ON DIVISORS OF FACTORS

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In the theory of rings of operators, it is interesting to study the algebraical structure of such rings. Many authors have investigated that structure, but it remains obscure except for some special ones. F. J. Murray and J. von Neumann [6] have made some interesting works; in fact, they introduced the concepts of fundamental groups, genera and the property Γ for finite factors and obtained many results. The purpose of this paper is to give a trial to study the algebraical structure theory of factors. In [3], we have studied the direct product of W^*-algebras, which will play the fundamental rôle throughout this paper.

A factor N is called a divisor of a factor M if M is the direct product of N and some factor P. The set of all divisors of M is called the divisor set of M. The first section will be devoted to the elementary properties of divisors. In the second section, we shall study some relations between factors and its divisors with respect to the concepts of the normalcy and the property Γ. Our results are as follows: If a given factor contains any divisor which is not normal (resp. has the property Γ), then the factor itself is not normal (resp. has the property Γ). In the final section, we shall show that the restricted infinite direct product of finite factors by traces in the sense of Takeda [8] has property Γ and its fundamental group contains all positive numbers.

Throughout this paper, by an isomorphism we mean a *-isomorphism and any two factors are identified if they are isomorphic to each other.

1. Definitions and preliminary considerations. Let M and N be two W^*-algebras on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively. By the *direct product* $M \otimes N$ of M and N, we shall mean the weak closure of the algebraical direct product $M \odot N$ on the Hilbert space $\mathfrak{H} \otimes \mathfrak{K}$ in the sense of F. J. Murray and J. von Neumann [5]. In [3], we have proved that $M \otimes N$ depends on M and N only, but not on the choice of underlying Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$. Moreover we have proved that $M \otimes N$ is a factor whenever M and N are factors. Therefore, the set of all factors may be considered as a (commutative) semi-group by the prouct $\otimes$.

LEMMA 1. *Let N, P be two W^*-algebras and M be the direct product of N and P. If M is a factor then N and P are factors.*

PROOF. If N is not a factor, then there exists an element x in the center of M which is not a scalar multiple of the identity. It is clear that $x \otimes e$ is contained in the center of M and not a scalar multiple of the identity where e is the identity of P. This is a contradiction since M is a factor, so that N is a factor. Analogously, P is a factor.

We shall give the following definition:

DEFINITION. A factor N is said to be a *divisor*[1] of a factor M, if M is the direct product of N and some factor P. The set of all divisors of a factor M is said the *divisor set* of M and is denoted by $D(M)$. A factor M is called *idempotent* if $M = M \otimes M$.

LEMMA 2. *Let N be a divisor of a factor M and P be a divisor of N, then P is a divisor of M.*

PROOF. By the assumptions, we have

$$M = N \otimes N_1 \text{ and } N = P \otimes P_1$$

where N_1 and P_1 are suitable divisors of M and N respectively. This implies

$$M = P \otimes P_1 \otimes N_1.$$

Thus P is a divisor of M.

Next we shall introduce the following order relation $\succ$ in the set of all factors: For given two factors M and N, by $M \succ N$ we mean that N is a divisor of M. Then, by the preceding lemma, the relation $\succ$ is a quasi-order.

LEMMA 3. *A finite factor has no infinite factor as its divisor.*

PROOF. Let M be a finite factor and N be its divisor, then we have $M = N \otimes P$ for a suitable divisor P of M. If N is infinite, then there exists a family $\{e_i\,;\, i = 1, 2, \ldots.\}$ of non-zero projections in N which are mutually orthogonal and equivalent. Then for any non-zero projection f in P, $\{e_i \otimes f\,;\, i = 1, 2, \ldots.\}$ is a family of non-zero projections in M which are mutually orthogonal and equivalent. This contradicts to the finiteness of M. This proves the lemma.

