}(l, n-l),\ \mathfrak{sp}(l) \oplus \mathfrak{sp}(n-l),\ \mathfrak{sp}(i) \oplus \mathfrak{sp}(l-i) \oplus \mathfrak{sp}(n-l-i))$ $(2i \leq 2l \leq n-1)$	$(\mathfrak{bc}_l, \{\lambda_i\})$	II 14
			$(\mathfrak{sp}(l, l),\ \mathfrak{sp}(l) \oplus \mathfrak{sp}(l),\ \mathfrak{sp}(i) \oplus \mathfrak{sp}(l-i) \oplus \mathfrak{sp}(l-i))$ $(n = 2l)$	$(\mathfrak{c}_l, \{\lambda_i\})$	II 14
$\mathfrak{so}(2n)$ $(n \geq 4)$	$\mathfrak{u}(i) \oplus \mathfrak{so}(2n-2i)$ $(2 \leq i \leq n-2)$	H_i	$(\mathfrak{so}(n, n), \mathfrak{so}(n) \oplus \mathfrak{so}(n),\ \mathfrak{so}(i) \oplus \mathfrak{so}(n-i) \oplus \mathfrak{so}(n-i))$	$(\mathfrak{d}_n, \{\lambda_i\})$	II 12
			$(\mathfrak{so}(2n-l, l),\ \mathfrak{so}(2n-l) \oplus \mathfrak{so}(l),\ \mathfrak{so}(i) \oplus \mathfrak{so}(2n-l-i) \oplus \mathfrak{so}(l-i))$ $(i \leq l \leq n-1)$	$(\mathfrak{b}_l, \{\lambda_i\})$	II 12
	$\mathfrak{u}(2i) \oplus \mathfrak{so}(2n-4i)$ $\left(1 \leq i < \left[\frac{n}{2}\right]\right)$	H_{2i}	$(\mathfrak{so}^*(4l),\ \mathfrak{u}(2l),\ \mathfrak{sp}(i) \oplus \mathfrak{u}(n-2i))$ $(n = 2l)$	$(\mathfrak{c}_l, \{\lambda_i\})$	II 15
			$(\mathfrak{so}^*(4l+2), \mathfrak{u}(2l+1),\ \mathfrak{sp}(i) \oplus \mathfrak{u}(n-2i))$ $(n = 2l+1,\ 1 \leq i \leq l)$	$(\mathfrak{bc}_l, \{\lambda_i\})$	II 15

TABLE 1-continued. TRG-triples with dim $Z(H) = 1$.

