THEOREMS ON POWER SERIES OF THE CLASS H^{p} *)

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1. Introduction. Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1.1}$$

be a function regular for $|z| < 1$. If, for some $p > 0$, the expression

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \tag{1.2}$$

is bounded as $r \to 1$, then the function $f(z)$ and its power series are said to belong to the class H^p. It is well known that, if $f(z)$ belongs to the class H^p, then $f(z)$ has a boundary function

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}), \quad 0 \leqq \theta \leqq 2\pi \tag{1.3}$$

for almost all θ and $f(e^{i\theta})$ is integrable L^p. Moreover if $p \geqq 1$ a necessary and sufficient condition for the function $f(z)$ to belong to the class H^p is that the series

$$\sum_{n=0}^{\infty} c_n e^{ni\theta} \tag{1.4}$$

is the Fourier series of its boundary function $f(e^{i\theta})$. Hence, in virture of M. Riesz's theorem, if $p > 1$, the class H^p is isomorphic to the class L^p. In this case, the series (1.4) is summable (C, ε), $\varepsilon > 0$, to the boundary functions $f(e^{i\theta})$ at almost all θ. The problem whether in this result we may replace summability (C, ε) by ordinary convergence remains open, but if $p = 1$, the answer is negative (Sunouchi [7]).

On the behaviour of power series of class H^p on the circle of convergence, important results were obtained by Littlewood and Paley [6] and Zygmund [11] [12]. The main tool of Littlewood and Paley was an auxiliary function

$$g^*(\theta) = g^*(\theta; f) = \left(\int_0^1 (1 - r) \chi^2(r, \theta)\, dr \right)^{1/2}, \quad 0 \leqq \theta \leqq 2\pi \tag{1.5}$$

where

$$\chi(r, \theta) = \left(\frac{1}{\pi} \int_0^{2\pi} |f'(re^{i(\theta+\varphi)})|^2 P(r, \varphi)\, d\varphi \right)^{1/2}$$

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and

$$P(r,\varphi)=\frac{1}{2}\frac{(1-r^2)}{|1-re^{i\varphi}|^2}.$$

But they proved an inequality theorem concerning to $g^*(\theta)$ only for H^p, $p=2k$ ($k=$ integer) and their proof is very difficult. Later Zygmund [13] gave a complete and simple proof. In the other papers [10] [11], he used another inequality theorem of Littlewood and Paley concerning to

$$g(\theta)=g(\theta;f)=\left(\int_0^1(1-r)|f'(re^{i\theta})|^2dr\right)^{1/2},\quad 0\leqq\theta\leqq 2\pi \tag{1.6}$$

and gave a simple proof of the main result of Littlewood-Paley and many interesting generalizations. The purpose of this paper is to give the generalized theorem on $g^*(\theta)$ and systematic treatment and generalization of theorems on the power series of the class H^p.

2. The function $g_\alpha^*(\theta)$ for the class H^p $(0<p\leqq 2)$.

The definition of the function $g_\alpha^*(\theta)$ is slightly less simple. It is given by the formula

$$g_\alpha^*(\theta)=g_\alpha^*(\theta;f)=\left(\int_0^1(1-r)^{2\alpha}dr\int_0^{2\pi}\frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}d\varphi\right)^{1/2} \tag{2.1}$$

If $\alpha=1$, $g_\alpha^*(\theta)$ reduces to the function $g^*(\theta)$ of Littlewood and Paley excepting constant factor. So we don't distinguish between $g_1^*(\theta)$ and $g^*(\theta)$. It is known that $g^*(\theta)$ is a majorant of many important functions. Especially $g^*(\theta)$ intervenes for the partial sums of the series (1.4).

Let us denote

$$s_n(\theta)=\sum_{\nu=0}^n c_\nu e^{i\nu\theta},\qquad t_n(\theta)=nc_ne^{in\theta},$$

$$\sigma_n^\alpha(\theta)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}s_\nu(\theta),\qquad (\alpha>-1)$$

$$\tau_n^\alpha(\theta)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^n A_{n-\nu}^{\alpha-1}t_\nu(\theta),\qquad (\alpha>0)$$

where

$$A_n^\alpha=\binom{n+\alpha}{n}\sim\frac{n^\alpha}{\Gamma(\alpha+1)},\qquad \text{as } n\to\infty,$$

then

$$\tau_n^\alpha(\theta)=n\{\sigma_n^\alpha(\theta)-\sigma_{n-1}^\alpha(\theta)\}=\alpha\{\sigma_n^{\alpha-1}(\theta)-\sigma_n^\alpha(\theta)\}. \tag{2.2}$$

Further put

$$h_\alpha(\theta)=h_\alpha(\theta;f)=\left(\sum_{n=1}^\infty\frac{|\tau_n^\alpha(\theta)|^2}{n}\right)^{1/2}, \tag{2.3}$$

then we have the following relation. The right half was given by Chow [1] and the left half for $\alpha = 1$ was remarked by Koizumi [5].

LEMMA 1. *For the function $f(z) \in H^p$ $(0 < p < \infty)$,*

$$A_\alpha h_\alpha(\theta) \leqq g_\alpha^*(\theta) \leqq B_\alpha h_\alpha(\theta). \tag{2.4}$$

REMARK 1. Throughout this paper, A_α, B_α, ... are positive constants depending only on α, and may be different from one occurence to another.

REMARK 2. For the definiteness of conjugate series, we put $\mathfrak{J}c_0 = 0$.

Remark 3. In proving theorems in this paper, we can suppose without loss of generarity that $f(z)$ has no zeros inside the unite circle. And in many case, it is enough to prove theorems for $f(z)$ regular on $|z| \leqq 1$. For, proving theorem for $f(Rz)$ $(0 < R < 1)$ and making R tend to 1, we get the theorem.

PROOF. If we write

$$\Phi_\alpha(r, \theta) = \sum_{n=1}^{\infty} (A_n^\alpha)^2 |\tau_n^\alpha(\theta)|^2 r^{2n}, \tag{2.5}$$

then

$$\begin{aligned}\{h_\alpha(\theta)\}^2 &= A_\alpha \sum_{n=1}^{\infty} \frac{(A_n^\alpha)^2}{(2n + 2\alpha + 1)(A_{2n}^{2\alpha})^2} |\tau_n^\alpha(\theta)|^2 \\ &= A_\alpha \int_0^1 (1 - r)^{2\alpha} \Phi_\alpha(r, \theta)\, dr.\end{aligned} \tag{2.6}$$

Since

$$\sum_{n=1}^{\infty} A_n^\alpha \tau_n^\alpha(\theta) z^n = \frac{z e^{i\theta} f'(z e^{i\theta})}{(1 - z)^\alpha}, \tag{2.7}$$

we have, by Parseval's theorem

$$\Phi_\alpha(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|r f'(r e^{i(\theta+\varphi)})|^2}{|1 - r e^{i\varphi}|^{2\alpha}}\, d\varphi, \tag{2.8}$$

so that

$$\begin{aligned}\{h_\alpha(\theta)\}^2 &= B_\alpha \int_0^1 (1 - r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|r f'(r e^{i(\theta+\varphi)})|^2}{|1 - r e^{i\varphi}|^{2\pi}}\, d\varphi \\ &\leqq C_\alpha \{g_\alpha^*(\theta)\}^2,\end{aligned} \tag{2.9}$$

by the definition (2.1).

On the other hand, for $0 \leqq r < 1/2$

$$\frac{(1 - r)^{2\alpha}}{|1 - r e^{i\varphi}|^{2\alpha}}$$

are limited above and below by positive numbers, and

$$\int_0^{2\pi} |f'(re^{i(\varphi+\theta)})|^2 \, d\varphi$$

is non-decreasing function of r. Thus

$$\int_0^{1/4} (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|f'(r^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi$$

$$\leqq D_\alpha \int_{1/4}^{1/2} (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi$$

$$\leqq E_\alpha \int_{1/4}^{1/2} (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|rf'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi$$

$$\leqq F_\alpha \int_0^{1} (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|rf'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi.$$

Consequently

$$\{g_\alpha^*(\theta)\}^2 = \int_0^1 (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi$$

$$= \int_0^{1/4} + \int_{1/4}^{1}$$

$$= G_\alpha \int_0^1 (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{|rf'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi$$

$$\leqq H_\alpha \{h_\alpha(\theta)\}^2$$

by (2.9). Thus we have (2.4).