PROPOSITION 1. *Any divisor of a factor of type* I_m $(m < \infty)$ *is a finite factor of type* I. *A factor of type* I_n $(n < \infty)$ *is a divisor of a factor of type* I_m $(m < \infty)$ *if and only if m is divisible by n.*

PROOF. Let M be a factor of type I_m and N be its divisor. Then N is a finite factor by Lemmas 1, 2 and we can put $M = N \otimes P$ for a divisor P of M. If N is non-discrete, then there exist infinitely many non-zero projections $\{e_i\,;\, i = 1, 2, \ldots.\}$ which are mutually orthogonal. Let f be any non-zero projection in P, then $\{e_i \otimes f\,;\, i = 1, 2, \ldots.\}$ are non-zero projections in M which are mutually orthogonal. This is a contradiction since M is a factor of type I_m $(m < \infty)$, that is, N is a finite factor of type I.

Analogously P is of type I_p for some positive integer p. Let $e_1, \ldots., e_m$ be mutually orthogonal minimal projections in N and $f_1, \ldots., f_p$ be the same ones in P. Then it is clear that

1) Our definition of a divisor is different from that of Murray and von Neumann [6].

$$e_i \otimes f_j\,;\ i = 1, \ldots, n\,;\ j = 1, \ldots, p$$

are mutually orthogonal minimal projections in M with the identity as their union. This shows that $m = np$, that is, m is divisible by n.

Conversely, if m is divisible by n, say $m = np$. Let N, P be factors of type I_n, I_p respectively, then their direct product is a factor of type I_m. In other words, a factor of type I_n is a divisor of a factor of type I_m.

Thus the proposition is completely proved.

LEMMA 4. *Let M be a factor which is infinite or of type* II_1, *then, for every integer n, $M \succ N$ where N is any factor of type* I_n.

PROOF. At first, assume that M is infinite and let N be any factor of type I_n. Let e, f be the identities of M, N respectively and f_1 be a minimal projection in N. Then $M \otimes N$ is an infinite factor and $e \otimes f$ is its identity and moreover $e \otimes f_1$ is an infinite projection. Hence $e \otimes f$ is equivalent to $e \otimes f_1$ and the contraction of $M \otimes N$ on the range of $e \otimes f_1$ is isomorphic to $M \otimes N$. On the other hand, we can easily see that the contraction of $M \otimes N$ on the range of $e \otimes f_1$ is isomorphic to M since $e \otimes f_1$ is equivalent to the identity. These show that $M = M \otimes N$, that is, N is a divisor of M.

When M is of type II_1 the lemma is an immediate consequence of a theorem of Murray and von Neumann [6; Theorem 6].

LEMMA 5. *If M is an idempotent factor, then the divisor set of M is a semi-group.*

PROOF. Let M_1 and M_2 be two divisors of M, then there exist two divisors N_1 and N_2 such that $M = M_i \otimes N_i$ $(i = 1, 2)$. Since M is idempotent, we have

$$M = M \otimes M = M_1 \otimes N_1 \otimes M_2 \otimes N_2 = (M_1 \otimes M_2) \otimes (N_1 \otimes N_2).$$

In other words, $M_1 \otimes M_2 \in D(M)$. This proves the lemma.

LEMMA 6. *Let M and N be idempotent factors, then $M = N$ if and only if $M \succ N$ and $N \succ M$.*

PROOF. As the necessity is obvious, we shall show only the sufficiency. By the assumption, we have $M = N \otimes N_1$ and $N = M \otimes M_1$. Hence we have

$$M \otimes N = N \otimes N_1 \otimes N = N \otimes N_1 = M.$$

Analogously we have $N = M \otimes N$. Therefore $M = N$.

As an immediate consequence of the above lemma, we have

LEMMA 7. *The set of all idempotent factors is a directed set by the relation* $\succ$.

The following lemma is an immediate consequence of Theorem 8 in [3].

LEMMA 8. *Let N be a divisor of a finite factor M, then the fundamental group of M contains that of N.*

2. **The normalcy and the property Γ.** