ON A REGULAR FUNCTION WHICH IS OF CONSTANT ABSOLUTE VALUE ON THE BOUNDARY OF AN INFINITE DOMAIN

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1. A metric in a circle. First we will introduce a metric in $|w| \leqq R$ as follows. We define the distance $(w, 0)$ of a point w ($|w| \leqq R$) from $w = 0$ by

$$(1) \qquad (w,0) = \frac{2R|w|}{R^2 + |w|^2}, \quad (0 \leqq (w,0) \leqq 1).$$

Let

$$(2) \qquad U_a(w) = \frac{R^2(w-a)}{R^2 - \bar{a}w} \qquad (|a| < R)$$

be a linear transformation, which transforms $|w| < R$ into itself, such that $U_a(a) = 0$. We define the distance (a, b) of any two points a, b in $|w| < R$ by

$$(3) \qquad (a, b) = (U_a(b), 0) = \frac{2R|U_a(b)|}{R^2 + |U_a(b)|^2} = \frac{2R\left|\dfrac{b-a}{R^2 - \bar{a}b}\right|}{1 + R^2\left|\dfrac{b-a}{R^2-\bar{a}b}\right|^2},$$

so that

$$(4) \qquad (a, b) = (b, a).$$

It is easily seen that for any linear transformation $U(w)$, which transforms $|w| < R$ into itself,

$$(5) \qquad (U(a), U(b)) = (a,b)$$

and a circle $(w, a) = \rho$ in our metric is an ordinary circle and the locus of points, which are equidistant from two given points is a circle, which cuts $|w| = R$ orthogonally.

In our metric, the triangle inequality

$$(6) \qquad (a, c) < (a, b) + (b, c)$$

holds.

PROOF. We may assume that $R = 1$ and $a = 0$, $0 < b < 1$ by (5), so that it suffices to prove:

$$(7) \qquad \frac{|c|}{1+|c|^2} < \frac{b}{1+b^2} + \frac{\left|\dfrac{b-c}{1-bc}\right|}{1+\left|\dfrac{b-c}{1-bc}\right|^2}.$$

Since $\frac{|c|}{1+|c|^2} \leqq \frac{|b|}{1+|b|^2}$ for $|c| \leqq |b|$, (7) holds for $|c| \leqq b$. Hence we assume that $0 < b < |c| < 1$. If c moves on a circle $|c| = \text{const.}$, then $\left|\frac{b-c}{1-bc}\right|$ becomes minimum, when c lies on the positive real axis and since $F(t) = t/(1+t^2)$ is an increasing function of t for $0 \leqq t \leqq 1$, the second term of the right-hand side of (7) becomes minimum, when c lies on the positive real axis. Hence to prove (7), it suffices to prove the following inequality for $0 < b < c < 1$:

$$\frac{c}{1+c^2} < \frac{b}{1+b^2} + \frac{\frac{c-b}{1-bc}}{1+\left(\frac{c-b}{1-bc}\right)^2} = \frac{b}{1+b^2} + \frac{(c-b)\,(1-bc)}{(1+b^2)(1+c^2)-4bc},$$

which holds evidently. Hence (6) holds in general, q. e. d.

The most important porperty of our metric is the following one. Since

$$\frac{1}{(w,\,a)} = \frac{R^2+|U_a(w)|^2}{2R|U_a(w)|} = 1 + \frac{(R-|U_a(w)|)^2}{2R|U_a(w)|} \geqq 1$$

and $|U_a(w)| = R$ on $|w| = R$, $\log\frac{1}{(w,\,a)}$ and its normal derivative vanish on $|w| = R$ and if $w = w(z)$ is a regular function of z, then

$$(8) \qquad \Delta \log \frac{1}{(w(z),\,a)} = 4R^2 \frac{|U_a'(w)|^2|w'(z)|^2}{(R^2+|U_a(w)|^2)^2} \geqq 0,$$

where Δ is the Laplacian, so that $\log(w(z),\,a)^{-1}$ is a subharmonic function of z.

2. Some notations. Let K be the Riemann sphere of diameter 1, which touches the w-plane at $w = 0$ and

$$(9) \qquad [a, b] = \frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$$

be the spherical distance of a, b.

Let Δ be an infinite domain on the z-plane, whose boundary Γ consists of at most a countable number of analytic curves. Let $w = w(z)$ be one-valued and meromorphic in Δ and on Γ, such that the value $w(z)$ in Δ belongs to a certain spherical disc $[w, w_0] < \delta$ and the value $w(z)$ on Γ belongs to $[w, w_0] = \delta$, so that the inverse function $z = z(w)$ of $w = w(z)$ is defined on a Riemann surface F, spread over $[w, w_0] < \delta$. If $z = z(w)$ has a transcendental singularity α in $[w, w_0] < \delta$, then w tends to α along a certain curve γ, when z tends to infinity along a certain curve l. In this case, the behaviour of $w(z)$ in Δ was first treated by K. Noshiro by applying Ahlfors' theory of covering surfaces. His research was followed by K. Kunugui, Y. Tumura and the present author[1]. In this paper, I will

1) K. Noshiro, On the singularities of analytic functions. Jap. Jour. Math. 17(1940).
K. Kunugui, Sur l'allure d'une fonction analytique uniforme au voisinage d'un point frontière de son domain de définition. Jap. Jour. Math. 18 (1943). Sur la théorie des fonctions méromorphes et uniformes. Jap. Jour. Math. 18 (1643).
Y. Tumura, Recherches sur la distribution des valeurs des fonctions analytiques. Jap. Jour. Math. 18 (1943).
M. Tsuji, Nevanlinna's fundamental theorems and Ahlfors' theorem on the number of asymptotic values. Jap. Jour. Math. 18 (1943).

prove the theorems obtained by these authors simply by means of the metric introduced in § 1.

First we will introduce some notations.

Let Δ_r be the part of Δ, which lies in $|z| \leqq r$ and F_r be its image on K and θ_r be the part of $|z| = r$ contained in Δ and $L(r)$ be the length of its image on K:

$$L(r) = \int_{\theta_r} \frac{|w'(re^{i\theta})|}{1+|w(re^{i\theta})|^2} r d\theta, \tag{10}$$

$A(r)$ be the area of F_r:

$$A(r) = \iint_{\Delta_r} \frac{|w'(re^{i\theta})|^2}{(1+|w(re^{i\theta})|^2)^2} r dr d\theta, \tag{11}$$

$$S(r) = A(r)/\pi\delta^2 \text{ (mean number of sheets of } F_r\text{)}, \tag{12}$$

where $\pi\delta^2$ is the area of $[w, w_0] \leqq \delta$,

$$T(r) = \int_0^r \frac{S(r)}{r} dr, \tag{13}$$

$\lambda(r)$ be the number of holes in Δ_r, and

$$\Lambda(r) = \int_{r_0}^r \frac{\lambda(r)}{r} dr, \quad (r_0 > 0). \tag{14}$$

We will prove

LEMMA 1.[2)] *If $z = z(w)$ has a transcendental singularity in $[w, w_0] < \delta$, then*

$$\lim_{r\to\infty} S(r) = \infty, \quad \lim_{r\to\infty} \frac{T(r)}{\log r} = \infty.$$

PROOF. Let α be a transcendental singularity of $z = z(w)$ in $[w, w_0] < \delta$, then w tends to α along a certain curve γ, when z tends to infinity along a certain curve l, so that $|z| = r$ meets l at a point z_r. If the boundary Γ of Δ contains a curve extending to infinity, then $|z| = r$ meets Γ for $r \geqq r_0$, so that the image of θ_r on K contains an arc, which connects a point $w_r = w(z_r)$ on γ to a point on $[w, w_0] = \delta$, so that $L(r) \geqq \eta > 0$ $(r \geqq r_0)$. Since

$$\eta^2 \leqq L(r)^2 \leqq 2\pi r \int_{\theta_r} \frac{|w'(re^{i\theta})|^2}{(1+|w(re^{i\theta})|^2)^2} r d\theta = 2\pi r \frac{dA(r)}{dr},$$

$$\eta^2 \log(r/r_0) \leqq 2\pi(A(r) - A(r_0)),$$

so that $\lim_{r\to\infty} A(r) = \infty$, hence $\lim_{r\to\infty} S(r) = \infty$, $\lim_{r\to\infty} T(r)/\log r = \infty$. If Γ does not contain a curve extending to infinity, then Γ consists of infinitely many closed curves and so $\lim_{r\to\infty} \lambda(r) = \infty$. By Ahlfors' first covering

2) K. NOSHIRO, l.c. 1).

theorom,[3] we have

(15) $$\lambda(r) \leqq S(r) + h\, L(r),$$

where h is a constant and it is easily proved that

$$L(r) \leqq S(r)^{1/2+\varepsilon} \qquad (\varepsilon > 0),$$

except certain intervals I_n, such that $\sum_n \int_{I_n} d \log r < \infty$. Hence we have

$$\lim_{r\to\infty} S(r) = \infty.$$

LEMMA 2. $\displaystyle\int_{r_0}^{r} \frac{L(r)}{r} dr = O(\sqrt{T(2r) \log r})$ *for all* r,

$$= O(\sqrt{T(r)} \log T(r)),$$

except crtain intervals I_n, *such that* $\sum_n \int_{I_n} d \log \log r < \infty$.

PROOF. From $L(r)^2 \leqq 2\pi r dA(r)/dr$, we have

$$\int_{r_0}^{r} \frac{L(r)}{r} dr \leqq \sqrt{\log \frac{r}{r_0} \int_{r_0}^{r} \frac{L(r)^2}{r} dr} \leqq \sqrt{2\pi \log \frac{r}{r_0} (A(r) - A(r_0))}.$$

Since $$T(2r) \geqq \int_{r}^{2r} \frac{S(r)}{r} dr \geqq \log 2\, S(r),$$

we have

$$\int_{r_0}^{r} \frac{L(r)}{r} dr = O(\sqrt{T(2r) \log r}).$$

The second part is proved by Dinghas.[4]

From (15), we have

LEMMA 3. $$A(r) \leqq T(r) + O\Big(\int_{r_0}^{r} \frac{L(r)}{r} dr\Big).$$

3. Main theorems. Under the same assumption as §2, we may assume that $w_0 = 0$ by a suitable rotation of K, so that $w(z)$ is regular in Δ and on Γ, such that $|w(z)| < R$ in Δ and $|w(z)| = R$ on Γ for some R. We assume, for the sake of simplicity, that $z = 0$ belongs to Δ.

Since $|w| \leqq R$ is projected on a disc $[w, 0] \leqq \delta$ $(\delta = R/\sqrt{1+R^2})$ on K and $\pi\delta^2 = \pi R^2/(1 + R^2)$ is its area, we have

(16) $$S(r) = \frac{1+R^2}{\pi R^2} \iint_{\Delta_r} \frac{|w'(re^{i\theta})|^2}{(1 + |w(re^{i\theta})|^2)^2} r dr d\theta.$$

3) L. AHLFORS, Zur Theorie der Überlagerungsflächen. Acta Math. 65 (1935).

4) A. DINGHAS, Eine Bemerkung zur Ahlforsschen Theorie der Überlagerungsflächen. Math. Zeits. 44 (1936).

Let $n(r,a)$ be the number of zero points of $w(z)-a\,(|a|<R)$ in Δ_r and

(17) $$N(r,a)=\int_0^r \frac{n(r,a)-n(0,a)}{r}dr+n(0,a)\log r,$$

(18) $$m(r,a)=\frac{1}{2\pi}\int_{\theta_r}\log\frac{1}{(w(re^{i\theta}),a)}d\theta,$$

where (w,a) is the metric of § 1 and

(19) $$T(r,a)=m(r,a)+N(r,a).$$

Then we will prove the following theorems, which are analogues of Nevanlinna's fundamental theorems for meromorphic functions for $|z|<\infty$.

THEOREM 1. *Let $w(z)$ be regular in an infinite domain Δ and on its boundary Γ, such that $|w(z)|<R$ in Δ and $|w(z)|=R$ on Γ. Then $T(r,a)$ is an increasing convex function of* $\log r$, *such that*

$$T(r,a)=\int_0^r \frac{S(r,a)}{r}dr+\text{const.}=T(r)+O\Big(\int_{r_0}^r \frac{L(r)}{r}dr\Big),$$

where

$$S(r,a)=\frac{2}{\pi}\iint_{\Delta_r}\frac{|v'|^2}{(1+|v|^2)^2}\,rdrd\theta,\quad (v=U_a(w)/R),$$

$$T(r)=\int_0^r\frac{S(r)}{r}dr,$$

$$S(r)=S(r,\Delta)=\frac{1+R^2}{\pi R^2}\iint_{\Delta r}\frac{|w'(re^{i\theta})|^2}{(1+|w(re^{i\theta}|)^2}\,rdrd\theta,$$

$S(r)$ being the mean number of sheets of the Riemann surface generated by $w=w(z)$ on the w-sphere.

$$\int_{r_0}^r\frac{L(r)}{r}dr=O(\sqrt{T(2r)\log r})\ \textit{for all } r,$$

$$=O(\sqrt{T(r)}\log T(r)),$$

except certain intervals I_n, such that $\sum_n\int_{I_n}d\log\log r<\infty.$

THEOREM 2.[5]

$$(q-1)T(r)\leqq\sum_{i=1}^q N(r,a_i)+\Lambda(r)+O\Big(\int_{r_0}^r\frac{L(r)}{r}dr\Big),\quad (q\geqq 2),$$

where

5) K. Noshiro, K. Kunugi. l. c. 1).

$$\Lambda(r) \leqq T(r) + O\Big(\int_{r_0}^{r} \frac{L(r)}{r}\, dr\Big).$$

We call $T(r) = T(r, \Delta)$ the chracteristic function of $w(z)$ in Δ and

$$\varlimsup_{r\to\infty} \frac{\log T(r)}{\log r} = \rho$$

its order.

REMARK. Since $\log[w(re^{i\theta}),\ a]^{-1} = \log(w(re^{i\theta}), a)^{-1} + O(1)$, Theorem 1 becomes

$$\frac{1}{2\pi}\int_{\theta_r} \log \frac{1}{[w(re^{i\theta}), a]}\, d\theta + N(r,\ a) = T(r) + O\Big(\int_{r_0}^{r} \frac{L(r)}{r} dr\Big).$$

In this form, Theorem 1 was proved by Tumura and the present author previously.[6)]

4. Proof of Theorem 1. Let a be any point in $|w| < R$ and

$$m(r,\ a) = \frac{1}{2\pi}\int_{\theta_r} \log \frac{1}{(w(re^{i\theta}),\ a)}\, d\theta,$$

then since $\log \dfrac{1}{(w, a)}$ vanishes at the end points of θ_r

$$rm'(r,\ a) = \frac{1}{2\pi}\int_{\theta_r} \frac{\partial}{\partial r} \log \frac{1}{(w,\ a)}\, r d\theta.$$

Let Γ_r be the part of Γ contained in $|z| \leqq r$ and ν be its outer normal and ds its line element and $z_1,\ \cdots,\ z_n$ be zero points of $w(z) - a$ in Δ_r. We assume that z_i does not lie on θ_r. We enclose z_i by a small circle γ_i and we take off the inside of $\{\gamma_i\}$ from Δ_r and Δ_r^0 be the remaining domain.

Then applying Green's formula for Δ_r^0, we have

$$\int_{\theta_r} \frac{\partial}{\partial \nu} \log \frac{1}{(w,\ a)}\, ds + \int_{\Gamma_r} \frac{\partial}{\partial \nu} \log \frac{1}{(w,\ a)}\, ds + \sum_i \int_{\gamma_i} \frac{\partial}{\partial \nu} \log \frac{1}{(w,\ a)}\, ds$$
$$= \iint_{\Delta_r^0} \Delta \log \frac{1}{(w, a)}\, r dr d\theta.$$

Since as remarked in §1, the normal derivative of $\log \dfrac{1}{(w, a)}$ vanishes on Γ_r, we have

$$2\pi rm'(r, a) + \sum_i \int_{\gamma_i} \frac{\partial}{\partial \nu} \log \frac{1}{(w, a)}\, ds = \iint_{\Delta_r^0} \Delta \log \frac{1}{(w, a)}\, r dr d\theta.$$

If we make the radius of γ_i tend to zero, we have from (8),

$$2\pi rm'(r, a) + 2\pi n(r,\ a) = \iint_{\Delta_r} \Delta \log \frac{1}{(w, a)}\, r dr d\theta$$

6) TUMURA, TSUJI, l.c.1).

$$= 4R^2 \iint_{\Delta_r} \frac{|U_a'(w)|^2 |w'|^2}{(R^2 + |U_a(w)|^2)^2} r dr d\theta.$$

Hence if we put

(20) $$S(r, a) = \frac{2R^2}{\pi} \iint_{\Delta_r} \frac{|U_a'(w)|^2 |w'|^2}{(R^2 + |U_a(w)|^2)^2} r dr d\theta,$$

then

(21) $$m'(r,a) + n(r,a)/r = S(r,a)/r,$$

so that

(22) $$T(r,a) = m(r,a) + N(r,a) = \int_0^r \frac{S(r,a)}{r} dr + \text{const.}$$

Hence $T(r, a)$ is an increasing convex function of $\log r$. $S(r, a)$ has the following geometrical meaning.

If we put

(23) $$v(z) = U_a(w(z))/R,$$

then $|v(z)| < 1$ and $S(r,a)$ becomes

(24) $$S(r, a) = \frac{2}{\pi} \iint_{\Delta_r} \frac{|v'|^2}{(1 + |v|^2)^2} r dr d\theta.$$

Since $|v| \leqq 1$ is projected on the lower half of the v-sphere and $\pi/2$ is its area, $S(r, a)$ is the mean number of sheets of the Riemann surface generated by $v = v(z)$ on the v-sphere.

Next we will prove

(25) $$S(r, a) - S(r, 0) = O(L(r)).$$

Since by (21),

$$\frac{1}{2\pi} \int_{\theta_r} \frac{\partial}{\partial r} \log \frac{1}{(w, 0)} r d\theta + n(r, 0) = S(r, 0),$$

$$\frac{1}{2\pi} \int_{\theta_r} \frac{\partial}{\partial r} \log \frac{1}{(w, a)} r d\theta + n(r, a) = S(r, a),$$

we have

(26) $$\frac{1}{2\pi} \int_{\theta_r} \frac{\partial}{\partial r} \log \frac{(w, a)}{(w, 0)} r d\theta = S(r, 0) - S(r, a) + n(r, a) - n(r, 0).$$

It is easily seen that

$$\frac{\partial}{\partial r} \log \frac{(w, a)}{(w, 0)} = \frac{\partial}{\partial r} \log \frac{|w - a|}{|w|} + O\left(\frac{|w'|}{1 + |w|^2}\right),$$

so that

$$\frac{1}{2\pi} \int_{\theta_r} \frac{\partial}{\partial r} \log \frac{(w, a)}{(w, 0)} r d\theta = \frac{1}{2\pi} \int_{\theta_r} \frac{\partial}{\partial r} \log \frac{|w - a|}{|w|} r d\theta + (O(L(r))$$

$$= \frac{1}{2\pi} \int_{\theta_r} d \arg \frac{w - a}{w} + O(L(r)),$$

hence by (26),

$$\frac{1}{2\pi}\int_{\theta_r} d\arg\frac{w-a}{w} = S(r,0) - S(r,a) + n(r,a) - n(r,0) + O(L(r)).$$

By the argument principle,

$$n(r,a) - n(r,0) = \frac{1}{2\pi}\int_{\Gamma_r} d\arg\frac{w-a}{w} + \frac{1}{2\pi}\int_{\theta_r} d\arg\frac{w-a}{w}$$

$$= \frac{1}{2\pi}\int_{\Gamma_r} d\arg\frac{w-a}{w} + S(r,0) - S(r,a) + n(r,a) - n(r,0) + O(L(r)),$$

so that

$$S(r,a) - S(r,0) = \frac{1}{2\pi}\int_{\Gamma_r} d\arg\frac{w-a}{w} + O(L(r)). \tag{27}$$

We will prove

$$\int_{\Gamma_r} d\arg\frac{w-a}{w} = O(L(r)). \tag{28}$$

Now Γ_r consists of a finite number of separate curves $\Gamma_r = \sum_i \gamma_r^{(i)} + \sum_i \lambda_r^{(i)}$, where $\gamma_r^{(i)}$ is a closed curve, which is the boundary of a hole in Δ_r and $\lambda_r^{(i)}$ is a curve, which meets θ_r. Since $\gamma_r^{(i)}$ is mapped on $|w| = R$, we have

$$\int_{\gamma_r^{(i)}} d\arg\frac{w-a}{w} = 0.$$

Consider one $\lambda_r^{(i)}$ and let $\theta_r^{(i)}$ be the part of θ_r, which meets $\lambda_r^{(i)}$ and $L_i(r)$ be the length of the image of $\theta_r^{(i)}$ on K, then since, if w makes one turn on $|w| = R$, $\int d\arg\frac{w-a}{w} = 0$, it is easily seen that

$$\int_{\lambda_r^{(i)}} d\arg\frac{w-a}{w} = O(L_i(r)).$$

Hence

$$\int_{\Gamma_r} d\arg\frac{w-a}{w} = \sum_i \int_{\gamma_r^{(i)}} d\arg\frac{w-a}{w} + \sum_i \int_{\lambda_r^{(i)}} d\arg\frac{w-a}{w}$$

$$= \sum_i O(L_i(r)) = O(L(r)),$$

which proves (28), so that by (27), we have (25). Hence from (22), (25), we have

$$T(r, a) = T(r, 0) + O\left(\int_{r_0}^{r} \frac{L(r)}{r}\, dr\right),$$

so that for any two points a, b in $|w| < R$,

$$T(r,a) = T(r,b) + O\left(\int_{r_0}^{r} \frac{L(r)}{r}\, dr\right). \tag{29}$$

From the proof, we see that if a, b lies in $[w, 0] \leqq \delta_0 < \delta$, then

$$O\left(\int_{r_0}^{r} \frac{L(r)}{r}\, dr\right) \leqq \alpha(\delta_0)\int_{r_0}^{r} \frac{L(r)}{r}\, dr,$$

where $\alpha(\delta_0)$ depends on δ_0 only. Let $d\omega(b)$ be the surface element of K at b, then multiplying $d\omega(b)$ on the both sides of (29) and taking the integral mean over $[w, 0] \leqq \delta_0$, we have from (22),

$$T(r, a) = \frac{1}{\pi\delta_0^2}\iint_{[b,0]\leqq\delta_0} T(r,b)d\omega(b) + O\left(\int_{r_0}^{r} \frac{L(r)}{r}\, dr\right)$$
$$= \int_0^r \frac{S_0(r)}{r} dr + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right),$$

where $S_0(r) = A_0(r)/(\pi\delta_0^2)$ is the mean number of sheets of the part of F_r, which lies above $[w, 0] \leqq \delta_0$.

Since by Ahlfors' first covering theorem[3])

$$S(r) - S_0(r) = O(L(r)),$$

we have

$$T(r, a) = \int_0^r \frac{S(r)}{r} dr + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right).$$

Hence Theorem 1 is proved.

5. Proof of Theorem 2. Let F be the Riemann surface generated by $w = w(z)$ over $[w, 0] < \delta$ and F_r be the part of F, which corresponds to Δ_r. Let $a_1, \cdots, a_q$ $(q \geqq 2)$ be q points in $[w, 0] < \delta$. We take off these q points from $[w, 0] < \delta$ and F^0 be the remaining domain and we take off from F_r points, which lie above $a_1, \cdots, a_q$ and F_r^0 be the remaining surface. Then F_r^0 is a covering surface of the basic domain F^0. By Ahlfors' fundamental theorem on covering surfaces,[3])

$$\rho^+(F_r^0) \geqq \rho(F^0)S(r) - hL(r),$$

where ρ is the Euler's characteristic and $\rho^+ = \mathrm{Max}\,(\rho, 0)$ and h is a constant depending on F^0 only.

Since $\rho(F^0) = q - 1$, we have

$$\rho^+(F_r^0) \geqq (q-1)\, S(r) - hL(r).$$

Since

$$\rho^+(F_r^0) \leqq \sum_{i=1}^{q} n(r, a_i) + \lambda(r),$$

we have

$$(q-1)\, S(r) \leqq \sum_{i=1}^{q} n(r, a_i) + \lambda(r) + hL(r),$$

so that

$$(q-1)T(r) \leqq \sum_{i=1}^{q} N(r, a_i) + \Lambda(r) + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right).$$

Hence Theorem 2 is proved.

REMARK. If the inverse function $z = z(w)$ of $w = w(z)$ has a transcendental singularity in $[w, 0] < \delta$, then by Lemma 1,

(30) $$\lim_{r \to \infty} T(r)/\log r = \infty.$$

Suppose that Δ is simply connected, then $\Lambda(r) = 0$, so that if we take $q = 2$ in Theorem 2,

$$T(r) \leqq \sum_{i=1}^{2} N(r, a_i) + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right).$$

If $w(z)$ takes $a_i (i = 1,2)$ only finite times in Δ, then $N(r, a_i) = O(\log r)$ $(i = 1,2)$, so that

$$T(r) \leqq O(\log r) + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right),$$

which contradicts Lemma 2 in virtue of (30).

