ON THE INVARIANTS OF W^*-ALGEBRAS

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The present paper has two purposes. The first one is to study the representation of normal states of W^*-algebras with connection to their invariants. Our results will make more complete the connection, subsisting between the invariants and the representation character of normal states and the topologies, which has been shown by R. Pallu de la Barriere [8] and E. L. Griffin [5]. The second purpose is to discuss the question when an algebraic isomorphism between purely infinite W^*-algebras arises spatially. The answer for this has been announced by E. L. Griffin, for which we will state an independent proof. The same question in the case of semi-finite W^*-algebras was completely discussed in [5] and [8].[1)]

1. Definitions and notations. By a W^*-algebra M, we shall always understand a weakly closed self-adjoint operator algebra with the identity on a Hilbert space H. We denote by $M^\natural$ the center of M. Let φ be a vector in H, $M(\varphi)$ means the closed linear manifold generated by a set $[a\varphi\,;\ a \in M]$. The projections correspond to $M'(\varphi)$, $M(\varphi)$ are called *cyclic projections* in M, M' respectively. A projection e is said to be *countably decomposable* for M if every family of orthogonal projections in M bounded by e is at most countable. In particular, M is said to be *countably decomposable* if the identity is countably decomposable for M. A vector φ in H is said a *separating vector* for M if $M'(\varphi) = H$, and a *generating vector* for M if $M(\varphi) = H$. Let a be an operator in M and e a projection in M (or M'), we denote by a_{eH} an operator on eH defined by $a_{eH}\varphi = ea\varphi$ for all $\varphi \in eH$, and by M_{eH} a W^*-algebra formed by all these a_{eH}. A state ρ of M means a positive linear functional on M such that $\rho(1) = 1$. A state τ of M is said a trace if it satisfies $\tau(ab) = \tau(ba)$ for $a, b \in M$. We use the term *a normal state* in the sense of J. Dixmier [2]. Let σ, ρ be two states of M, the notations $\sigma << \rho$ will be used in the sense of [4].

The invariant $C(\mathcal{X})$ of a semi-finite W^*-algebra means the invariant defined by [8 ; Chap. III. Definition 3], and we denote by $C_{eH}(\mathcal{X})$ the invariant of M_{eH}. M is said to be *purely infinite* if 1 is purely infinite for M. By an isomorphism, we understand a *-isomorphism.

2. The representation theorems of normal states. In [8 ; Chap. II, Theorem 4] was proved the representation theorem for normal traces of a finite W^*-algebra with the invariant $C(\mathcal{X}) \geqq 1$. We first prove this representation theorem, independently of the commutative case, for normal states.

It is well known in the commutative case that M has a separating vector

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if and only if it is countably decomposable. We notice that this fact is generalized in a finite W^*-algebra with $C(\mathcal{X}) \geqq 1$.

LEMMA 1. *Let M be a finite W^*-algebra with the invariant $C(\mathcal{X}) \geqq 1$. If the center $M^\natural$ has a separating vector, M has also a separating vector.*

This lemma is an immediate consequence of [8; Chap. II. Theorem 2 and Cor. of Prop. 2] and we can prove directly it, but omit its proof.

LEMMA 2. *Let M be a finite W^*-algebra with the invariant $C(\mathcal{X}) \geqq 1$ on a Hilbert space H. Then for each normal state ρ of M there exists a vector $\varphi \in H$ such that $\rho(a) = <a\varphi, \varphi>$ for all $a \in M$.*

PROOF. By Lemma 1, there exists a family $\{e_\alpha\}$ of mutually orthogonal projections in $M^\natural$ such that $\sum_\alpha e_\alpha = 1$ and $M_{e_\alpha H}$ has a separating vector. Therefore using [3; Prop. 6], for each normal state σ_α of $M_{e_\alpha H}$, we can choose a vector $\psi_\alpha \in e_\alpha H$ such that

$$\sigma_\alpha(a_\alpha) = <a_\alpha\psi_\alpha, \psi_\alpha> \text{ for all } a_\alpha \in M_{e_\alpha H}.$$

