GENERALIZED APPROXIMATELY FINITE W^*-ALGEBRAS

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J. von Neumann has classified factors in some classes, type I, II, III and finite, infinite in his monumental works "Rings of operators". The really interesting is the theory of type II_1 (type II and finite). As a special one of such factors, F. J. Murray and J. von Neumann [8] investigated approximately finite factors and they have many interesting results. I. Kaplansky [6] generalized this theory to general W^*-algebras. In these theories, the separability condition for the underlying Hilbert space is essential. The purpose of this paper is to generalize these theories in non-separable cases, on the basis of the theory of direct products of W^*-algebras in the preceding paper[7].

In the first section we shall study some preliminary lemmas. The second section will be devoted to the study of factors. A factor will be called to be approximately finite if it is of type II_1 and generated by a family of subfactors of type I which mutually commute. Then two approximately finite factors are algebraically *-isomorphic to each other if and only if the cardinals of families of subfactors mentioned above are identical. Murray and von Neumann's approximately finite factor is considered as a special one.

In the final section, we shall generalize the above considerations for factors to general W^*-algebras. A W^*-algebra will be called to be approximately finite if it is of type II_1 and generated by W^*-subalgebras of type I which mutually commute. Especially if these W^*-subalgebras have no commutative part in every central decomposition, then it is called to be uniformly approximately finite. Then every approximately finite W^*-algebra can be represented as a direct sum of uniformly approximately finite W^*-subalgebras, and a uniformly approximately finite W^*-algebra is a direct product of an approximately finite factor and a commutative W^*-algebra.

1. Preliminaries. In this paper, a W^*-subalgebra of W^*-algebra will be meant a weakly closed self-adjoint subalgebra. Let $\mathbf{S}_\lambda(\lambda\in\Lambda)$ be a family of sets of operators, by $R(\mathbf{S}_\lambda\,;\,\lambda\in\Lambda)$ we shall mean the smallest W^*-subalgebra of full operator algebra which contains all $\mathbf{S}_\lambda$. A W^*-algebra is called *σ-finite* if any family of projections which are mutually orthogonal is at most countable. A W^*-algebra $\mathbf{M}$ is *finite* if $U^*U=I$ implies $UU^*=I$ for any unitary operator $U\in\mathbf{M}$. J. Dixmier [1] showed that in a finite W^*-algebra there exists a unique center valued trace $\natural$.

Let $\mathbf{M}$ be a finite W^*-algebra of type II, then one can represent $\mathbf{M}$ faithfully as a *standard* W^*-algebra (in the sense of I. E. Segal [11]) on a suitable Hilbert space $\mathfrak{H}$. Moreover if $\mathbf{M}$ is σ-finite, then there exists a vector $x\in\mathfrak{H}\,(\|x\|=1)$ with following properties: If we define a positive linear functional τ on $\mathbf{M}$ by

$$\tau(A) = (Ax, x),$$

then τ is a complete trace, that is, $\tau(A^*A) = 0$ if and only if $A = 0$ and $\tau(AB) = \tau(BA)$ for all $A, B \in \mathbf{M}$. Put

$$<A, B> = \tau(B^*A) \text{ and } [[A]] = <A, A>^{\frac{1}{2}},$$

then $<, >$ satisfies the common properties of the inner product and $\mathbf{M}$ becomes a prehilbert space. Let $\mathfrak{K}$ be the completion of this prehilbert space, then $\mathbf{M}$ can be represented as a W^*-algebra on $\mathfrak{K}$ faithfully. It is known that $\mathbf{M}$ on $\mathfrak{H}$ is unitarily equivalent to $\mathbf{M}$ on $\mathfrak{K}$ (cf. [4], [10], [11]).

The next lemma is due to F. J. Murray and J. von Neumann [8; Theorem 1] in the case of factors.

LEMMA 1.1. *Let* $\mathbf{M}$ *be a* σ*-finie finite standard* W^**-algebra and* $[[\cdot]]$ *be the metric as above. Then the strong (weak) closure of a self-adjoint *-algebra in* $\mathbf{M}$ *coincides with the metric closure of it.*

PROOF. Let $\mathfrak{H}$ be the underlying Hilbert space and x be the vector which define the metric $[[\ \cdot\ ]]$. Let $\mathbf{N}$ be a self-adjoint *-algebra in $\mathbf{M}$ and $\mathbf{N}_1$, $\mathbf{N}_2$ be the closure of $\mathbf{N}$ by the strong topology and the metric topology respectively. It is obvious that $\mathbf{N}_1 \subseteqq \mathbf{N}_2$. Let A be an arbitrary element in $\mathbf{N}_1$, then there exists a directed set A_α in $\mathbf{N}$ which converges to A in the metric topology. We can easily choose a subsequence A_{α_i} which converges to A in the metric topology. By an analogous way to the proof of [8, Lemma 1.5.4], we can assume without loss of generality that $\|A_{\alpha_i}\| \leqq K$ for all i. For any $B \in \mathbf{M}'$

$$\|(A_{\alpha_i} - A)Bx\| \leqq \|B\| \, [[A_{\alpha_i} - A]] \to 0 \text{ as } i \to \infty.$$

By the standardness of $\mathbf{M}$, $[\mathbf{M}x] = \mathfrak{H}$. Hence for all $y \in \mathfrak{H}$

$$\|(A_{\alpha_i} - A)y\| \to 0 \quad \text{as } i \to \infty.$$

In other wards, A_{α_i} converges to A in the strong topology. This shows that $\mathbf{N}_1 \supseteqq \mathbf{N}_2$. Thus we have proved that $\mathbf{N}_1 = \mathbf{N}_2$.

LEMMA 1.2. *Let* $\mathbf{M}$ *be a* σ*-finite finite* W^**-algebra and* $\{\mathbf{N}_\lambda;\ \lambda \in \Lambda\}$ *be a family of factors of type* I *in* $\mathbf{M}$ *which commute with each other. Then* $R(\mathbf{N}_\lambda;\ \lambda \in \Lambda)$ *is a factor too.*

PROOF. We can assume without loss of generality that $\mathbf{M}$ is standard on the acting space. Let $\mathbf{S}$ be the algebra which is generated algebraically by all $\mathbf{N}_\lambda$. Then $\mathbf{S}$ is weakly dense in $R(\mathbf{N}_\lambda;\ \lambda \in \Lambda)$. Now we shall define τ and $[[\ \cdot\ ]]$ as above. Let A be an arbitrary central element in $R(\mathbf{N}_\lambda;\ \lambda \in \Lambda)$, then, by the preceding lemma, for any $\varepsilon > 0$ there exists a $B \in \mathbf{S}$ with

$$[[A - B]] < \varepsilon.$$

Then there exist $\mathbf{N}_{\lambda_1}, \ldots, \mathbf{N}_{\lambda_n}$ such that B is contained in the algebra $\mathbf{N} = R(\mathbf{N}_{\lambda_i};\ i = 1, \ldots, n)$. It is clear that $\mathbf{N}$ is a factor of type I. For any $X \in \mathbf{N}$, we have

$$[[(B-\tau(B)I)X-X(B-\tau(B)I)]]=[[(B-A)X-X(B-A)]]<2\varepsilon\|X\|.$$

Hence we have

$$[[B-\tau(B)I]]<2\varepsilon$$

(cf. [2, Lemma 4.7.1]), moreover we have

$$|\tau(B)-\tau(A)|=|\tau(B-A)|\leqq[[A-B]]<\varepsilon.$$

Accordingly

$$[[A-\tau(A)I]]\leqq[[A-B]]+[[B-\tau(B)I]]+[[(\tau(B)-\tau(A))I]]<4\varepsilon.$$

Since ε is arbitrary, we have $[[A-\tau(A)I]]=0$, which shows that $A=\tau(A)I$ and $R(\mathbf{N}_\lambda;\ \lambda\in\Lambda)$ is a factor.