Hence $w(z)$ takes any value in $[w, 0] < \delta$ infinitely often, with one possible exception. This is due to K. Noshiro.[1]

If Δ is not simply connected, we take $q = 3$ and taking account of Lemma 2, we have

$$T(r) \leqq \sum_{i=1}^{3} N(r, a) + O\left(\int_{r_0}^{r} \frac{L(r)}{r} dr\right).$$

From this we see as above, that $w(z)$ takes any value in $[w, 0] < \delta$ infinitely often, with two possible exceptions. This is due to K. Kunugui.[1]

6. Extension of Ahlfors' theorem. Let $w = w(z)$ be a transcendental meromorphic function for $|z| < \infty$ and w_0 be a direct transcendental singularity of the inverse function $z = z(w)$ of $w = w(z)$, such that $[w, w_0] < \delta$ is mapped on a domain Δ on the z-plane, where $w(z) \neq w_0$ in Δ. Let Δ_0 be the smallest simply connected domain, which contains Δ and θ_r be the part of $|z| = r$ contained in Δ_0, which separates a point z_0 of Δ from $z = \infty$ and $r\theta(r)$ be its length.

Then Ahlfors[5] proved that

(31) $$\log T(2r) \geqq \pi \int_{r_0}^{r} \frac{dr}{r\theta(r)} - \text{const.},$$

where $T(r)$ is the Nevanlinna's characteristic function of $w(z)$. From this follows easily the well known theorem on the number of direct transcendental singularities of the inverse function of a meromorphic

5) L. Ahlfors, Über die asymptotischen Werte der meromorphen Funktionen endlicher Ordnung. Acta Acad. Aboensis. Math. et Phys. 6 Nr.9(1932).

function of finite order. We will prove an extension of this theorem by means of Theorem 1. By a suitable rotation of the Riemann sphere K, we may assume that $w_0 = \infty$, so that $w(z)$ is regular in an infinite domain Δ, such that $|w(z)| > R$ in Δ and $|w(z)| = R$ on its boundary Γ. Then $v(z) = 1/w(z)$ is regular in Δ, such that $|v(z)| < 1/R$ in Δ and $|v(z)| = 1/R$ on Γ. Since $v(z) \neq 0$, $N(r, 0) = 0$, hence if we apply Theorem 1 on $v(z)$, we have

$$(32) \qquad T(r, 0) = m(r, 0) = \int_0^r \frac{S(r, 0)}{r} dr + \text{const.}$$
$$= \int_0^r \frac{S(r)}{r} dr + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right),$$

where

$$(33) \qquad m(r, 0) = \frac{1}{2\pi}\int_{\theta_r} \log\frac{1}{(v, 0)} d\theta = \frac{1}{2\pi}\int_{\theta_r} \log \frac{R^2 + |w|^2}{2R|w|} d\theta,$$

$$(34) \qquad S(r, 0) = \frac{2R^{-2}}{\pi}\iint_{\Delta_r} \frac{|v'|^2}{(R^{-2} + |v|^2)^2} rdrd\theta = \frac{2R^2}{\pi}\iint_{\Delta_r} \frac{|w'|^2}{(R^2 + |w|^2)^2} rdrd\theta,$$

$$(35) \qquad S(r) = \frac{1 + R^{-2}}{\pi R^{-2}}\iint_{\Delta_r} \frac{|v'|^2}{(1 + |v|^2)^2} rdrd\theta = \frac{1 + R^2}{\pi}\iint_{\Delta_r} \frac{|w'|^2}{(1 + |w|^2)^2} rdrd\theta,$$

$S(r)$ being the mean number of sheets of the Riemann surface generated by $w = w(z)$ on the w-sphere.

Since for $|w| \geqq R$,

$$\frac{2R^2}{1 + R^2}(1 + |w|^2) \geqq R^2 + |w|^2 \geqq 1 + |w|^2, \text{ if } R \geqq 1,$$
$$1 + |w|^2 \geqq R^2 + |w|^2 \geqq \frac{2R^2}{1 + R^2}(1 + |w|^2), \text{ if } R \leqq 1,$$

if we put

$$(36) \qquad \alpha(R) = \frac{2R^2}{1 + R^2} (R \geqq 1), \ = \frac{1 + R^2}{2R^2} (R \leqq 1),$$

then we have from (34), (35),

$$\frac{1}{\alpha(R)} S(r) \leqq S(r, 0) \leqq \alpha(R) S(r),$$

so that from (32), (33),

$$(37) \qquad \frac{1}{\alpha(R)} T(r) + \text{const.} \leqq \frac{1}{2\pi}\int_{\theta_r} \log \frac{R^2 + |w|^2}{2R|w|} d\theta \leqq \alpha(R) T(r) + \text{const.}$$

If we put $M(r) = \underset{\theta_r}{\text{Max.}} |w(z)|$, then

$$\log \frac{R^2 + |w|^2}{2R|w|} \leqq \log \frac{M(r)}{R} \quad \text{on } \theta_r,$$

so that from (37),

$$(38) \qquad \frac{1}{\alpha(R)} T(r) + \text{const.} \leqq \log \frac{M(r)}{R}.$$

