NOTES ON FOURIER ANALYSIS (XL): ON THE ABSOLUTE SUMMABILITY OF THE FOURIER SERIES

NOBORU MATSUYAMA

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1. Let $\{\lambda_n\}$ be a positive and increasing sequence, and let

$$R(\omega) = \omega^{-1} \sum_{\lambda_n < \omega} (\omega - \lambda_n) a_n \tag{1.1}$$

be the $(R, \lambda_n, 1)$-mean of the series $\sum a_n$. If $R(\omega)$ is of bounded variation in the interval (λ_1, ∞), that is

$$\int_{\lambda_1}^{\infty} |dR(\omega)| = \int_{\lambda_1}^{\infty} \omega^{-2} \left| \sum_{\lambda_n < \omega} \lambda_n a_n \right| d\omega < \infty,$$

then $\sum a_n$ is said to be absolutely $(R, \lambda_n, 1)$-summable, or simply $|R, \lambda_n, 1|$-summable.

Let $f(t)$ be an L-integrable function in the interval $(0, 2\pi)$, and its Fourier series be

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

For the absolute summability of the Fourier series, following theorems are known:

THEOREM A. [1] *If for any* $\beta > 0$

$$\varphi(t)(\log t^{-1})^{\beta} = O(1) \quad (t \to 0),$$

then the Fourier serier of $f(t)$ *is summable* $|R, \log n, 1|$ *at* $t = x$, *where*

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}.$$

THEOREM B. [2] *If* $\varphi(t)$ *is of bounded variation in* $(0, \pi)$, *then the Fourier series of* $f(t)$ *is summable* $|R, n, \varepsilon|$ *at* $t = x$, *where* $\varepsilon > 0$.

THEOREM C. [3] *If* $\varphi(t) \log 1/t$ *is of bouuded variation in* $(0, \pi)$ *then the Fourier series of* $f(t)$ *is summable* $|R, \exp(n^{\alpha}), 1|$ *at* $t = x$, *where* $0 < \alpha < 1$.

In this paper we consider the summability $|R, \exp((\log n)^{\alpha}), 1|$, where $\alpha > 0$, and prove the following theorems:

THEOREM 1. *If* $\varphi(t)(\log 1/t)^{\beta} = O(1)$, *then the Fourier series of* $f(t)$ *is summable* $|R, \lambda_n, 1|$ *at* $t = x$, *where*

$$\lambda_n = \exp((\log n)^{\alpha}), \quad 0 < \alpha < \beta \textit{ and } \alpha < 1.$$

THEOREM 2. *If* $\varphi(t)(\log \log 1/t)^{\beta} = O(1)$, *then the Fourier series of* $f(t)$ *is summable* $|R, \log n, 1|$, *where* $\beta > 1$.

THEOREM 3. *If $\varphi(t)(\log 1/t)^{\alpha-1}$ is of bounded variation in $(0,\pi)$, then the Fourier series of $f(t)$ is summable $|R, \lambda_n, 1|$ at $t = x$, where* $\lambda_n = \exp((\log n)^\alpha), \alpha > 1$.

We can suppose $0 < \beta < 1$, in Theorem 1 and $1 < \alpha < 2$ in Theorem 3. Furthermore we can suppose that

$$f(t) = f(-t), \quad \int_0^\pi f(t)dt = 0 ,$$

and $x = 0$; consequently it leads to consider the series $\sum_{n=1}^\infty a_n$.

2. PROOF OF THEOREM 1. Let

$$\lambda_n = \exp((\log n)^\alpha) \qquad (n = 2, 3, \cdots\cdot)$$

and let $\omega > 0$. There is an m such that

$$\lambda_m \leqq \omega < \lambda_{m+1}.$$

Now,

$$\begin{aligned} R(\omega) &= \omega^{-1}\sum_{n=1}^{m} (\omega - \lambda_n)\, a_n = -\omega^{-1}\sum_{n=1}^{m-1} S_n\Delta\lambda_n + S_m(\omega - \lambda_m) \\ &= -\omega^{-1}\left\{\sum_{n=1}^{m-2} n\sigma_n\Delta^2\lambda_n + (m-1)\sigma_{m-1}\Delta\lambda_{m-1} - S_m(\omega - \lambda_m)\right\}, \end{aligned}$$

where

$$S_n = a_1 + a_2 + \cdots\cdot + a_n, \quad \text{and } \sigma_n = n^{-1}(S_1 + S_2 + \cdots\cdot + S_n).$$

Hence

$$\begin{aligned} \int_{\lambda_1}^\infty |dR(\omega)| &= \sum_{m=1}^\infty \int_{\lambda_m}^{\lambda_{m+1}} |dR(\omega)| \\ &= \sum_{m=1}^\infty \int_{\lambda_m}^{\lambda_{m+1}} \omega^{-2} \left|\sum_{n=1}^m \lambda_n a_n\right| \leqq A \sum_{m=1}^\infty (\lambda_m\lambda_{m+1})^{-1}\Delta\lambda_m \left|\sum_{n=1}^m \lambda_n a_n\right| \\ &= A\sum_{m=1}^\infty |R(\lambda_m) - R(\lambda_{m+1})|. \end{aligned}$$

Thus, it is sufficient to prove the convergence of the last series for which we have

$$\begin{aligned} &\sum_{m=1}^\infty |R(\lambda_m) - R(\lambda_{m+1})| \\ &\leqq \sum_{m=1}^\infty \left|\lambda_m^{-1}\sum_{n=1}^{m-2} n\sigma_n\Delta^2\lambda_n - \lambda_{m+1}^{-1}\sum_{n=1}^{m-1} n\sigma_n\Delta^2\lambda_n\right| \\ &+ \sum_{m=1}^\infty \left|\lambda_m^{-1}(m-1)\sigma_{m-1}\Delta\lambda_{m-1} - \lambda_{m+1}^{-1} m\sigma_m\Delta\lambda_m\right| \equiv I_1 + I_2, \end{aligned}$$

say. Then from Lemma 1 of [1]

$$I_1 \leqq \sum_{m=1}^\infty \left|\Delta(\lambda_m^{-1})\sum_{n=1}^{m-2} n\sigma_n\Delta^2\lambda_n\right| + \sum_{m=1}^\infty \left|\lambda_{m+1}^{-1}(m-1)\sigma_{m-1}\Delta^2\lambda_{m-1}\right|$$

$$= \sum_{m=1}^{\infty} \frac{(\log m)^{\alpha-1}}{m\lambda_m} \left| \sum_{n=1}^{m-2} \frac{n}{(\log n)^{\beta}} \frac{\lambda_n}{n^2} (\log n)^{\alpha-1} \right|$$

$$+ \sum_{m=1}^{\infty} \left| \frac{m}{\lambda_m} \frac{1}{(\log m)^{\beta}} \frac{\lambda_m}{m^2} (\log m)^{\alpha-1} \right| \equiv I_{11} + I_{12},$$

say. Then

$$I_{12} = \sum_{m=1}^{\infty} 1/m\ (\log m)^{1+(\beta-\alpha)} < \infty, \tag{2.1}$$

and

$$I_{11} = \sum_{m=1}^{\infty} \frac{(\log m)^{\alpha-1}}{m\lambda_m} \sum_{n=1}^{m-2} \lambda_n/n\,(\log n)^{1+(\beta-\alpha)}$$

$$\leqq \sum_{n=1}^{\infty} \frac{\lambda_n}{n(\log n)^{1+(\beta-\alpha)}} \sum_{m=n}^{\infty} \frac{1}{m\lambda_m (\log m)^{1-\alpha}}$$

$$= \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{\lambda_n}{n(\log n)^{1+(\beta-\alpha)}} \frac{1}{\lambda_n} < \infty. \tag{2.2}$$

From (2.1) and (2.2), we obtain $I_1 = O(1)$. We shall next consider I_2.

$$I_2 = \sum_{m=1}^{\infty} \left| \frac{(m-1)\sigma_{m-1}\Delta\lambda_{m-1}}{\lambda_m} - \frac{m\sigma_m\Delta\lambda_m}{\lambda_{m+1}} \right|$$

$$= \sum_{m=1}^{\infty} \left| \frac{(m-1)\sigma_{m-1}\Delta\lambda_{m-1}}{\lambda_m} - \frac{\{(m-1)\sigma_{m-1} + S_m\}\Delta\lambda_m}{\lambda_{m+1}} \right|$$

$$\leqq \sum_{m=1}^{\infty} \left| \frac{S_m\Delta\lambda_m}{\lambda_{m+1}} \right| + \sum_{m=1}^{\infty} \left| \Delta\left(\frac{\Delta\lambda_{m-1}}{\lambda_m}\right)(m-1)\sigma_{m-1} \right| \equiv I_{21} + I_{22},$$

say.