LEMMA 2. *If $\alpha > 1/2$ and $0 < r < 1$, then*

$$\int_0^{2\pi} \frac{d\varphi}{|1-re^{i\varphi}|^{2\alpha}} \leqq A_\alpha (1-r)^{1-2\alpha} \tag{2.10}$$

PROOF. Since

$$\frac{|1-re^{i\varphi}|^{2\alpha}}{(\delta^2+\varphi^2)^\alpha}, \quad -\pi \leqq \varphi \leqq \pi \tag{2.11}$$

where $\delta = 1-r$, are bound above and below by positive numbers,

$$\int_0^{2\pi} \frac{d\varphi}{|1-re^{i\varphi}|^{2\alpha}} \leqq A_\alpha \int_{-\pi}^{\pi} \frac{d\varphi}{(\delta^2+\varphi^2)^\alpha} = A_\alpha \int_0^\delta + A_\alpha \int_\delta^\pi$$

$$\leqq A_\alpha \int_0^\delta \frac{d\varphi}{\delta^{2\alpha}} + A_\alpha \int_\delta^\pi \frac{d\varphi}{\varphi^{2\alpha}}$$

$$\leqq A_\alpha \delta^{1-2\alpha} + A_\alpha [\varphi^{1-2\alpha}]_\delta^\pi \leqq A_\alpha \delta^{1-2\alpha},$$

for $1-2\alpha<0$.

LEMMA 3. *If we put*

$$f^*(\theta)=\sup_{0<|h|\leqq\pi}\left|\frac{1}{h}\int_0^h |f(e^{i(\theta+u)})|\,du\right| \tag{2.12}$$

and

$$f_k^*(\theta)=\sup_{0<|h|\leqq\pi}\left|\frac{1}{h}\int_0^h |f(e^{i(\theta+u)})|^k\,du\right|^{1/k} \tag{2.13}$$

then

$$|f(re^{i(\theta+\varphi)})|\leqq Af^*(\theta)(1+|\varphi|/\delta) \qquad \text{for } f(e^{i\theta})\in L \tag{2.14}$$

and

$$|f(re^{i(\theta+\varphi)})|\leqq Bf_k^*(\theta)(1+|\varphi|/\delta)^{1/k} \quad \text{for } f(e^{i\theta})\in L^k \quad (k>1) \tag{2.15}$$

where $\delta=1-r$, *and* $f_k^*(\theta)\in L^2$, *if* $f(e^{i\theta})\in L^2$ *and* $k<2$.

This is essentially due to Hardy and Littlewood [3]. Since

$$f(re^{i(\theta+\varphi)})=\frac{1}{\pi}\int_0^{2\pi} f(e^{i(\theta+u)})\,P(r,u-\varphi)\,du,$$

where $P(r,t)$ is the Poisson kernel, by partial integration, we can easily get (2.14).

To deduce (2.15) from (2.14) it is enough to note that, by Jensen's in equality

$$\left\{\frac{1}{\pi}\int_0^{2\pi} |f(e^{i(\theta+t)})|\,P(r,t)\,dt\right\}^{1/k}\leqq\frac{1}{\pi}\int_0^{2\pi} |f(e^{i(\theta+t)})|^k P(r,t)\,dt.$$

THEOREM 1. *If* $f(z)\in H^p$ $(0<p\leqq 2)$, *and* $\alpha>1/p$, *then*

$$\int_0^{2\pi} (g_\alpha^*(\theta))^p\,d\theta\leqq A_{p,\alpha}\int_0^{2\pi} |f(e^{i\theta})|^p\,d\theta. \tag{2.16}$$

This was given in my previous paper [9]. We shall give a slightly simpler proof.

PROOF. We shall begin with the case $p=2$.

(1) $p=2$. Then for $2\alpha>1$,

$$\int_0^{2\pi}\{g_\alpha^*(\theta)\}^2\,d\theta\leqq A_\alpha\int_0^1 (1-r)^{2\alpha}\,dr\int_0^{2\pi}\frac{d\varphi}{|1-re^{i\varphi}|^{2\alpha}}\int_0^{2\pi} |f'(re^{i(\theta+\varphi)})|^2\,d\theta$$

$$\leqq B_\alpha\int_0^1 (1-r)^{2\alpha}\sum_{n=1}^\infty n^2|c_n|^2 r^{2(n-1)}\,dr\int_0^{2\pi}\frac{d\varphi}{|1-re^{i\varphi}|^{2\alpha}}$$

$$\leqq C_\alpha \sum_{n=1}^{\infty} n^2 |c_n|^2 \int_0^1 (1-r)^{2\alpha}(1-r)^{1-2\alpha} r^{2(n-1)}\, dr \qquad \text{(by Lemma 2)}$$

$$\leqq D_\alpha \sum_{n=1}^{\infty} n^2 |c_n|^2 \int_0^1 (1-r)\, r^{2(n-1)}\, dr$$

$$\leqq E_\alpha \sum_{n=1}^{\infty} |c_n|^2 \leqq G_\alpha \int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta.$$

Hence the case $p = 2$ is proved.

(2) Suppose that $0 < p < 2$ and put

$$F(z) = \{f(z)\}^{p/2}$$

then $F(z)$ is regular for $|z| < 1$ and belongs to H^2. Since

$$f'(z) = \frac{2}{p}\{F(z)\}^{(2-p)/p} F'(z),$$

so that

$$\frac{f'(z)}{(1-z)^\alpha} = \frac{2}{p}\left\{\frac{(F(ze^{i\theta}))^{(2-p)/p} F'(ze^{i\theta})}{(1-z)^\alpha}\right\}$$

we have, by Parseval's relation

$$\{g_\alpha^*(\theta\,;f)\}^2 \leqq A_{p,\alpha} \int_0^1 (1-r)^{2\alpha}\, dr \int_0^{2\pi} \frac{\{|F(re^{i(\theta+\varphi)})|^2\}^{(2-p)/p} |F'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi.$$

By Lemma 3, for $k < 2$ this is smaller than

$$B_{p,\alpha}\{F_k^*(\theta)\}^{2(2-p)/p} \int_0^1 (1-r)^{2\alpha}\, dr \int_{-\pi}^{\pi} \left\{\frac{\delta + |\varphi|}{\delta}\right\}^{\frac{2}{k}(\frac{2}{p}-1)} \frac{|F'(re^{i(\theta+\varphi)})|^2}{(\delta^2+\varphi^2)^\alpha}\, d\varphi$$

$$\leqq C_{p,\alpha}\{F_k^*(\theta)\}^{2(2-p)/p} \int_0^1 (1-r)^{2\alpha}\, dr \int_{-\pi}^{\pi} \frac{(\delta^2+\varphi^2)^{1/k(2/p-1)} |F'(re^{i(\theta+\varphi)})|^2}{\delta^{2/k(2/p-1)}(\delta^2+\varphi^2)^\alpha}\, d\varphi$$

$$= C_{p,\alpha}\{F_k^*(\theta)\}^{2(2-p)/p} \int_0^1 (1-r)^{2\alpha - \frac{2}{k}(\frac{2}{p}-1)}\, dr \int_{-\pi}^{\pi} \frac{|F'(re^{i(\theta+\varphi)})|^2}{(\delta^2+\varphi^2)^{\alpha-1/k(2/p-1)}}\, d\varphi.$$