Let z ($|z| = r$) be any point of Δ. Let $U(z)$ be a harmonic function in $|z| < kr$ ($k > 1$), with the boundary value $U(z) = \log \frac{R^2 + |w|^2}{2R|w|}$ on θ_{kr} and $U(z) = 0$ on the complementary arcs of θ_{kr} on $|z| = kr$. Then since $\log \frac{R^2 + |w|^2}{2R|w|}$ is subharmonic and vanishes on Γ,

$$(39) \qquad \log \frac{|w|}{R} + \log \frac{1}{2} \leqq \log \frac{R^2 + |w|^2}{2R|w|} \leqq U(z) \text{ in } \Delta_{kr}.$$

Since $U(z) > 0$ in $|z| < kr$, we have for $|z| \leqq r$, by (37),

$$U(z) \leqq \frac{k+1}{k-1} \frac{1}{2\pi} \int^{2\pi} U(kre^{i\theta})\, d\theta$$

$$= \frac{k+1}{k-1} \frac{1}{2\pi} \int_{\theta_{kr}} \log \frac{R^2 + |w|^2}{2R|w|} d\theta \leqq \frac{k+1}{k-1} \alpha(R) T(kr) + \text{const.},$$

so that from (39),

$$(40) \qquad \log \frac{M(r)}{R} \leqq \frac{K+1}{K-1} \alpha(R)\, T(kr) + \text{const.}$$

From (38), (40), we have

$$(41) \quad \frac{1}{\alpha(R)} T(r) + \text{const} \leqq \log \frac{M(r)}{R} \leqq \alpha(R) \frac{k+1}{k-1} T(kr) + \text{const.}, (k > 1).$$

Hence

$$(42) \qquad \overline{\lim_{r\to\infty}} \frac{\log \log M(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r}.$$

Now Δ_r consists of a finite number of connected domains. Let Δ_r^0 be the connected one, which contains $z = 0$.

We define $\bar{\theta}(r)$ as follows. If the circle $|z| = r$ meets Γ, then the part of $|z| = r$, which belongs to the boundary of Δ_r^0 consists of a finite number of arcs $\{\theta_r^{(i)}\}$ on $|z| = r$. Let $r\theta^{(i)}(r)$ be the length of $\theta_r^{(i)}$. Then we put $\bar{\theta}(r) = \sup_i \theta^{(i)}(r)$.

If $|z| = r$ does not meet Γ and is contained entirely in Δ, then we put $\bar{\theta}(r) = \infty$. Then I have proved[6] that

$$\log \log \frac{M(r)}{R} \geqq \pi \int_0^{\beta r} \frac{dr}{r\bar{\theta}(r)} - \text{const.} \qquad (0 < \beta < 1),$$

where β is any positive number less than 1.

Hence if we put $\alpha = \frac{\beta}{k}$ $(0 < \alpha < 1)$, then we have from (41),

$$\log T(r) \geqq \pi \int_0^{\alpha r} \frac{dr}{r\bar{\theta}(r)} - \text{const.} \quad (0 < \alpha < 1).$$

Hence we have

6) M. TSUJI, A theorem on the majoration of harmonic measure and its applications (which will appear in this Journal).

THEOREM. 3. *Let $w(z)$ be one-valued and regular in an infinite domain Δ and on its boundary Γ, such that $|w(z)| > R$ in Δ and $|w(z)| = R$ on Γ and $M(r) = \operatorname{Max}_{\theta r} |w(z)|$,*

$$T(r) = T(r,\Delta) = \int_0^r \frac{S(r)}{r}\,dr, \qquad (S(r) = S(r, \Delta)).$$

Then

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r},$$

$$\frac{1}{\alpha(R)} T(r) + \text{const.} \leqq \log\frac{M(r)}{R} \leqq \alpha(R)\,\frac{k+1}{k-1} T(kr) + \text{const.} \quad (k>1),$$

$$\log T(r) \geqq \pi \int_0^{\alpha r} \frac{dr}{r\theta(r)} - \text{const.} \quad (0 < \alpha < 1),$$

where

$$\alpha(R) = \frac{2R^2}{1+R^2} (R \geqq 1), \ = \frac{1+R^2}{2R^2} (R \leqq 1).$$

7. Extension of Theorem 1 and 2. In § 1, we introduced a metric in $|w| < 1$ by

$$(w,\ 0) = 2|w|/(1 + |w|^2).$$

Now

$$g(w,\ 0) = \log(1/|w|), \quad |w| = e^{-g(w,0)}$$

is the Green's function of $|w| < 1$, with $w = 0$ as its pole, so that

$$(w,\ 0) = 2e^{-g(w,0)}/(1 + e^{-2g(w,0)}).$$

Suggested by this form, we will introduce a metric in a domain D on the Riemann sphere K, which is bounded by p analytic Jordan curves $C_1, \cdots, C_p$. Let $g(w,\ a)$ be the Green's function of D, with a as its pole. We define the distance $(a,\ b)$ of any two points a, b of D by

