ON AHLFORS' DISCS THEOREM AND ITS APPLICATION

KAZUO IKOMA

(Received in revised form, February 16, 1956)

Recently J. Dufresnoy [3], M. Tsuji [8] and Z. Yûjôbô [9] have proved a generalization of Ahlfors' discs theorem [1] by use of Ahlfors' theory [2] of covering surfaces. On the other hand, A. Pfluger [6] and Y. Juve [4] have obtained an extension of Koebe's distortion theorem to some univalent pseudo-regular functions.

From the results mentioned above, we are motivated to write this paper. First, in **1**, we define the functions which are called pseudo-analytic (K) and $\{K\}$ by following S. Kakutani and A. Pfluger. In **3**, by the method due to Z. Yûjôbô and by the aid of lemmas which are stated in **2**, we establish a theorem for our function which is pseudo-analytic $\{K\}$ corresponding to the above Dufresnoy-Tsuji-Yûjôbô's theorem for the analytic function. Further, as its application, an extension of Bloch's theorem is proved in **4**.

1. Let $w = f(z) = u(x,y) + iv(x,y)$ be one-valued continuous in a connected domain D and suppose that it satisfies the following conditions:
(i) u_x, u_y, v_x, v_y exist and are continuous in $D - E_1$, where E_1 is at most enumerable and closed with respect to D;
(ii) $J(z) = u_x v_y - u_y v_x > 0$ in $D - E_2$, where E_2 has the same property as E_1;
then $w = f(z)$ is called pseudo-regular in D. It is well-known that such a pseudo-regular function is an inner transformation in the sense of Stoïlow.

If $f(z)$ is pseudo-regular in a neighbourhood of z_0, except at z_0, and $\lim_{z\to z_0} f(z) = \infty$, then z_0 is called a pole of $f(z)$. If $f(z)$ is pseudo-regular in D except at poles, then $f(z)$ is called pseudo-meromorphic in D. When $f(z)$ is pseudo-regular, pseudo-meromorphic or a constant, it is called pseudo-analytic.

It is well-known that an infinitesimal circle with center at each point z belonging to $D - E_1 - E_2$ is transformed by $f(z)$ into an infinitesimal ellipse with center at $f(z)$, if we neglect infinitesimals of higher orders. The magnitude of the ratio of the major and minor axes of the infinitesimal ellipse is called a dilatation quotient of $f(z)$ at z, and we denote it by $q(z)$. If a pseudo-regular (-meromorphic) function $f(z)$ satisfies the condition:
(iii) the dilatation quotient $q(z)$ of $f(z)$ is bounded in $D - E_1 - E_2$: $q(z) \leqq K$ $(K \geqq 1)$;
then it is called pseudo-regular (-meromorphic) (K) in D. Furthermore, if it satisfies the condition:
(iv) $\lim_{z\to 0} \dfrac{|f(z) - f(0)|}{|z|^{1/K}}$ exists in a domain D which contains $z = 0$, then we call it pseudo-regular (-meromorphic) $\{K\}$ in D.

In this paper, we consider the functions which are pseudo-analytic (K) and $\{K\}$.

2. For later use, first we state the following three lemmas.

LEMMA 1. *Let $w=f^*(z)$ be pseudo-meromorphic (K) in $|z|<\rho$ and suppose that $f^*(0)=0$. For any r such that $0<r<\rho$, we denote by $L(r)$, $A(r)$ respectively the length and the area of the Riemannian images of $|z|=r$ and $|z|<r$ by $w=f^*(z)$ on the w-sphere. If $L(r)<\pi/2$ and $A(r)<\pi/2$, then we have $|f^*(z)|<1$ for $|z|\leqq r$.*

REMARK. Since Z. Yûjôbô's proof [9] for the case that $f^*(z)$ is meromorphic in $|z|<\rho$ can be applied without any modification for the case that $f^*(z)$ is pseudo-meromorphic (K) in $|z|<\rho$, we omit here the proof of Lemma 1.