Let $\alpha = (1+\varepsilon)/p$ $(\varepsilon > 0)$, $\beta = (1+\varepsilon)/p - 1/k(2/p-1)$ and k be sufficiently near to 2, then

$$2\beta - 1 = \left(\frac{2}{p}-1\right)\left(1-\frac{2}{k}\right) + \frac{2\varepsilon}{p} > 0,$$

and the above formula is

$$C_{p,\alpha}\{F_k^*(\theta)\}^{2(2/p-1)} \int_0^1 (1-r)^{2/\beta} dr \int_{-\pi}^{\pi} \frac{|F'(re^{i(\theta+\varphi)})|^2}{(\delta^2+\varphi^2)^\beta}\, d\varphi$$

$$\leqq D_{p,\alpha}\{F_k^*(\theta)\}^{2(2/p-1)}\int_0^1 (1-r)^{2\beta}dr\int_0^{2\pi}\frac{|F'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\beta}}d\varphi$$

$$\leqq E_{p,\alpha}\{F_k^*(\theta)\}^{2(2/p-1)}\{g_\beta^*(\theta\,;\,F)\}^2, \qquad (2\beta>1).$$

Thus

$$\int_0^{2\pi}\{g_\alpha^*(\theta\,;f)\}^p d\theta \leqq F_{p,\alpha}\int_0^{2\pi}\{F_k^*(\theta)\}^{2-p}\{g_\beta^*(\theta\,;\,F)\}^p d\theta$$

$$\leqq F_{k,\alpha}\left\{\int_0^{2\pi}(F_k^*(\theta))^2\,d\theta\right\}^{(2-p)/2}\left\{\int_0^{2\pi}(g_\beta^*(\theta\,;\,F))^2\,d\theta\right\}^{p/2}$$

by the inequality of Hölder. From the maximal theorem of Hardy and Littlewood, we have

$$\int_0^{2\pi}\{F_k^*(\theta)\}^2\,d\theta \leqq A_k\int_0^{2\pi}|F(e^{i\theta})|^2\,d\theta$$

and so by the case (1),

$$\int_0^{2\pi}(g_\alpha^*(\theta\,;f))^p\,d\theta \leqq G_{p,\alpha}\int_0^{2\pi}|F(e^{i\theta})|^2 d\theta \leqq K_{p,\alpha}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta.$$

Thus we proved Theorem 1 completely.

COROLLARY 1. *If $f(z)$ belongs to the class H^p $(0<p\leqq 2)$, then*

$$\int_0^{2\pi}\{H_2^\alpha(\theta)\}^p\,d\theta \leqq A_{p,\alpha}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta \qquad (\alpha>1/p)$$

where

$$H_2^\alpha(\theta)=\left\{\sup_{0<n<\infty}\frac{1}{n}\sum_{k=1}^{n}|\sigma_k^{\alpha-1}(\theta)-f(e^{i\theta})|^2\right\}^{1/2}.$$

This is a maximal theorem concerning with strong summability.

PROOF. Let us put n_θ is the index n when $H_2^\alpha(\theta)$ attains the supremum, then

$$\sup_{0<n<\infty}\frac{1}{n}\sum_{k=1}^{n}|\sigma_k^{\alpha-1}(\theta)-\sigma_k^\alpha(\theta)|^2=\frac{1}{n_\theta}\sum_{k=1}^{n_\theta}|\sigma_k^{\alpha-1}(\theta)-\sigma_k^\alpha(\theta)|^2$$

$$\leqq\sum_{k=1}^{n_\theta}\frac{|\sigma_k^{\alpha-1}(\theta)-\sigma_k^\alpha(\theta)|^2}{k}\leqq\sum_{k=1}^{\infty}\frac{|\sigma_k^{\alpha-1}(\theta)-\sigma_k^\alpha(\theta)|^2}{k}=\{h_\alpha(\theta)\}^2,$$

by (2.3). Thus Corollary is immediate from Theorem 1 and Lemma 1.

COROLLARY 2. *If $f(z)$ belongs to the class H^p $(0<p\leqq 2)$, then*

$$\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n} \qquad (\alpha>1/p)$$

is convergent for almost all θ.

This is immediate from Theorem 1 and Lemma 1. This Corollary has been ever proved by Chow [1].

From this, we can get the following Corollaries by the well known method.

COROLLARY 3. *If* $f(z)$ *belongs to* H^p $(0 < p \leqq 2)$, *then the series* $\sum \lambda_n c_n e^{ni\theta}$, *where* $\sum (\lambda_n)^2/n$ *converges, is summable* $|C, \alpha|, (\alpha > 1/p)$ *for almost all* θ.

For the proof, see Chow [1].

COROLLARY 4. *If* $f(z)$ *belongs to* H^p $(0 < p \leqq 2)$, *then for almost all* θ, *the sequence* $\{n\}$ *can be divided into two complementary subsequences* $\{n_k\}$ *and* $\{m_k\}$, *depending in general on* θ, *and such that* $\sigma_{n_k}^{\alpha-1}(\theta)$ *tends to* $f(\theta)$ *and the series* $\sum 1/m_k$ *converges, where* $\alpha > 1/p$.

For the reduction of this Corollary, see Zygmund [10].

3. The function $g_\alpha^*(\theta)$ for the class H^p $(\infty > p > 2)$.

For the class H^p $(\infty > p > 2)$, we have the following theorem. This is essentially due to Zygmund [13]. The proof is also repetition of his argument.

THEOREM 2. *If* $f(z) \in H^p$ $(\infty > p > 2)$ *and* $\alpha > 1/2$, *then*

$$\int_0^{2\pi} (g_\alpha^*(\theta))^p \, d\theta \leqq A_{p,\alpha} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta. \tag{3.1}$$

PROOF. Let μ be a positive number such that

$$\frac{1}{p/2} + \frac{1}{\mu} = 1$$

and let $\xi(\theta)$ be any positive function such that

$$\left\{ \int_0^{2\pi} \xi^\mu(\theta) \, d\theta \right\}^{1/\mu} \leqq 1. \tag{3.2}$$

Then it is known that

$$\left\{ \int_0^{2\pi} (g_\alpha^*(\theta))^p \, d\theta \right\}^{1/p} = \left[\left\{ \int_0^{2\pi} (g_\alpha^{*2}(\theta))^{p/2} \, d\theta \right\}^{2/p} \right]^{1/2} \tag{3.3}$$

$$= \sup_{\xi(\theta)} \left\{ \int_0^{2\pi} (g_\alpha^*(\theta))^2 \xi(\theta) \, d\theta \right\}^{1/2},$$

and the inner integral

$$\int_0^{2\pi} (g_\alpha^*(\theta))^2 \xi(\theta)\, d\theta \leqq A_\alpha \int_0^{2\pi} \xi(\theta)\, d\theta \int_0^1 \delta^{2\alpha}\, dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\, d\varphi \quad (\delta = 1-r)$$

$$\leqq B_\alpha \int_0^{2\pi} \xi(\theta)\, d\theta \int_0^1 \delta^{2\alpha}\, dr \int_0^{2\pi} |f'(re^{i\varphi})|^2 \frac{d\varphi}{|1-re^{i(\varphi-\theta)}|^{2\alpha}}$$

$$\leqq B_\alpha \int_0^1 \delta\, dr \int_0^{2\pi} |f'(re^{i\varphi})|^2\, d\varphi\, \delta^{2\alpha-1} \int_0^{2\pi} \frac{\xi(\theta)}{|1-re^{i(\varphi-\theta)}|^{2\alpha}}\, d\theta.$$

