ON THE STRONG SUMMABILITY OF POWER SERIES AND FOURIER SERIES

GEN-ICHIRÔ SUNOUCHI

(Received September 10, 1954)

Let

$$f(z)=\sum_{n=0}^{\infty} c_n z^n=\sum_{n=0}^{\infty} c_n r^n e^{in\theta}$$

be a function analytic for $r=|z|<1$. If for some $\lambda>0$, the integral

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta$$

remains bounded when $r\to 1$, the function $f(z)$ is said to belong to the class H^{λ}. If $\lambda>1$, a necessary and sufficient condition for the function $f(z)$ to belong to the class H^{λ}, is that the real part of the series

$$\sum_{n=0}^{\infty} c_n e^{in\theta}$$

is the Fourier series of a function of the class L^{λ}.

H. C. Chow [1] proved that, if $f(z)\in H^{\lambda}$ $(1<\lambda\leqq 2)$, then for almost every θ

$$\sum_{\nu=0}^{n} |\sigma_{\nu}^{\alpha-1}(\theta)-f(e^{i\theta})|^{q}=o(n)$$

where $\alpha>1/\lambda$, $0<q\leqq\lambda/(\lambda-1)$ and $\sigma_n^{\alpha}(\theta)$ is the n-th (C,α)-means of the series

$$\sum_{n=0}^{\infty} c_n e^{in\theta}.$$

In the present note, we shall prove a more precise theorem:

THEOREM. *If* $f(z)\in H^{\lambda}$ $(1\leqq\lambda<2)$, *then*

$$\sum_{\nu=0}^{n} |\sigma_{\nu}^{\alpha-1}(\theta)-f(e^{i\theta})|^{q}=o(n), \qquad \text{a. e.}$$

where $\alpha=1/\lambda, 0<q<\lambda/(\lambda-1)$.

The proof of this theorem depends closely upon the argument of Zygmund [2]. The following lemmas are proved in the cited Zygmund's paper. For the sake of simplicity, we write by $f(\theta)$ the real part of $f(e^{i\theta})$.

LEMMA 1. *Let* $f(\theta)$ *be a function of the class* $L^{\lambda}(\lambda\geqq 1)$, *and let* $U(r,\theta)$ *be the Poisson integral of* $f(\theta)$. *If*

$$\left|\int_{0}^{\theta} |f(t)|^{\lambda} dt\right| \leqq \mu|\theta| \qquad \textit{for } |\theta|\leqq\pi,$$

then

$$|U(r,\theta)| \leqq C\left\{\mu\left(1+\frac{|\theta|}{\delta}\right)\right\}^{1/\lambda}, \tag{1}$$

and

$$\left|\int_0^\theta |U(rt)|^\lambda\, dt\right| \leqq C\mu|\theta|, \tag{2}$$

where $\delta = 1-r$.

LEMMA 2. *Let* $\mathfrak{E}$ *be an arbitrary perfect set of positive measure and of period* 2π, *and let* $\varphi(\theta)$ *denote the function equal to o in* $\mathfrak{E}$ *and to d if* θ *belongs to an interval contiguous to* $\mathfrak{E}$ *and of length d. Then for almost every point* $\theta \in \mathfrak{E}$ *the integral*

$$\int_{-\pi}^{\pi} \frac{\varphi^{\alpha}(\theta+u)}{|u|^{\alpha+1}}\,du$$

is finite for any positive number a.

LEMMA 3. *Let* $U(r,\theta)$ *and* $V(r,\theta)$ *be the Poisson integral of* $f(\theta)$ *and the harmonic function conjugate to* $U(r,\theta)$ *respectively. If* $f \geqq 0$, *then*

$$\left|\frac{d}{d\psi}U(r,\psi)\right| \leqq \frac{C}{\delta}U(r,\psi), \quad \left|\frac{d}{d\psi}V(r,\psi)\right| \leqq \frac{C}{\delta}U(r,\psi),$$

where $\delta = 1-r$.