Now, if ρ is any normal state of M, ρ defines the normal state ρ_α of $M_{e_\alpha H}$ by $\rho_\alpha(a_\alpha) = \rho(e_\alpha a e_\alpha)$, where a_α is the operator in $M_{e_\alpha H}$ induced by $a \in M$. From the preceding fact, there exists a vector $\varphi_\alpha \in e_\alpha H$ such that

$$\rho_\alpha(a_\alpha) = <a_\alpha\varphi_\alpha, \varphi_\alpha> \text{ for all } a_\alpha \in M_{e_\alpha H}.$$

As $a = \sum_\alpha a e_\alpha = \sum_\alpha e_\alpha a e_\alpha$, we obtain

$$\rho(a) = \sum_\alpha \rho(e_\alpha a e_\alpha) = \sum_\alpha \rho_\alpha(a_\alpha) = \sum_\alpha <a_\alpha\varphi_\alpha, \varphi_\alpha> = \sum_\alpha <a\varphi_\alpha, \varphi_\alpha>.$$

But, $\rho(1) = 1$ or $\sum_\alpha ||\varphi_\alpha||^2 = 1$, so that $\{\varphi_\alpha\}$ is summable. Set $\varphi = \sum_\alpha \varphi_\alpha$, then $<a\varphi, \varphi> = \sum_\alpha <a\varphi_\alpha, \varphi_\alpha> = \rho(a)$ as desired.

COROLLARY. *Let M be a commutative W^*-algebra. For each normal state ρ of M, there exists a vector $\varphi \in H$ such that $\rho(a) = <a\varphi, \varphi>$ for all $a \in M$* [8; Chap. *I*, Prop. 1].

In fact, M is a finite W^*-algebra with $C(\mathcal{X}) \geqq 1$.

The generalization of the above lemma has been already established in [5; Theorem 7]. Now, we will give a simple proof for this statement with the aid of the method in [8; Chap. III. Theorem 2] and obtain the more precise result.

THEOREM 1. *Let M be a finite W^*-algebra on a Hilbert space H. Then the following conditions are equivalent;*

(*A*) *For each normal state ρ of M, there exist vectors $\{\varphi_i\}_{i=1,2,\dots,n}$ in H such that*

$$\rho(a) = \sum_{i=1}^n <a\varphi_i, \varphi_i> \quad \text{for all } a \in M.$$

(*A'*) *For each normal trace τ of M, there exist vectors $\{\xi_i\}_{i=1,2,\dots,n}$ in H such that*

$$\tau(a) = \sum_{i=1}^n <a\xi_i\, \xi_i> \quad \text{for all } a \in M.$$

(*B*) *M has the invariant* $\inf_{\chi\in\Omega} C(\chi) \geqq 1/n$.

PROOF. It is clear that (*A*) implies (*A'*). We first observe that (*A'*) implies (*B*). We may assume that M is countably decomposable and M' is finite. Therefore, there exists a faithful normal trace τ of M. By the assumption of (*A'*), we obtain

$$\tau(a) = \sum_{i=1}^{n} <a\xi_i, \xi_i> \text{ for all } a \in M$$

where $\{\xi_i\}_{i=1,2,\ldots,n}$ are vectors in H. Let $\mathfrak{M}$ be the closed linear manifold generated by $\{\xi_i\}_{i=1,2,\ldots,n}$. Since $\tau(a)$ is faithful, we have $M'(\mathfrak{M}) = H$ and $1 = \bigvee_{i=1}^{n} P_{M'(\xi_i)}$. As $M_{M'(\xi_i)}$ has a separating vector ξ_i, it follows that $C_{M'(\xi_i)}(\chi) \geqq 1$. Applying [8; Chap. I, Prop. 2] to M', we obtain