Then, by the Hölder inequality,

$$I_{21} \leqq \sum_{m=2}^{\infty} |S_m| \frac{1}{m(\log m)^{1-\alpha}} \leqq \left(\sum_{m=2}^{\infty} m^{-1}|S_m|^p\right)^{1/p} \left(\sum_{m=2}^{\infty} m^{-1}(\log m)^{-q(1-\alpha)}\right)^{1/q} \tag{2.3}$$

where

$$p^{-1} + q^{-1} = 1 \text{ and } \alpha < p^{-1} < \beta.$$

Since $(1-\alpha)q = (1-\alpha)p/(p-1) > 1$, the second factor of the right-hand side of (2.3) converges. After Hardy and Littewood, [4]

$$\left(\sum_{m=2}^{\infty} |S_m|^p/m\right)^{1/p} \leqq K\left(\int_0^{\pi} |f(t)|^p t^{-1} dt\right)^{1/p}$$

$$\leqq K\left(\int_0^{\pi} \frac{dt}{t(\log t^{-1})^{\beta p}}\right) = O(1).$$

Hence

$$I_{21} = O(1). \tag{2.4}$$

We have

$$I_{22} \leqq \sum_{m=2}^{\infty} \frac{m}{(\log m)^{\beta}} \left\{ \frac{|\Delta^2\lambda_{m-1}|}{\lambda_m} + |\Delta\lambda_m| \left|\Delta\left(\frac{1}{\lambda_m}\right)\right| \right\} \tag{2.5}$$

$$\leqq \sum_{m=2}^{\infty} \frac{m}{(\log m)^{\beta}} \left\{ m^{-2}(\log m)^{-(1-\alpha)} + m^{-2}(\log m)^{-2(1-\alpha)} \right\} = O(1).$$

From (2.4) and (2.5), $I_2 = O(1)$. Thus the theorem is proved.

The proof of Theorem 2 runs similarly as that of Theorem 1.

3. PROOF OF THEOREM 3.

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt\, dt = \frac{2}{\pi} f(\pi)(\log \pi^{-1})^{\beta} \int^{\pi} (\log 1/t)^{-\beta} \cos nt\, dt$$

$$- \frac{2}{\pi} \int_0^{\pi} d(f(t)(\log 1/t)^{\beta}) \int_0^{t} (\log 1/u)^{-\beta} \cos nu\, du,$$

where $0 < \beta < 1$.

Let

$$(\log 1/t)^{-\beta} \sim \sum \alpha_n \cos nt,$$

then

$$\alpha_n = \frac{2}{\pi} \int_0^{\pi} \cos nt\, (\log 1/t)^{-\beta}\, dt,$$

and $\Sigma \alpha_n$ converges absolutely. Hence

$$\int_{\lambda_1}^{\infty} |dR(\omega)| = \int_{\lambda_1}^{\infty} \omega^{-2} \left| \sum_{\lambda_n < \omega} \lambda_n a_n \right| d\omega$$

$$\leqq \int_{\lambda_1}^{\infty} \omega^{-2} f(\pi)(\log 1/\pi)^{\beta} \left| \sum_{\lambda_n < \omega} \lambda_n \alpha_n \right| d\omega$$

$$+ \frac{2}{\pi} \int_{\lambda_1}^{\infty} \omega^{-2} d\omega \left| \sum_{\lambda_n < \omega} \lambda_n \int_0^{\pi} d(f(t)(\log 1/t)^{\beta}) \int_0^{t} (\log 1/u)^{-\beta} \cos nu\, du \right|$$

$$(3.1) \qquad = O(1) + \frac{2}{\pi} \int_0^{\pi} |d(f(t)(\log 1/t)^{\beta})| \int_{\lambda_1}^{\infty} \omega^{-2} d\omega \left| \int_0^{t} (\log 1/u)^{-\beta} \left| \sum_{\lambda_n < \omega} \lambda_n \cos nu \right| du. \right.$$

If we put

$$I(\omega, t) \equiv \int_0^{t} (\log 1/u)^{-\beta}\, du \sum_{\lambda_n < \omega} \lambda_n \cos nu,$$

then we have

$$I(\omega, t) \equiv \int_0^{t} (\log 1/u)^{-\beta} \left(\sum_{n=1}^{m} \lambda_n \cos nu \right) du$$

$$= \int_0^{t} (\log 1/u)^{-\beta}\, du \left\{ \sum_{n=1}^{m-1} D_n(u) \Delta\lambda_n + \lambda_m D_m(u) \right\} du.$$

$$(3.2) \qquad |I(\omega, t)| \leqq \int_0^{t} (\log 1/u)^{-\beta} \left\{ \sum_{n=1}^{m-1} n \frac{\lambda_n}{n (\log n)^{1-\alpha}} + m\lambda_m \right\} du$$

$$\leqq t(\log 1/t)^{-\beta}\Big\{m\lambda_m + \sum_{n=1}^{m-1} \lambda_n(\log n)^{\alpha-1}\Big\}.$$