PROOF OF THE THEOREM. Let $\mathfrak{E}$ be a perfect set situated in the interval $(-\pi,\pi)$ such that

$$\left|\int_\theta^{\theta+h} |f(u)|^\lambda\, du\right| \leqq \mu|h| \qquad \text{for } \theta\in\mathfrak{E} \text{ and } |h| \leqq \pi.$$

We can assume for simplicity that the point $\theta = 0$ belongs to $\mathfrak{E}$, and we investigate for $\theta = 0$.

Let

$$f(z) = \sum_{\nu=0}^{n} c_\nu z^\nu,$$

and $\sigma_n^\alpha(\theta)$ be the n-th (C,α)-means of the power series of $f(e^{i\theta})$. Further let

$$t_n^\alpha(\theta) \equiv \sigma_n^{\alpha-1}(\theta) - \sigma_n^\alpha(\theta) = \frac{1}{\alpha A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\nu c_\nu e^{i\nu\theta},$$

and

$$t_n^\alpha = t_n^\alpha(0), \quad \sigma_n^\alpha = \sigma_n^\alpha(0).$$

Then we have

$$\sum_{\nu=1}^{\infty} \alpha A_\nu^\alpha t_\nu^\alpha z^\nu = \frac{zf'(z)}{(1-z)^\alpha},$$

and applying the Hausdorff-Young theorem,

$$\left\{\sum_{\nu=1}^{\infty}\alpha^q(A_\nu^\alpha)^q\,|t_\nu^\alpha|^q\,r^{\nu q}\right\}^{1/q}\leqq\left\{\frac{r^p}{2\pi}\int_{-\pi}^{\pi}\left|\frac{f'(re^{i\psi})|}{1-re^{i\psi}|^\alpha}\right|^p d\psi\right\}^{1/p},$$

where $z = re^{i\psi}$ and $p = q/(q-1)$.

Let $\delta = 1 - r$, and by lemma 3

$$\left\{\sum_{\nu=1}^{\infty}(A_\nu^\alpha)^q|t_\nu^\alpha|^q\gamma^{\nu q}\right\}^{1/q}\leqq\frac{C}{\delta}\left\{\int_{-\pi}^{\pi}\frac{U^p(r,\psi)}{\Delta_r^{\alpha p}(\psi)}d\psi\right\}^{1/p}$$

where

$$\Delta_r(\psi) = (1 - 2r\cos\psi + r^2)^{1/2}.$$

Without loss of generality, we may assume $f(\theta) \geqq 0$.

If $P_r(\psi)$ is the Poisson kernel, then

$$\int_{-\pi}^{\pi}\frac{U^p(r,\psi)}{\Delta_r^{\alpha p}(\psi)}d\psi = \int_{-\pi}^{\pi}\frac{U^{p-1/\alpha}(r,\psi)}{\Delta_r^{\alpha p}(\psi)}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)P_r(\psi-u)du\right\}^{1/\alpha}d\psi.$$

Since $1/\alpha = \lambda \geqq 1$, applying Jensen's inequality, the last term is less than

$$C\int_{-\pi}^{\pi}\frac{U^{p-1/\alpha}(r,\psi)}{\Delta_r^{\alpha p}(\psi)}\int_{-\pi}^{\pi}f^\lambda(u)P_r(\psi-u)\,du d\psi$$

$$\leqq C\int_{-\pi}^{\pi}f^\lambda(u)du\int_{-\pi}^{\pi}\frac{U^{p-1/\alpha}(r,\psi)P_r(\psi-u)}{\Delta_r^{\alpha p}(\psi)}d\psi,$$

which we decompose into four integrals, extended over the four squares

$$0\leqq\pm\psi\leqq\pi,\ \ 0\leqq\pm u\leqq\pi.$$

It is sufficient to consider one of them, since the other may be estimated simillarly. For instance, let us consider the integral