$\mathfrak{e}_6$	$\mathfrak{su}(6) \oplus \sqrt{-1}\boldsymbol{R}$	H_2	$(\mathfrak{e}_{6(2)},\ \mathfrak{su}(6) \oplus \mathfrak{su}(2),\ \mathfrak{s}(\mathfrak{u}(3) \oplus \mathfrak{u}(3)))$	$(\mathfrak{f}_4, \{\lambda_1\})$	II 17
			$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1)))$	$(\mathfrak{bc}_2, \{\lambda_1\})$	II 19
			$(\mathfrak{e}_{6(6)},\ \mathfrak{sp}(4),\ \mathfrak{so}(6))$	$(\mathfrak{e}_6, \{\lambda_2\})$	II 16
$\mathfrak{e}_6$	$\mathfrak{s}(\mathfrak{u}(5) \oplus \mathfrak{u}(1)) \oplus \mathfrak{su}(2)$	H_3	$(\mathfrak{e}_{6(6)},\ \mathfrak{sp}(4),\ \mathfrak{so}(5) \oplus \mathfrak{so}(2))$	$(\mathfrak{e}_6, \{\lambda_3\})$	II 16
$\mathfrak{e}_7$	$\mathfrak{so}(12) \oplus \mathfrak{so}(2)$	H_1	$(\mathfrak{e}_{7(7)},\ \mathfrak{su}(8),\ \mathfrak{so}(6) \oplus \mathfrak{so}(6))$	$(\mathfrak{e}_7, \{\lambda_1\})$	II 21
			$(\mathfrak{e}_{7(-5)},\ \mathfrak{so}(12) \oplus \mathfrak{su}(2),\ \mathfrak{u}(6))$	$(\mathfrak{f}_4, \{\lambda_1\})$	II 23
			$(\mathfrak{e}_{7(-25)},\ \mathfrak{e}_6 \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{so}(10) \oplus \mathfrak{so}(2))$	$(\mathfrak{c}_3, \{\lambda_1\})$	II 25
$\mathfrak{e}_7$	$\mathfrak{s}(\mathfrak{u}(7) \oplus \mathfrak{u}(1))$	H_2	$(\mathfrak{e}_{7(7)},\ \mathfrak{su}(8),\ \mathfrak{so}(7))$	$(\mathfrak{e}_7, \{\lambda_2\})$	II 22
$\mathfrak{e}_7$	$\mathfrak{su}(2) \oplus \mathfrak{so}(10) \oplus \mathfrak{so}(2)$	H_6	$(\mathfrak{e}_{7(7)},\ \mathfrak{su}(8),\ \mathfrak{so}(5) \oplus \mathfrak{so}(5) \oplus \mathfrak{so}(2))$	$(\mathfrak{e}_7, \{\lambda_6\})$	II 21
			$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2),\ \mathfrak{so}(3) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2))$	$(\mathfrak{f}_4, \{\lambda_4\})$	II 24
			$(\mathfrak{e}_{7(-25)},\ \mathfrak{e}_6 \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{so}(9) \oplus \mathfrak{so}(2))$	$(\mathfrak{c}_3, \{\lambda_2\})$	II 25
$\mathfrak{e}_8$	$\mathfrak{so}(14) \oplus \mathfrak{so}(2)$	H_1	$(\mathfrak{e}_{8(8)},\ \mathfrak{so}(16),\ \mathfrak{so}(7) \oplus \mathfrak{so}(7))$	$(\mathfrak{e}_8, \{\lambda_1\})$	II 26
			$(\mathfrak{e}_{8(-24)},\ \mathfrak{e}_7 \oplus \mathfrak{su}(2),\ \mathfrak{so}(3) \oplus \mathfrak{so}(11))$	$(\mathfrak{f}_4, \{\lambda_4\})$	II 29
$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \sqrt{-1}\boldsymbol{R}$	H_8	$(\mathfrak{e}_{8(8)},\ \mathfrak{so}(16),\ \mathfrak{su}(8))$	$(\mathfrak{e}_8, \{\lambda_8\})$	II 27
			$(\mathfrak{e}_{8(-24)},\ \mathfrak{e}_7 \oplus \mathfrak{su}(2),\ \mathfrak{e}_6 \oplus \sqrt{-1}\boldsymbol{R})$	$(\mathfrak{f}_4, \{\lambda_1\})$	II 28
$\mathfrak{f}_4$	$\mathfrak{sp}(3) \oplus \sqrt{-1}\boldsymbol{R}$	H_1	$(\mathfrak{f}_{4(4)},\ \mathfrak{sp}(3) \oplus \mathfrak{su}(2),\ \mathfrak{u}(3))$	$(\mathfrak{f}_4, \{\lambda_1\})$	II 30
$\mathfrak{f}_4$	$\mathfrak{so}(7) \oplus \sqrt{-1}\boldsymbol{R}$	H_4	$(\mathfrak{f}_{4(4)},\ \mathfrak{sp}(3) \oplus \mathfrak{su}(2),\ \mathfrak{so}(3) \oplus \mathfrak{so}(4))$	$(\mathfrak{f}_4, \{\lambda_4\})$	II 31
			$(\mathfrak{f}_{4(-20)},\ \mathfrak{so}(9),\ \mathfrak{so}(7))$	$(\mathfrak{bc}_1, \{\lambda_1\})$	II 32
$\mathfrak{g}_2$	$\mathfrak{u}(2)$	H_2	$(\mathfrak{g}_{2(2)},\ \mathfrak{su}(2) \oplus \mathfrak{su}(2),\ \mathfrak{so}(2))$	$(\mathfrak{g}_2, \{\lambda_2\})$	II 33

TABLE 2. TRG-triples with $\dim Z(H) = 2$.

TRG-triple $((G/H, \langle\ ,\ \rangle, \sigma),\ I,\ N = K \cdot o = K/K_0)$.
$\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{k}_0$ are Lie algebras of G, H, K and K_0, respectively.
$\sigma = \mathrm{Ad}(\exp\{(2\pi\sqrt{-1}/3)Z\})$, $\quad I = -\varphi \circ J$.
$(\mathfrak{g}^*, \theta)$ corresponds to a GLA defined by a partition $\{\Pi_0, \Pi_1\}$ of Π.