The next lemma can be proved by an analogous way to the proof of [7; Theorem 5] and we shall omit its proof.

LEMMA 1.3. *Let* $\mathbf{N}$ *be a factor of type* I *in a* $\mathbf{W}^*$*-algebra* $\mathbf{M}$, *the latter being not of type* I. *Then* $\mathbf{M}$ *is *-isomorphic to the direct product of* $\mathbf{N}$ *and a* $\mathbf{W}^*$*-algebra* $\mathbf{M}_1$ *which is *-isomorphic to the contraction of* $\mathbf{M}$ *to the range of a minimal projection in* $\mathbf{N}$.

LEMMA 1.4. *Let* $\mathbf{M}$ *be a* $\mathbf{W}^*$*-algebra which is not of type* I, *then, for any positive interger* p, *we have*

$$\mathbf{M}=\mathbf{M}_1\otimes\mathbf{M}_2$$

where $\mathbf{M}_1$ *is a factor of type* I.

PROOF. Consider p families of projections $\{P_{1\lambda}\},\ \ldots,\{P_{p\lambda}\}$ such that all projections are mutually orthogonal and $P_{1\lambda},\ \ldots,\ P_{p\lambda}$ are mutually equivalent for every λ. Choose a maximal p families $\{P_{1\lambda}\},\ \ldots,\ \{P_{p\lambda}\}$ with this property by Zorn's lemma and put

$$P_i=\sum_\lambda P_{i\lambda}\qquad\text{for } i=1,\ \ldots,\ p.$$

Then it is clear that they are mutually orthogonal and equivalent. If $Q=I-\sum_{i=1}^{p}P_i$ is not zero, then $Q\mathbf{M}Q$ is considered as a $\mathbf{W}^*$-algebra on the range of Q which is not of type I. Hence we can easily choose orthogonal equivalent non-zero projections $Q_1,\ \ldots\ldots,\ Q_p$ which are contained in Q. This contradicts to the maximality of the sets $\{P_{1\lambda}\},\ \ldots,\{P_{p\lambda}\}$ and we have $I=\sum_{i=1}^{p}P_i$.

Since P_i are mutually orthogonal and equivalent, we can easily construct a system of matrix units in $\mathbf{M}$ whose diagonals are P_i. The W^*-algebra generated by these matrix units is of type I, hence the lemma is the immediate consequence of the preceding lemma.

LEMMA 1.5. *Let* $\mathbf{N}$ *be a factor of type* I_p *in a* $\mathbf{W}^*$*-algebra* $\mathbf{M}$, *the latter being not of type* I. *Then, for any* $q\geqq p$ *which is divisible by* p, *there exists a factor* $\mathbf{N}_1$ *of type* I_q *which contains* $\mathbf{N}$.

PROOF. Let $r = q/p$. By the preceding lemma, we have

$$\mathbf{M} = \mathbf{N} \otimes \mathbf{M}_1.$$

It is clear that $\mathbf{M}_1$ is not of type I. Hence, we have

$$\mathbf{M}_1 = \mathbf{N}_2 \otimes \mathbf{M}_2,$$

where $\mathbf{N}_2$ is a factor of type I_r, so that

$$\mathbf{M} = \mathbf{N} \otimes \mathbf{N}_2 \otimes \mathbf{M}_2.$$

Put $\mathbf{N}_1 = \mathbf{N} \otimes \mathbf{N}_2$, then $\mathbf{N}_1$ is a factor of type I_q and contains $\mathbf{N}$.

REMARK. In Lemmas 1.3, 1.4 and 1.5, we have assumed that $\mathbf{M}$ is not of type I. In the case of type I, we can state analogous ones under suitable conditions.

Let $\mathbf{M}$ be a W^*-algebra and E be any projection in $\mathbf{M}$, then we shall denote the contraction of $\mathbf{M}$ on the range of E as $\mathbf{M}_E$.

LEMMA 1.6. *Let* $\mathbf{N}$ *be a factor in a finite* $\mathbf{W}^*$*-algebra* $\mathbf{M}$, *then*

$$R(\mathbf{N}, \mathbf{Z}) = \mathbf{N} \otimes \mathbf{Z}$$

where $\mathbf{Z}$ *is the center of* $\mathbf{M}$.

PROOF. As $R(\mathbf{N}, \mathbf{Z})$ is a W^*-subalgebra in a finite W^*-algebra, $R(\mathbf{N}, \mathbf{Z})$ is also finite. Suppose that $R(\mathbf{N}, \mathbf{Z})$ is a σ-finite, then there exists a complete trace τ such as at the beginning of this section. Let τ_1 and τ_2 be its contractions on $\mathbf{N}$ and $\mathbf{Z}$, then τ_1 and τ_2 are traces in these W^*-algebras. As $\mathbf{Z}$ is contained in the center of $R(\mathbf{N}, \mathbf{Z})$, we have

$$(AB)^\natural = A^\natural B$$

where A, B are arbitrary elements in $\mathbf{N}, \mathbf{Z}$ respectively and $\natural$ is the center valued trace in $R(\mathbf{N}, \mathbf{Z})$. It is known that $\tau(X) = \tau(X^\natural)$ in $R(\mathbf{N}, \mathbf{Z})$ and then

$$\tau(AB) = \tau((AB)^\natural) = \tau(A^\natural B) = \tau(\tau_1(A) IB) = \tau_1(A)\ \tau_2(B).$$

Therefore, by an analogous way to [8; Theorem 1], we can prove the lemma, since $\mathbf{N} \cap \mathbf{Z} = \{\alpha I\}$.

Now we shall consider the general case. We can choose a family E_λ of central projections in $\mathbf{M}$ which are mutually orthogonal such that each contraction $R(\mathbf{N}, \mathbf{Z})_{E_\lambda}$ is σ-finite. It is clear that $\mathbf{N}_{E_\lambda}$ are factors. Therefore, we have $R(\mathbf{N}, \mathbf{Z})_{E_\lambda} = \mathbf{N}_{E_\lambda} \otimes \mathbf{Z}_{E_\lambda}$ and this implies that

$$R(\mathbf{N}, \mathbf{Z}) = \sum_\lambda \mathbf{N}_{E_\lambda} \otimes \mathbf{Z}_{E_\lambda}.$$

On the other hand $(\mathbf{N} \otimes \mathbf{Z})_{E_\lambda} = \mathbf{N}_{E_\lambda} \otimes \mathbf{Z}_{E_\lambda}$ and this implies that

$$\mathbf{N} \otimes \mathbf{Z} = \sum_\lambda \mathbf{N}_{E_\lambda} \otimes \mathbf{Z}_{E_\lambda}.$$

Hence we have $R(\mathbf{N}, \mathbf{Z}) = \mathbf{N} \otimes \mathbf{Z}$.