$$S(r) = \int_0^\pi f^\lambda(u)du\int_0^\pi\frac{U^{p-1/\alpha}(r,\psi)P_r(\psi-u)}{\Delta_r^{\alpha p}(\psi)}d\psi$$

$$= \int_0^\delta\int_0^\pi + \int_\delta^\pi\int_0^{u/2} + \int_\delta^\pi\int_{u/2}^\pi \tag{3}$$

$$= S_1(r) + S_2(r) + S_3(r),$$

say. If we put

$$\int_0^\pi\frac{U^{p-1/\alpha}(r,\psi)P_r(\psi-u)}{\Delta_r^{\alpha p}(\psi)}d\psi = \int_0^{2\delta} + \int_{2\delta}^\pi = A(r,u) + B(r,u),$$

then, by the inequalities

$$\Delta_r(\psi)\geqq\delta,\ \ \Delta_r(\psi)\geqq C\psi\ \ (0\leqq\psi\leqq\pi)$$

and lemma 1, we obtain

$$A(r,u)\leqq C\mu^{\alpha(p-1/\alpha)}\delta^{-\alpha p}\int_0^\delta P_r(\psi-u)d\psi$$

$$\leqq C\mu^{\alpha p-1}\,\delta^{-\alpha p}.$$

Since in the interval $0 \leqq \mu \leqq \delta,\ 2\delta \leqq \psi \leqq \pi$,

$$P_r(\psi - u) \leqq P_r\left(\frac{1}{2}\psi\right) \leqq C\delta\psi^{-2},$$

we have, again making use of lemma 1,

$$B(r, u) \leqq C\int_{2\delta}^{\pi} \frac{(\mu\psi^{\delta-1})^{\alpha(p-1/\alpha)}\delta\psi^{-2}}{\psi^{p\alpha}} d\psi$$

$$\leqq C\mu^{\alpha p-1}\delta^{2-\alpha p}\int_{2\delta}^{\infty} \psi^{-3} d\psi$$

$$\leqq C\mu^{\alpha p-1}\delta^{2-\alpha p}\delta^{-2} \leqq C\mu^{\alpha p-1}\delta^{-\alpha p},$$

and hence

$$\text{(4)} \qquad S_1(r) \leqq C\mu^{\alpha p-1}\delta^{-\alpha p}\int_0^{\delta} f^{\lambda}(u)du$$

$$\leqq C\mu^{\alpha p-1}\delta^{-\alpha p}\mu^{\delta} \leqq C\mu^{\alpha p}\delta^{1-\alpha p}.$$

The inner integral of $S_2(r)$ is

$$\int_0^{u/2} \frac{U^{p-1/\alpha}(r, \psi)P_r(\psi - u)}{\Delta_r^{\alpha p}(\psi)} d\psi$$

$$\leqq P_r\left(\frac{1}{2}u\right)\int_0^{u/2} \frac{U^{p-1/\alpha}(r, \psi)}{\Delta_r^{\alpha p}(\psi),} d\psi,$$

$$\leqq P_r\left(\frac{1}{2}u\right)\left\{\int_0^{u/2} U^{\lambda}(r, \psi)d\psi\right\}^{p\alpha-1}\left\{\int_0^{u/2} \frac{1}{\Delta_r^{p\alpha/(2-p\alpha)}(\psi)} d\psi\right\}^{2-p\alpha}.$$

by Hölder's inequality. From lemma 1, the last expression does not exceed

$$\leqq C\delta u^{-2}(\mu u)^{p\alpha-1}\left(\int_0^{\delta/2} + \int_{\delta/2}^{u/2}\right)^{2-p\alpha}$$

$$\leqq C\mu^{p\alpha-1}\delta\, u^{p\alpha-3}\left(\delta^{\frac{2-2p\alpha}{2-p\alpha}}\right)^{2-p\alpha}$$