LEMMA 2. *Let $w=\varphi(z)$ be pseudo-regular (K) in $|z|<r^*$ such that $\varphi(0)=0$ and suppose that $w=\varphi(z)$ maps $|z|<r^*$ one to one pseudo-conformally onto a Jordan domain D_w which contains $|w|<r^*$, and whose contour contains at least one point on $|w|=r^*$. Then we have for $|z|<r^*e^{-4\pi K}$*

$$|\varphi(z)|<e^{8\pi}r^{*1-1/K}|z|^{1/K}.$$

PROOF. First we map $|z|<r^*$ and D_w, each cut along the negative axes, conformally on the parallel-strip-domains S and T, which are respectively contained in $|\mathfrak{J}(s)|<\pi$ and $|\mathfrak{J}(t)|<\pi$, by $s=\log z$ and $t=\log w$ respectively. Then we have a branch of $t=\log\{\varphi(e^s)\}$ pseudo-regular (K), which maps S pseudo-conformally on T. We denote this branch by $t=t_0(s)$ for simplicity. The image of $|z|=r<r^*$ by $s=\log z$ is a segment θ_r in S which lies on $\mathfrak{R}(s)=\log r$, and the length $\theta(r)$ of θ_r is 2π. The image of $|z|=r<r^*$ by $w=\varphi(z)$ is a Jordan closed curve L_r surrounding $w=0$, and the image of L_r by $t=\log w$ is a Jordan arc Λ_r in T whose end points lie on $\mathfrak{J}(t)=\pi$ and $\mathfrak{J}(t)=-\pi$. Then θ_r is transformed into Λ_r by $t=t_0(s)$.

On the other hand, by Kakutani's theorem [5], we can see that the dilatation quotient of $t=t_0(s)$ on θ_r is equal to the dilatation quotient $q(r)$ of $\varphi(z)$ on $|z|=r$. Moreover, if we put $\max\limits_{|z|=r}|\varphi(z)|\equiv M(r)$, then we have $\max\limits_{s\in\theta_r}\mathfrak{R}\{t_0(s)\}=\log M(r)$.

Now we suppose that r satisfies the inequality

$$\begin{aligned}\int_{\log r}^{\log r^*}\frac{d\{\mathfrak{R}(s)\}}{q(r)\theta(r)}&=\frac{1}{2\pi}\int_{r}^{r^*}\frac{dr}{rq(r)}\\&\geqq\frac{1}{2\pi K}\int_{r}^{r^*}\frac{dr}{r}\\&=\frac{1}{2\pi K}\log\frac{r^*}{r}\\&>2\end{aligned}$$

i. e. $r<r^*e^{-4\pi K}$. Then, by an extension of Ahlfors' distortion theorem [5], for any r such that $r<r^*e^{-4\pi K}$, we get

$$\log r^*-\log M(r)>\int_{\log r}^{\log r^*}\frac{dr}{rq(r)}-8\pi\geqq\frac{1}{K}\log\frac{r^*}{r}-8\pi.$$

so that

$$M(r) < e^{8\pi} r^{*1-1/K} r^{1/K}.$$

Hence we have for $|z| < r^* e^{-4\pi K}$

$$|\varphi(z)| < e^{8\pi} r^{*1-1/K} |z|^{1/K}.$$

LEMMA 3. *Let $w = f^*(z)$ be pseudo-regular $\{K\}$ in $|z| < \rho$ such that $f^*(0) = 0$, and denote by $\frac{df^*(z)}{dz}$ the derivative of $f^*(z)$ along $|z| = r^* < \rho$, then we have*

$$\frac{1}{e^{8\pi} r^{*1-1/K}} \lim_{z\to 0} \frac{|f^*(z)|}{|z|^{1/K}} < \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{df^*(z)}{dz}\right| d\theta.$$