2. Generalized approximately finite factors. In this section, we shall concern with factors.

LEMMA 2.1. *Let* $\mathbf{M}$ *be a factor of type* II_1. *Suppose that there exists a family* $\{\mathbf{M}_\lambda\,;\ \lambda \in \Lambda\}$ *with the following properties*:

(1) *each* $\mathbf{M}_\lambda$ *is of type* I (*necessary finite*),

(2) $\mathbf{M}_\lambda'$*s commute with each other in elmentwise,*

(3) $\mathbf{M} = R(\mathbf{M}_\lambda\,;\ \lambda \in \Lambda)$.

Then there exists a family $\{\mathbf{N}_\mu\,;\ \mu \in \mathrm{M}\}$ *which satisfies the conditions* (2), (3) *and*

(4) *each* $\mathbf{N}_\mu$ *is of type* I_2.

Moreover the cardinal of $\{\mathbf{M}_\lambda\,;\ \lambda \in \Lambda\}$ *as a set is identical with that of* $\{\mathbf{N}_\mu\,;\ \mu \in \mathrm{M}\}$.

PROOF. Split Λ into mutually disjoint subsets $\{\Lambda_\alpha\,;\ \alpha \in A\}$ such that $\Lambda = \bigcup_{\alpha \in A} \Lambda_\alpha$ and each Λ_α is countably infinite. Let $\mathbf{M}^\alpha = R(\mathbf{M}_\lambda\,;\ \lambda \in \Lambda_\alpha)$, then it is clear that there exists an increasing sequence $\{P_n\}$ of factors of type I in $\mathbf{M}$ with $\mathbf{M} = R(\mathbf{P}_n\,;\ n = 1, 2, \dots)$. Hence there exists an increasing sequence $\{\mathbf{Q}_n\}$ of factors of type I_2 in $\mathbf{M}^\alpha$ with $\mathbf{M}^\alpha = R(\mathbf{Q}_n\,;\ n = 1, 2, \dots)$ (cf., [8, §4. 4]). Put $\mathbf{M}_1^\alpha = \mathbf{Q}_1$. By induction, we shall define $\mathbf{M}_n^\alpha$ with following properties

(i) $R(\mathbf{M}_i^\alpha\,;\ i = 1, 2, \dots, n) = \mathbf{Q}_n$,

(ii) $\mathbf{M}_i^\alpha$'s commute with each other.

Suppose that $\mathbf{M}_1^\alpha, \dots, \mathbf{M}_{n-1}^\alpha$ are already defined. By assumptions

$$R(\mathbf{M}_i^\alpha\,;\ i = 1, 2, \dots,\ n-1) = \mathbf{Q}_{n-1} \subset \mathbf{Q}_n$$

and $\mathbf{Q}_{n-1}$, $\mathbf{Q}_n$ are of type $\mathrm{I}_{2^{n-1}}$, I_{2^n} respectively. By an analogous way to the proof of [7, Lemma 4. 1. 2], we can take a factor $\mathbf{M}_{n+1}^\alpha$ of type I_2 such that $R(\mathbf{M}_i^\alpha\,;\ i = 1, 2, \dots, n) = \mathbf{Q}_n$ and $\mathbf{M}_n^\alpha$ commute with $\mathbf{Q}_{n-1}$. Hence the sequence $\{\mathbf{M}_n^\alpha\}$ has the desired properties. Moveover, we have

$$R(\mathbf{M}_n^\alpha\,;\ n = 1, 2, \dots) = R(\mathbf{Q}_n\,;\ n = 1, 2, \dots) = \mathbf{M}^\alpha.$$

If we write the family $\{\mathbf{M}_n^\alpha\,;\ \alpha \in \Lambda_\alpha\,;\ n = 1, 2, \dots\}$ by $\{\mathbf{N}_\mu\,;\ \mu \in M\}$, then it is obvious that this family has the disired properties.

By the construction of $\{\mathbf{N}_\mu\,;\ \mu \in M\}$, the cardinal of this family equals to that of $\{\mathbf{M}_\lambda\,;\ \lambda \in \Lambda\}$.

The following lemma is a slight generalization of a lemma due to M. Nakamura [9] and we shall omit its proof.

LEMMA 2. 2. *Let* $\mathbf{M}$ *be a finite factor which is generated by a finite number of subfactors* $\mathbf{M}_1, \dots, \mathbf{M}_n$ *where* $\mathbf{M}_i$*'s commute with each other. Then the trace on* $\mathbf{M}$ *is multiplicative in the sense of*

$$\tau(A_1 A_2 \dots A_n) = \tau(A_1)\tau(A_2) \dots \tau(A_n)$$

for $A_i \in \mathbf{M}_i$ $(i = 1, 2, \dots, n)$.

THEOREM 2. 1. *Let* $\mathbf{M}$ *be a finite factor. If there exists a family* $\{\mathbf{M}_\lambda\,;\ \lambda \in \Lambda\}$ *of subfactors satisfying the conditions* (1), (2) *and* (3) *in Lemma* 2. 1, *then the cardinal of the family is uniquely determined by* $\mathbf{M}$.

PROOF. Let $\mathfrak{H}$ be the Hilbert space on which $\mathbf{M}$ acts as a standard factor in the sense of I. E. Segal [11]. By Lemma 2. 1, we may assume that each $\mathbf{M}_\lambda$ is of type I_2 and there exists a system of matrix units $\{W_{i,j}^{(\lambda)}\,;\ i, j = 1, 2\}$

in each $\mathbf{M}_\lambda$. Put $U_\lambda = W_{1,2}^{(\lambda)} + W_{2,1}^{(\lambda)}$, then U_λ is unitary and $\mathbf{M}_\lambda$ is generated by $W_{1,1}^{(\lambda)}$ and U_λ. Let $\mathfrak{G}$ be the group generated by all U_λ and $\mathbf{N}_0$ be the commutative *-algebra generated by all $W_{1,1}^{(\lambda)}$ and $\mathbf{N}$ be its weak closure. Then $\mathbf{N}$ is a commutative W^*-algebra in $\mathbf{M}$. We can easily show that $\mathbf{M} = R(\mathbf{N}, \mathfrak{G})$.

Let τ be the trace of $\mathbf{M}$, then by the standardness of $\mathbf{M}$ on $\mathfrak{H}$, there exists a vector $x \in \mathfrak{H}$ such that $\tau(\cdot) = (\cdot\, x, x)$. It is obvious that the contraction of τ on each $\mathbf{M}_\lambda$ is a trace of $\mathbf{M}_\lambda$, hence $\tau(W_{1,1}^{(\lambda)}) = 0$ and $\tau(U_\lambda) = 0$.