$$C_{M'(\xi_i)}(\chi)^{-1} = P^{\natural}_{M'(\xi_i)}(\chi)C(\chi)^{-1}$$

for all $\chi \in \Omega_{M'(\xi_i)}$such that $C(\chi) \neq 0$. Hence $C(\chi) = P^{\natural}_{M'(\xi_i)}(\chi)C_{M'(\xi_i)}(\chi) \geqq P^{\natural}_{M'(\xi_i)}(\chi)$, $nC(\chi) \geqq \left(\sum_{i=1}^{n} P^{\natural}_{M'(\xi_i)}\right)(\chi) \geqq 1$ for all $\chi \in \Omega$ such that $C(\chi) \neq 0$. Since the set $\{\chi\,;\, C(\chi) = 0\}$ is non-dense, $\inf_{\chi\in\Omega} C(\chi) \geqq 1/n$.

Now, we shall complete the proof by showing that (*B*) implies (*A*). By the assumption, there exists a family $\{e_i\}_{i=1,2,\ldots,n}$ of mutually orthgonal and equivalent projections in M such that $\sum_{i=1}^{n} e_i = 1$ and $C_{e_iH}(\chi) \geqq 1$. In fact, again, applying [8; Chap. I, Prop. 2] to M'

$$C_{e_iH}(\chi)^{-1} = n^{-1}C(\chi)^{-1}$$

and hence $C_{e_iH}(\chi) = nC(\chi)$ or $C_{e_iH}(\chi) \geqq 1$.

Let ρ be any normal state of M, then ρ defines a normal state ρ_i of M_{e_iH} by $\rho_i(a_i) = \rho(e_iae_i)$ where a_i is the operator in M_{e_iH} induced by $a \in M$. By Lemma 2, we can choose a vector $\psi_i \in e_iH$ such that $\rho_i(a_i) = <a_i\psi_i, \psi_i>$ for all $a_i \in M$. Thus

$$\sum_{i=1}^{n} \rho(e_iae_i) = \sum_{i=1}^{n} \rho_i(a_i) = \sum_{i=1}^{n} <a_i\psi_i, \psi_i> = \sum_{i=1}^{n} <a\psi_i, \psi_i>.$$

But $\rho(a) << \sum_{i=1}^{n} \rho(e_iae_i)$, in fact, for all positive $a \in M$

$$\rho(a) = \sum_{i=1}^{n} \rho(ae_i) \leqq \sum_{i=1}^{n} |\rho(a^{1/2}a^{1/2}e_i)|$$
$$\leqq \sum_{i=1}^{n} \rho(a)^{1/2}\rho(e_iae_i)^{1/2} = \rho(a)^{1/2}\Big[\sum_{i=1}^{n} \rho(e_iae_i)^{1/2}\Big]$$

hence

$$\rho(a) \leqq \Big[\sum_{i=1}^{n} \rho(e_iae_i)^{1/2}\Big]^2 \leqq n\Big(\sum_{i=1}^{n} \rho(e_iae_i)\Big).$$

As in [4; § 5. (5. 6)], by the representation Theorem of Segal, we get, vectors $\{\varphi_i\}_{i=1,2,\ldots,n}$ such that

$$\rho(a) = \sum_{i=1}^{n} <a\varphi_i, \varphi_i>.$$

REMARK. It is easy to see that if M is an arbitrary finite W^*-algebra, for each normal state of M, there exists a sequence $\{\varphi_i\}$ of vectors such that $\{M'(\varphi_i)\}$ are mutually orthogonal and

$$\rho(a) = \sum_{i=1}^{\infty} <a\varphi_i, \varphi_i> \quad \text{for all } a \in M.$$

In fact, there exists a family $\{e_i\}_{i\in I}$ of mutually orthogonal projections in $M^\natural$ such that $\sum_{i\in I} e_i = 1$ and $\inf_{\chi\in\Omega} C_{e_i H}(\chi) > 0$. This result was established in [4].

THEOREM 2. *Let M be a finite W^*-algebra, then the following conditions are equivalent*;

(A) *M has the invariant* $\inf_{\chi\in\Omega} C(\chi) > 0$.

