THE ABSOLUTE VALUE OF W*-ALGEBRAS OF FINITE TYPE

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1. Introduction. In the theory of W^*-algebras, many classical order-properties in the commutative case have been extended to the algebra, and in particular, R. Kadison [5] showed that the algebraic structure is determined by the order structure in a sense. However, in the commutative case, we had known the stronger fact that the algebra can be characterised as a vector lattice. Considering these facts, we can raise a more general, interesting question as follows: *Is it possible to characterise a W^*-algebra as a vector space with an order structure?* Moreover, it seems that a suitable settlement of this question will be useful for the study of the algebraical type which will occupy a central position in future studies of the algebra.

In the management of this question there is a pathology that owing to the non-coincidence of right ideals and left ones in the non-commutative case, right ideals can not be explanated by the order structure only of the self-adjoint portion. But, there is a useful notion for the elimination of this pathology; let M be a W^*-algebra, x an element of M and $x = u|x|(|x| = (x^*x)^{1/2})$ be the polar decomposition of x, then we shall call $|x|$ the absolute value of x and a mapping $x \to |x|$ the absolute-value mapping. The absolute value has many interesting properties, and we can easily show that the non-coincidence of right ideals with left ones can be explanated by the absolute value. Therefore our question can be reformed as follows: *Is it possible to characterise a W^*-algebra as a vector space with an absolute value mapping?*

The question is comparatively manageable in the semi-finite case; for, J. Dixmier [3] and I. E. Segal [7] have given a non-commutative extension of abstract integration, introduced generalized L^p-spaces, and extended classical properties in L^p-spaces, and in particular, they have shown that the absolute value in generalized L^p-spaces inherits many classical properties. Therefore the classical theories for the (AL^p)-spaces by S. Kakutani [6] and H. Bohnenblust [1] will offer a useful model for our intension.

From these points of view, the purpose of this paper is to try an axiomatical dealings of a generalized L^2-space, which is most manageable in generalized L^p-spaces, using the absolute value.

Moreover, since the semi-finite algebra can, in a suitable sense, be reduced to the finite one, our object will be restricted to the finite case in this paper

Then, the principal purpose of this paper is to show that *any hilbert space with an absolute value-mapping satisfying some axioms (Axioms I-II,* § 3 *below) is a generalized L^2-space.*

2. Preliminaries. In this section, we shall consider remarkable properties of a generalized L^2-space.

Let M be a W^*-algebra of finite type with a σ-weakly continuous, complete trace φ, then the space M with an inner product $(x,y)=\varphi(y^*x)$ is a pre-hilbert space and the hilbert space obtained by the completion is denoted by $L^2(M,\varphi)$ and is called a generalized L^2-space, associated with the algebra M and a trace φ [cf. 3, 7].

Then, I. E. Segal [7] had presented a more concrete realization of a generalized L^2-space to us: let us represent M as an operator ring on a hilbert space $\mathfrak{H}$, then $L^2(M,\varphi)$ is considered a hilbert space composed of all linear operators on $\mathfrak{H}$ which are square-integrable with respect to the gage induced by φ.

Throughout this paper, we shall consider the generalized L^2-space $L^2(M,\varphi)$ in the sense of Segal. We can point out the following remarkable properties $(\alpha)-(\eta)$ of the adjoint operation, the order of the self-adjoint portion, the absolute value and the unit.

Let P be the positive portion of $L^2(M,\varphi)$, then we can easily show the following:

(α) if and only if $a\geqq 0$, $(a,b)\geqq 0$ for all $b\in P$.

Let x be an element of $L^2(M,\varphi)$ and $x=u|x|(|x|=(x^*x)^{1/2})$ be the polar decomposition of x, then a correspondence $x\rightarrow|x|$ is a mapping of $L^2(M,\varphi)$ on P, and $|x|$ is called the absolute value of x. Then, J. Dixmier [3, lemma 3] had extended the following classical inequality to $L^2(M,\varphi)$.

(β) $\quad |(a,b)|\leqq(|a^*|,|b^*|)^{1/2}(|a|,|b|)^{1/2}$ for a, $b\in L^2(M,\varphi)$.

Moreover we shall point out the classical properties on $L^2(M,\varphi)$.

(γ) for any $a,b\in L^2(M,\varphi)$ there is an element c such that $(|a|,|b|)=(a,c)$ and $|c|=|b|$.

For, put $a=\widetilde{u}|a|$ ($\widetilde{u}$: unitary) and $c=\widetilde{u}|b|$, then $(a,c)=\varphi(c^*a)=\varphi(|b|\widetilde{u}^*\widetilde{u}|a|)=(|a|,|b|)$ and $|c|=|b|$.[1)]

(δ) $\quad |a|\perp|b|$ implies $|a^*|\leqq|(a+b)^*|$,

where $|a|\perp|b|$ means that $|a|$ is orthogonal to $|b|$.

For, $|a|\perp|b|$ implies that there is a projection e of M such that $|a|=e|a|e$ and $|b|e=0$, where the product of two elements is the extended product of measurable operators [cf. 7]; hence $e(b^*b)e=0$, so that $be=0$, and analogously $a(I-e)=0$, where I is the unit of M. Therefore $a=ae$ and $b=b(I-e)$. Then,

$$\begin{aligned}|(a+b)^*|^2&=(ae+b(I-e))(ea^*+(I-e)b^*)\\&=aea^*+b(I-e)b^*\geqq aea^*=aa^*=|a^*|^2,\end{aligned}$$

hence by the Heinz' theorem [4, Satz 3], $|a^*|\leqq|(a+b)^*|$.

Finally we shall point out the following properties of the unit:

(ε) $\quad$ for $x\geqq 0$, $x\perp I$ implies $x=0$.

(ζ) $\quad |x|\leqq\alpha I$ implies $|x^*|\leqq\alpha I$.

1) In the finite algebra, we can extend any partially isometric operator to a unitary one.

(η) $|x| \leqq \alpha I$ and $|y| \leqq \beta I$ imply $|x+y| \leqq (\alpha+\beta)I$.

3. Axioms and the main theorem. In this section, we shall consider a hilbert space L as follows: (1) there is an adjoint operation $*$ such that $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$ and $(x^*)^* = x$ for $x, y \in L$ and α, β complex numbers. (2) the real subspace L_s composed of all self-adjoint elements ($x^* = x$) is a *real* hilbert space[2] with a partially ordering given by a set of positive elements, P, the so-called "positive cone" of L_s, and the following further condition is satisfied; if and only if $a \geqq 0$, $(a, b) \geqq 0$ for all $b \in P$, where $(\ ,\ )$ is the inner product of L.

Moreover we shall set the following axioms.

Axiom I. Absolute value.

There is a mapping $a \to |a|$ of L on P as follows:

$\mathbf{I}_1$: $\|a\| = \|\,|a|\,\|$ and $|(a, b)| \leqq (|a|, |b|)^{1/2} (|a^*|, |b^*|)^{1/2}$ for $a, b \in L$, where $\|\cdot\|$ is the norm in L.

$\mathbf{I}_2$: For any $a, b \in L$, there is an element c such that $(|a|, |b|) = (a, c)$ and $|c| = |b|$.

$\mathbf{I}_3$: $|a| \perp |b|$ implies $|a^*| \leqq |(a+b)^*|$.

Axiom $\mathbf{I}_1$ is natural. Axiom $\mathbf{I}_2$ is an abstraction of the polar decomposition. It will be desirable to find a more simple formulation, if possible. Axiom $\mathbf{I}_3$ is an abstraction of the algebraic properties of the absolute value. It will be natural that an abstraction of such type is necessary, though there may be another formulation.

Axiom II. Unit.

There is an element $I > 0$ with the properties:

$\mathbf{II}_1$: For $x \geqq 0$, $x \perp I$ implies $x = 0$.

$\mathbf{II}_2$: $|x| \leqq \alpha I$ implies $|x^*| \leqq \alpha I$ for α a positive number.

$\mathbf{II}_3$: $|x| \leqq \alpha I$ and $|y| \leqq \beta I$ imply $|x+y| \leqq (\alpha+\beta)I$ for α, β positive numbers.

We shall say an element I as above a unit of L.

Then we shall show the following theorem.

Main Theorem. *For any hilbert space L satisfying the axioms I-II with a unit I, there are a finite W^*algebra M with a σ-weakly continuous, complete tace φ, and a linear isometry ρ of L on the generalized L^2-space $L^2(M, \varphi)$ associated with M and φ such that:*

(1) $\rho(I/\|I\|) = \tilde{I}$,

where $\tilde{I}$ is the unit of M,

(2) $\rho(a) \geqq 0$, *if and only if* $a \geqq 0$,

(3) $\rho(|x|) = |\rho(x)|$ *for all* $x \in L$,

where $|\rho(\cdot)|$ is the absolute value of $L^2(M, \varphi)$.

In order to prove the theorem, we shall provide some preparatory

2) By a real hilbert space, we mean that (x, y) is real for $x, y \in L_s$.

considerations in the following sections.

4. Characteristic elements. We shall introduce a notion of the characteristic elements (projections) in L.

LEMMA 4.1. *The following properties are satisfied*:

(1) $(x^*, y^*) = (y, x)$ *for* $x, y \in L$,

(2) $\pm a \leqq |a|$ *for* $a \in L_s$, *and*

(3) $(|a+b|, |c|) \leqq N_1(a,b,c)(|a|, |c|)^{1/2} + N_2(a,b,c)(|b|, |c|)^{1/2}$

for a, b *and* $c \in L$, *where* N_1 *and* N_2 *are positive numbers depending on* a, b *and* c.

PROOF. (1) is easily obtained.

(2) Let $a \in L_s$ and $h \geqq 0$, then by the axiom $\mathbf{I}_1$,

$$|(a,h)| \leqq (|a|, h)^{1/2} \cdot (|a|, h)^{1/2} = (|a|, h);$$

hence $(|a| \pm a, h) \geqq 0$, so that $\pm a \leqq |a|$.

(3) By the axiom $\mathbf{I}_2$,

$$\begin{aligned}(|a+b|, |c|) &= (a+b, c_1) \leqq |(a, c_1)| + |(b, c_1)| \\ &\leqq (|a^*|, |c_1^*|)^{1/2}(|a|, |c|)^{1/2} + (|b^*|, |c_1^*|)^{1/2}(|a|, |c|)^{1/2}\end{aligned}$$

This completes the proof.

DEFINITION 1. For $x \in L_s$, if it is expressible such that $x = x_1 - x_2, x_1 \geqq 0$, $x_2 \geqq 0$ and $x_1 \perp x_2$, we say, it is *orthogonally decomposable* and $x_1 - x_2$ is an *orthogonal decomposition* of x.

LEMMA 4.2. *Any element of* L_s *is orthogonally decomposable in the unique way.*

PROOF. By Lemma 4.1, $\pm x \leqq |x|$ for $x \in L_s$. Put $\mathfrak{F}(x) = \{h \mid h \geqq 0, \pm x \leqq h\}$ then $\mathfrak{F}(x)$ is a closed convex set. Let x_0 be the point, which uniquely exists by the uniform convexity of L, that realizes the minimum of $\|h\|$ on $\mathfrak{F}(x)$.

Then $\|x\| \leqq \|x_0\| \leqq \||x|\|$; hence $\|x_0\| = \||x|\|$ and so $x_0 = |x|$. Moreover $(|x| + x, |x| - x) = \||x|\|^2 - \|x\|^2 + (x, |x|) - (|x|, x) = 0$. This means that $(|x| + x) \perp (|x| - x)$, so that $x = \frac{1}{2}(|x| + x) - \frac{1}{2}(|x| - x)$ is an orthogonal decomposition. Now let $x = x_1 - x_2$ be an orthogonal decomposition, then $\pm x \leqq x_1 + x_2$ and $\|x_1 + x_2\| = \|x\|$; hence by the unicity of x_0, $x_1 + x_2 = x_0 = |x|$. This completes the proof.

A unique element x_1 (resp. x_2) in the above lemma is denoted by $x_1 = x^+$ (resp. $x_2 = x^-$).

DEFINITION 2. For any $x \geqq 0$ and $y \geqq 0$, $x \succ y$ means that we have $y \perp u$ for any $u \geqq 0$ with $x \perp u$. Moreover if $x \succ y$ and $x \prec y$, we shall denote $x \approx y$.

Then it is clear that $x \prec y$ and $y \prec z$ imply $x \prec z$.

DEFINITION 3. A set $\mathfrak{J}$ of elements of L is said to be a *right ideal* if the following conditions are satisfied:

(1) $x, y \in \mathfrak{J}$ implies $\alpha x + \beta y \in \mathfrak{J}$ for α, β complex numbers,
(2) $x \in \mathfrak{J}$ and $|y| \prec |x|$ imply $y \in \mathfrak{J}$,
(3) $x_n \in \mathfrak{J}$ $(n = 1, 2, \dots)$ and $x_n \to x$ (strongly) imply $x \in \mathfrak{J}$.

LEMMA 4.3. *Put* $[x] = \{y \mid |y| \prec |x|,\ y \in L\}$. $[x]$ *is a right ideal.*

PROOF. Suppose that $y_1, y_2 \in [x]$ and $|x| \perp u$, then by Lemma 4.1,

$$(|y_1 + y_2|,\ u) \leqq N_1(y_1, y_2, u)\ (|y_1|, u)^{1/2} + N_2(y_1, y_2, u)\ (|y_2|, u)^{1/2} = 0;$$

hence $y_1 + y_2 \in [x]$.

$(|\alpha y_1|,\ u) = (\alpha y_1,\ u_1) = |\alpha|\,|(y_1, u_1)| \leqq |\alpha|\,(|y_1^*|,\ |u_1^*|)^{1/2}(|y_1|,\ |u_1|)^{1/2} = |\alpha|$ $(|y|_1^*,\ |u_1^*|)^{1/2}(|y_1|, u)^{1/2} = 0$; hence $\alpha y_1 \in [x]$.

Let $y_n \in [x]$ and $y_n \to y_0$ (strongly), then by the axiom I_2

$$\begin{aligned}(|y_0|, u) &= (y_0, u_2) = (y_0 - y_n + y_n,\ u_2) \\ &\leqq |(y_0 - y_n, u_2)| + |(y_n, u_2)| \\ &\leqq \|y_0 - y_n\|\ \|u_2\| + (|y_n^*|,\ |u_2^*|)^{1/2}(|y_n|,\ |u_2|)^{1/2} \\ &= \|y_0 - y_n\|\ \|u_2\| + (|y_n^*|,\ |u_2^*|)^{1/2}(|y_n|, u)^{1/2} \\ &= \|y_0 - y_n\|\ \|u_2\| \text{ for all } n.\end{aligned}$$

Hence $(|y_0|,\ u) = 0$, so that $y_0 \in [x]$. This completes the proof.

DEFINITION 4. The right ideal $[x]$ $(x \geqq 0)$ is called a ***principal right ideal*** generated by x.

LEMMA 4.4. *Put* $O(x) = \{y \mid |y| \perp |x|,\ y \in L\}$, *then* $O(x)$ *is a right ideal and* $O(x) = [x]^\perp$, *where* $[\cdot]^\perp$ *is the orthogonal complement of a closed subspace* $[\cdot]$.