Next, for any $U \in \mathfrak{G}$, we shall define the manifold $\mathfrak{M}_U$ by

$$\mathfrak{M}_U = [UAx;\ A \in N].$$

We shall show that if $U \neq I$ then $\mathfrak{M}_U$ is orthogonal to $\mathfrak{M}_I$. Let A, B be arbitrary elements in $\mathbf{N}$, then for any $\varepsilon > 0$ there exist A_0, B_0 in $\mathbf{N}_0$ such that

$$|(UAx,\ Bx) - (UAx, B_0x)| < \varepsilon/2$$

and

$$|(UAx, B_0x) - (UA_0x, B_0x)| < \varepsilon/2.$$

For these U, A_0, B_0, we can find indices $\lambda_1, \ldots, \lambda_n$ such that U, A_0, B_0 are contained in $R(\mathbf{M}_{\lambda_i};\ i = 1, 2, \ldots, n)$. Therefore, we can write as follows:

$$U = U_{\lambda_1}^{\varepsilon_1} \cdots U_{\lambda_n}^{\varepsilon_n},$$

$$A_0 = \sum \alpha_{\lambda_1,\ldots,\lambda_n} W_{1,1}^{(\lambda_1),\lambda_{1,p}} \cdots W_{1,1}^{(\lambda_n),\lambda_{n,p}}, \quad B_0 = \sum \beta_{\alpha_1,\ldots,\alpha_n} W_{1,1}^{(\lambda_1)\lambda_{1,q}} \cdots W_{1,1}^{(\lambda_n)\lambda_{n,q}}$$

where $\varepsilon_i, \lambda_{i,p}, \lambda_{i,q}$ are 0 or 1. Then by using Lemma 2. 2, we have

$$\begin{aligned}(UA_0x, B_0x) &= \sum\sum \alpha_{\lambda_1,\ldots,\lambda_n}\beta_{\lambda_1,\ldots,\lambda_n}(U_{\lambda_1}^{\varepsilon_1}\cdots U_{\lambda_n}^{\varepsilon_n} W_{1,1}^{(\lambda_1),\lambda_{1,p}}\cdots W_{1,1}^{(\lambda_n),\lambda_{n,q}}x,\ W_{1,1}^{(\lambda_1),\lambda_{1,q}}\cdots W_{1,1}^{(\lambda_n),\lambda_{n,q}}x) \\ &= \sum\sum \alpha_{\lambda_1,\ldots,\lambda_n}\beta_{\lambda_1,\ldots,\lambda_n}(U_{\lambda_i}^{\varepsilon_i} W_{1,1}^{(\lambda_i),\lambda_{i,p}}x,\ W_{1,1}^{(\lambda_i),\lambda_{i,q}}x) = 0.\end{aligned}$$

Accordingly, we have

$$\begin{aligned}|(UAx, Bx)| &= |(UAx,\ Bx) - (UA_0x,\ B_0x)| \\ &= (UAx,\ Bx) - (UAx,\ B_0x) - (UAx, B_0x) - (UA_0x,\ B_0x)| \\ &< \varepsilon.\end{aligned}$$

Since ε is arbitrary, this shows that $(UAx,\ Bx) = 0$. This implies that if $U \neq I$, $\mathfrak{M}_U$ is orthogonal to $\mathfrak{M}_I$ and, by this fact, we can easily prove that $\mathfrak{M}_U$ is orthogonal to $\mathfrak{M}_V$ for $U, V \in \mathfrak{G}$ with $U \neq V$.

According to the standardness of $\mathbf{M}$ on $\mathfrak{H}$, $[\mathbf{M}x] = \mathfrak{H}$ and, as we noticed in the above, $\mathbf{M} = R(\mathbf{N}, \mathfrak{G})$. Hence

$$\mathfrak{H} = \sum_{U \in \mathfrak{G}} \oplus \mathfrak{M}_U.$$

It is clear that $\mathfrak{M}_U$ is the image of $\mathfrak{M}_I$ under U. Therefore $\mathfrak{M}_U$ have the same dimension for all $U \in \mathfrak{G}$. Let α_0 be this dimension and α be the cardinal of $\mathfrak{G}$, then $\mathfrak{H}$ has $\alpha_0\alpha$ as its dimension. By the elementary calculations of sets theory, it is easily shown that α is the cardinal of $\{\mathbf{M}_\lambda;\ \lambda \in \Lambda\}$.

Now $\{\mathbf{M}_\mu^0;\ \mu \in M\}$ be any family satisfying the conditions (2), (3), (4) in Lemma 2. 1 and β be its cardinal. Then according to the procedure mentioned above we can constract a unitary group $\mathfrak{G}^0$ and manifolds $\mathfrak{M}_V^0$ for $V \in \mathfrak{G}^0$ such that

$$\mathfrak{H} = \sum_{\Gamma \in \mathfrak{G}^0} \mathfrak{M}^0_{\Gamma},$$

If $\beta > \alpha$, then, by the construction of $\mathfrak{M}_I$ and $\mathfrak{M}^0_I$, it is easily shown that the dimension β_0 of $\mathfrak{M}^0_I$ is greater than the dimension α_0 of $\mathfrak{M}_I$. Hence we have $\beta_0\beta > \alpha_0\alpha$. On the other hand $\beta_0\beta$ equals to $\alpha_0\alpha$ since both are the dimension of $\mathfrak{H}$. This is a contradiction, that is, $\alpha = \beta$.

REMARKS. In the proof of the preceding theorem, we have employed the abelian W^*-algebra $\mathbf{N}$ in $\mathbf{M}$. We show that $\mathbf{N}$ is maximal abelian and moreover regular in the sense of J. Dixmier [2]. Such property of $\mathbf{N}$ is of some interest, but we do not go into its detailed investigations.

By the above theorem, we will give the following definition:

DEFINTION 2. 1. A finite factor $\mathbf{M}$ which is of type II is called to be α-*approximately finite* if there exists a family of subfactors satisfying the conditions (1)-(2) in Lemma 2. 1 and all such factors are called as *approximately finite factors.*

Then we have following corollaries:

COROLLARY 2. 1. *A factor is $\aleph_0$-approximately finite if and only if it is approximately finite in the sense of F. J. Murray and J. von Neumann* [8].

The following is an immediate consequence of Lemma 1. 2 and Theorem 2. 1.

COROLLARY 2. 2. *The condition* (1) *in Lemma* 2. 1 *can be replaced by* (1′) *each $\mathbf{M}_\lambda$ is $\aleph_0$-npproximately finite.*

Now we shall prove the following theorem:

THEOREM 2. 2. $\mathbf{M}$ *and* $\mathbf{N}$ *be α and β-approximately finite factors respectively, then* $\mathbf{M}$ *is *-isomorphic to* $\mathbf{N}$ *if and only if* $\alpha = \beta$.

PROOF. Since the necessity is clear, we shall show the sufficiency. Let $\{\mathbf{M}_\lambda;\lambda \in \Lambda\}$ and $\{\mathbf{N}_\lambda;\lambda \in \Lambda\}$ be the families of factors of $\mathbf{M}$ and $\mathbf{N}$ respectively which satisfy the conditions (1)-(3) in Lemma 2. 1. (By the assumption we may use the same set of indices Λ). By Lemma 2. 1, we can assume that $\mathbf{M}_\lambda$ and $\mathbf{N}_\lambda$ are of type I_2 for all λ. Hence, there exists a *-isomorphism θ_λ from $\mathbf{M}_\lambda$ onto $\mathbf{N}_\lambda$.