(B) *The σ-weak and the weak topologies coincide on M.*

(C) *The strongest and the strong topologies coincide on M.*

(D) *The $\natural$-application is continuous in the weak* (*strong*) *topology.*

PROOF. From Theorem 1 and [2; prop. 2] it follows that $(A) \rightarrow (B) \leftrightarrows (C)$. $(C) \rightarrow (D)$ is obvious. Thus ws have only to prove that (D) implies (A). Let τ_α be a normal trace of M, then there exists a vector $\xi \in H$ such that $\tau_\alpha(a) = <a^\natural\xi, \xi>$ for all $a \in M$. Now, as the $\natural$-application is weakly continuous, τ_α is also weakly coninuous. Hence we can pick up vectors $\{\xi_i\}_{i=1,2,\dots,n}$ such that $\tau_\alpha(a) = \sum_{i=1}^{n} <a\xi_i, \xi_i>$ for all $a \in M$. Now let n_α be the minimum of all integers n such that τ_α is expressible as above (it is said the order of τ_α). Then we get $\sup_\alpha n_\alpha < +\infty$ for the order n_α of all normal traces τ_α on M. In fact, otherwise we can select a sequence $\{\tau_i\}$ of normal traces on M such that $n_i \uparrow \infty$. If we define $\tau_0(a) = \sum_{i=1}^{\infty} 2^{-i}\tau_i(a)$ for all $a \in M$, then τ_0 is also a normal trace of M and $\tau_i(a) << \tau_0(a)$ for all i. Hence, for all i, there exist vectors $\{\eta_{ij}\}_{j=1,2,\dots,n_0}$ such that $\tau_i(a) = \sum_{i=1}^{n_0} <a\eta_{ij}, \eta_{ij}>$ for all $a \in M$, where n_0 is the order of τ_0. But we have $n_0 < n_i$ for some i as $n_i \uparrow \infty$. This is a contradiction. Set $n = \sup_\alpha n_\alpha$, then by Theorem 2, we have $\inf_{\chi\in\Omega} C(\chi) \geqq 1/n > 0$.

If the $\natural$-application is strongly continuous, the application $a \rightarrow <a^\natural\xi, \xi>$ is strongly continuous, and hence weakly continuous from the result in [1].

Concerning this theorem, in [8; Chap. III, Theorem 7] was proved the fact that $(A) \rightarrow (D)$, and $(A) \rightleftarrows (D)$ in [5; Theorem 8]. And also for semi-finite W^*-algebra $(A) \rightarrow (B)$, (C) was proved in [5; Theorem 12].

In order to complete our information concerning the representation of normal states, we need the following details for the infinite case.

THEOREM 3. *Let M be a purely infinite W^*-algebra on a Hilbert space H, then*

(A) *For every normal state ρ of M, there exists a vector φ in H such that* $\rho(a) = <a\varphi, \varphi>$ *for all* $a \in M$.

(B) *The weak and the σ-weak topologies coincide on M.*

(C) *The strong and the strongest topologies coincide on M.*

From [2; prop. 3 and its proof], we can see that the above statement (A), (B) and (C) are true if M' is a properly infinite W^*-algebra. Therefore the theorem follows as a special case from this fact, but now we will give the direct proof for this.

Proof. It is well known that there exists a projection e in M such that $\rho(1-e)=0$ and such that ρ is faithful on M_{eH}, i. e. $\rho(f)=0$ for f a projection in M_{eH} implies $f=0$. Then M_{eH} is countably decomposable. Since M is purely infinite, there exists a vector ψ in eH such that $M'_{eH}(\psi)=eH$. By [3; prop. 6], we obtain a vector φ in eH such that $\rho(eae) = <eae\varphi, \varphi>$ for all $a \in M$. But then $|\rho(a(1-e))|^2 \leqq \rho(a^*a)\rho(1-e)=0$ (similarly $\rho((1-e)a)=0$) yield that for all $a \in M$,

$$\begin{aligned}\rho(a) &= \rho(eae)+\rho(ea(1-e))+\rho((1-e)ae)+\rho((1-e)a(1-e))\\ &= \rho(eae) = <eae\varphi, \varphi> = <a\varphi, \varphi>.\end{aligned}$$

(B) and (C) are the immediate consequences of [2; prop. 2] and (A).