PROOF. By an analogous method with the above lemma, it is easily shown that $O(x)$ is a right ideal $\subset [x]^\perp$.

Conversely suppose that $y \in [x]^\perp$, then by the axiom $\mathbf{I}_2$, there is an element c $(|c| = |x|)$ such that

$$(|y|,\ |x|) = (y, c).$$

Since c belongs to $[x]$, $(|y|, |x|) = 0$, so that $y \in O(x)$. This completes the proof.

Let $E(x)$ be the orthogonal projection from L on $[x]$. Moreover, put $[x]_* = \{y \mid |y^*| \prec |x^*|,\ y \in L\}$, then by the property $(x, y) = (y^*, x^*)$ and an analogous method as above, we can show that $[x]_*$ is a closed subspace and $[x]_*^\perp = \{y \mid |y^*| \perp |x^*|, y \in L\}$.

Let $F(x)$ be the orthogonal projection of L on $[x]_*$. Then we obtain the following lemma.

LEMMA 4.5. *For any* $x, y \in L$, $E(x)$ *commutes with* $F(y)$.

PROOF. Let $z = z_1 + z_2$, $z_1 \in [x]$ and $z_2 \in O(x)$, then $|z_1| \perp |z_2|$; hence by the axiom $\mathbf{I}_3$, $|z_1^*| \leqq |(z_1 + z_2)^*|$.

Now suppose that $z \in [y]_*$ (resp. $\in [y]_*^\perp$), then $|z_i^*| \leqq |z^*| (i = 1, 2)$ mean

$z_1, z_2 \in [y]_*$ (resp. $\in [y]_*^\perp$): hence $L = [x] \cap [y]_* + [x]^\perp \cap [y]_* + [x] \cap [y]_*^\perp + [x]^\perp \cap [y]_*^\perp$. Therefore $E(x)F(y)$ and $F(y)E(x)$ are projections of L on $[x] \cap [y]_*$, so that $E(x)F(y) = F(y)E(x)$. This completes the proof.

LEMMA 4.6. *For $x \geqq 0$ and $h \geqq 0$, $E(x)F(x)h \geqq 0$.*

PROOF. At first, we shall show that $y = y_1 + iy_2$ $(y_i \in L_s) \in [x] \cap [x]_*$ implies y_i $(i = 1, 2) \in [x]$, so that $[x] \cap [x]_* = [x] \cap L_s + i[x] \cap L_s$.

Since $y \in [x] \cap [x]_*$, $|y| \prec x$ and $|y_*| \prec x$, so that $\left|\dfrac{y+y^*}{2}\right| \prec x$ and $\left|\dfrac{iy^*-iy}{2}\right| \prec x$; hence $y_1, y_2 \in [x]$. Therefore $y \in [x] \cap [x]_* \cap L_s$ implies y^+, $y^- \in [x] \cap [x]_*$.

Now put $E(x)F(x)h = y_1 + iy_2$ $(y_i \in [x] \cap [x]_* \cap L_s)$, then

$$(E(x)F(x)h, p) = (h, E(x)F(x)p) = (h, p) \geqq 0 \text{ for all } p \in P \cap [x] \cap [x]_*;$$

hence $y_2 = 0$. Moreover let $y_1 = y_1^+ - y_1^-$ be the orthogonal decomposition, then

$$(y_1, y_1^-) = -(y_1^-, y_1^-) \leqq 0;$$

hence $y_1^- = 0$, so that $E(x)F(x)h \geqq 0$. This completes the proof.

LEMMA 4.7. *For any $x \geqq 0$, $E(x)F(x)I \leqq I$.*

PROOF. $I = E(x)F(x)I + E(x)(1 - F(x))I + (1 - E(x))F(x)I + (1 - E(x))(1 - F(x))I$, where 1 is the identity operator on L.

Since $|E(x)F(x)I + E(x)(1 - F(x))I| \perp |(1 - E(x))F(x)I + (1 - E(x))(1 - F(x))I|$, by the axiom $\mathbf{I}_3$, $|(E(x)F(x)I + E(x)(1 - F(x))I)^*| \leqq I$, so that by the axiom $\mathbf{II}_2$, $|E(x)F(x)I + E(x)(1 - F(x))I| \leqq I$.

Moreover $|(E(x)F(x)I)^*| \perp |(E(x)(1 - F(x))I)^*|$, so that $|E(x)F(x)I| \leqq |E(x)F(x)I + E(x)(1 - F(x))I| \leqq I$; hence $E(x)F(x)I = |E(x)F(x)I| \leqq I$. This completes the proof.

LEMMA 4.8. *$(E(x)y)^* = F(x)y^*$ for any $x \geqq 0$ and $y \in L$.*

PROOF. Since $[x]_* = \{z \mid |z^*| \prec |x^*| = |x| = x, z \in L\}$, $[x]_* = \{u^* \mid u \in [x]\}$; hence $(E(x)y)^* \in [x]_*$. Analogously $((1 - E(x))y)^* \in [x]_*^\perp$, hence $F(x)y^* = F(x)((E(x)y)^* + ((1 - E(x))y)^*) = (E(x)y)^*$. This completes the proof.

LEMMA 4.9. *For any $x \geqq 0$, $E(x)I = F(x)I = E(x)F(x)I = \sup\limits_{0 \leqq z \leqq I, z \in [x]} z$ and $[x] = [E(x)I]$.*

PROOF. Let $0 \leqq z \leqq I$ and $z \in [x]$, then by Lemma 4.6, $E(x)F(x)z = z \leqq E(x)F(x)I$, so that $E(x)F(x)I = \sup\limits_{0 \leqq z \leqq I, z \in [x]} z$.

Next we shall show that $E(x)F(x)I \approx x$. Suppose that there is an element $u(\geqq 0)$ such that $E(x)F(x)I \perp u$ and x is not orthogonal to u; since $x \in [x] \cap [x]_*$, $(x, u) = (E(x)F(x)x, u) = (x, E(x)F(x)u) > 0$, so that $E(x)F(x)u > 0$.

On the other hand, $(I, E(x)F(x)u) = (E(x)F(x)I, u) = 0$; hence by the axiom $\mathbf{II}_1$, $E(x)F(x)u = 0$. This contradicts to the above inequality; hence $E(x)F(x)I \approx x$. Therefore we obtain that $[x] = [E(x)F(x)I]$. Moreover from the proof of Lemma 4.7,

$$E(x)F(x)I \leqq |E(x)F(x)I + E(x)\,(1 - F(x))I| = |E(x)I| \leqq I;$$
hence $E(x)F(x)I = |E(x)I|$. Then,

$$(E(x)I,\ E(x)I) = (|E(x)I|,\ |E(x)I|) = (E(x)F(x)I, E(x)F(x)I).$$

On the other hand,

$(E(x)I, E(x)I) = (E(x)F(x)I + E(x)\,(1 - F(x))I,\ E(x)F(x)I + E(x)(1 - F(x))I) = (E(x)F(x)I,\ E(x)F(x)I) + (E(x)\,(1 - F(x))I, E(x)\,(1 - F(x))I)$; hence $E(x)\,(1 - F(x))\,I = 0$. Finally $E(x)F(x)I = E(x)I = (E(x)I)^* = F(x)I$. This completes the proof.

DEFINITION 5. For any $x \geqq 0$, put $\mathrm{e}(x) = E(x)I$ and we shall call $\mathrm{e}(x)$ a *characteristic element* corresponding to x.

By the axiom $\mathbf{II}_1$, L is a principal right ideal and $L = [I]$. Moreover by the above lemma, to any principal right ideal there corresponds a unique characteristic element, and if $[x_1] \subseteqq [x_2]$, $\mathrm{e}(x_1) \leqq \mathrm{e}(x_2)$.

LEMMA 4.10. *Let Γ be a linearly ordered set of indices and (y_α) be a monotone increasing subset of P (i.e. $y_\alpha \geqq y_\beta$ for $\alpha \geqq \beta$) and suppose that $y_\alpha \leqq z$ for all $\alpha \in \Gamma$ and an element z, then it has a least upper bound y_0. Moreover there is a monotone increasing subsequence (y_{α_n}) of (y_α) such that $y_{\alpha_n} \to y_0$ (strongly).*

PROOF. Put $m = \sup_{\alpha\in\Gamma} \|y_\alpha\|$. Since $\|y_\alpha\| \geqq \|y_\beta\|$ for $\alpha \geqq \beta$ and $m \leqq \|z\|$, there is a monotone increasing subsequence (y_{α_n}) of (y_α) such that $\lim_n \|y_{\alpha_n}\| = m$. Then,

$$\|y_\alpha - y_\beta\|^2 \leqq \|y_\alpha\|^2 - \|y_\beta\|^2 \quad \text{for } \alpha \geqq \beta;$$

hence $\lim_n y_{\alpha_n} = y'$ (strongly) with $0 \leqq y' \leqq z$. Now we shall show that y' has the property of the lemma. Since (y_α) is monotone increasing, for any $y_\alpha \in (y_\alpha)$ either $y_\alpha \leqq y_{\alpha_{n_0}}$ for some n_0 or $y_\alpha \geqq y_{\alpha_n}$ for all n.

If $y_\alpha \leqq y_{\alpha_{n_0}}$ then $y_\alpha \leqq y_{\alpha_{n_0}} \leqq y'$, and if $y_\alpha \geqq y_{\alpha_n}$ for all n, then $y_\alpha \geqq y'$; hence $\|y_\alpha\| \geqq \|y'\| = \sup_{\alpha\in\Gamma} \|y_\alpha\|$, so that $\|y_\alpha\| = \|y'\|$. On the other hand, $\|y_\alpha - y'\|^2 + \|y'\|^2 \leqq \|y_\alpha\|^2$; hence $\|y_\alpha - y'\| = 0$ and $y_\alpha = y'$. Therefore $y_\alpha \leqq y'$ for all $\alpha \in \Gamma$.

Now let $h \geqq y_\alpha$ for all $\alpha \in \Gamma$ then $h \geqq \lim_n y_{\alpha_n} = y'$. This means that y' is a l. u. b. of (y_α). This completes the proof.

LEMMA 4.11. *Any right ideal is principal.*

PROOF. Let M be a right ideal and put $F = \{z \mid 0 \leqq z \leqq I,\ z \in M\}$, then by Lemma 4.10 there is a maximal element e in F, which is characteristic by Lemma 4.9. Let e_1 and e_2 be two maximal elements, then $e_1 \prec \frac{e_1 + e_2}{2}$ and $e_2 \prec \frac{e_1 + e_2}{2}$; hence $e_1 \leqq \mathrm{e}\left(\frac{e_1 + e_2}{2}\right)$ and $e_2 \leqq \mathrm{e}\left(\frac{e_1 + e_2}{2}\right)$. By the maximality of e_1 and e_2, $e_1 = e_2 = \mathrm{e}\left(\frac{e_1 + e_2}{2}\right)$.

Suppose that $[e] \subsetneqq M$, then there is an element y of M such that $y \overline{\in} [e]$ and $y \geqq 0$, so that $\mathrm{e}(e+y) > e$, this contradicts to the maximality of e; hence $[e] = M$. This completes the proof.

Let $\mathbf{E}$ be the totality of characteristic elements of L, then the following theorem is immediately obtained from the above lemma.

THEOREM. 1. $\mathbf{E}$ *is a complete lattice.*

For, let $\{M_\alpha\}$ be a family of right ideals, then $\bigcap_\alpha M_\alpha$ and $O(\bigcap_\alpha O(M_\alpha))$ are right ideals, and moreover if $[x]$ is a right ideal such that $M_\alpha \subseteqq [x]$ for all α, then $O(\bigcap_\alpha O(M_\alpha)) \subseteqq O(O(x)) = [x]$.

LEMMA 4.12. *In order that an element e belongs to $\mathbf{E}$, it is necessary and sufficient that $0 \leqq e \leqq I$ and $e \perp (I-e)$.*

PROOF. Let $\mathrm{e}(e)$ be the characteristic element corresponding to e, then $\mathrm{e}(e) \geqq e$. Now suppose that $e < \mathrm{e}(e)$, then $I - e = (I - \mathrm{e}(e)) + (\mathrm{e}(e) - e)$; hence $e \perp (I-e)$ means that $(e, I-e) = (e, \mathrm{e}(e) - e) = 0$ and so $\mathrm{e}(e) - e \in O(e)$, this contradicts to $\mathrm{e}(e) - e \in [e]$. This completes the proof.

Since $O(x) = [x]^\perp$, the characteristic element of $O(x)$ is $(1 - E(x))I = I - \mathrm{e}(x)$.

From this fact and the above lemma, we can immediately obtain the following.

LEMMA 4.13.

(1) $e_1, e_2 \in \mathbf{E}$ *and* $e_1 \perp e_2$ *imply* $e_1 + e_2 \in \mathbf{E}$.
(2) $e_1 \geqq e_2$ *implies* $e_1 - e_2 \in \mathbf{E}$.
(3) $e_n \in \mathbf{E}$ $(n = 1, 2, \ldots.)$ *and* $e_n \to e$ *(strongly) imply* $e \in \mathbf{E}$.

5. Integral representations. In this section, we shall show the following theorem.

THEOREM 2. *For any $x \geqq 0$ there is a system of characteristic elements $(e(\lambda))$ $(0 \leqq \lambda \leqq \infty)$, called the resolution of unity such that*

(1) $\lambda \leqq \mu$ *implies* $e(\lambda) \leqq e(\mu)$,
(2) $\lambda_n \leqq \lambda$ $(n = 1, 2, \ldots.)$ *and* $\lambda_n \to \lambda$ *imply* $e(\lambda_n) \to e(\lambda)$ *(strongly)*,
(3) $e(0) = 0$, $\lim\limits_{\lambda\to\infty} e(\lambda) = e(\infty) = I$,
(4) $x = \int_0^\infty \lambda \, de(\lambda)$

where the integration is of abstract Radon-Stieltjes type, and

(5) $\|x\|^2 = \int_0^\infty |\lambda|^2 d\|e(\lambda)\|^2$.

To prove the theorem, we shall need some lemmas.

LEMMA 5.1. *Put* $e(\lambda) = e((\lambda I - x)^+)$, *then* $e(\lambda) \leqq e(\mu)$ *for* $0 \leqq \lambda \leqq \mu$.

PROOF. $\mu I - x = (\mu - \lambda)I + (\lambda I - x)$
$= (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ + (\mu - \lambda)(I - e(\lambda))$
$- (\lambda I - x)^-$.

Since $\{(\mu - \lambda)e(\lambda) + (\lambda I - x)^+\} \perp |(\mu - \lambda)\ (I - e(\lambda)) - (\lambda I - x)^-|$, by the uniqueness of orthogonal decomposition $(\mu I - x)^+ \geqq (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ \geqq (\lambda I - x)^+$; hence $e(\lambda) \leqq e(\mu)$. This completes the proof.

LEMMA 5.2. $\lambda_n \leqq \lambda (n = 1, 2, \ldots.)$ *and* $\lambda_n \to \lambda$ *imply* $e(\lambda_n) \to e(\lambda)$ (*strongly*).