Let $\mathbf{M}^0$ and $\mathbf{N}^0$ be *-algebras generated by $\{\mathbf{M}_\lambda;\lambda \in \Lambda\}$ and $\{\mathbf{N}_\lambda;\gamma \in \Lambda\}$ respectively. Let A be an arbitrary element in $\mathbf{M}^0$, then A is expressed in finite sum:

$$A = \sum \alpha_{\lambda_1,\ldots,\lambda_n} A_{\lambda_1} A_{\lambda_2} \ldots A_{\lambda_n}$$

where $A_{\lambda_p} \in \mathbf{M}_{\lambda_p}$. Put

$$\theta(A) = \sum \alpha_{\lambda_1,\ldots,\lambda_n} \theta_{\lambda_1}(A_{\lambda_1})\, \theta_{\lambda_2}(A_{\lambda_2}) \ldots \theta_{\lambda_n}(A_{\lambda_n}).$$

Then we can prove without difficulties that θ is a *-isomorphism between $\mathbf{M}^0$ and $\mathbf{N}^0$. Moreover, by Lemma 2. 2, we have

$$\begin{aligned}\tau(A) &= \sum \alpha_{\lambda_1,\ldots,\lambda_n} \tau(A_{\lambda_1})\, \tau(A_{\lambda_2}) \ldots \tau(A_{\lambda_n}) \\ &= \sum \alpha_{\lambda_1,\ldots,\lambda_n} \tau'(\theta(A_{\lambda_1}))\tau'(\theta(A_{\lambda_2})) \ldots \tau'(\theta(A_{\lambda_n}))\end{aligned}$$

$$= \tau'(\theta(A))$$

where τ and τ' are traces of $\mathbf{M}$ and $\mathbf{N}$ respectively. Accordingly the mapping θ is extended to a mapping from $\mathbf{M}$ onto $\mathbf{N}$ and we can easily prove that this extended mapping is a *-isomorphism from $\mathbf{M}$ onto $\mathbf{N}$(cf. [8, p. 760]). This prove the theorem.

The following theorem is proved in [7] only in the separable case. Now we shall extend it in the non-separable case.

THEOREM 2.3. *The direct product of two approximately finite factors is approximately finite.*

PROOF. Let $\mathbf{M}$ and $\mathbf{N}$ be two approximately finite factors and $\{\mathbf{M}_\lambda;\ \lambda \in \Lambda\}$ and $\{\mathbf{N}_\mu;\ \mu \in M\}$ be the families in $\mathbf{M}$ and $\mathbf{N}$ such as Theorem 2.1. Since $\mathbf{M} \otimes \mathbf{N}$ is a factor (cf. [7, Lemma 14]), we must show that it is approximately finite. It is clear that

$$\mathbf{M} \otimes \mathbf{N} = R(\mathbf{M}_\lambda \otimes \mathbf{N}_\mu;\ \lambda \in \Lambda,\ \mu \in M)$$

and $\mathbf{M}_\lambda \otimes \mathbf{N}_\mu$ are of type I and commute to each other. This shows that $\mathbf{M}_\lambda \otimes \mathbf{N}_\mu$ is approximately finite.

Recently Z. Takeda [12] has introduced the notion of the infinite direct product of operator algebras. We shall give brief considerations to the infinite direct product of approximately finite factors. Let $\{\mathbf{M}_\lambda\}$ be the family of finite factors and τ_λ be their traces. Then formal expression $\otimes_\lambda \tau_\lambda$ can be considered as a positive functional on the algebraical direct product of $\mathbf{M}_\lambda$. By the usual way we can construct a Hilbert space $\mathfrak{H}$ by them. By the *restricted direct product* of $\mathbf{M}_\lambda$, we shall mean the weak closure of the algebraical direct product of $\mathbf{M}_\lambda$ on $\mathfrak{H}$.

By Corollary 2.2, an approximately finite factor $\mathbf{M}$ can be generated by infinite (countable) subfactors $\mathbf{M}_n$ which commute with each other and each of which is approximately finite. By Lemma 2.2, the trace of $\mathbf{M}$ is multiplicative and so $\mathbf{M}$ is the restricted infinite direct product of $\mathbf{M}_n$. Conversely we can easily show that the restricted infinite direct product of approximately finite factors is an approximately finite factor. Thus we have the following:

THEOREM 2.4. *A factor is approximately finite if and only if it is *-isomorphic to the restricted infinite direct product of approximately finite factors.*

Now we shall consider examples. For a given discrete group $\mathfrak{G}$, we can construct the Hilbert space $\mathfrak{H} = L_2(\mathfrak{G})$ as a usual way, i.e., $\mathfrak{H}$ is the set of all complex valued functions $f(x)$ on $\mathfrak{G}$ such that $\sum_{x\in\mathfrak{G}} |f(x)|^2$ is finite and $(f(x), g(x)) = \sum_{x\in\mathfrak{G}} f(x)\overline{g(x)}$ for any $f, g \in \mathfrak{H}$. For any $a \in \mathfrak{G}$, we shall define the operator U_a on $\mathfrak{H}$ as following:

$$U_a f(x) = f(a^{-1}x) \qquad \text{for all } f \in \mathfrak{H}.$$

Then it is known that U_a is unitary. By $\mathbf{W}(\mathfrak{G})$ we shall mean the W^*-algebra generated by $\{U_a;\ a \in \mathfrak{G}\}$. As R. Godement [4] has shown, $\mathbf{W}(\mathfrak{G})$ is finite.

Especially we shall take the group $\mathfrak{G}$ of all those permutations of (1, 2, ...) which move only a finite number of elements. Then F. J. Murray and J. von Neumann [9] proved that $\mathbf{W}(\mathfrak{G})$ is an $\aleph_0$-approximately finite factors. Let $\{\mathfrak{G}_\lambda;\ \lambda \in \Lambda\}$ be a family of such permutation groups and α be the cardinal of Λ. We shall construct a infinite direct product $\mathfrak{G}$ of $\{\mathfrak{G}_\lambda;\ \lambda \in \Lambda\}$, then we can easily show that $\mathbf{W}(\mathfrak{G})$ is an α-approximately finite factor. Thus our α-approximately finite factor exists for every cardianl α.

3. Approximately finite W*-algebras in the large. In this section, we shall introduce the notion of approximate finiteness for general **W***-algebras and study the properties of such **W***-algebras.

DEFINITION 3. 1. A **W***-algebra $\mathbf{M}$ of type II_1 is *approximately finite* if there exists a family $\{\mathbf{M}_\lambda;\lambda \in \Lambda\}$ of **W***-subalgebras with following properties
(1) each $\mathbf{M}_\lambda$ is of type I and contains the center of $\mathbf{M}$ as its center,
(2) $\mathbf{M}_\lambda$ commute with each other,
(3) $\mathbf{M} = R(\mathbf{M}_\lambda;\ \lambda \in \Lambda)$.