Now

$$\sum_{n=1}^{m-1} \lambda_n(\log n)^{\alpha-1} \leqq \int_1^m (\log x)^{\alpha-1} \exp((\log x)^\alpha)\, dx$$

$$= \int_0^{\log m} t^{\alpha-1}e^{t+t^\alpha}\, dt \leqq m\int_0^{\log m} t^{\alpha-1}e^{t^\alpha}\, dt = O\ (m\lambda_m).$$

From (3.2) and the above estimations,

$$|I(\omega,t)| \leqq At(\log 1/t)^{-\beta}m\lambda_m$$

$$\leqq At(\log 1/t)^{-\beta}\omega \exp((\log \omega)^{1/\alpha}).$$

Now

$$\int_{\lambda_1}^\infty \omega^{-2}d\omega \left|\int_0^t (\log 1/u)^{-\beta}\Big(\sum_{\lambda_n<\omega} \lambda_n \cos nu\Big)du\right| = \int_{\lambda_1}^{\exp(\log 1/t)^\alpha} + \int_{\exp(\log 1/t)\omega}^\infty \equiv I_1 + I_2,$$

say. Then

$$I_1 = \int_{\lambda_1}^{\exp(\log 1/t)^\alpha} \omega^{-2}|I(\omega,t)|\,d\omega = \int_{\lambda_1}^{\exp(\log 1/t)^\alpha} t(\log 1/t)^{-\beta}\omega^{-1} \exp(\log\omega)^{1/\alpha}\, d\omega$$

$$= t(\log 1/t)^{-\beta}\int_1^{(\log 1/t)^\alpha} e^{x^{1/\alpha}}dx \leqq O(t(\log 1/t)^{-\beta}t^{-1}(\log 1/t)^{\alpha-1})$$

$$= O((\log 1/t)^{(\alpha-1)-\beta}).$$

On the other hand

$$I_2 = \int_{\exp(\log 1/t)^\alpha}^\infty \omega^{-2}|\, I(\omega,\ t)\,|\, d\omega$$

$$\leqq \int_{\exp(\log 1/t)^\alpha}^\infty \omega^{-2}\, |I(\omega,\pi)|\, d\omega + \int_{\exp(\log 1/t)^\alpha}^\infty \omega^{-2}\, d\omega \left|\int_t^\pi (\log 1/u)^{-\beta}\sum_{\lambda_n<\omega} \lambda_n \cos nu\, du\right|.$$

If we put

$$J(\omega,t) \equiv \int_t^\pi (\log 1/u)^{-\beta}\Big(\sum_{\lambda_n<\omega} \lambda_n \cos nu\Big)\, du,$$

then by the similar estimation as $I(\omega,t)$, we have

$$|\,J(\omega,t)| \leqq t^{-1}(\log 1/t)^{-\beta}\omega \exp(-(\log\omega)^{1/\alpha}).$$

We have also

$$\int_{\exp(\log 1/t)^\alpha}^\infty \omega^{-2}|I(\omega,\ \pi)|\,d\omega < \infty.$$

Hence

$$\int_{\exp(\log 1/t)^{\alpha}}^{\infty} \omega^{-2}|J(\omega, t)|\, d\omega$$
$$= t^{-1}(\log 1/t)^{-\beta}\int_{\exp(\log 1/t)^{\alpha}}^{\infty} \exp(-(\log \omega)^{1/\alpha})\omega^{-1}\, d\omega$$
$$= t^{-1}(\log 1/t)^{-\beta}\int_{(\log 1/t)^{\alpha}}^{\infty} e^{-x^{1/\alpha}}\, dx = O(t^{-1}(\log 1/t)^{-\beta} t(\log 1/t)^{\alpha-1})$$
$$= O((\log 1/t)^{(\alpha-1)-\beta}).$$

Lastly we have

$$\int_{\lambda_1}^{\infty} |dR(\omega)| \leqq O\left(\int_{0}^{\infty} |d(f(t)(\log 1/t)^{\beta}|\ (\log 1/t)^{(\alpha-1)-\beta}\right).$$

Hence if $\beta = \alpha - 1$, then by the hypothesis

$$\int_{\lambda_1}^{\infty} |dR(\omega)| = O(1).$$

Thus the theorem is proved.

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.