(43) $$(a,\ b) = 2e^{-g(a,b)}/(1 + e^{-2g(a,b)}), \quad (0 \leqq (a,\ b) < 1).$$

Since $g(a,\ b) = g(b,\ a)$,

$$(a,\ b) = (b,\ a).$$

Similarly as § 1, we see that

$$\log(1/(w,\ a)) = \log((1 + e^{-2g(w,a)})/2e^{-g(w,a)}) \geqq 0$$

and its normal derivatives vanish on C_i and if $w = w(z)$ is a regular function of z, then

(44) $$\Delta \log\frac{1}{(w(z),\ a)} = \frac{4e^{2g}}{(1+e^{2g})^2} D[g] \geqq 0, \qquad (g = g(w,\ a)),$$

where

$$D[g] = \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2, \qquad (z = x + iy).$$

Hence $\log(1/(w(z),\ a)) \geqq 0$ is a subharmonic function of z. Let Δ be an infinite domain on the z-plane and $w(z)$ be one-valued and meromorphic in Δ and on its boundary Γ, such that the value $w(z)$ in Δ belongs to a domain D on the w-sphere, bounded by p analytic Jordan curves $C_1, \cdots, C_p$

and the value $w(z)$ on Γ belongs to one of C_i. We define θ_r, Δ_r, $L(r)$, $\lambda(r)$ as §2 and let

$$S(r) = \frac{1}{I(D)} \iint_{\Delta_r} \frac{|w'(re^{i\theta})|^2}{(1+|w(re^{i\theta})|^2)^2} r dr d\theta \tag{45}$$

be the mean number of sheets of the Riemann surface generated by $w=w(z)$ on the w-sphere, where $I(D)$ is the area of D.

We put for any a in D,

$$m(r, a) = \frac{1}{2\pi} \int_{\theta_r} \log \frac{1}{(w(re^{i\theta}), a)} d\theta,$$

$$T(r, a) = m(r, a) + N(r, a),$$

where (w, a) is the metric in D. Then as §3, we have

$$m'(r, a) + \frac{n(r, a)}{r} = \frac{1}{2\pi r} \iint_{\Delta_r} \Delta \log \frac{1}{(w, a)} r dr d\theta \tag{46}$$

$$= \frac{2}{\pi r} \iint_{\Delta_r} \frac{e^{2g}}{(1+e^{2g})^2} D[g] r dr d\theta, \qquad (g = g(w, a)).$$

Hence if we put

$$S(r, a) = \frac{2}{\pi} \iint_{\Delta_r} \frac{e^{2g}}{(1+e^{2g})^2} D[g] r dr d\theta, \tag{47}$$

then

$$m'(r, a) + \frac{n(r, a)}{r} = \frac{S(r, a)}{r},$$

$$T(r, a) = m(r, a) + N(r, a) = \int_0^r \frac{S(r, a)}{r} dr + \text{const.} \tag{48}$$

Hence $T(r, a)$ is an increasing convex function of $\log r$. $S(r, a)$ has the following geometrical meaning. Let $h(w, a)$ be the conjugate harmonic function of $g(w, a)$, then

$$v(z) = e^{-(g+ih)}, \qquad (g = g((w, a), \; h = h(w, a)) \tag{49}$$

is a regular function of z, which is many-valued in general. By a suitable corss cuts, we change Δ_r into a simply connected domain Δ_r^0. Since $|v(z)| < 1$, Δ_r is mapped by $v = v(z)$ on the lower half of the v-sphere, whose area is $\pi/2$. Since

$$S(r, a) = \frac{2}{\pi} \iint_{\Delta_r} \frac{|v'|^2}{1+|v|^2)^2} r \, dr \, d\theta,$$

$S(r, a)$ is the mean number of sheets of the Riemann surface generated by $v = v(z)$ on the v-sphere.

Similarly as Theorem 1 and 2, we can prove the following theorems.

THEOREM 4. *Let $w(z)$ be one-valued and meromorphic in an infinite domain Δ and on its boundary Γ, such that the value $w(z)$ in Δ belongs to* *a domain D on the w-sphere, which is bounded by p analytic Jordan curves, $C_1, \cdots, C_p$ and the value $w(z)$ on Γ belongs to one of C_i. Then $T(r, a)$ is an increasing convex function of* $\log r$, *such that*

$$T(r, a) = \int_0^r \frac{S(r, a)}{r} dr + \text{const.} = T(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right),$$

where $S(r, a)$ is the mean number of sheets of the Riemann surface generated by $v = e^{-(g+ih)}$ on the v-sphere and

$$T(r) = \int_0^r \frac{S(r)}{r} dr,$$

where $S(r)$ is the mean number of sheets of the Riemann surface generated by $w = w(z)$ on the w-sphere.

THEOREM 5.[7]

$$(p + q - 2)T(r) \leqq \sum_{i=1}^{q} N(r, a_i) + \Lambda(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right).$$

REMARK. We see easily

$$\log(1/[w(re^{i\theta}), a]) = \log(1/(w(re^{i\theta}), a)) + O(1),$$

so that Theorem 4 becomes

$$\frac{1}{2\pi}\int_{\theta_r} \log \frac{1}{[w(re^{i\theta}), a]} d\theta + N(r, a) = T(r) + O\left(\int_{r_0}^r \frac{L(r)}{r} dr\right).$$

In this form, Theorem 4 was proved by Y. Tumura.[4]

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7) Y. TUMURA, l. c. 1).