DEFINITION 3. 2. Let $\mathbf{M}$ be a **W***-algebra of type I. The type of $\mathbf{M}$ is *uniformly greater than* 1 if, for any non-zero central projection E, there exists a non-zero non-central projection P with $P < E$.

DEFINITION 3. 3. An approximately finite W^*-algebra $\mathbf{M}$ is *uniformly α-approximately finite* if there exists a family $\{\mathbf{M}_\lambda;\ \lambda \in \Lambda\}$ of W^*-subalgebras which satisfies the conditions (1), (2), (3) and
(4) the type of each $\mathbf{M}_\lambda$ is uniformly greater than 1,
(5) the cardinal of $\{\mathbf{M}_\lambda;\ \lambda \in \Lambda\}$ as a set is α.

The unicity of the cardinal of such family will be proved in the Corollary of Theorem 3. 1.

THEOREM 3. 1. *A W^*-algebra $\mathbf{M}$ is uniformly α-approximately finite if and only if $\mathbf{M}$ is *-isomorphic to the direct product of an α-approximately finite factor and a commutative W^*-algebra.*

PROOF OF SUFFICIENCY. We can assume that $\mathbf{M}$ is the direct product of an α-approximately finite factor $\mathbf{A}$ and a commutative W^*-algebra $\mathbf{N}$. There exists a family $\{\mathbf{A}_\lambda\}$ of factors in $\mathbf{A}$ satisfying the conditions (1)-(3) in Lemma 2. 1 and whose cardinal is α. Let $\mathbf{M}_\lambda$ be the direct product of $\mathbf{A}_\lambda$ and $\mathbf{N}$, then it is obvious that the family $\{\mathbf{M}_\lambda\}$ satisfies the conditions (1)-(5) of Definitions in this section. In other words, $\mathbf{M}$ is uniformly α-approximately finite.

To prove the necessity of the theorem, we shall give some lemmas.

LEMMA 3. 1. *In the theorem, we can assume that $\mathbf{M}$ is σ-finite.*

PROOF. There exists a family $\{E_\mu\}$ of central projections which are mutually orthogonal and $\sum E_\mu = I$ and each contraction $\mathbf{M}_{E_\mu}$ of $\mathbf{M}$ on the range of E_μ is σ-finite. Now we shall assume that the theorem is valid for all $\mathbf{M}_{E_\mu}$. Then, for each μ, there exists a family $\{\mathbf{M}_{\mu,\lambda}\}$ of W^*-subalgebras of $\mathbf{M}_{E_\mu}$ satisfyng the conditions (1)-(5). Let

$$\mathbf{M}_\lambda = \sum_\mu E_\mu \mathbf{M}_{\mu,\lambda} E_\mu,$$

then it is clear that $\{\mathbf{M}_\lambda\}$ satisfies the same conditions, that is, $\mathbf{M}$ is uniformly α-approximately finite.

Let $\mathbf{M}$ be a fixed σ-finite α-approximately finite W^*-algebra, then we can define a complete normal trace τ and a metric $[[\ \cdot\ ]]$ as in §1. In the follwoing, these τ, $[[\ \cdot\ ]]$ and a family $\{\mathbf{M}_\lambda;\ \lambda \in \Lambda\}$ of W^*-subalgebras satisfying (1)-(5) will be fixed. Under these notations, we shall prove the following.

LEMMA 3.2. *For given $A_1, \dots, A_m \in \mathbf{M}$ and $\varepsilon > 0$, there exists an integer $p = p(A_1, \dots, A_m, \varepsilon)$ and W^*-subalgebra $\mathbf{N}$ such that:*

(6) *$\mathbf{N}$ is of type I_p,*

(7) *there exist $B_1, \dots, B_m \in \mathbf{M}$ such that*

$$[[A_i - B_i]] < \varepsilon \qquad \text{for } i = 1, \dots, m.$$

PROOF. Let $\mathbf{S}$ be the *-algebra which is generated by all $\mathbf{M}_\lambda$ in algebraical sense, then $\mathbf{S}$ is weakly dense in $\mathbf{M} = R(\mathbf{M}_\lambda;\ \lambda \in \Lambda)$. Therefore, by Lemma 1.1, for any $\varepsilon > 0$ there exist $A_1', \dots, A_m'$ in $\mathbf{S}$ such that

$$[[A_i - A_i']] < \varepsilon/2 \text{ for } i = 1, \dots, m.$$

By the definition of $\mathbf{S}$, there exist $\mathbf{M}_{\lambda_1}, \dots, \mathbf{M}_{\lambda_n}$ such that $A_1', \dots, A_m'$ are contained in the algebra $\mathbf{P} = R(\mathbf{M}_{\lambda_i};\ i = 1, \dots n)$. Since every $\mathbf{M}_{\lambda_i}$ is of type I, $\mathbf{P}$ is of type I (necessary finite). Hence there exists a family $\{E_n\}$ of central projections which are mutually orthogonal and $\sum E_n = I$ and each $\mathbf{P}_{E_n}$ is of type I_n. There exists n_0 such that

$$\left[\left[\sum_{n=n_0}^{\infty} E_n\right]\right] < \varepsilon K/2$$

where

$$K = \operatorname*{Max}_{1 \leqq i \leqq m} (\|A_i\|).$$

Put $E = E_1 + \dots + E_{n_0}$ and $B_i = EA_i'$, then

$$[[A_i - B_i]] = [[A_i - EA_i']] \leqq [[(I-E)A_i]] + [[E(A_i - A_i')]] < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Let $p = n_0!$, then by Lemma 1.4 there exists a W^*-subalgebra $\mathbf{N}_1$ in $\mathbf{M}_E$ which contains $\mathbf{P}_E$ and is of type I_p. By Lemma 1.3, there exists a W^*-subalgebra $\mathbf{N}_2$ in $\mathbf{M}_{(I-E)}$ which is of type I_p. Let

$$\mathbf{N} = \mathbf{N}_1 E + \mathbf{N}_2 (I-E),$$

then $\mathbf{N}$ is of type I_p and $B_i \in \mathbf{N}$. This proves the lemma.

LEMMA 3.3. *If $\alpha = \aleph_0$, then there exists a sequence $\{A_n\}$ of operators in $\mathbf{M}$ such that*

$$\mathbf{M} = R(A_n;\ n = 1, 2, \dots)$$

where $\mathbf{Z}$ is the center of $\mathbf{M}$.

PROOF. Let $\{\mathbf{M}_n;\ n = 1, 2, \ldots\}$ be a family of W^*-subalgebras in the Definition. For each n, there exists a family $\{E_{n,m}\}$ of central projections in $\mathbf{M}_n$ which are mutually orthogonal, $\sum_{m=1}^{\infty} E_{n,m} = I$ and each contraction $\mathbf{M}_{n,m}$ of $\mathbf{M}_n$ on the range of $E_{n,m} (\neq 0)$ is of type I. It is clear that if $E_{n,m} \neq 0$, then

$$\mathbf{M}_{n,m} = \mathbf{P}_{n,m} \otimes \mathbf{Z}_{n,m}$$

where $\mathbf{Z}_{n,m}$ is a factor of type I_m. There exist linear basis in $\mathbf{P}_{n,m}$ and we can choose a countable set $\{A_{n,m,p};\ p = 1, 2, \ldots\}$ which are weakly dense in $\mathbf{P}_{n,m}$. This shows that

$$\mathbf{N}_{n,m} = R(A_{n,m,p},\ \mathbf{Z}_{n,m};\ n, m, p = 1, 2, \ldots).$$

Since the set $\{E_{n,m} A_{n.m,p}\ E_{n,m};\ n, m, p = 1, 2, \ldots\}$ is countable, we can describe them as $A_1, A_2, \ldots$. By the above considerations, it is obvious that

$$\mathbf{M} = R(\mathbf{Z}, A_i,;\ i = 1, 2, \ldots).$$

This proves the lemma.