3. The invariant of a purely infinite W^*-algebra. In this section, we shall introduce the invariant of a purely infinite W^*-algebra, which plays the essential role for the problem of a spatial isomorphism, as in the semi-finite case.

Lemma 3. *Let $\{e_i\}_{i\in I}$, $\{e'_j\}_{j\in J}$ be infinite families of orthogonal, cyclic projections in a purely infinite W^*-algebra M, and let e a projection in $M^\natural$. If $\sum_{i\in I} e_i = \sum_{j\in J} e'_j = e$, then the Cardinal of I equals the Cardinal of J.*

Proof. It is sufficient to prove the lemma in the case $e=1$. Now let $S_i=\{j;\ e_ie'_je_i \neq 0\}$, then S_i is a countable set. In fact, if $e_i = P_{M'(\varphi_i)}$, φ_i is a separating vector for M_{e_iH}. Since $e_i = \sum_{j\in J} e_ie'e_i$, $<e_i\varphi_i, \varphi_i> = \sum_{j\in J} <e_ie'_j e_i\varphi_i, \varphi_i> < +\infty$. Therefore, $<e_ie'_je_i\varphi_i, \varphi_i> = 0$ except countably many of $j \in J$. This means that all but countably many of $e_ie'_je_i (j\in J)$ are zero.

For any $j \in J$, if $e_ie'_je_i = 0$ for all $i \in I$, then $e_ie'_j = 0$ for all $i\in I$, which implies $\sum_{i\in I} e_ie'_j = e'_j = 0$. This contradiction proves that for each $j\in J$, there exists $i_j \in J$ such that $e_{i_j}e'_je_{i_j} \neq 0$; but then $j \in S_{i_j}$, yielding that

$$J = \bigcup_i S_i.$$

Thus, the Cardinal of $J \leqq$ the Cardinal of $I \times \aleph_0 =$ the Cardinal of I. By symmetry, the Cardinal of I equals to the Cardinal of J.

This lemma admits the following definition.

Definition 1. *Let M be a purely infinite W^*-algebra. A nonzero projection e in $M^\natural$ is said a homogeneous projection if for every projection f in $M^\natural$ bounded by e and countably decomposable for $M^\natural$, there exists a family of fixed transfinite Cardinal of orthogonal, equivalent, cyclic projections $\{f_i\}_{i\in I}$ in M such that $f = \sum_{i\in I} f_i$. The Cardinal of I (determined uniquely by Lemma 3) is said the order* *of* e.

We notice that if e is a homogeneous projection of the order $\aleph_0$, every projection f in $M^\natural$ bounded by e and countably decomposable for M is itself cyclic. Now we shall show the structure theorem of a purely infinite W^*-algebra, which yield the definition of the invariant.

THEOREM 4. *Let M be a purely infinite W^*-algebra on a Hilbert space H and let π be a family of all transfinite Cardinals α for which there are homogeneous projections of the order α. Then there exists a family of orthogonal projections $\{e_\alpha\}_{\alpha\in\pi}$ in $M^\natural$ such that $1=\sum_{\alpha\in\pi}e_\alpha$ and each e_α is a homogeneous projection of the order α.*