Let us put

$$\Xi(u,\varphi) = \int_0^u \{\xi(\varphi+\theta) + \xi(\varphi-\theta)\}\, d\theta$$

and

$$\xi^*(\varphi) = \sup_{0<u\leqq\pi} \frac{1}{u}\, |\Xi(u,\varphi)|, \tag{3.4}$$

then the last integral

$$\delta^{2\alpha-1} \int_{-\pi}^{\pi} \frac{\xi(\theta)\, d\theta}{|1-re^{i(\varphi-\theta)}|^{2\alpha}} = \delta^{2\alpha-1} \int_0^{\pi} \{\xi(\varphi+u)+\xi(\varphi-u)\} \frac{du}{|1-re^{iu}|^{2\alpha}}$$

$$\leqq C_\alpha\, \delta^{2\alpha-1} \int_0^{\pi} \{\xi(\varphi+u)+\xi(\varphi-u)\} \frac{du}{(\delta^2+u^2)^\alpha}$$

$$\leqq C_\alpha\, \delta^{2\alpha-1} \int_0^{\delta} + C_\alpha\, \delta^{2\alpha-1} \int_\delta^{\pi} = I + J,$$

say. Then

$$I \leqq D_\alpha\, \delta^{2\alpha-1} \int_0^{\delta} \{\xi(\varphi+u)+\xi(\varphi-u)\} \frac{du}{\delta^{2\alpha}}$$

$$\leqq D_\alpha\, \xi^*(\varphi)$$

and

$$J \leqq \delta^{2\alpha-1} \int_\delta^{\pi} \{\xi(\varphi+u)+\xi(\varphi-u)\} \frac{du}{u^{2\alpha}}$$

$$= \delta^{2\alpha-1} \Big[\Xi(u,\varphi) u^{-2\alpha} \Big]_\delta^\pi + 2\alpha\, \delta^{2\alpha-1} \int_\delta^{\pi} \Xi(u,\varphi)\, u^{-2\alpha-1}\, du$$

$$\leqq \delta^{2\alpha-1}\, \Xi(\pi,\varphi)\, \pi^{-1}\pi^{-2\alpha+1} + 2\alpha\, \delta^{2\alpha-1}\, \xi^*(\varphi) \int_\delta^{\infty} u^{-2\alpha}\, du$$

$$\leqq \xi^*(\varphi)\left\{2\,\delta^{2\alpha-1}+\frac{2\alpha}{2\alpha-1}\right\}\leqq E_\alpha\,\xi^*(\varphi).$$

Thus the left-hand side of (3.3)

$$\begin{aligned}\int_0^{2\pi}\{g_\alpha^*(\theta)\}^2\,\xi(\theta)\,d\theta &\leqq F_\alpha\int_0^1\delta\,dr\int_0^{2\pi}|f'(re^{i\varphi})|^2\,\xi^*(\varphi)\,d\varphi\\ &= F_\alpha\int_0^{2\pi}g^2(\varphi)\,\xi^*(\varphi)\,d\varphi\\ &\leqq F_\alpha\left\{\int_0^{2\pi}(\xi^*(\varphi))^\mu\,d\varphi\right\}^{1/\mu}\left\{\int_0^{2\pi}g^p(\varphi)\,d\varphi\right\}^{2/p}\\ &\leqq G_\alpha\left\{\int_0^{2\pi}(\xi(\varphi))^\mu\,d\varphi\right\}^{1/\mu}\left\{\int_0^{2\pi}|f(e^{i\varphi})|^p\,d\varphi\right\}^{2/p}\end{aligned}$$

from the maximal theorem of Hardy-Littlewood and the theorem of Littlewood-Paley [6] (simple proof; Zygmund [14]). Thus we get

$$\left\{\int_0^{2\pi}(g_\alpha^*(\theta))^p\,d\theta\right\}^{1/p}\leqq A_{p,\alpha}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{1/p},\quad(\infty>p>2)$$

and the theorem is proved. From this, we can derive easily that if $f(z)$ belongs to H^p $(\infty>p>2)$, and $\alpha>1/2$, then

$$\int_0^{2\pi}\left\{\sum_{n=1}^\infty\frac{|\tau_n^\alpha(\theta)|^2}{n}\right\}^{p/2}d\theta\leqq A_{p,\alpha}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta,$$

and

$$\int_0^{2\pi}\left\{\sup_{0<n<\infty}\frac{1}{n}\sum_{k=1}^n|\sigma_n^{\alpha-1}(\theta)-f(e^{i\theta})|^2\right\}^{p/2}d\theta\leqq B_{p,\alpha}\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta$$

4. A proof of the theorem of Littlewood and Paley. From Theorems 1 and 2, we have especially,

THEOREM 3. *If $f(z)$ belongs to H^p $(1<p<\infty)$, then*

$$\int_0^{2\pi}(g^*(\theta))^p\,d\theta\leqq A'_p\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta. \tag{4.1}$$

From this, we have the following

$$\int_0^{2\pi}\{S(\theta)\}^p\,d\theta\leqq B_p\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta,\qquad(p>1) \tag{4.2}$$

and

$$\int_0^{2\pi} \{\mu(\theta)\}^p \, d\theta \leqq C_p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \qquad (p > 1) \tag{4.3}$$

where $S(\theta)$ is the function of Lusin and $\mu(\theta)$ is the function of Marcinkiewicz. The function of $g^*(\theta)$ is essenitally a majorant of these functions. The reduction of (4.2) is done by a moment's consideration, but the reduction of (4.3) is somewhat difficult. For the detailed definitions and proofs, see Zygmund [13].

The main theorem of Littlewood-Paley is condensed in the following theorem.

THEOREM 4. *If $f(z)$ belongs to H^p $(1 < p < \infty)$, then*

$$A_p \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \leqq \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{|s_n(\theta) - \sigma_n(\theta)|^2}{n} \right\}^{p/2} d\theta \leqq A_p' \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{4.4}$$

$$B_{p,\alpha,\beta} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \leqq \int_0^{2\pi} \left\{ \sum_{k=1}^{\infty} |s_{n_k}(\theta) - \sigma_{n_k}(\theta)|^2 \right\}^{p/2} d\theta \leqq B_{p,\alpha,\beta}' \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{4.5}$$

$$C_{p,\alpha,\beta} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \leqq \int_0^{2\pi} \left\{ \sum_{k=1}^{\infty} |\Delta_k(\theta)|^2 \right\}^{p/2} d\theta \leqq C_{p,\alpha,\beta}' \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta, \tag{4.6}$$

where $\displaystyle \Delta_k(\theta) = \sum_{n_{k-1}+1}^{n_k} c_n e^{ni\theta}$ *and* $\beta > n_{k+1}/n_k > \alpha > 1$.

(4.4) is a consequence of Theorem 3 and Lemma 1*). The left-halves of (4.5) and (4.6) are proved by the following results of Zygmund [11]. That is

$$\int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{|s_n(\theta) - \sigma_n(\theta)|^2}{n} \right\}^{p/2} d\theta \leqq D_{p,\alpha,\beta} \int_0^{2\pi} \left\{ \sum_{k=1}^{\infty} |s_{n_k}(\theta) - \sigma_{n_k}(\theta)|^2 \right\}^{p/2} d\theta$$
$$\leqq E_{p.\alpha.\beta} \int_0^{2\pi} \left\{ \sum_{k=1}^{\infty} |\Delta_k(\theta)|^2 \right\}^{p/2} d\theta$$

for $1 < p < \infty$. For the proof of the reverse inequalities, we need

LEMMA 4. *Let $\{f_n(z)\}$ $(n = 1, 2 \ldots.)$ be a sequence of the function of H^p $(1 < p < \infty)$, and let $s_{n,k}(\theta)$ denote the k-th partial sum of the boundary series of $f_n(z)$. Then*

$$\int_0^{2\pi} \left(\sum_{n=1}^{\infty} |s_{n,k_n}(\theta)|^2 \right)^{p/2} d\theta \leqq A_p \int_0^{2\pi} \left(\sum_{n=1}^{\infty} |f_n(e^{i\theta})|^2 \right)^{p/2} d\theta.$$

A comparatively simple proof was given by Zygmund [10], using Rademacher's function.