PROOF. $(\lambda I - x) - (\lambda_n I - x) = (\lambda - \lambda_n)I$
$= \{(\lambda I - x)^+ - (\lambda_n I - x)^+\} + \{(\lambda_n I - x)^- - (\lambda I - x)^-\}$.
Since $(\lambda I - x)^+ \geqq (\lambda_n I - x)^+$ and analogously $(\lambda_n I - x)^- \geqq (\lambda I - x)^-$, $\|(\lambda - \lambda_n) I\| \geqq \|(\lambda I - x)^+ - (\lambda_n I - x)^+\|$; hence $(\lambda_n I - x)^+ \to (\lambda I - x)^+$ (strongly). Suppose that $e(\lambda_n) \nrightarrow e(\lambda)$, then there are a sequence (n_k) and $\varepsilon > 0$ such that $\|e(\lambda) - e(\lambda_{n_k})\| > \varepsilon$ for all n_k. On the other hand, for any n_{k0} there is an $n_{k1} \in (n_k)$ such that $e(\lambda_{n_{k0}}) < e(\lambda_{n_{k1}})$ and $n_{k1} > n_{k0}$. For if $e(\lambda_{n_{k0}}) \not< e(\lambda_{n_k})$ for $n_k > n_{k0}$, then $e(\lambda_{n_{k0}}) \geqq e(\lambda_{n_k})$ for all $n_k \geqq n_{k0}$, so that $\bigvee_{n_k \geqq n_{k0}} e(\lambda_{n_k}) = e(\lambda_{n_{k0}}) < e(\lambda)$, since $\|e(\lambda) - e(\lambda_{n_{k0}})\| > \varepsilon$. Then $(\lambda_{n_k} I - x)^+ \in [e(\lambda_{n_{k0}})]$ for all n_k, so that its limit $(\lambda I - x)^+$ belongs to $[e(\lambda_{n_{k0}})]$, this means that $e(\lambda) \leqq e(\lambda_{n_{k0}})$ and a contradiction. Therefore there is a subsequence (m_j) of (n_k) such that $e(\lambda m_j) > e(\lambda m_h)$ for $m_j > m_h$. Put $e' = \bigvee_{m_j} e(\lambda_{m_j})$, then $e' = \lim_j e(\lambda_{m_j})$, so that $\|e(\lambda) - e'\| \geqq \varepsilon$; $e' < e(\lambda)$.

On the other hand, by an analogous reason as above, $(\lambda I - x)^+$ belongs to $[e']$, this is a contradiction. This completes the proof.

LEMMA 5.3. $e(0) = 0$ *and* $\lim_{\lambda \to \infty} e(\lambda) = e(\infty) = I$.

PROOF. Put $\lim_{\lambda \to \infty} e(\lambda) = e(\infty)$, which surely exists and belongs to $\mathbf{E}$ by Lemmas 4.10 and 4.13. In general, if $f \in \mathbf{E}$ and $f \leqq I - e(\lambda)$, then $x \geqq \lambda f$, for $x = \lambda I - (\lambda I - x)^+ + (\lambda I - x)^- = \{\lambda e(\lambda) - (\lambda I - x)^+\} + \{\lambda(I - e(\lambda)) + (\lambda I - x)^-\}$, so that $E(e(\lambda))F(e(\lambda))x = \lambda e(\lambda) - (\lambda I - x)^+ \geqq 0$ means that $x \geqq \lambda(I - e(\lambda)) + (\lambda I - x)^- \geqq \lambda(I - e(\lambda)) \geqq \lambda f$. Since $I - e(\infty) \leqq I - e(\lambda)$ for all λ, $x \geqq \lambda (I - e(\infty))$ for all λ; hence $\|x\| \geqq \lambda \|I - e(\infty)\|$ for all λ, so that $e(\infty) = I$. This completes the proof.

LEMMA 5.4. $$x = \int_0^\infty \lambda \, de(\lambda).$$

PROOF. In general, $\lambda \leqq \mu$ implies that $\lambda(e(\mu) - e(\lambda)) \leqq \{\mu e(\mu) - (\mu I - x)^+\} - \{\lambda e(\lambda) - (\lambda I - x)^+\} \leqq \mu(e(\mu) - e(\lambda))$, for $(\mu I - x) = (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ + (\mu - \lambda)(e(\mu) - e(\lambda)) + (\mu - \lambda)(I - e(\mu)) - (\lambda I - x)^-$.

On the other hand,

$$\begin{aligned}\lambda I - x &= \mu I - x - (\mu - \lambda)I \\ &= (\mu I - x)^+ - (\mu - \lambda)e(\mu) - \{(\mu I - x)^- + (\mu - \lambda)(I - e(\mu))\} \\ &= h - k,\end{aligned}$$

where $h = (\mu I - x)^+ - (\mu - \lambda)e(\mu)$ and $k = (\mu I - x)^- + (\mu - \lambda)(I - e(\mu))$.

Since $|h| \perp k$, $(\lambda I - x)^- = k + h^-$. Hence

$$\begin{aligned}(\mu I - x) &= (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ + (\mu - \lambda)(e(\mu) - e(\lambda)) \\ &\quad + (\mu - \lambda)(I - e(\mu)) - k - h^- \\ &= (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ + (\mu - \lambda)(e(\mu) - e(\lambda)) - h^- - (\mu I - x)^- \\ &= (\mu I - x)^+ - (\mu I - x)^-.\end{aligned}$$

Therefore,

$(\mu I - x)^+ = (\mu - \lambda)e(\lambda) + (\lambda I - x)^+ + (\mu - \lambda)(e(\mu) - e(\lambda)) - h^-$. Hence,

$$\begin{aligned}&\{\mu e(\mu) - (\mu I - x)^+\} - \{\lambda e(\lambda) - (\lambda I - x)^+\} \\ &\quad = \lambda(e(\mu) - e(\lambda)) + (\mu - \lambda)e(\mu) - (\mu - \lambda)e(\lambda) \\ &\quad - (\lambda I - x)^+ - (\mu - \lambda)(e(\mu) - e(\lambda)) + h^- + (\lambda I - x)^+ \\ &\quad = \lambda(e(\mu) - e(\lambda)) + h^-.\end{aligned}$$

This means that $\{\mu e(\mu) - (\mu I - x)^+\} - \{\lambda e(\lambda) - (\lambda I - x)^+\} \geqq \lambda(e(\mu) - e(\lambda))$.

Moreover,

$$\begin{aligned}&\{\mu e(\mu) - (\mu I - x)^+\} - \{\lambda e(\lambda) - (\lambda I - x)^+\} \\ &\quad = \mu e(\mu) - \lambda e(\lambda) - (\mu I - x)^+ + (\lambda I - x)^+ \\ &\quad = \mu(e(\mu) - e(\lambda)) + (\mu - \lambda)e(\lambda) - \{(\mu I - x)^+ - (\lambda I - x)^+\} \\ &\quad \leqq \mu(e(\mu) - e(\lambda)) + (\mu - \lambda)e(\lambda) - \{(\mu - \lambda)e(\lambda) + \\ &\qquad\qquad (\lambda I - x)^+ - (\lambda I - x)^+\} = \mu(e(\mu) - e(\lambda)).\end{aligned}$$

For any division $\Delta: 0 = \lambda_0 < \lambda_1 < \ldots. < \lambda_n = \Lambda < \infty$ of the interval $(0, \Lambda)$ with $0 < \lambda_i - \lambda_{i-1} < \varepsilon$ $(i = 1, 2, \ldots, n)$, we have

$$\begin{aligned}m(\Delta) &\equiv \sum_{i=1}^{n} \lambda_{i-1}(e(\lambda_i) - e(\lambda_{i-1})) \leqq \sum_{i=1}^{n} [\{\lambda_i e(\lambda_i) - (\lambda_i I - x)^+\} - \\ &\{\lambda_{i-1}e(\lambda_{i-1}) - (\lambda_{i-1}I - x)^+\}] = \{\lambda_0 e(\lambda_0) - (\lambda_0 I - x)^+\} + \{\lambda_n e(\lambda_n) - (\lambda_n I - x)^+\} \\ &= \Lambda e(\Lambda) - (\Lambda I - x)^+ \\ &\leqq \sum_{i=1}^{u} \lambda_i(e(\lambda_i) - e(\lambda_{i-1})) \equiv M(\Delta),\end{aligned}$$

and

$$\begin{aligned}M(\Delta) - m(\Delta) &= \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(e(\lambda_i) - e(\lambda_{i-1})) \\ &\leqq \varepsilon \sum_{=1i}^{n} (e(\lambda_i) - e(\lambda_{i-1})) = \varepsilon(e(\Lambda) - e(0)) \leqq \varepsilon I;\end{aligned}$$

hence by making $\varepsilon \to 0$, $\Lambda e(\Lambda) - (\Lambda I - x)^+ = \displaystyle\int_0^\Lambda \lambda\, de(\lambda)$.

Moreover,

$$\begin{aligned}\Lambda e(\Lambda) - (\Lambda I - x)^+ &= \Lambda e(\Lambda) - (\Lambda I - x) - (\Lambda I - x)^- \\ &= x - \Lambda(I - e(\Lambda)) - (\Lambda I - x)^- \\ &= x - (\Lambda(I - e(\Lambda)) + (\Lambda I - x)^-) \leqq x\end{aligned}$$

Therefore by Lemma 4.10, there is $\lim_{\Lambda\to\infty}\int_0^\Lambda \lambda\, de(\lambda)$, so that $\Lambda(I-e(\Lambda)) + (\Lambda I - x)^- \to x_0$ (strongly).

Now we shall show that $\lim_{\Lambda\to\infty}(\Lambda(I-e(\Lambda)) + (\Lambda I - x)^-) = 0$. For any λ, $\{\Lambda(I-e(\Lambda)) + (\Lambda I - x)^-\} \in O(e(\lambda))$ for $\Lambda > \lambda$, and so $x_0 \in O(e(\lambda))$; hence $x_0 \in \bigcap_{\lambda\geqq 0} O(e(\lambda)) = (0)$, for if $\bigcap_{\lambda\geqq 0} O(e(\lambda)) \neq (0)$, $L \neq O(\bigcap_{\lambda\geqq 0} O(e(\lambda))) \supseteqq [e(\lambda)]$ for all λ, so that $I > \bigvee_{\lambda\geqq 0} e(\lambda) = e(\infty)$, this is a contradiction.

Therefore $x = \int_0^\infty \lambda\, de(\lambda)$, this completes the proof.

The above lemmas will complete the proof of Theorem 2.

6. Construction of W*-algebra. Let M be the totality of elements of L as follows: if $x \in M$, there is a positive number α such $|x| \leqq \alpha I$. Then we shall construct a W^*-algebra of finite type, using M.

LEMMA 6.1. *$|\alpha x| = |\alpha|\,|x|$ for $x \in L$ and α a complex number.*

PROOF. Suppose that $(|\alpha x|, |x|) \geqq (|(\alpha x)^*|, |x^*|)$, then $|(\alpha x, x)| \leqq (|(\alpha x)^*|, |x^*|)^{1/2}(|\alpha x|, |x|)^{1/2} \leqq (|\alpha x|, |x|)$; hence $0 \leqq (|\alpha x| - |\alpha|\,|x|, |\alpha x| - |\alpha|\,|x|) \leqq 2|\alpha|^2(x,x) - 2|\alpha|\cdot|(\alpha x, x)| = 0$, so that $|\alpha x| = |\alpha|\,|x|$ and moreover $|(\alpha x, x)| = |\alpha|(|x|, |x|) \leqq (|(\alpha x)^*|, |x^*|)^{1/2}(|\alpha x|, |x|)^{1/2} = (|(\alpha x)^*|, |x^*|)^{1/2}|\alpha|(|x|, |x|)^{1/2}$; hence $|\alpha|(|x|, |x|) = (|(\alpha x)^*|, |x^*|) = (|\alpha x|, |x|)$.

Next suppose that $(|\alpha x|, |x|) \leqq (|(\alpha x)^*|, |x^*|)$, then by an analogous discussion as above, we obtain that $(|\alpha x|, |x|) = (|(\alpha x)^*|, |x^*|)$; hence by the above discussion $|\alpha x| = |\alpha|\,|x|$. This completes the proof.

By the axiom **II** and the above lemma, M is a self-adjoint subspace of L.

LEMMA 6.2. *If $x \in L$ and $y \in M$, $E(x)y \in M$.*

PROOF. Put $y = E(x)y + (1-E(x))y$, then $|(E(x)y)^*| \leqq |y^*| \leqq \alpha I$, so that $E(x)y \in M$. This completes the proof.

LEMMA 6.3. *If $x \in M \cap L_s$, and $(e(\lambda))$ be the resolution of unity corresponding to x, then there is a positive number K such that $x = \int_{-K}^{K} \lambda\, de(\lambda)$. Moreover the family $\{E(e(\lambda))\}$ of projections is a spectral resolution.*

PROOF. Since the first part is clear, we shall prove the second part. It is clear that $\lambda \leqq \mu$ implies $E(e(\lambda)) \leqq E(e(\mu))$.

Suppose that $\lambda_n \leqq \lambda$ $(n = 1, 2, \ldots.)$ and $\lambda_n \to \lambda$, then by the Theorem 2, $e(\lambda_n) \to e(\lambda)$ (strongly); hence

$\|(E(e(\lambda)) - E(e(\lambda_n)))F(y)I\|^2$

$$= ((E(e(\lambda)) - E(e(\lambda_n)))F(y)I,\ (E(e(\lambda)) - E(e(\lambda_n)))F(y)I)$$
$$= ((E(e(\lambda)) - E(e(\lambda_n)))I,\ (E(e(\lambda)) - E(e(\lambda_n)))F(y)I)$$
$$= (e(\lambda) - e(\lambda_n),\ (E(e(\lambda)) - E(e(\lambda_n)))F(y)I)$$
$$\leqq |e(\lambda) - e(\lambda_n)|\ \|I\| \to 0 \text{ for all } y \in L.$$

Since, by Theorem 2, linear combinations of $\{F(y)I | y \in L\}$ are dense in L, the above property means that $E(e(\lambda_n))$ converges to $E(e(\lambda))$ with the strong operator topology. This completes the proof.

From the above lemma, we can define a bounded operator $T(x) = \int_{-K}^{K} \lambda\, dE(e(\lambda))$, then it is clear that $T(x)I = \int_{-K}^{K} \lambda\, de(\lambda) = x.$

Let $y \in M$ and $y = y_1 + iy_2\ (y_1, y_2 \in M \cap L_s)$, we shall consider the correspondence $y \to T(y) = T(y_1) + iT(y_2)$; then it is clear that $T(y)I = y_1 + iy_2 = y$.