PROOF. Let α be a transfinite Cardinal in π. By Zorn's lemma, choose a maximal family $\{e_i\}_{j\in I}$ of orthogonal homogeneous projections of the order α in $M^\natural$. Set $e_\alpha=\sum_{i\in I}e_i$. At first, we shall show that any projection e in $M^\natural$ such that $e\leqq e_\alpha$ is a homogeneous projection of the order α. In fact, if a projection f in $M^\natural$ is countably decomposable for $M^\natural$ with $f\leqq e$, then $f\leqq e_\alpha$ and $f=e_\alpha f=\sum_{i\in I}e_i f=\sum_{i\in I'}e_i f$ (I' ; countable set), but then there exists a family of orthogonal, equivalent, cyclic projection $\{f_{ij}\}_{j\in J}$ in M such that $e_i f=\sum_{i\in J}f_{ij}$ and the Cardinal of J is α. Put $f_j=\sum_{i\in I'}f_{ij}$. $\{f_j\}_{j\in J}$ is a family of orthogonal, equivalent, cyclic projections in M such that $f=\sum_{j\in J}f_j$. Thus we know that e_α and every projection in $M^\natural$ bounded by e^α are homogeneous projection of the order α. If e in $M^\natural$ is a homogeneous projection of the order α, then $e\leqq e_\alpha$, otherwise the fact that $e-ee_\alpha$ is a homogeneous projection of the order α contradicts to our maximal $\{e_i\}_{i\in I}$.

Next, let α,β be transfinite Cardinals in π such that $\alpha\neq\beta$, then e_α and e_β defined as above are mutually orthogonal. Otherwise $e_\alpha\cdot e_\beta$ is a homogeneous projection of the order α and of the order β, which is impossible.

We define $p=\sum_{\alpha\in\pi}e_\alpha$. If $1-p\neq 0$, take a vector φ in $(1-p)H$, $e_{i_0}=p_{M'(\varphi)}\leqq 1-p$. From Zorn's lemma, there exists a maximal family of orthogonal projections $\{e_i\}_{i\in I}$ containing e_{i_0} such that $e_i\sim e_{i_0}$ for each i. Set $e=\sum_{i\in I}e_i$, it is clearly impossible that $e_{i_0}\prec(1-e)(1-p)$, and also if $(1-e)(1-p)=0, e=1-p$ shows that $1-p$ is a homogeneous projection. This contradiction yields $1=p$. Assume that $(1-e)\ (1-p)\neq 0$, then there exists a projection h in $M^\natural$ such that $h(1-e)(1-p)\prec he_{i_0}$. Now $h(1-p)=\sum_{i\in I}he_i+h(1-e)(1-p)$, with the he_i all orthogonal, equivalent, cyclic projections in M. Since I is infinite, $\sum_{i\in I}he_i\sim\sum_{i\in I-i_0}he_i$. If follows that $\sum_{i\in I}he_i\succ\sum_{i\in I-i_0}he_i+h(1-e)(1-p)\sim hp$. Thus $h(1-p)\sim\sum_{i\in I}he_i$. Then it is easy to see that $h(1-p)$ is a homogeneous projection. It follows that

$1-p$ contains a homogeneous projections. This contradicts our choice of p. Thus $1=p$.

DEFINITION 2. *Let M be a purely infinite W^*-algebra and let $\{e_\alpha\}_{\alpha\in\pi}$ a family defined by Theorem 4, and K_α the open and closed set in the spectrum Ω of $M^\natural$ correspond to e_α. We define the algebraic invariant of M, as the function $p(X)$ defined on $\bigcup_{\alpha\in\tau} K_\alpha$ by the relation $p(X)=\alpha$ for $X\in K_\alpha$.*

Let $p(X)$, $p'(X)$ be the algebraic invariants of M, M' respectively. We define the invariant of M, as the function $C(X)$ defined on a dense open set of Ω by the relation $C(X)=(p(X),\ p'(X))$.

REMARK. $p(X)$ is invariant by isomorphisms.

4. Spatial isomorphisms of purely infinite W^*-algebras. Let M, $\widetilde{M}$ be purely infinite W^*-algebras on Hilbert spaces $H, \widetilde{H}$ respectively. We shall find the condition under which the following statement holds:

(S) If θ is an isomorphism of M onto $\widetilde{M}$, then θ is a spatial isomorphism.

In fact, the invariant defined in the previous section will give the complete answer to our question as in the case where $M, \widetilde{M}$ are semifinite. Now, if we identify the spectrums of $M, \widetilde{M}$ by the isomorphism, the main theorem is stated as follows:

THEOREM 5. *Let $\widetilde{M}$ and M be purely infinite W^*-algebras both with the same invariant $C(X)$, then (S) holds.*

To prove the theorem, we shall proceed as in the semi-finite case.

LEMMA 4. *Let M and $\widetilde{M}$ be purely infinite W^*-algebras with the same invariant $(\aleph_0, \aleph_0)$, then (S) holds.*[2]

It is sufficient to prove the lemma in the case where $M^\natural$ is countably decomposable. Then M, $\widetilde{M}$ have separating and generating vectors φ, $\widetilde{\varphi}$ respectively. Now let $\rho(a)=<\theta(a)\widetilde{\varphi}, \widetilde{\varphi}>$ for all $a\in M$, ρ is a normal state of M and hence, by Theorem 3 there exists a vector ψ in H such that $\rho(a) = <a\psi, \psi>$. Here, ψ is also a separating vector of M, in fact, since $||a\psi||^2=\rho(a^*a)=||\theta(a)\widetilde{\varphi}||^2, a\psi=0$ implies $\theta(a)\widetilde{\varphi}=0, \theta(a)=0$, finally $a=0$. Thus $M(\psi)\sim M(\varphi)=H$. Let v' be the partial isometry in M' defined by $v'M(\psi)=M(v'\psi)=H$. Now, set $\eta=v'\psi$, $M(\eta)=H$ and $||a\eta||^2=||av'\psi||^2= <a^*av'\psi, v'\psi> = <a^*av'^*v'\psi, \psi> = <a^*a\psi, \psi> = ||a\psi||^2 = ||\theta(a)\widetilde{\varphi}||^2$, so that we can define a linear isometry u from H onto $\widetilde{H}$ such that $ua\eta=\theta(a)\widetilde{\varphi}$. If $\widetilde{b}$ arbitrary in $\widetilde{M}$, then $uau^{-1}\widetilde{b}\widetilde{\varphi}=ua\theta^{-1}(\widetilde{b})\eta=\theta(a)\widetilde{b}\widetilde{\varphi}$ proves that $\theta(a) = uau^{-1}$ for all $a\in M$.

2) If $H, \widetilde{H}$ are separable, $M, \widetilde{M}$ have both the invariant $(\aleph_0, \aleph_0)$ Therefore (S) always holds (cf. [7])

LEMMA 5. *Let M and $\widetilde{M}$ be purely infinite W^*-algebras with the same invariant* $(\aleph_0, \alpha)$, *then* (S) *holds.*

PROOF. As in lemma 4, we may assume that $M^\natural$ is countably decomposable. Since the invariant of M is $(\aleph_0, \alpha)$, there exists a vector $\varphi \in H$ such that $M'(\varphi) = H$, and also a family of orthogonal equivalent, cyclic projections $\{e'_i\}_{i\in I}$ in M' such that $1 = \sum_{i\in I} e'_i$ and the Cardinal of I is α. Then, since M is countably decomposable, for all $i \in I$ there exist vectors φ_i, ψ_i in e'_iH such that $e'_iH = M'_{e'_iH}(\varphi_i) = M_{e'_iH}(\psi_i)$. This proves that $M_{e'_iH}$ have all the invariant $(\aleph_0, \aleph_0)$. Furthermore, $M_{e'_iH}$ are all isomorphic to M, in fact, let u'_{ij} be partial isometries such that $u'^*_{ij}u'_{ij} = e'_i$ $u'_{ij}u'^*_{ij} = e'_j$, if $ae'_i = 0$, $ae'_j = a u'_{ij}e'_iu'^*_{ij} = u'_{ij}ae'_iu'^*_{ij} = 0$ for all j yield that $a = \sum_{i\in I} ae'_i = 0$. Similarly, we obtain a family of orthogonal, equivalent, cyclic projections $\{\widetilde{e'_i}\}_{i\in I}$ in $\widetilde{M}'$ such that $\widetilde{1} = \sum_{i\in I} \widetilde{e'_i}$, $\widetilde{M}_{\widetilde{e}'_i\widetilde{H}}$ have all the invariant $(\aleph_0, \aleph_0)$ and are all isomorphic to $\widetilde{M}$. Set $\theta_i = \theta_{\widetilde{e}'_i} \cdot \theta \cdot \theta_{e'_i}^{-1}$, where $\theta_{e'_i}$, $\theta_{\widetilde{e}'_i}$ are isomorphisms of M onto $M_{e'_iH}$, $\widetilde{M}$ onto $\widetilde{M}_{\widetilde{e}'_i\widetilde{H}}$ respectively. Then θ_i are isomorphisms of $M_{e'_iH}$ onto $\widetilde{M}_{\widetilde{e}'_i\widetilde{H}}$. From lemma 4, we obtain a linear isometries u_i from e'_iH onto $\widetilde{e'_i}\widetilde{H}$ such that $(\theta(a))_{\widetilde{e}'_i\widetilde{H}} = u_ia_{e'_iH}u_i^{-1}$. Now, let u be a linear isometry from H onto $\widetilde{H}$ such that the restriction to e'_iH is u_i, it is easy to see that $\theta(a) = uau^{-1}$.