PROOF OF THEOREM 4. From Lemma 4, we have

*) We suppose that the left-half of (4.4) is proved by another method.

$$\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|s_{n_k}(\theta)-\sigma_{n_k}(\theta)|^2\Big)^{p/2}d\theta$$

$$=\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}\frac{1}{n_k^2}\Big(\Big|\sum_{m=1}^{n_k}mc_me^{mi\theta}\Big|^2\Big)^{p/2}d\theta$$

$$\leqq A_{p,\alpha,\beta}\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}\Big|\sum_{m=1}^{n_k}m\,c_me^{mi\theta}\Big|^2\sum_{\nu=n_k+1}^{n_{k+1}}1/\nu^3\Big)^{p/2}d\theta$$

$$\leqq B_{p,\alpha,\beta}\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}\sum_{\nu=n_k+1}^{n_{k+1}}\frac{1}{\nu^3}\Big|\sum_{m=1}^{\nu}m\,c_m\,e^{mi\theta}\Big|^2\Big)^{p/2}d\theta$$

$$=B_{p,\alpha,\beta}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|s_n(\theta)-\sigma_n(\theta)|^2}{n}\Big)^{p/2}d\theta.$$

Thus (4. 5) is proved. On the other hand

$$|\sigma_{n_k+1}(\theta)-\sigma_{n_k}(\theta)|^2\leqq\Big\{\sum_{m=n_k+1}^{n_{k+1}}|\sigma_m(\theta)-\sigma_{m-1}(\theta)|^2\Big\}^2$$

$$\leqq\Big\{\sum_{m=n_k+1}^{n_{k+1}}m|\sigma_m(\theta)-\sigma_{m-1}(\theta)|^2\Big\}\Big\{\sum_{m=n_k+1}^{n_{k+1}}1/m\Big\}$$

$$\leqq\log\frac{n_{k+1}}{n_k+1}\Big\{\sum_{m=n_k+1}^{n_{k+1}}m|\sigma_m(\theta)-\sigma_{m-1}(\theta)|^2\Big\}$$

$$\leqq\log\beta\Big\{\sum_{m=n_k+1}^{n_{k+1}}|s_m(\theta)-\sigma_m(\theta)|^2/m\Big\}$$

by (4. 2). Since

$$|\Delta_{k+1}(\theta)|^2=|s_{n_{k+1}}(\theta)-s_{n_k}(\theta)|^2$$
$$\leqq|s_{n_{k+1}}(\theta)-\sigma_{n_{k+1}}(\theta)|^2+|s_{n_k}(\theta)-\sigma_{n_k}(\theta)|^2+|\sigma_{n_{k+1}}(\theta)-\sigma_{n_k}(\theta)|^2,$$

we get the required ineqnality

$$\sum_{k=1}^{\infty}|\Delta_k|^2\leqq A_{p,\alpha,\beta}\sum_{k=1}^{\infty}|s_{n_k}(\theta)-\sigma_{n_k}(\theta)|^2+B_{p,\alpha,\beta}\sum_{n=1}^{\infty}\frac{|s_n(\theta)-\sigma_n(\theta)|^2}{n}$$

and get the formula (4. 6). Thus the theorem is proved completely.

The following corollary is deduced by the well known method from Theorem 4.

COROLLARY 5. *If $f(z)$ belongs to the class H^p $(p>1)$, and if $\beta>n_{k+1}/n_k>\alpha>1$, then*

$$\int_0^{2\pi}\Big\{\sup_{1<k<\infty}|s_{n_k}(\theta)|\Big\}^p d\theta\leqq A_{p,\alpha,\beta}\int_0^{2\pi}|f(e^{i\theta})|^p d\theta$$

COROLLARY 6. *If $\{\varepsilon_k\}$ is any sequence of numbers of which each has one of the values* 1, -1, *and if* $f\in H^p$ $(p>1)$, $\beta > n_{k+1}/n_k > \alpha > 1$, *then*

$$\int_0^{2\pi} \left|\sum_{k=1}^{\infty} \varepsilon_k \Delta_k(\theta)\right|^p d\theta \leqq B_{p,\alpha,\beta} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta$$

where

$$\Delta_k(\theta) = \sum_{n=n_{k-1}+1}^{n_k} c_n e^{in\theta}.$$

5. The power series of *H*-class.

Concerning with the power series of H-class, A. Zygmund [11] proved that

$$\int_0^{2\pi} h_1(\theta)\, d\theta \leqq A \int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| \, d\theta + A' \tag{5.1}$$

and

$$\int_0^{2\pi} (h_1(\theta))^\mu \, d\theta \leqq B_\mu \left(\int_0^{2\pi} |f(e^{i\theta})| \, d\theta\right), \qquad (0<\mu<1). \tag{5.2}$$

So, from Lemma 1, we get

THEOREM 5. *If $f(z)$ belongs to the class H, then*

$$\int_0^{2\pi} g^*(\theta)\, d\theta \leqq A \int_0^{2\pi} |f(e^{i\theta})| \log^+ |f(e^{i\theta})| \, d\theta + A' \tag{5.3}$$

$$\int_0^{2\pi} (g^*(\theta))^\mu \, d\theta \leqq B_\mu \left(\int_0^{2\pi} |f(e^{i\theta})| \, d\theta\right)^\mu, \qquad (0<\mu<1) \tag{5.4}$$

The present author has not ever a simple and direct proof of this theorem.

THEOREM 6. *If $f(z)$ belongs to H, then*

$$\int_0^{2\pi} \{\mu(\theta)\}\, d\theta \leqq A \int_0^{2\pi} |(fe^{i\theta})| \log^+ |f(e^{i\theta})| \, d\theta + A' \tag{5.5}$$

$$\int_0^{2\pi} \{\mu(\theta)\}^r \, d\theta \leqq B_r \left(\int_0^{2\pi} |f(e^{i\theta})| \, d\theta\right)^r, \qquad (0<r<1) \tag{5.6}$$

where

$$\mu(\theta) = \left\{\int_0^{\pi} \frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^2}{t^3}\, dt\right\}^{1/2}$$

and

$$F(\theta) = c + \int_0^\theta f(e^{it})\,dt.$$

This is immediate from the fact

$$\mu(\theta) \leqq Cg^*(\theta)$$

which was proved by Zygmund [13].

6. **Power series of the class $H^p(0 < p < 1)$.** For the class $H^p(0 < p < 1)$, we have a more precise results than Theorem 2.

THEOREM 7. *If $f(z)$ belongs to H^p $(0 < p < 1)$ and $\alpha = 1/p$, then*

$$\int_0^{2\pi} \{g_\alpha^*(\theta)\}^p\,d\theta \leqq A_p \int_0^{2\pi} |f(e^{i\theta})|^p \log^+ |f(e^{i\theta})|\,d\theta + A_p' \tag{6.1}$$

$$\int_0^{2\pi} \{g_\alpha^*(\theta)\}^{p\mu}\,d\theta \leqq B_{p,\mu} \left\{\int_0^{2\pi} |f(e^{i\theta})|^p\,d\theta\right\}^\mu, \qquad (0 < \mu < 1). \tag{6.2}$$

PROOF. This case is reduced to the case $p = 1$.

If we take $0 < p < 1$, $\alpha = 1/p$ and p

$$G(z) = \{f(z)\}^p$$

then

$$G(z) \in H,$$

$$f'(z) = \alpha\{G(z)\}^{\alpha-1}G'(z),$$

and

$$(g_\alpha^*(\theta\,;f))^2 = \frac{\alpha^2}{2\pi}\int_0^1 (1-r)^{2\alpha}\,dr \int_0^{2\pi} \frac{|G(re^{i(\varphi+\theta)})|^{2(\alpha-1)}\,|G'(re^{i(\varphi+\theta)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\,d\varphi.$$