LEMMA 6.4. *Put* $\widetilde{M} = \{T(y) | y \in M\}$, *then* $\widetilde{M}$ *is a* W^*-*algebra.*

PROOF. Let $T(y_1),\ T(y_2) \in \widetilde{M}$, then $(\alpha T(y_1) + \beta T(y_2))I = \alpha y_1 + \beta y_2 = T(\alpha y_1 + \beta y_2)I$. Since $F(y)$ commutes with $T(x)$ for any $x, y \in M$, the above equality means that $(\alpha T(y_1) + \beta T(y_2))F(y)I = T(\alpha y_1 + \beta y_2)F(y)I$; hence $\alpha T(y_1) + \beta T(y_2) = T(\alpha y_1 + \beta y_2) \in \widetilde{M}$.

If $T(y) = T(y_1) + iT(y_2)\ (y_1,\ y_2 \in L_s \cap M)$, clearly $T(y)^* = T(y_1) - iT(y_2) = T(y^*)$; hence $\widetilde{M}$ is self-adjoint.

$E(y)T(x)I = E(y)x \in M$ by Lemma 6.2; hence $E(y)T(x)I = T(E(y)x)I$, so that $E(y)T(x) = T(E(y)x) \in \widetilde{M}$.

Let $\widetilde{M}^b$ be the uniform closure of $\widetilde{M}$, then $T(x)T(y) \in \widetilde{M}^b$ $(x, y \in M)$, for any element of $\widetilde{M}$ is uniformly approximated by finite linear combinations of elements of $\{E(x) | x \in M\}$; hence $\widetilde{M}^b$ is a C^*-algebra.

Now we shall introduce a new norm $|||\cdot|||$ on $M \cap L_s$ as follows: $|||x||| = \inf_{|x| \leqq \alpha I} \alpha$ for $x \in M \cap L_s$. Then $x = \int_{-|||x|||}^{|||x|||} \lambda\, de(\lambda)$ and $\int_{-|||x|||+\varepsilon}^{|||x|||-\varepsilon} \lambda\, de(\lambda) \neq x$ for any $\varepsilon > 0$.

Put $S = \{x |\ |||x||| \leqq 1,\ x \in M \cap L_s\}$, then S is bounded and closed, for $\|x\| = (|x|, |x|)^{1/2} \leqq (I, I)^{1/2}$, and $x_n \in S$ and $x_n \to x$ (strongly) imply $\pm x_n \leqq I$ and $\pm x \leqq I$; hence $|x| \leqq I$. Therefore S is weakly compact, so that it is complete by the norm $|||\cdot|||$ [cf. 2, lemma]. Since $|||x|||$ is equal to the operator norm of $T(x)$, the self-adjoint portion of $\widetilde{M}^b$ coincides with $\widetilde{M \cap L_s}$ and so $\widetilde{M}^b = \widetilde{M}$; hence $\widetilde{M}$ is a C^*-algebra, and moreover there is a locally convex topology, by which the unit sphere is compact, so that $\widetilde{M}$ is a W^*-algebra [cf. 8]. This completes the proof.

7. Proof of Main Theorem. Now we shall prove the main theorem.

The proof is devided into two parts, and the first part is devoted to (1), (2) of the theorem and the second to (3), that is, the unicity of the absolute value. By Lemma 6.1, it is enough to prove the theorem under the assumption "$\|I\| = 1$".

LEMMA 7.1. *Preserving the order structure, we can introduce a product into M such that it becomes a W^*-algebra of finite type with the unit I.*

PROOF. From the discussions of the last section, the mapping $x \rightarrow T(x)$ $(x \in M)$ is a linear isomorphism of M on $\widetilde{M}$, and moreover it is clear that $x \geqq 0$ is equivalent to $T(x) \geqq 0$; hence by this mapping, preserving the order structure, we can introduced canonically a product into M; then M is a W^*-algebra with the unit I.

Moreover,

$$\begin{aligned}(x, y) &= (xI, yI) = (T(x)I,\ T(y)I) = (T(y)^*T(x)I, I)\\ &= (T(y^*)T(x)I, I) = (T(y^*x)I, I) = (y^*x, I) \quad \text{for } x, y, \in M.\end{aligned}$$

Therefor, put $\varphi(x) = (x, I)$ for $x \in M$, then $\varphi(xy) = (y, x^*) = (x, y^*) = (yx, I) = \varphi(yx)$ for $x, y \in M$. Moreover, $\varphi(x^*x) = (x, x) = 0$ implies $x = 0$; hence M has a complete trace, so that it is a W^*-algebra of finite type. This completes the proof.

LEMMA 7.2. *Let $L^2(M, \varphi)$ be the generalized L^2-space, associated with the algebra M and the above trace φ, and ρ be the injection mapping of M (in L) on M (in $L^2(M, \varphi)$), then ρ is uniquely extended to the isometric mapping of L on $L^2(M, \varphi)$, and satisfies the following relations*: (1) $\rho(I) = I$ *and* (2) $a \geqq 0$ *is equivalent to* $\rho(a) \geqq 0$, *where I is the unit of M.*

This is clear.

By the above two lemmas, we complete the first part of the proof.

Next we shall show the unicity of the absolute value. Now put $\|x\| = \inf\limits_{|x| \leqq \alpha I} \alpha$ for all $x \in M$, then by Lemma 6.1, $\|\cdot\|$ is a norm on M, and moreover by this norm, M is a Banach space, for it coincides with the uniform norm $\|\cdot\|_\infty$ on the self-adjoint portion M_s, so that $\|a_n + ib_n\|_\infty \rightarrow 0$ $(a_n, b_n \in M_s)$ means $\|a_n\|_\infty \rightarrow 0$ and $\|b_n\|_\infty \rightarrow 0$; hence $\|a_n + ib_n\| \rightarrow 0$. Conversely, by the axiom II_2 $\|x^*\| = \|x\|$, so that $\|a_n + ib_n\| \rightarrow 0$ means analogously $\|a_n + ib_n\|_\infty \rightarrow 0$; hence $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. Now we shall identify L with $L^2(M, \varphi)$ under the mapping ρ and put $|x|_1 = (xx^*)^{1/2}$ for all $x \in L$, where the product is the extended one in $L^2(M, \varphi)$, then $\mathbf{e}(|x|) = \mathbf{e}(|x|_1)$, for $x = |x|_1 u$, where u is a unitary element; hence, put $u = \int_0^{2\pi} e^{i\theta} de(\theta)$, then $|x|_1 = xu^* = T(x)\int_0^{2\pi} e^{-i\theta} dF(e(\theta))I = \int_0^{2\pi} e^{-i\theta} dF(e(\theta))T(x)I = \int_0^{2\pi} e^{-i\theta} dF(e(\theta,$ ince $[x]$ is invariant under $F(y)$ $(y \in L)$, the above equality means that $|x|_1$

belongs to $[x]$; hence $|x|_1 \prec |x|$. Next suppose that $|x|_1 \perp v (v \geqq 0)$, then $(|x|, v) = (x, v_1) = (|x|_1 u, v_1) = (|x|_1, v_1 u^*)$, where $|v_1| = v$. Since v_1 belongs to $[v]$, so is $v_1 u^*$; hence $(|x|, v) = 0$, so that $|x| \prec |x|_1$.

LEMMA 7.3. *Let $\{e_i \mid i = 1, 2, \ldots, n\}$ be a finite family of mutually orthogonal projections and $\{\alpha_i \mid i = 1, 2, \ldots, n\}$ be a family of complex numbers. Then* $\left|\sum_{i=1}^n \alpha_i e_i\right| = \left|\sum_{i=1}^n \alpha_i e_i\right|_1 = \sum_{i=1}^n |\alpha_i| e_i$.