LEMMA 6. *Let $\widetilde{M}$ and M be purely infinite W^*-algebras with the same invariant* (α, β), then (S) holds.

Proof. We may assume that $M^\natural$ is countably decomposale. Since the invariant of M is (α, β), there exists a family of orthogonal, equivalent, cyclic projections $\{e_i\}_{i\in I}$ in M such that $1 = \sum_{i\in I} e_i$ and the Cardinal of I is α. Then the family $\{\theta(e_i)\}_{i\in I}$ is also a family of projections in $\widetilde{M}$ with properties similar to those of $\{e_i\}_{i\in I}$.

Now, we notice that θ induce an isomorphism of M_{e_iH} onto $\widetilde{M}_{\theta(e_i)\widetilde{H}}$, and M', $\widetilde{M}'$ are isomorphic M'_{e_iH}, $M'_{\theta(e_i)\widetilde{H}}$ respectively. Thus M_{e_iH}, $\widetilde{M}_{\theta(e_i)\widetilde{H}}$ have all the invariant $(\aleph_0, \beta)$. Let i_0 a fixed element of I, using lemma 5, we obtain a linear isometry u_0 of $e_{i_0}H$ onto $\theta(e_{i_0})\widetilde{H}$ such that $u_0a_{e_{i_0}H}u_0 = (\theta(a))_{\theta(e_{i_0})\widetilde{H}}$. Let v_i be partial isometries in M such that $v_i^*v_i = e_{i_0}$, $v_iv_i^* = e_i$, then $\theta(v_i)^*\theta(v_i) = \theta(e_{i_0})$, $\theta(v_i)\theta(v_i)^* = \theta(e_i)$. Now, we can define a linear isometry $u = \sum_{i\in I}\theta(v_i)u_0v_i^*$ which is a mapping of H onto $\widetilde{H}$, then

$$uau^{-1} = \sum_{i,}\theta(v_i)u_0v_i^*av_ju_0^{-1}\theta(v_j^*)$$
$$= \sum_{i,j}\theta(v_i)\theta(v_i^*av_j)\theta(v_j^*) \qquad (\text{since } v_i^*av_j \in M_{e_{i_0}H})$$

$$= \sum_{i,j}\theta(v_i v_i^* a v_j v_j^*) = \sum_{i,j}\theta(e_i a e_j) =$$
$$= \theta\left(\sum_{i,j} e_i a e_j\right) = \theta(a).$$

This complete the proof.

The proof of Theorem 2.

The general case, by projections in $M^\natural$, is reduced to the case where M, $\widetilde{M}$ have both the invariant (α, β); then, by applying lemma 6, the proof of the theorem is complete.

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