PROOF. Let W be the Riemann covering surface onto which $|z| < \rho$ is mapped by $w = f^*(z)$, then W is of hyperbolic type by Kakutani's theorem [5]. Hence W can be transformed one to one and conformally on $|\sigma| < \rho$ by a suitable function $\sigma = g^{-1}(w)$. Then it is seen that the composed function $\sigma = g^{-1}(w) = g^{-1}\{f^*(z)\} \equiv \psi(z)$ is pseudo-regular (K) in $|z| < \rho$ and maps $|z| < \rho$ one to one pseudo-conformally on $|\sigma| < \rho$. In particular, we choose $\psi(z)$ such that $\psi(0) = 0$. Then the image of $|z| < r^* < \rho$ by $\sigma = \psi(z)$ is a Jordan domain D_σ containning $\sigma = 0$. Further, we select a positive number k so that the image D_ζ of D_σ by $\zeta = k\sigma$ may be able to contain $|\zeta| < r^*$ and the contour Γ of D_ζ may be able to contain at least one point on $|\zeta| = r^*$. Then the composed function $w = g(\sigma) = g(\zeta/k) \equiv h(\zeta)$ is regular in $|\zeta| < k\rho$ and there holds $f^*(z) = h(\zeta)$ for z and ζ corresponding each other by $\zeta = k\psi(z) \equiv \varphi(z)$.

Now, by Cauchy's integral formula, we have

$$h'(0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}\frac{dh(\zeta)}{d\zeta}\, d\zeta = \frac{1}{2\pi i}\int_{|z|=r^*} \frac{1}{\varphi(z)}\frac{df^*(z)}{dz}\, dz,$$

so that

$$|h'(0)| \leqq \frac{1}{2\pi}\int_{|z|=r^*} \left|\frac{1}{\varphi(z)}\right| \left|\frac{df^*(z)}{dz}\right| |dz|$$

$$\leqq \frac{1}{2\pi r^*}\int_0^{2\pi} \left|\frac{df^*(z)}{dz}\right| r^*\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{df^*(z)}{dz}\right| d\theta.$$

On the other hand,

$$|h'(0)| = \lim_{\zeta\to 0}\left|\frac{h(\zeta)}{\zeta}\right| = \lim_{z\to 0}\left|\frac{f^*(z)}{\varphi(z)}\right|.$$

By Lemma 2, since $|\varphi(z)| < e^{8\pi} r^{*1-1/K}|z|^{1/K}$ for $|z| < r^* e^{-4\pi K}$, we get

$$|h'(0)| > \frac{1}{e^{8\pi} r^{*1-1/K}} \lim_{z\to 0} \frac{|f^*(z)|}{|z|^{1/K}}.$$

Hence it follows that

$$\frac{1}{e^{8\pi} r^{*1-1/K}} \lim_{z\to 0} \frac{|f^*(z)|}{|z|^{1/K}} < \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{df^*(z)}{dz} \right| d\theta.$$

3. Now, by the above lemmas, we can prove the following

THEOREM 1. *Let $w = f(z)$ be pseudo-meromorphic $\{K\}$ in $|z| < R$ and F be the Riemann surface generated by $w = f(z)$ on the w-sphere. Let $D_1, D_2, \ldots, D_q$ ($q \geqq 3$) be q disjoint simply connected closed domains on the w-sphere and suppose that every simply connected island of F which lies above D_i is of multiplicity $\geqq m_i$, m_i being positive integers or ∞. If $\sum_{i=1}^{q}\left(1 - \frac{1}{m_i}\right) > 2$, then*

$$R < \left(C \frac{1 + |f(0)|^2}{\lim_{z\to 0} |f(z) - f(0)|/|z|^{1/K}} \right)^K,$$

where C is a constant depending only on $D_1, D_2, \ldots, D_q$.

PROOF. For any r ($\leqq R$), let $f(z)$ ramify at least $m_i(r)$-ply ($m_i(r) \geqq m_i(R) \equiv m_i$) on D_i ($i = 1, 2, \ldots, q$) in $|z| \leqq r \leqq R$, there holds

$$\sum_{i=1}^{q}\left(1 - \frac{1}{m_i(r)}\right) - 2 \geqq \sum_{i=1}^{q}\left(1 - \frac{1}{m_i}\right) - 2 > 0.$$

It can be easily seen that the positive minimum value of $\sum_{i=1}^{q}\left(1 - \frac{1}{m_i}\right) - 2$ is $1/42$, so that

$$\sum_{i=1}^{q}\left(1 - \frac{1}{m_i(r)}\right) \geqq 2 + \frac{1}{42}. \tag{1}$$

On the other hand, we denote by $L(r)$, $A(r)$ respectively the length and the area of the Riemannian images of $|z| = r$ and $|z| < r$ by $w = f(z)$, i. e.

$$L(r) = \int_0^{2\pi} \frac{\left|\frac{df(re^{i\theta})}{rd\theta}\right|}{1 + |f(re^{i\theta})|^2} r\, d\theta, \quad A(r) = \int_0^r \int_0^{2\pi} \frac{J(re^{i\theta})}{(1 + |f(re^{i\theta})|)^2} r\, dr\, d\theta.$$

Then, we have the following inequality obtained by Ahlfors [2] from his theory of covering surfaces:

$$\sum_{i=1}^{q}\left(1 - \frac{1}{m_i(r)}\right) \leqq 2 + h\frac{L(r)}{A(r)}, \tag{2}$$

where $h(>0)$ depends only on D_i ($i = 1, 2, \ldots, q$). Hence we get from (1) and (2)

$$\frac{L(r)}{A(r)} \geqq \frac{1}{42h}. \tag{3}$$

By Schwarz's inequality, we can see

$$[L(r)]^2 \leqq \int_0^{2\pi} r\, d\theta \int_0^{2\pi} \frac{\left|\frac{df(re^{i\theta})}{rd\theta}\right|^2}{(1 + |f(re^{i\theta})|^2)^2} r\, d\theta$$

Using the well-known formula $|df(z)/dz|^2 \leqq q(z)J(z)$, it holds that

$$\frac{[L(r)]^2}{2\pi r} \leqq K\int_0^{2\pi} \frac{J(re^{i\theta})}{(1+|f(re^{i\theta})|^2)^2} r d\theta = K\frac{dA(r)}{dr},$$

so that

$$\frac{dr}{r} \leqq 2\pi K \frac{dA(r)}{[L(r)]^2}. \tag{4}$$

Integrating both members from r_0 to R and using (3), we have

$$\log\frac{R}{r_0} \leqq 2\pi K\int_{r_0}^{R} \frac{dA(r)}{[L(r)]^2} \leqq 2\pi(42h)^2K\int_{r_0}^{R} \frac{dA(r)}{[A(r)]^2} < \frac{3528\pi h^2K}{A(r_0)},$$

so that

$$A(r_0) < \frac{3528\pi h^2 K}{\log(R/r_0)}.$$

If we put $r_0 \equiv R\exp(-7056\,h^2K)$, then there holds

$$A(r_0) < \pi/2. \tag{5}$$

Hence, for any r_1 $(0 < r_1 < r_0)$, we have from (4) and (5)

$$\log\frac{r_0}{r_1} \leqq 2\pi K\int_{r_1}^{r_0} \frac{dA(r)}{[L(r)]^2} < \frac{2\pi K}{[L(r^*)]^2}A(r_0) < \frac{\pi^2 K}{[L(r^*)]^2}, \tag{6}$$

where r^* is the radius which minimizes $L(r)$ in $r_1 \leqq r \leqq r_0$. Put $r_1 = e^{-4K}r_0$, then it holds that for $r_1 = e^{-4K}r_0 \leqq r^* \leqq r_0$,

$$L(r^*) < \pi/2. \tag{7}$$

Moreover, $A(r)$ is the increasing function of r, thence we see from [5],

$$A(r^*) < \pi/2. \tag{8}$$

Now, we make the rotation of the Riemann sphere: $f^*(z) = (f(z) - f(0))/(1+\overline{f(0)}f(z))$. Evidently, $f^*(z)$ is pseudo-meromorphic $\{K\}$ in $|z| < R$, and $L(r)$ and $A(r)$ for $f^*(z)$ are the same as those for $f(z)$, so that (7) and (8) hold for $f^*(z)$, hence we have $|f^*(z)| \leqq 1$ in $|z| \leqq r^*$ by Lemma 1, and so $f^*(z)$ is pseudo-regular $\{K\}$ in $|z| \leqq r^*$. Then

$$\frac{\pi}{2} > L(r^*) = \int_{|z|=r^*} \frac{\left|\frac{df^*(z)}{dz}\right|}{1+|f^*(z)|^2}|dz| > \frac{r^*}{2}\int_0^{2\pi}\left|\frac{df^*(z)}{dz}\right| d\theta,$$

and by Lemma 3, we can get

$$\frac{r^*}{2}\int_0^{2\pi}\left|\frac{df^*(z)}{dz}\right| d\theta > \frac{\pi(r^*)^{1/K}}{e^{8\pi}}\lim_{z\to 0}\frac{|f^*(z)|}{|z|^{1/K}} \geqq \frac{\pi(r_1)^{1/K}}{e^{8\pi}}\lim_{z\to 0}\frac{|f^*(z)|}{|z|^{1/K}}$$

$$= \frac{\pi r_0^{1/K}}{e^{4+8\pi}}\lim_{z\to 0}\frac{|f^*(z)|}{|z|^{1/K}} = \frac{\pi R^{1/K}\exp(-7056\,h^2)}{e^{4+8\pi}}\lim_{z\to 0}\frac{|f^*(z)|}{|z|^{1/K}}.$$

Further we have

$$\lim_{z\to 0}\frac{|f^*(z)|}{|z|^{1/K}}=\frac{1}{1+|f(0)|^2}\lim_{z\to 0}\frac{|f(z)-f(0)|}{|z|^{1/K}}.$$

Therefore we obtain

$$R<\frac{1}{2^K}\exp\{4K(1+2\pi+1764h^2)\}\left(\frac{1+|f(z)|^2}{\lim\limits_{z\to 0}|f(z)-f(0)|/|z|^{1/K}}\right)^K,$$

so that we have the required result by putting

$$C\equiv 1/2\cdot\exp\{4(1+2\pi+1764h^2)\}.$$

4. If we use Theorem 1, then we can extend Bloch's theorem [7].

THEOREM 2 (*An extension of Bloch's theorem*). *Let $w=f(z)$ be pseudo-meromorphic $\{K\}$ in $|z|<1$ and suppose that $f(0)=0$, $\lim\limits_{z\to 0}|f(z)|/|z|^{1/K}\geqq 1$, then the Riemann surface generated by $w=f(z)$ on the w-sphere contains a schlicht spherical disc whose radius $\geqq\beta>0$, β being a constant independent of $f(z)$.*

PROOF. Let D_1, D_2, D_3, D_4, D_5 be five disjoint spherical discs on ζ-sphere and C be a constant which is decided depending only on D_1, D_2, D_3, D_4, D_5 as in Theorem 1. If we consider $\zeta=Cf(z)$ instead of $f(z)$ and apply Theorem 1 to $\zeta=Cf(z)$, then we have a schlicht island above at least one disc D_i of D_1, D_2, D_3, D_4, D_5, hence the schlicht domain B_i corresponding to this disc D_i is contained in $|z|<1$. If we transform the above five discs into the discs $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ on w-sphere by $w=\zeta/C$, then Δ_i corresponding to D_i is the range of $f(z)$ in B_i. Hence, it suffices to take the minimum of radii of Δ_1, $\Delta_2, \Delta_3, \Delta_4, \Delta_5$ to β.

