ON THE EULER SUMMABILITY OF A CLASS OF DIRICHLET SERIES

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The object of this paper is to study the Euler-Knopp summability, denoted $\mathfrak{E}(r)$, of Dirichlet series $\sum_{n=0}^{\infty} a_n e^{-r_n z}$ for the case in which $r_n = (n+1)^\alpha$ for $0 < \alpha < 1$. If $\alpha = 1$ we have a Taylor series under the substitution $\xi = e^{-z}$ and this problem has been studied extensibely by Agnew [1], Knopp [3] and many others. N. Obreschkoff [4] has made a study of this problem in the case $r_n = \log(n+1)$. The Euler-Knopp series-to-series transformation, carries the series $\sum_{n=0}^{\infty} a_n$ into the series $\sum_{n=0}^{\infty} A_n$ where A_n is defined by

$$A_n = \sum_{k=0}^{n} \binom{n}{k} r^{k+1}(1-r)^{n-k} a_k, \qquad 0 < r < 1 \tag{1}$$

for $n = 0, 1, 2, \ldots$. This transformation carries convergent series into series that converge to the same sum.

The following theorem is proved.

THEOREM. *Let* $A_n = \sum_{k=0}^{n} \binom{n}{k} r^{k+1}(1-r)^{n-k} a_k \, e^{-(k+1)^\alpha z_0}$. *Suppose* $\sum_{p=0}^{\infty} A_p r^{-p}$ *converges for some* $0 < r_0 < 1$. *Then* $\sum_{n=0}^{\infty} a_n e^{-(n+1)^\alpha z}$ *is absolutely summable* $\mathfrak{E}(r)$ *for* $r > r_0$ *for those* z *satisfying the inequality* $\pi(1-\alpha)/2 > |\mathrm{Arg}(z - z_0)|$.

PROOF. As is well known we can solve equations (1) for the a_k to obtain

$$a_k = \sum_{p=0}^{k} \binom{k}{p} r^{-p-1}\left(1 - \frac{1}{r}\right)^{k-p} A_p \tag{2}$$

for $k = 0, 1, 2, \ldots$.

As applied to the series $\sum_{n=0}^{\infty} a_n e^{-(n+1)^\alpha z_0}$ this implies that

$$a_k e^{-(k+1)^\alpha z_0} = \sum_{p=0}^{k} \binom{k}{p} r^{-p-1}\left(1 - \frac{1}{r}\right)^{k-p} A_p$$

and the series whose convergence is to be established can be written in the form

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$$\sum_{n=0}^{\infty}(1-r)^n\sum_{k=0}^{n}\binom{n}{k}r^{k+1}(1-r)^{-k}a_k e^{-(k+1)^{\alpha}z}\equiv\sum_{n=0}^{\infty}(1-r)^n B_n \tag{3}$$

where

$$B_n=\sum_{k=0}^{n}\binom{n}{k}r^{k+1}(1-r)^{-k}e^{-(k+1)^{\alpha}(z-z_0)}\sum_{p=0}^{k}\binom{k}{p}r^{-p-1}\left(1-\frac{1}{r}\right)^{k-p}A_p. \tag{4}$$

Formally reversing the order of summation in the right member of (4) yields the result

$$B_n=\sum_{p=0}^{n}r^{-p-1}\left(1-\frac{1}{r}\right)^{-p}A_p\sum_{k=p}^{m}\binom{n}{k}\binom{k}{p}(1-r)^{-k}\left(1-\frac{1}{r}\right)^{k}e^{-(k+1)^{\alpha}(z-z_0)} \tag{5}$$