PROOF. Since $|(\alpha_1 e_1)^*|_1$ is orthogonal to $\left|\left(\sum_{i=2}^n \alpha_i e_i\right)^*\right|_1$, $|(\alpha_1 e_1)^*|$ is orthogonal to $\left|\left(\sum_{i=2}^n \alpha_i e_i^*\right)\right|$; hence by the axiom $\mathbf{I}_3$, $|\alpha_1 e_1| = |\alpha_1| e_1 \leqq \left|\sum_{i=1}^n \alpha_i e_i\right|$ and so $|\alpha_1 e_1| \leqq e_1 \left|\sum_{i=1}^n \alpha_i e_i\right| e_1$.

Analogously, $|\alpha_j e_j| \leqq e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_j$ for all j, so that $\sum_{j=1}^n |\alpha_j e_j| \leqq \sum_{j=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i.\right| e_j$

On the other hand,

$$\begin{aligned}
\left(\sum_{i=1}^n \alpha_i e_i, \sum_{i=1}^n \alpha_i e_i\right) &= \left(\sum_{i=1}^n |\alpha_i e_i|, \sum_{i=1}^n |\alpha_i e_i|\right) \\
&\leqq \left(\sum_{j=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_j, \sum_{j=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_j\right) \\
&\leqq \left(\sum_{j,k=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_k, \sum_{j,k=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_k\right) \\
&= \left(\sum_{j=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_j, \sum_{j=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_j\right) \\
&\quad + \sum_{\substack{j \neq k \\ j,k=1}}^n \left(e_j \left|\sum_{i=1}^n\right| \alpha_i e_i \left| e_k, e_j \right|\sum_{i=1}^n \alpha_i e_i\right| e_k\Big) \\
&= \left(\left|\sum_{i=1}^n \alpha_i e_i\right|, \left|\sum_{i=1}^n \alpha_i e_i\right|\right) = \left(\sum_{i=1}^n \alpha_i e_i, \sum_{i=1}^n \alpha_i e_i\right);
\end{aligned}$$

hence $e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_k = 0$ for $j \neq k$ and $\sum_{i=1}^n \left|\alpha_i e_i\right| = \sum_{j,k=1}^n e_j \left|\sum_{i=1}^n \alpha_i e_i\right| e_k = \left|\sum_{i=1}^n \alpha_i e_i\right|$, so that $\left|\sum_{i=1}^n \alpha_i e_i\right| = \sum_{i=1}^n |\alpha_i| e_i = \left|\sum_{i=1}^n \alpha_i e_i\right|_1$.

This completes the proof.

LEMMA 7.4. *For any unitary element u of M, $|u| = |u|_1 = I$.*

PROOF. For any unitary element u, there is a sequence$\left(\sum_{i(p)=1}^{n(p)} \alpha_{i(p)} e_{i(p)}\right)$ as the above lemma such that $\lim_{p\to\infty} \left\| u - \sum_{i(p)=1}^{n(p)} \alpha_{i(p)} e_{i(p)} \right\| = 0$; hence $\| u \| = \lim_{p\to\infty} \left\| \sum_{i(p)=1}^{n(p)} \alpha_{i(p)} e_{i(p)} \right\| = \lim_{p\to\infty} \left\| \sum_{i(p)=1}^{n(p)} \alpha_{i(p)} e_{i(p)} \right\|_\infty = \| u \|_\infty$, so that $|u| \leqq I$. On the other hand, $(|u|, |u|) = (u, u) = (I, I)$; hence $|u| = I$. This completes the proof.

LEMMA 7.5. *Let h be a positive element of L, u a unitary element and e be a projection which commutes with h, then $|ehu| = e|hu|e$ and $|hu| = e|hu|e + (I-e)|hu|(I-e)$.*

PROOF. $e(|(ehu)^*|) = e(|(ehu)^*|_1) = e((u^*eh^2eu)^{1/2}) \leqq u^*eu$ and $e(|\{(I-e)hu\}^*|) \leqq u^*(I-e)u$; hence $|(ehu)^*| \perp |\{(I-e)hu\}^*|$, so that by the axiom I_3, $|ehu| \leqq |hu|$ and $|(I-e)hu| \leqq |hu|$, and so $e\,|ehu|e = |ehu| \leqq e|hu|e$ and $(I-e)|(I-e)hu|(I-e) = |(I-e)hu| \leqq (I-e)|hu|(I-e)$.

Then,

$$\begin{aligned}
(|hu|, |hu|) &= (hu, hu) = (ehu + (I-e)hu, ehu + (I-e)hu) \\
&= (|ehu|, |ehu|) + (|(I-e)hu|, |(I-e)hu|) \\
&\leqq (|ehu| + |(I-e)hu|,\ |ehu| + |(I-e)hu|) \\
&\leqq (e|hu|e + (I-e)|hu|(I-e), e|hu|e + (I-e)|hu|(I-e)) \\
&\leqq (|hu|, |hu|) = (e|hu|e + (I-e)|hu|(I-e),\ e|hu|e + (I-e)|hu|(I-e)) \\
&\quad + ((I-e)|hu|e,\ (I-e)|hu|e) + (e|hu|(I-e), e|hu|(I-e));
\end{aligned}$$

hence by an analogous method with the proof of Lemma 7.1, $|ehu| = e|hu|e$ and $|hu| = e|hu|e + (I-e)\,|hu|(I-e)$. This completes the proof.

LEMMA 7.6. $\| x \| = \| x \|_\infty$ *for all* $x \in M$.

PROOF. At first, let $\{e_i | i = 1, 2, \ldots, n\}$ be a finite family of mutually orthogonal projections, $\{\alpha_i | i = 1, 2, \ldots, n\}$ a family of positive numbers and u be a unitary element, then $\left|\left(\sum_{i=1}^{n} \alpha_i e_i\right) u\right| = \sum_{i=1}^{n} \alpha_i e_i = \left|\left(\sum_{i=1}^{n} \alpha_i e_i\right) u\right|_1$, for by Lemma 6.2, $|u| = I$ means $|e_i u| \leqq I$; hence $|e_i u| \leqq e_i$ and by an analogous method as Lemma 7.4, $|e_i u| = e_i$, so that by Lemma 7.5,

$$\left|\left(\sum_{i=1}^{n} \alpha_i e_i\right) u\right| = \sum_{i=1}^{n} \alpha_i |e_i u| = \sum_{i=1}^{n} \alpha_i e_i.$$

Since elements as the above forms are uniformly dense in M, we can show, by an analogous method with Lemma 7.4 that $\| x \| = \| x \|_\infty$ for $x \in M$. This completes the proof.

Finally, we shall complete the proof of (3) of the main theorem. Put $x = hu$ ($h > 0$, u unitary), then by Lemma 7.5 $|x|$ commutes with $h = |x|_1$. Now suppose that $|x| \neq |x|_1$, then there exist a projection e and a positive number $\varepsilon\ (>0)$ such that it commutes with $|x|$ and $|x|_1$, and $e|x|e > e|x|_1 e + \varepsilon e$ or $e|x|e + \varepsilon e < e|x|_1 e$, and $e|x|e$ and $e|x|_1 e$ belongs to M.

We shall assume that $e|x|e > e|x|_1 e + \varepsilon e$, then by Lemma 7.5, $e|x|e = |ex|$ and moreover $|ex|_1 = (ehuu^*he)^{1/2} = ehe = e|x|_1 e$, so that $|ex| > |ex|_1 + \varepsilon e$; hence $\| ex \| \geqq \| ex \|_\infty + \varepsilon$. This contradicts to Lemma 7.6; hence $e|x|e \not> e|x|_1 e + \varepsilon e$. Under the assumption "$e|x|e + \varepsilon e < e|x|_1 e$", we can obtain an analogous contradiction; hence $|x| = |x|_1$ for all $x \in L$. Though we show $|x| = (xx^*)^{1/2}$, by the trivial modification of the product, we can also consider $|x| = (x^*x)^{1/2}$.

This completes the proof.

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