$\mathfrak{g}$	$\mathfrak{h}$	Z	φ	$(\mathfrak{g}^*,\ \mathfrak{k},\ \mathfrak{k}_0)$	$(\Pi,\ \Pi_1)$	$(\mathfrak{g}^*, \theta)$
$\mathfrak{su}(n)$ $(n \geq 3)$	$\mathfrak{s}(\mathfrak{u}(i) \oplus \mathfrak{u}(j-i) \oplus \mathfrak{u}(n-j))$ $\left(1 \leq i \leq \left[\frac{n-1}{2}\right],\ i < j \leq n-1\right)$	$H_i + H_j$	φ_0	$(\mathfrak{sl}(n, \boldsymbol{R}),\ \mathfrak{so}(n),\ \mathfrak{so}(i) \oplus \mathfrak{so}(j-i) \oplus \mathfrak{so}(n-j))$	$(\mathfrak{a}_{n-1}, \{\lambda_i, \lambda_j\})$	I 7
			$\varphi_\pm$	$(\mathfrak{sl}(n, \boldsymbol{R}),\ \mathfrak{so}(n),\ \mathfrak{so}(i) \oplus \mathfrak{so}(j-i) \oplus \mathfrak{so}(n-j))$	$(\mathfrak{a}_{n-1}, \{\lambda_i, \lambda_j\})$	I 7
		$H_i + H_{n-i}$ $(j = n-i)$	φ_0	$(\mathfrak{su}(l, n-l),\ \mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{u}(n-l)),$ $\mathfrak{su}(i) \oplus \mathfrak{s}(\mathfrak{u}(l-i) \oplus \mathfrak{u}(n-l-i)) \oplus \sqrt{-1}\boldsymbol{R})$ $\left(i \leq l \leq \left[\frac{n-1}{2}\right]\right)$	$(\mathfrak{bc}_l, \{\lambda_i\})$	II 11
				$(\mathfrak{su}(l, l), \mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{u}(l)),$ $\mathfrak{su}(i) \oplus \mathfrak{s}(\mathfrak{u}(l-i) \oplus \mathfrak{u}(l-i)) \oplus \sqrt{-1}\boldsymbol{R})$ $(n = 2l)$	$(\mathfrak{c}_l, \{\lambda_i\})$	II 11
	$\mathfrak{s}(\mathfrak{u}(2i) \oplus \mathfrak{u}(2j-2i) \oplus \mathfrak{u}(n-2j))$ $(1 \leq i < j \leq l,$ $n = 2l+2)$	$H_{2i} + H_{2j}$	φ_0	$(\mathfrak{su}^*(n),\ \mathfrak{sp}(l+1),\ \mathfrak{sp}(i) \oplus \mathfrak{sp}(j-i) \oplus \mathfrak{sp}(n-j))$	$(\mathfrak{a}_l, \{\lambda_i, \lambda_j\})$	I 9
			$\varphi_\pm$	$(\mathfrak{su}^*(n),\ \mathfrak{sp}(l+1),\ \mathfrak{sp}(i) \oplus \mathfrak{sp}(j-i) \oplus \mathfrak{sp}(n-j))$	$(\mathfrak{a}_l, \{\lambda_i, \lambda_j\})$	I 9

TABLE 2-continued. TRG-triples with dim $Z(H) = 2$.

$\mathfrak{so}(2n)$ $(n \geq 4)$	$\mathfrak{u}(n-1) \oplus \mathfrak{so}(2)$	$H_{n-1} + H_n$	φ_0	$(\mathfrak{so}(n,n),\ \mathfrak{so}(n) \oplus \mathfrak{so}(n),\ \mathfrak{so}(n-1))$	$(\mathfrak{d}_n, \{\lambda_{n-1}, \lambda_n\})$	I 10
				$(\mathfrak{so}(n+1, n-1),\ \mathfrak{so}(n+1) \oplus \mathfrak{so}(n-1),\ \mathfrak{so}(n-1) \oplus \mathfrak{so}(2))$	$(\mathfrak{b}_{n-1}, \{\lambda_{n-1}\})$	II 12
				$(\mathfrak{so}^*(4l+2),\ \mathfrak{u}(2l+1),\ \mathfrak{sp}(l) \oplus \mathfrak{u}(1))$ $(n = 2l+1)$	$(\mathfrak{bc}_l, \{\lambda_l\})$	II 15
			$\varphi_\pm$	$(\mathfrak{so}(n,n),\ \mathfrak{so}(n) \oplus \mathfrak{so}(n),\ \mathfrak{so}(n-1))$	$(\mathfrak{d}_n, \{\lambda_{n-1}, \lambda_n\})$	I 10
$\mathfrak{e}_6$	$\mathfrak{so}(8) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2)$	$H_1 + H_6$	φ_0	$(\mathfrak{e}_{6(6)},\ \mathfrak{sp}(4),\ \mathfrak{so}(4) \oplus \mathfrak{so}(4))$	$(\mathfrak{e}_6, \{\lambda_1, \lambda_6\})$	I 15
				$(\mathfrak{e}_{6(2)},\ \mathfrak{su}(6) \oplus \mathfrak{su}(2),\ \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \sqrt{-1}\boldsymbol{R})$	$(\mathfrak{f}_4, \{\lambda_4\})$	II 18
				$(\mathfrak{e}_{6(-14)},\ \mathfrak{so}(10) \oplus \sqrt{-1}\boldsymbol{R},\ \mathfrak{so}(7) \oplus \sqrt{-1}\boldsymbol{R})$	$(\mathfrak{bc}_2, \{\lambda_2\})$	II 20
				$(\mathfrak{e}_{6(-26)},\ \mathfrak{f}_4,\ \mathfrak{so}(8))$	$(\mathfrak{a}_2, \{\lambda_1, \lambda_2\})$	I 16
			$\varphi_\pm$	$(\mathfrak{e}_{6(6)},\ \mathfrak{sp}(4),\ \mathfrak{so}(4) \oplus \mathfrak{so}(4))$	$(\mathfrak{e}_6, \{\lambda_1, \lambda_6\})$	I 15
				$(\mathfrak{e}_{6(-26)},\ \mathfrak{f}_4,\ \mathfrak{so}(8))$	$(\mathfrak{a}_2, \{\lambda_1, \lambda_2\})$	I 16