Now $\binom{n}{k}\binom{k}{p}=\binom{n}{p}\binom{n-p}{k}$. By use of the Mellin transform [2, p. 116] we may show that if $s=\sigma+it$

$$e^{-(k+p+1)^{\alpha}(z-z_0)}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(z-z_0)^{-s}\frac{\Gamma(s)}{(k+p+1)^{\alpha s}}\,ds$$

if $\sigma>0$ and $\Re(z-z_0)>0$, the latter inequality holding by our hypothesis. Also by the theory of the Gamma function it follows that

$$(k+p+1)^{-\alpha s}=\frac{1}{\Gamma(\alpha s)}\int_0^{\infty}\xi^{\alpha s-1}e^{-(k+p+1)\xi}d\xi$$

valid if $\Re(s)>0$. Combining there two results we have

$$e^{-(k+p+1)^{\alpha}(z-z_0)}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(z-z_0)^{-s}\frac{\Gamma(s)}{\Gamma(\alpha s)}\int_0^{\infty}\xi^{\alpha s-1}e^{-(k+p+1)}d\xi\,ds$$

valid if $\sigma>0$ and $\Re(z)>\Re(z_0)$. Thus after some simplifications equation (5) may be rewritten in the form

$$B_n=\sum_{p=0}^{n}A_p\binom{n}{p}(1-r)^{-p}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(z-z_0)^{-s}\left(\frac{\Gamma(s)}{\Gamma(\alpha s)}\right)\int_0^{\infty}\xi^{\alpha s-1}e^{-(p+1)\xi}(1-e^{-\xi})^{n-p}\,d\xi\,ds. \tag{6}$$

Let us now consider the series

$$\sum_{n=0}^{\infty}(1-r)^n B_n, \tag{7}$$

Under the summation sign in (6), let A_p be replaced by $|A_p|$ and let the integrands be replaced by their absolute values. Let the resulting expression be substituted for B_n in the series (7); and let the order of summation in the result of this substitution be reversed. The resulting series omitting the factor $(2\pi)^{-1}$ then is of the form

$$\sum_{p=0}^{\infty}|A_p|\sum_{n=p}^{\infty}\binom{n}{p}(1-r)^{n-p}\int_{\sigma-i\infty}^{\sigma+i\infty}|(z-z_0)^{-s}|\left|\frac{\Gamma(s)}{\Gamma(\alpha s)}\right|$$
$$\cdot\int_0^{\infty}\xi^{\alpha\sigma-1}e^{-(p+1)\xi}(1-e^{-\xi})^{n-p}\,d\xi\,dt.$$

Since $(1-r)\,(1-e^{-\xi})<1$ for $0<r<1$ and all real ξ it follows that the order of the second summation sign and the integral may be reversed to obtain

$$(8)\qquad \sum_{p=0}^{\infty}|A_p|\int_{\sigma-i\infty}^{\sigma+i\infty}|(z-z_0)^{-s}|\left|\frac{\Gamma(s)}{\Gamma(\alpha s)}\right|\int_0^{\infty}\xi^{\alpha\sigma-1}[1+r(e^{\xi}-1)]^{-p-1}d\xi\,dt.$$

If now the series (8) converges this will imply the absolute convergence of the series (7) and complete the proof of the theorem. To this end we note that a simple calculation shows

$$\int_0^{\infty}\xi^{\alpha\sigma-1}[1+r(e^{\xi}-1)]^{-p-1}\,d\xi=O(r^{-(p+1)}),$$

as p becomes infinite. Also we recall that [2, p. 116] $s=\sigma+it$.

$$|\Gamma(s)|=e^{-(\pi/2)|t|}|t|^{\sigma-1/2}(\sqrt{2\pi}+O(|t|)),$$

If this is taken into account it is easily seen that the first integral in the expression (8) converges for those z for which

$$(9)\qquad \pi(1-\alpha)/2>|\operatorname{Arg}(z-z_0)|.$$

Therefore the series (8) will converge and the series (7) converge absolutely for those z satisfying (9) if

$$(10)\qquad \sum_{p=0}^{\infty}|A_p|r^{-p}\cdot$$

converges. Since $\sum_{p=0}^{\infty}A_p\,r^{-p}$ converges by hypothesis for some $0<r_0<1$ this completes the proof of the theorem.

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