If $\alpha = \aleph_0$ then, by Lemma 3. 2, $\mathbf{M}$ can be considered to be approximately finite (B) in the sense of F. J. Murray and J. von Neumann [8]. Hence, by an analogous way to [7; §§ 4. 3, 4. 4, 4, 5], we can show the following lemma. We shall omit its proof.

LEMMA 3. 4. *If* $\alpha = \aleph_0$ *then for given* $A_1, \ldots, A_m \in \mathbf{M}$ *any* $p = 1, 2, \ldots$ *and* $\varepsilon > 0$, *there exists an* $n = n(A_1, \ldots, A_{m,p,}\varepsilon)$ *such that for every* $q \geqq n$ *which is divisible by* p *and every* W^**-subalgebra* $\mathbf{N}_0$ *of type* I_p, *there exists a* W^**-subalgebra* $\mathbf{N}$ *with following properties*:

(8) $\mathbf{N}$ *is of type* I_q,

(9) *there exist* $B_1, \ldots, B_m \in \mathbf{M}$ *with* $[[B_i - A_i]] < \varepsilon$,

(10) $\mathbf{N}_0 \subseteqq \mathbf{N}$.

LEMMA 3. 5. *The theorem is valid in the case* $\alpha = \aleph_0$.

PROOF. Let $\{A_n\}$ be a sequence of operators in $\mathbf{M}$ which satisfies the condition in Lemma 3. 3. For A_1 and $m = 1$, we shall define a W^*-subalgebra $\mathbf{N}_1$ after Lemma 3. 2 and we assume that it is of type I_p. By a repeted application of the preceding lemma, we can easily choose the W^*-subalgebras $\mathbf{N}_2, \mathbf{N}_3, \ldots$, such that

(11) each $\mathbf{N}_i$ is of type $\mathrm{I}_{p,i}$

(12) $\mathbf{N}_1 \subset \mathbf{N}_2 \subset \ldots \subset \mathbf{M}$,

(13) for every n, there exist $B_1, \ldots, B_n \in \mathbf{N}_n$ with $[[B_i - A_i]] < 1/n$.

Let $\mathbf{S}$ be the algebra which is the algebraical union of $\mathbf{N}_n$. It is obvious that every A_i can be approximated by elements in $\mathbf{S}$ in the metric topology. Since $\mathbf{S}$ contains the cente $\mathbf{Z}$ of $\mathbf{M}$, the weak closure of $\mathbf{S}$ coincides with $\mathbf{M}$. It is clear that

$$\mathbf{N}_n = \mathbf{M}_n \otimes \mathbf{Z}$$

where $\mathbf{M}$ is a factor of type I_{np}. Let $\mathbf{M}^0 = R(\mathbf{N}_n; n = 1, 2, \ldots.)$, then $\mathbf{M}^0$ is a factor by Lemma 1.2 and $\aleph_0$-approximately finite. Since $\mathbf{M} = R(\mathbf{N}_n \otimes \mathbf{Z}; n = 1, 2, \ldots.)$, we have

$$\mathbf{M} = \mathbf{M}^0 \otimes \mathbf{Z}$$

by Lemma 1.6. This proves the lemma.

PROOF OF NECESSITY. There exists a family $\{\mathbf{N}_\lambda; \lambda \in \Lambda\}$ which satisfies the conditions of Definition 3.1. Split Λ to subsets...., Λ_μ, ... such that every Λ_μ is countably infinite. Now we put

$$\mathbf{M}_\mu = R(\mathbf{N}_\lambda; \lambda \in \Lambda_\mu),$$

Then $\mathbf{M}_\mu$ is $\aleph_0$-approximately finite. It is clear that the cardinal of $\{\mathbf{M}_\mu\}$ is α and they commute with each other. Hence we can assume without loss of generality that each $\mathbf{N}_\lambda$ is $\aleph_0$-approximately finite.

By the preceding lemma, we have

$$\mathbf{N}_\lambda = \mathbf{M}_\lambda \otimes \mathbf{Z}$$

where $\mathbf{M}_\lambda$ is an $\aleph_0$-approximately finite and $\mathbf{Z}$ is the center of $\mathbf{N}$ (which coincides with that of $\mathbf{M}$). Put $\mathbf{M}^0 = R(\mathbf{M}_\lambda; \lambda \in \Lambda)$ and we shall show that $\mathbf{M}^0$ is an α-approximately finite factor. Since each $\mathbf{M}_\lambda$ can be generated by factors which commute with each other and are of type I, $\mathbf{M}^0$ is generated by such factors. Hence, by Lemma 1.2, $\mathbf{M}^0$ is a factor and so it is clear that $\mathbf{M}^0$ is an α-approximately finite factor. By Lemma 1.6, we have

$$\mathbf{M} = R(\mathbf{N}_\lambda; \lambda \in \Lambda) = R(\mathbf{M}_\lambda \otimes \mathbf{Z}; \lambda \in \Lambda) = R(\mathbf{M}^0, \mathbf{Z}) = \mathbf{M}^0 \otimes \mathbf{Z}.$$

This proves the theorem.

Now we shall consider general approximately finite W^*-algebras.

THEOREM 3.2. *Let $\mathbf{M}$ be an approximately finite W^*-algebra, then, for any cardinal $\alpha \geqq \aleph_0$, there exists a central projection E_α such that E_α are mutually orthogonal and have the union equal to I and each $\mathbf{M}_{E_\alpha}$ is α-uniformly approximately finite.*

To prove the theorem, we shall prepare for some lemmas. In the following, we shall assume that $\mathbf{M}$ is an approximately finite W^*-algebra. Let $\{\mathbf{N}_\lambda; \lambda \in \Lambda\}$ be as in Definition 3.1. We shall say that a subfamily $\{\mathbf{N}_\lambda; \lambda \in \Lambda'\}$ of this family is uniformly distributed if $R(\mathbf{N}_\lambda; \lambda \in \Lambda')$ has no commutative part, that is, for any central projection in $\mathbf{M}$ the contraction of $R(\mathbf{N}_\lambda; \lambda \in \Lambda')$ on the range of this projection is not commutative.