REFERENCES

[1] S. ARAKI, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ. 13 (1962), 1–34.

[2] A. BOREL AND F. HIRZEBRUCH, Characteristic classes and homogeneous spaces, I, Amer. J. Math. 80 (1958), 458–538.

[3] N. BOURBAKI, Groupes et Algèbres de Lie, Chapitre 4, 5 et 6, Masson, Paris, 1981.

[4] J. FARAUT, S. KANEYUKI, A. KORÁNYI, Q. LU AND G. ROOS, Analysis and geometry on complex homogeneous domains, Progr. Math. 185, Birkhäuser, Boston, MI, 2000.

[5] A. GRAY, Riemannian manifolds with geodesic symmetries of order 3, J. Differential Geom. 7 (1972), 343–369.

[6] S. HELGASON, Differential geometry, Lie groups, and symmetric spaces, Academic Press, New York-London, 1978.

[7] J. HILGERT AND G. ÓLAFSSON, Causal symmetric spaces: Geometry and harmonic analysis, Academic Press, San Diego, 1997.

[8] S. KANEYUKI, On the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_{ev}$ of semisimple graded Lie algebras, J. Math. Soc. Japan 45 (1993), 1–19.

[9] S. KANEYUKI, Signatures of roots and a new characterization of causal symmetric spaces, Progr. Nonlinear Differential Equations Appl. 20, 213–230, Birkhäuser Boston, Boston, MA, 1996.

[10] S. KANEYUKI AND H. ASANO, Graded Lie algebras and generalized Jordan triple systems, Nagoya Math. J. 112 (1988), 81–115.

[11] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry Vol. II, Interscience Tracts in Pure and Applied Mathematics No.15, Interscience Publishers John Wiley & Sons, Inc., New York, 1969.

[12] M. NISHIYAMA, Classification of invariant complex structures on irreducible compact simply connected coset spaces, Osaka J. Math. 21 (1984), 39–58.

[13] K. NOMIZU, Invariant affine connections on homogeneous spaces, Amer. J. Math. 76 (1954), 33–65.

[14] T. OSHIMA AND J. SEKIGUCHI, Eigenspaces of invariant differential operators on an affine symmetric spaces, Invent. Math. 57 (1980), 1–81.

[15] T. OSHIMA AND J. SEKIGUCHI, The restricted root system of a semisimple symmetric pair, Adv. Stud. Pure Math. 4 (1984), 433–497.

[16] M. TAKEUCHI, Stability of certain minimal submanifolds of compact Hermitian symmetric spaces, Tôhoku Math. J. 36 (1984), 293–314.

[17] K. TOJO, Totally real totally geodesic submanifolds of compact 3-symmetric spaces, Tôhoku Math. J. 53 (2001), 131–143.

[18] K. TOJO, Classification of totally real and totally geodesic submanifolds of compact 3-symmetric spaces, J. Math. Soc. Japan 58 (2006), 17–53.

[19] H. C. WANG, Closed manifolds with homogeneous complex structure, Amer. J. Math. 76 (1954), 1–32.

[20] A. WOLF AND A. GRAY, Homogeneous spaces defined by Lie group automorphisms I, II, J. Differential Geom. 2 (1968), 77–114, 115–159.

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