$$\leqq C\mu^{p\alpha-1}\delta^{3-2p\alpha}u^{p\alpha-3}.$$

Since $\theta = 0$ is the Lebesgue point of $f^{\lambda}(u)$, we obtain

$$S_2(r) \leqq C\mu^{p\alpha-1}\delta^{3-2p\alpha}\int_{\delta}^{\pi} \frac{f^{\lambda}(u)}{u^{3-p\alpha}} du$$

$$\text{(5)} \qquad \leqq C\mu^{p\alpha-1}\delta^{3-2p\alpha}\mu\delta^{-2+p\alpha}$$

$$\leqq C\mu^{p\alpha}\delta^{1-p\alpha}.$$

Concerning the integral $S_3(r)$, we have

$$S_3(r) = C\int_{\delta}^{\pi} \frac{f^{\lambda}(u)}{\Delta_r^{\alpha p}(u/2)} du\int_{u/2}^{\pi} U^{p-1/\alpha}(r, \psi)P_r(\psi - u)d\psi$$

$$\leqq C\int_{\delta}^{\pi} \frac{f^{\wedge}(u)}{u^{\alpha p}} du \left\{\int_{-\pi}^{\pi} U(r, \psi)P_r(\psi - u)d\psi\right\}^{p-1/\alpha}$$

(since $0 < 1/\alpha \leqq 1$ by Hölder's inequality)

$$\leqq C\int_{\delta}^{\pi} \frac{f^{\wedge}(u)}{u^{\alpha p}} U^{p-1/\alpha}(r^2, u)\, du$$

$$\leqq C\Big(\int_{\mathfrak{E}(\delta)} + \int_{\mathfrak{F}(\delta)}\Big) = J(r) + K(r),$$

say, where $\mathfrak{E}$ (δ) is the portion of $\mathfrak{E}$ contained in the interval (δ, π) and $\mathfrak{F}(\delta) = (\delta, \pi) - \mathfrak{E}(\delta)$. Since

$$U(r^2, u) \leqq C\mu^{\alpha} \quad \text{for } u \in \mathfrak{E}(\delta),$$

we obtain by lemma 1

$$\begin{aligned} J(r) &\leqq C\mu^{\alpha(p-1/\alpha)} \int_{\delta}^{\pi} \frac{f^{\wedge}(u)}{u^{\alpha p}} du \\ &\leqq C\mu^{\alpha p-1}\mu\delta^{1-\alpha p} \\ &\leqq C\mu^{\alpha p}\delta^{1-\alpha p}. \end{aligned}$$

The set $\mathfrak{F}(\delta)$ consists of an enumerable sequence of intervals $d_i = (a_i, b_i)$, and hence, applying lemma 1 to the function $U(r^2, u)$, we get

$$\begin{aligned} K(r) &\leqq C\,\mu^{\alpha p-1}\sum_i \int_{a_i}^{b_i} \frac{f^{\wedge}(u)}{u^{\alpha p}}\left(1 + \frac{d_i}{\delta}\right)^{p\alpha-1} du \\ &\leqq C\,\mu^{\alpha p-1}\left\{\sum_i \int_{a_i}^{b_i} \frac{f^{\wedge}(u)}{u^{\alpha p}} du + \sum_i \int_{a_i}^{b_i} \frac{f^{\wedge}(u)}{u^{\alpha p}}\left(\frac{d_i}{\delta}\right)^{p\alpha-1} du\right\} \\ &\leqq C\,\mu^{\alpha p-1}\left\{\int_{\delta}^{\pi} \frac{f^{\wedge}(u)}{u^{\alpha p}} du + \delta^{(p\alpha-1)} \int_{\delta}^{\pi} \frac{f^{\wedge}(u)\varphi^{p\alpha-1}(u)}{u^{\alpha p}} du\right\} \\ &\leqq C\,\mu^{\alpha p-1}\left\{\mu\delta^{1-p\alpha} + \delta^{1-p\alpha}\int_{\delta}^{\pi} \frac{f^{\wedge}(u)\varphi^{p\alpha-1}(u)}{u^{\alpha p}} du\right\} \\ &\leqq C\mu^{\alpha p}\delta^{1-\alpha p} + C\,\mu^{\alpha p-1}\delta^{1-p\alpha}\int_{\delta}^{\pi} \frac{f^{\wedge}(u)\varphi^{\alpha p-1}(u)}{u^{\alpha p}} du. \end{aligned}$$