LEMMA 3.6. *Suppose that $\mathbf{M}$ is σ-finite, then there exists a uniformly distributed countable subfamily of $\{\mathbf{N}_\lambda; \lambda \in \Lambda'\}$.*

PROOF. Let $\mathfrak{F}$ be the set of all countable subfamilies of $\{\mathbf{N}_\lambda; \lambda \in \Lambda\}$. For $\{\mathbf{N}_\lambda; \lambda \in \Lambda_1\}$, $\{\mathbf{N}_\lambda; \lambda \in \Lambda_2\} \in \mathfrak{F}$, we shall define

$$\{\mathbf{N}_\lambda; \lambda \in \Lambda_1\} \succ \{\mathbf{N}_\lambda; \lambda \in \Lambda_2\}$$

if the commutative part of the former is contained in that of the latter. Then it is obvious that this relation satisfies conditions of the usual order ralation. Let $\mathfrak{F}$ be a linearly ordered subset of $\mathfrak{F}$, then we can find a countable subset $\mathfrak{F}_2$ of $\mathfrak{F}_1$ which is cofinal with $\mathfrak{F}_1$, since $\mathbf{M}$ is σ-finite. Let

$$\Lambda_0 = \bigcup_{\{\mathbf{N}_\lambda ; \lambda \in \Lambda_2\} \in_2 \mathfrak{F}} \Lambda_i$$

then $\{\mathbf{N}_\lambda ; \lambda \in \Lambda\}$ is countable and $\{\mathbf{N}_\lambda ; \lambda \in \Lambda_0\} \succ \{\mathbf{N}_\lambda ; \lambda \in \Lambda_j\}$ for all $\{\mathbf{N}_\lambda ; \lambda \in \Lambda_j\} \in \mathfrak{F}_1$. Therefore, by Zorn's lemma, there exists a $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$ in $\mathfrak{F}$ which is maximal under the relation $\succ$. If $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$ has a commutative part, then we can easily choose $\mathbf{N}_\mu$ such that

$$\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\} \cup \{\mathbf{N}_\mu\} \succ \{\mathbf{N}_\mu ; \in \Lambda\}$$

properly. This contradicts to the maximality of $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$, that is, this is uniformly distributed.

LEMMA 3.7. *Let $\mathbf{M}$ be σ-finite and $\mathfrak{F}$ be as above and $\mathfrak{F}_1$ be a subset of $\mathfrak{F}$, then following conditions are equivalent*:

(13) *there is no uniformly distributed countable subset in $\mathfrak{F}$,*

(14) *$R(\mathbf{N}_\lambda ; \lambda \in \Lambda)$ has a commutative part.*

PROOF. It is clear that (14) implies (13) and we shall show the converse. As in the preceding lemma, we can find a countable subfamily $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$ which is maximal under the relation $\succ$. As $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$ is not uniformly distributed, there exists a central projection E in $\mathbf{M}$ such that the contraction of $R(\mathbf{N}_\lambda ; \lambda \in \Lambda')$ on the range of E is commutative. By the maximality of $\{\mathbf{N}_\lambda ; \lambda \in \Lambda'\}$, there is no $\mathbf{N}_\lambda$ in $\{\mathbf{N}_\lambda ; \lambda \in \Lambda\}$ such that $\mathbf{N}_{\lambda E}$ is non-commutative. This shows that $R(\mathbf{N}_\lambda ; \lambda \in \Lambda)$ has a commutative part.

LEMMA 3.8. *Let $\mathfrak{F}_0 = \{\{\mathbf{N}_\lambda ; \lambda \in \Lambda_\gamma\} ; \gamma \in \Gamma\}$ be a set of uniformly distributed countable families and $\Lambda_0 = \bigcup_{\gamma \in \Gamma} \Lambda_\gamma$, then*

$$R(\mathbf{N}_\lambda ; \lambda \in \Lambda_0)$$

is uniformly approximately finite.

PROOF. Split Γ into subsets $\{\Gamma_\mu\}$ which are mutually disjoint and each of which is countable. Then, by an analogous way to the proof of Lemma 3.4, we can prove that $R(\mathbf{N}_\lambda ; \lambda \in \bigcup_{\gamma \in \Gamma_\mu} \Lambda_\gamma)$ is uniformly $\aleph_0$-approximately finite. Hence we can show that $R(\mathbf{N}_\lambda ; \lambda \in \Lambda)$ is uniformly approximately finite by a similar way to the proof of Theorem 3.1.

LEMMA 3.9. *For any central projection E in $\mathbf{M}$, there exists a central projection $F \leqq E$ such that $\mathbf{M}_E$ is uniformly approximately finite.*

PROOF. There exists a central projection $E_1 \leqq E$ such that $\mathbf{M}_{E_1}$ is σ-finite and so we can assume that $\mathbf{M}_E$ is σ-finite. Hence it is sufficient to show the lemma under the assumption that $\mathbf{M}$ is σ-finite and $E = I$.

Let $\{\{\mathbf{N}_\lambda ; \lambda \in \Lambda_\gamma\} ; \gamma \in \Gamma\}$ be the set of all uniformly distributed countable subfamilies of $\{\mathbf{N}_\lambda ; \lambda \in \Lambda\}$. By Zorn's lemma, we can find a maximal subfamily $\{\{\mathbf{N}_\lambda ; \lambda \in \Lambda_\gamma\} ; \gamma \in \Gamma_0\}$ such that $\Lambda_\gamma (\gamma \in \Gamma_0)$ are mutually disjoint. Let Λ' be the family of complement of $\bigcup_{\gamma\in\Gamma_0}\Lambda_\gamma$ in Λ, then, by the maximality of $\{\{\mathbf{N}_\lambda ; \lambda\in\Lambda_\gamma\} ; \gamma\in\Gamma_0\}$, $R(\mathbf{N}_\lambda ; \lambda\in\Lambda')$ satisfies the condition (1) in Lemma 3.8. Hence there exists a projection $F\in\mathbf{M}$ which is central in $R(\mathbf{N}_\lambda ; \lambda\in\Lambda')$ and the contraction of $R(\mathbf{N}_\lambda ; \lambda\in\Lambda')$ on the range of F is commutative. It is clear that F is central in $\mathbf{M}$. This shows that

$$\mathbf{M}_F = R(\mathbf{N}_{\lambda F} ; \lambda\in\bigcup_{\gamma\in\Gamma_0}\Lambda_\gamma)$$

and $\mathbf{M}_F$ is uniformly approximately finite by the preceding lemma.

PROOF OF THEOREM. By the preceding lemma, there exist central projections such that contractions of $\mathbf{M}$ on their ranges are uniformly approximately finite. Let $\{\mathbf{P}_\mu\}$ be a maximal family of such projections which are mutually orthogonal. Then we have $\sum P_\mu = I$ by the preceding lemma.

Let E_α be the union of P_μ such that $\mathbf{M}_{p\mu}$ is uniformly α-approximately finite. Clearly $\mathbf{M}_{E_\alpha}$ is uniformly α-approximately finite and $\sum E_\alpha = I$. This proves the theorem.

COROLLARY. *Let $\mathbf{M}, \mathbf{N}$ be two approximately finite W^*-algebras and $\{E_\alpha\}$, $\{F_\alpha\}$ be the families of central projections in $\mathbf{M}, \mathbf{N}$ respectively which are determine by Theorem* 3.2. *Then $\mathbf{M}$ is *-isomorphic to $\mathbf{N}$ if and only if the center of $\mathbf{M}$ is *-isomorphic to the one of $\mathbf{N}$ and by this isomorphism E_α is mapped to E_α for all α.*

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