REMARK. In the particular case when $w=f(z)$ is meromorphic, M. Tsuji proved Theorem 1 [8] by use of Theorem 2 [7].

THEOREM 3. *Let $w=f(z)$ be pseudo-meromorphic $\{K\}$ in $|z|<R$ and suppose that the Riemann surface F generated by $w=f(z)$ on the w-sphere is a covering surface of a closed Riemann surface Φ whose genus $p\geqq 2$, then we have*

$$R<\left(C\frac{1+|f(0)|^2}{\lim\limits_{z\to 0}|f(z)-f(0)|/|z|^{1/K}}\right)^K,$$

where C is a positive constant depending only on Φ.

PROOF. Put, as in the preceding case,

$$L(r)=\int_0^{2\pi}\frac{\left|\dfrac{df(re^{i\theta})}{rd\theta}\right|}{1+|f(re^{i\theta})|^2}r\,d\theta \quad\text{and}\quad A(r)=\int_0^r\int_0^{2\pi}\frac{J(re^{i\theta})}{(1+|f(re^{i\theta})|^2)^2}r\,dr\,d\theta.$$

Let $\rho(r)$ be the Euler's characteristic of the Riemann surface F_r generated by $w=f(z)$ when z varies on $|z|\leqq r$, ρ_0 be the Euler's characteristic of Φ, and n be the number of sheets of Φ. Then by Ahlfors' fundamental theorem [2] on covering surfaces, we have

$$\rho^+(r)\geqq\frac{\rho_0}{n\pi}A(r)-hL(r),$$

where $\rho^+(r)$ means $\mathrm{Max}(\rho(r), 0)$ and h is a positive constant depending only on Φ.

It is easily seen that ρ_0 equals to $2(p-1)$, so that ρ_0 is positive. Since F_r is simply connected, there holds $\rho^+(r)=0$. Hence we have

$$\frac{L(r)}{A(r)} \geqq \frac{\rho_0}{n\pi h}.$$

From this, we can proceed similarly as in the proof of Theorem 1 and get the present theorem.

REMARK. In the particular case that $K=1$ i. e. when our pseudo-analytic functions reduce to analytic functions, Theorem 1 reduces to Dufresnoy-Yûjôbô-Tsuji's theorem [3], [9], [8], and Theorems 2 and 3 reduce to Tsuji's theorems [7] and [8].

REFERENCES

[1] L. AHLFORS, Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant à une région donnée, Acta Soc. Sci. Fenn. N. s. 2, Nr. 2(1933).

[2] L. AHLFORS, Zur Theorie der Überlagerungsflächen, Acta Math., 65(1935).

[3] J. DUFRESNOY, Sur les domaines couverts par les valeurs d'une fonction méromorphe ou algébroide. (Thèse, 1935).

[4] Y. JUVE, Über gewisse Verzerrungseigenschaften konformer und quasikonformer Abbildungen, Ann. Acad. Sci. Fenn., A, I, 174(1954).

[5] S. KAKUTANI, Applications of the theory of pseudo-regular functions to the type-problem of Riemann surfaces, Japanese J. Math., 13(1936).

[6] A. PFLÜGER, Quasikonforme Abbildungen und logarithmische Kapazität, Ann. de L'Inst. Fourier, 2(1950-1951).

[7] M. TSUJI, On an extension of Bloch's theorem, Proc. Acad. Tôkyô, 18(1942).

[8] M. TSUJI, An extension of Bloch's theorem and its application to normal family, Tôhoku Math., J., 4(1952).

[9] Z. YÛJÔBÔ, An application of Ahlfors' theory of covering surfaces, J. Math. Soc. Japan, 4(1952).

DEPARTMENT OF MATHEMATICS, FACULTY OF LIBERAL ARTS AND SCIENCE,
YAMAGATA UNIVERSITY.