Thus we obtain

$$\left\{\sum_{\nu=1}^{\infty}(A_{\nu}^{\alpha})^{q}\,|t_{\nu}^{\alpha}|^{q}\, r^{\nu q}\right\}^{1/q}$$

$$\leqq \frac{C}{\delta}\left\{C\,\mu^{p\alpha}\delta^{1-p\alpha} + C\,\mu^{\alpha p-1}\,\delta^{1-p\alpha}\int_{-\pi}^{\pi} \frac{f^{\wedge}(u)\varphi^{p\alpha-1}(u)}{|u|^{p\alpha}} du\right\}^{1/q},$$

or more generally

$$\sum_{\nu=1}^{\infty}(A_\nu^\alpha)^q\,|t_\nu^\alpha(\theta)|\;r^{\nu q}$$

$$\leqq \left\{C_1\mu^{p\alpha}\delta^{1-p\alpha-p}+C_2\mu^{p\alpha-1}\delta^{1-p\alpha-p}\int_{-\pi}^{\pi}\frac{f^\lambda(u+\theta)\varphi^{p\alpha-1}(u+\theta)}{|u|^{p\alpha}}du\right\}^{q/p}.$$

Since $p\alpha-1>0$, by lemma 2

$$\int_{-\pi}^{\pi}\frac{\varphi^{p\alpha-1}(u+\theta)}{|u|^{p\alpha}}du<\infty,$$

and we can prove (see Zygmund [2])

$$\int_{-\pi}^{\pi}\frac{\varphi^{p\alpha-1}(u+\theta)f^\lambda(u+\theta)}{|u|^{p\alpha}}du<\infty.$$

Put $r=1-1/(n+1)$, then

$$\sum_{\nu=1}^{n}A_\nu^{\alpha q}\,|\sigma_n^{\alpha-1}(\theta)-\sigma_n^\alpha(\theta)|^q\leqq Ce^q\sum_{\nu=1}^{n}A_\nu^{\alpha q}\,|\sigma_n^{\alpha-1}(\theta)-\sigma_n^\alpha(\theta)|^q\,r^{\nu q}$$

$$\leqq\left\{C_1\mu^{p\alpha}n^{p(\alpha+1)-1}+C_2\,\mu^{p\alpha-1}\,n^{p(\alpha+1)-1}\int_{-\pi}^{\pi}\frac{\varphi^{p-1}(u+\theta)f^\lambda(u+\theta)}{|u|^{p\alpha}}du\right\}^{q/p}$$

$$=O(n^{q(\alpha+1-1/p)})=O(n^{q\alpha+1}),$$

by lemma 2.

Therefore by Abel's transformation,

$$\sum_{\nu=1}^{n}|\sigma_n^{\alpha-1}(\theta)-\sigma_n^\alpha(\theta)|^q=O(n),\qquad\text{a. e.}$$

in which we can replace $O(n)$ by $o(n)$, using the argument due to Zygmund [2]. Thus we have proved the theorem completely.

LITERATURES

[1] H. C. CHOW, Theorems on power series and Fourier series, Proc. London Math. Soc., 1(1951), 206-216.

[2] A. ZYGMUND, On the convergence and summability of power series on the circle of convergence (II), Proc. London Math. Soc., 47(1942), 326-356.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.