From (2.14) of Lemma 3,

$$G(re^{i(\varphi+\theta)}) \leqq AG^*(\theta)\left\{1 + \frac{|\varphi|}{\delta}\right\}, \qquad (\delta = 1-r)$$

where

$$G^*(\theta) = \sup_{0<|h|\leqq\pi} \left|\frac{1}{h}\int_0^h |G(e^{i(\theta+u)})|\,du\right|,$$

the right-hand side is smaller than

$$A_p\{G^*(\theta)\}^{2(\alpha-1)}\int_0^1 (1-r)^{2\alpha}\,dr \int_{-\pi}^{\pi} \frac{(\delta^2+\varphi^2)^{\alpha-1}}{(1-r)^{2(\alpha-1)}(\delta^2+\varphi^2)^{\alpha-1}}\,\frac{|G'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\,d\varphi$$

$$\leqq A_p\{G^*(\theta)\}^{2(\alpha-1)}\int_0^1 (1-r)^{2\alpha}\,dr \int_{-\pi}^{\pi} \frac{|G'(r\,e^{i(\varphi+\theta)})|^2}{|1-re^{i\varphi}|^2}\,d\varphi$$

$$\leqq A_p\{G^*(\theta)\}^{2(\alpha-1)}\{g^*(\theta;\ G)\}^2.$$

$$\int_0^{2\pi}\{g_\alpha^*(\theta;\ f)\}^p\,d\theta \leqq A_p\int_0^{2\pi}\{G^*(\theta)\}^{1-p}\{g^*(\theta;\ G)\}^p\,d\theta$$

$$\leqq A_p\left\{\int_0^{2\pi}G^*(\theta)\,d\theta\right\}^{1-p}\left\{\int_0^{2\pi}g^*(\theta;\ G)\,d\theta\right\}^p$$

by Hölder's inequality. From the maximal theorem and Theorem 5, we get

$$\int_0^{2\pi}\{g_\alpha^*(\theta)\}^p\,d\theta \leqq B_p\int_0^{2\pi}|G(e^{i\theta})|\log^+|G(e^{i\theta})|\,d\theta + B_p'$$

$$\leqq B_p\int_0^{2\pi}|f(e^{i\theta})|^p\log^+|f(e^{i\theta})|\,d\theta + B_p'$$

Analogously

$$\int_0^{2\pi}\{g^{*\alpha}(\theta;\ f)\}^{p\mu}\,d\theta \leqq A_p\int_0^{2\pi}\{G^*(\theta)\}^{\mu(1-p)}\{g^*(\theta;\ G)\}^{p\mu}\,d\theta$$

$$\leqq A_p\left[\int_0^{2\pi}\{G^*(\theta)\}^{\mu}\,d\theta\right]^{1-p}\left[\int_0^{2\pi}\{g^*(\theta;\ G)\}^{\mu}\,d\theta\right]^p$$

$$\leqq B_{p,\mu}\left\{\int_0^{2\pi}|G(e^{i\theta})|\,d\theta\right\}^{\mu}$$

$$\leqq B_{p,\mu}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\mu}.$$

Thus Theorem is proved completely.

THEOREM 8, *If $f(z)$ belongs to H^p $(0<p\leqq 1)$, and $\alpha=1/p$, then*

$$\int_0^{2\pi}\{h_\alpha(\theta)\}^p\,d\theta \leqq A_p\int_0^{2\pi}|f(e^{i\theta})|^p\log^+|f(e^{i\theta})|\,d\theta + A_p' \tag{6.3}$$

$$\int_0^{2\pi}\{h_\alpha(\theta)\}^{p\mu}\,d\theta \leqq B_{p,\mu}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\mu}. \tag{6.4}$$

This is immediate from Theorem 7 and Lemma 1. This formula was given by the present author [9] by more complicated method.

7. Maximal theorems of the Cesàro mean of the power series of the class H^p $(0<p<1)$.

THEORFM 9. *If $f(z)$ belongs to H^p $(0<p\leqq 1)$ and $\alpha>1/p-1$, then*

$$\int_0^{2\pi} \{ \sup_{0<n<\infty} |\sigma_n^\alpha(\theta)| \}^p d\theta \leqq A_{p,\alpha} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta, \tag{7.1}$$

THEOREM 10. *If $f(z)$ belongs to $H^p (0 < p \leqq 1/2)$, and $\alpha = 1/p - 1$, then*

$$\int_0^{2\pi} \{ \sup_{0<n<\infty} |\sigma_n^\alpha(\theta)| \}^p d\theta \leqq B_p \int_0^{2\pi} |f(e^{i\theta})|^p \log^+ |f(e^{i\theta})| d\theta + B_p' \tag{7.2}$$

$$\int_0^{2\pi} \{ \sup_{0<n<\infty} |\sigma_n^\alpha(\theta)|^p \}^{p\mu} d\theta \leqq C_{p,\mu} \Big(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \Big)^\mu, \qquad (0 < \mu < 1). \tag{7.3}$$

Theorem 9 is a generalization of classical results of Hardy-Littlewood [4] and Gwilliam [2]. (7.2) of Theorem 10 is an affirmative answer of a problem of Zygmund [12]. From (7.3) we can easily see that $\sigma_n^\alpha(\theta)$ ($\alpha = 1/p - 1$) converges to $f(e^{i\theta})$ for almost all θ. This was proved by Zygmund [12] for $0 < p < 1$. For the case $1/2 < p < 1$, the maximal theorem is left open, but convergence of $\sigma_n^\alpha(\theta)$ is proved in the next section.

The present author [9] deduced Theorem 9 and Theorem 10 from Lemma 1 and Theorem 8 with the aid of the following lemma.

LEMMA 5. *If $f(z) = g^2(z)$ and $\alpha > 0$, then*

$$|\sigma_n^\alpha(\theta; f)| \leqq A_\alpha \cdot \frac{1}{n^\alpha} \sum_{k=0}^n \frac{|\sigma_k^{(\alpha-1)/2}(\theta; g) - \sigma_k^{(\alpha+1)/2}(\theta; g)|^2}{(k+1)^{1-\alpha}}$$
$$+ A_\alpha' \frac{1}{n^\alpha} \sum_{k=0}^n \frac{|\sigma_k^{(\alpha+1)/2}(\theta; g)|^2}{(k+1)^{1-\alpha}}$$

where $\sigma_n^\alpha(\theta; f)$ is the α-th Cesàro mean of the boundary series of $f(z)$.

For the detailed argument, see my previous paper [9].

8. Strong summability and ordinary summability of the power series.

In Corollary 1, we have proved the maximal theorem of the strong summability of $\sigma_n^{\alpha-1}(\theta)$ ($\alpha > 1/p$) for H^p ($1 < p \leqq 2$). But if we give up the maxinal theorem, then we can prove the more precise result.

THEOREM 11. *If $f(z)$ belongs to H^p ($1 \leqq p < 2$) and $\alpha = 1/p$, then*

$$\sum_{k=0}^n |\sigma_k^{\alpha-1}(\theta) - f(e^{i\theta})|^q = o(n)$$

for almost all θ, where $0 < q < p/(p-1)$.

This was proved in the author's paper [8]. The method of the proof depends closely upon the paper of Zygmund [12]. In his paper, Zygmund proved the strong summability theorem of the function of L and the Cesàro summability theorem of the series of the class H^p ($0 < p < 1$). The proofs of both theorems have many features in common, but the the details of proofs are different. After proving Theorem 11, we can deduce from that the following Cesàro summability theorem.

THEOREM 12. *If $f(z)$ belongs to H^p $(1/2 < p < 1)$ and $\alpha = 1/p - 1$, then the series $\sum c_n e^{in\theta}$ is summable (C, α) to $f(e^{i\theta})$ for almost all θ.*

PROOF. If we put

$$f(z) = g^2(z),$$

then $g(z)$ belongs to $H^\lambda(\lambda = 2p,\ 1 < \lambda < 2)$, and $(\alpha + 1)/2 = 1/2p = 1/\lambda$. From the formula in [8], p. 225, we have

$$\sum_{k=1}^{n} A_k^{2\lambda} |\sigma_k^{1/\lambda - 1}(\theta\,;\,g) - \sigma_k^{1/\lambda}(\theta\,;\,g)|^2 = o(n^{2/\lambda+1}). \qquad \text{a. e.}$$

that is

$$\sum_{k=1}^{n} A_k^{\alpha+1} |\sigma_k^{(\alpha-1)/2}(\theta\,;\,g) - \sigma_k^{(\alpha+1)/2}(\theta\,;\,g)|^2 = o(n^{\alpha+2}), \qquad \text{a. e.}$$

By Abel's transformation,

$$\sum_{k=1}^{n} \frac{|\sigma_k^{(\alpha-1)/2}(\theta\,;\,g) - \sigma_k^{(\alpha+1)/2}(\theta\,;\,g)|^2}{k^{1-\alpha}} = o(n^\alpha), \qquad \text{a. e.}$$

From Lemma 5, we have

$$|\sigma_n^\alpha(\theta\,;\,f)| \leqq o(1) + \frac{1}{n^\alpha} \sum_{k=0}^{n} \frac{|\sigma_k^{(\alpha+1)/2}(\theta\,;\,g)|^2}{(k+1)^{1-\alpha}}, \qquad \text{a. e.}$$

Since evidently $\sigma_n^{(\alpha+1)/2}(\theta\,;\,g)$ tends to $g(e^{i\theta})$ for almost all θ, if we take polynomial $f_\epsilon(z)$ near to $f(z)$, then we can conclude

$$\sigma_n^\alpha(\theta\,;\,f) \to f(e^{i\theta}), \qquad \text{a. e.} \qquad \text{as } n \to \infty.$$

Thus we get Theorem 12 from Theorem 11.

9. The affirmative answer to a problem of Zygmund.

On another conjecture of Zygmund, we can prove the following theorem.

THEOREM 13. *If $f(z)$ belongs to H^p, then*

$$(9.1) \quad \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2/p}} \right\}^{p/2} d\theta \leqq A_p \int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta, \ (0 < p \leqq 2, \alpha = 1/p)$$

$$(9.2) \int_0^{2\pi} \left\{ \sup_{0<n<\infty} \frac{|\sigma_n^\alpha(\theta)|}{(\log(n+2))^{1/p}} \right\}^p d\theta \leqq B_p \int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta, \ (0 < p \leqq 1, \alpha = 1/p - 1).$$

(9.2) is deduced from (9.1) by the usual method. (9.2) is an answer to the problem raised by Zygmund [12]. But there is a slip in his original paper, so I proved (9.2) in the case $1/2 < p \leqq 1$, since this case is better than Zygmund's original conjecture. After that, I noticed his correction [14] in such a form as (9.2), so we will prove Theorem [13] completely. For the proof of theorem we need a lemma which was given in [9].

LEMMA. (A) *For positive* α,

$$A_\alpha \sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2\alpha}} \leqq \int_0^1 \frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2\alpha}} dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}} d\varphi$$
$$\leqq A'_\alpha \sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2\alpha}}.$$

(B) *If we put*

$$\left[\sum_{n=1}^{\infty} \frac{|\tau_n^{1/2}(\theta)|^2}{n\{\log(n+1)\}}\right]^{1/2} = f^{**}(\theta), \quad \left\{\sup_{0<r<1} \frac{1}{|\log(1-r)|} \sum_{n=1}^{\infty} |s_n^{-1/2}(\theta)|^2 r^{2n}\right\}^{1/2} = f^\dagger(\theta).$$

then the integrals of $\{f^{**}(\theta)\}^2$ *and* $\{f^\dagger(\theta)\}^2$ *are majorated by the integral of* $|f(e^{i\theta})|^2$, *if* $f(z) \in H^2$.

PROOF OF THEOREM. The case $1 < p \leqq 2$ of (9.1) was proved in my previous paper [9]. So we begin with the case $p = 1$.

(1) $p = 1$. Let us put

$$F(z) = \{f(z)\}^{1/2}$$

then $F(z) \in H^2$. Denote by $s_n^{*-1/2}(\theta)$ and $\tau_n^{*1/2}(\theta)$ the corresponding $(C, -1/2)$ sums etc. of the boundary series of $F(z)$. If we put

$$\Phi(r,\theta) = \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2} d\varphi$$
$$= 2\int_0^{2\pi} \frac{|F(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|} \frac{|F'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|} d\varphi$$
$$= 2\int_0^{2\pi} \left| \sum_{n=0}^{\infty} s_n^{*-1/2}(\theta) r^n e^{in\varphi} \sum_{n=0}^{\infty} A_n^{1/2} \tau_n^{*1/2}(\theta) r^n e^{in\varphi} \right|^2 d\varphi$$
$$= 2\int_0^{2\pi} \left| \sum_{n=0}^{\infty} \left\{ \sum_{\nu=0}^{n} A_\nu^{1/2} \tau_\nu^{*1/2}(\theta)\, s_{n-\nu}^{*-1/2}(\theta) \right\} r^n e^{in\varphi} \right|^2 d\varphi$$
$$= 2\sum_{n=0}^{\infty} \left| \sum_{\nu=0}^{n} A_\nu^{1/2} \tau_\nu^{*1/2}(\theta) s_{n-\nu}^{*-1/2}(\theta) \right|^2 r^{2n}.$$

Applying Minkowski's inequality and Hölder's inequality successively, we obtain

$$\left(\sum_{n=0}^{\infty} \left| \sum_{\nu=0}^{n} A_\nu^{1/2}\, \tau_\nu^{*1/2}(\theta)\, s_{n-\nu}^{*-1/2}(\theta) \right|^2 r^{2n} \right)^{1/2}$$
$$\leqq A_\alpha \sum_{\nu=0}^{\infty} (\nu+1)^{1/2} |\tau_\nu^{*1/2}(\theta)| \left(\sum_{n=\nu}^{\infty} |s_{n-\nu}^{*-1/2}(\theta)|^2 r^{2n} \right)^{1/2}$$

$$\leqq A_\alpha \sum_{\nu=0}^{\infty} (\nu+1)^{1/2} |\tau_\nu^{*1/2}(\theta)|\, |\log(1-r)|^{1/2} r^\nu F^\dagger(\theta) \quad \text{(by (B) of Lemma 2)}$$

$$\leqq A_\alpha F^\dagger(\theta) |\log(1-r)|^{1/2} \left(\sum_{\nu=0}^{\infty} (\nu+1) |\tau_\nu^{*1/2}(\theta)|^2 r^\nu \right)^{1/2} \left(\sum_{\nu=0}^{\infty} r^\nu \right)^{1/2}$$

$$\leqq B_\alpha F^\dagger(\theta) (1-r)^{-1/2} |\log(1-r)|^{1/2} \left(\sum_{\nu=0}^{\infty} (\nu+1) |\tau_\nu^{*1/2}(\theta)|^2 r^\nu \right)^{1/2}.$$

So

$$\Phi(r,\theta) \leqq C_\alpha \{F^\dagger(\theta)\}^2 (1-r)^{-1} |\log(1-r)| \left(\sum_{\nu=0} (\nu+1) |\tau_\nu^{*1/2}(\theta)|^2 r^\nu \right).$$

By the formula of Lemma (A)

$$\sum_{n=1}^{\infty} \frac{|\tau_n^1(\theta)|^2}{n\{\log(n+1)\}^2}$$

$$\leqq D_\alpha \int_0^1 \frac{r^2(1-r)^2}{|\log(1-r)|^2} \Phi(r,\theta)\, dr$$

$$\leqq E_\alpha \{F^\dagger(\theta)\}^2 \int_0^1 \frac{r^2(1-r)}{|\log(1-r)|} \sum_{\nu=0}^{\infty} (\nu+1) |\tau_\nu^{*1/2}(\theta)|^2 r^\nu\, dr$$

$$\leqq F_\alpha \{F^\dagger(\theta)\}^2 \sum_{\nu=0}^{\infty} \frac{|\tau_\nu^{*1/2}(\theta)|^2}{(\nu+1)\log(\nu+2)}$$

$$\leqq G_\alpha \{F^\dagger(\theta)\}^2 \{F^{**}(\theta)\}^2.$$

Consequently

$$\int_0^{2\pi} \left\{ \frac{|\tau_n^1(\theta)|^2}{n\{\log(n+1)\}^2} \right\}^{1/2} d\theta \leqq G_\alpha \int_0^{2\pi} (F^\dagger(\theta))(F^{**}(\theta))\, d\theta$$

$$\leqq G_\alpha \left\{ \int_0^{2\pi} (F^\dagger(\theta))^2\, d\theta \right\}^{1/2} \left\{ \int_0^{2\pi} (F^{**}(\theta))^2\, d\theta \right\}^{1/2}$$

$$\leqq H_\alpha \int_0^{2\pi} |F(e^{i\theta})|^2\, d\theta \leqq H_\alpha \int_0^{2\pi} |f(e^{i\theta})|\, d\theta,$$

by Lemma (B).

The case $1/2 < p < 1$. put

$$G(z) = \{f(z)\}^p$$

then $G(z) \in H$. Denote by $s_n^*(\theta)$, $\sigma_n^*(\theta)$ and $\tau_n^*(\theta)$ the corresponding partial sums, Cesàro means and their differences of the boundary series of $G(z)$. Then we have

$$\int_0^{2\pi}\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^*(\theta)|^2}{n\{\log(n+1)\}^2}\right\}^{1/2}d\theta\leqq A\int_0^{2\pi}|G(e^{i\theta})|\,d\theta$$

by the case proved.

Moreover we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|G(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\,d\varphi=\sum_{n=0}^{\infty}|s_n^*(\theta)|^2r^{2n},$$

and let us put

$$\left\{\sup_{0<r<1}\frac{r^2(1-r)}{\{\log(1-r)\}^2}\sum_{n=1}^{\infty}|s_n^*(\theta)|^2r^{2(n-1)}\right\}^{1/2}=I(\theta),$$

then we can prove analogously to Lemma 3 in [9]

$$I(\theta)\leqq A\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^*(\theta)|^2}{n\{\log(n+1)\}^2}\right\}^{1/2}+B\sup_{1\leqq n<\infty}\frac{|\sigma_n^*(\theta)|}{\log(n+1)}$$

and

$$\int_0^{2\pi}I(\theta)\,d\theta\leqq\int_0^{2\pi}|G(e^{i\theta})|\,d\theta.$$

On the other hand, since $\alpha=1/p$, and $G(z)=\{f(z)\}^p$, we have

$$f'(z)=\alpha\{G(z)\}^{\alpha-1}G'(z)$$

and by applying Hölder's inequality,

$$\left\{\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2/p}}\right\}^{p/2}$$

$$\leqq\left\{\int_0^1\frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2/p}}\,dr\int_0^{2\pi}\frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\,d\varphi\right\}^{p/2}$$

$$\leqq C\left[\int_0^1\int_0^{2\pi}\left\{\frac{|G(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\right\}^{\alpha-1}\frac{|G'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\,\frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2/p}}\,d\varphi\,dr\right]^{p/2}$$

$$\leqq D\left[\int_0^1\frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2/p}}\,dr\left\{\int_0^{2\pi}\frac{|G(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\,d\varphi\right\}^{\alpha-1}\right.$$

$$\left.\cdot\left\{\int_0^{2\pi}\frac{|G'(re^{i(\theta+\varphi)})|^{2/(2-\alpha)}}{|1-re^{i\varphi}|^{2/(2-\alpha)}}\,d\varphi\right\}^{2-\alpha}\right]^{p/2}$$

$$\leqq D\left[\int_0^1\frac{r^2(1-r)^{1+\alpha}}{|\log(1-r)|^2}\,dr\left\{\int_0^{2\pi}\frac{|G(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^2}\,d\varphi\,\frac{r^2(1-r)}{|\log(1-r)|^2}\right\}^{\alpha-1}\right.$$

$$\left.\cdot\left\{\int_0^{2\pi}\frac{|G'(re^{i(\theta+\varphi)})|^{2/(2-\alpha)}}{|1-re^{i\varphi}|^{2/(2-\alpha)}}\,d\varphi\right\}^{2-\alpha}\right]^{p/2}$$

$$\leqq E\{I(\theta)\}^{(\alpha-1)p}\left\{\int_0^1 \frac{r^2(1-r)^{1+\alpha}}{|\log(1-r)|^2}\sum_{n=1}^\infty n^{\alpha+1}|\tau_n^*(\theta)|^2 r^{2n}\,dr\right\}^{p/2}$$

$$\leqq F\{I(\theta)\}^{(\alpha-1)p}\left\{\sum_{n=1}^\infty \frac{|\tau_n^*(\theta)|^2}{n\{\log(n+1)\}^2}\right\}^{p/2}$$

$$\leqq F\{H(\theta)\},$$

where

$$H(\theta)=\mathrm{Max}\left[I(\theta),\ \left\{\sum_{n=1}^\infty \frac{|\tau_n^*(\theta)|^2}{n\{\log(n+1)\}^2}\right\}^{1/2}\right]$$

and

$$\int_0^{2\pi} H(\theta)\,d\theta \leqq G\int_0^{2\pi} |G(e^{i\theta})|\,d\theta.$$

Thus we get

$$\int_0^{2\pi}\left\{\sum_{n=1}^\infty \frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2/p}}\right\}^{p\ 2} d\theta$$

$$\leqq A_p\int_0^{2\pi}|G(e^{i\theta})| \leqq A_p\int_0^{2\pi}|H(e^{i\theta})|^p\,d\theta.$$

For the case $p=1/m$ and $1/(m+1)<p<1/m$ $(m=2,3,\ldots.)$, we can prove similarly, cf. Chow [1].

The reduction of (9.2) from (9.1) is identical to the proof of Theorem 6 in [9]. That is from Lemma 5,

$$|\sigma_n^\alpha(\theta;f)| \leqq A_\alpha\frac{1}{n^\alpha}\sum_{k=0}^n \frac{|\sigma_k^{(\alpha-1)/2}(\theta;g)-\sigma_k^{(\alpha+1)/2}(\theta;g)|^2}{(k+1)^{1-\alpha}}$$

$$+A'_\alpha\frac{1}{n^\alpha}\sum_{k=0}^n\frac{|\sigma_k^{(\alpha+1)/2}(\theta;g)|^2}{(k+1)^{1-\alpha}}$$

where

$$f(z)=g^2(z),\quad\text{and}\quad \alpha=1/p-1.$$

This is smaller than

$$\leqq B_\alpha\sum_{k=0}^n\frac{|\sigma_k^{(\alpha-1)/2}(\theta;g)-\sigma_k^{(\alpha+1)/2}(\theta;g)|^2}{(k+1)}+B'_\alpha\left\{\sup_{0<n<\infty}|\sigma_k^{(\alpha+1)/2}(\theta;g)|\right\}^2$$

$$\leqq C_\alpha(\log n)^{1/p}\sum_{k=0}^n\frac{|\sigma_k^{(\alpha-1)/2}(\theta;g)-\sigma_k^{(\alpha+1)/2}(\theta;g)|^2}{(k+1)\{\log(k+2)\}^{1/p}}$$

$$+C'_\alpha\{\sup|\sigma_k^{(\alpha+1)/2}(\theta;g)|\}^2.$$

Consequently

$$\int_0^{2\pi} \left\{ \sup_{0<n<\infty} \frac{|\sigma_n^\alpha(\theta; f)|}{\{\log(n+2)\}^{1/p}} \right\}^p d\theta$$

$$\leqq C_\alpha \int_0^{2\pi} \left\{ \sum_{k=0}^n \frac{|\sigma_k^{(\alpha-1)/2}(\theta; g) - \sigma_k^{(\alpha+1)/2}(\theta; g)|^2}{(k+1)\{\log(k+2)\}^{1/p}} \right\}^{2p/2} d\theta$$

$$+ C'_\alpha \int_0^{2\pi} \left\{ \sup_{0<n<\infty} |\sigma_n^{(\alpha+1)/2}(\theta; g)| \right\}^{2p} d\theta$$

$$\leqq D_\alpha \int_0^{2\pi} |g(e^{i\theta})|^{2p} d\theta \leqq D_\alpha \int_0^{2\pi} |f(e^{i\theta})|^p d\theta$$

by (9.1) and $\alpha = 1/p - 1$. Thus we get the formula (8.2).

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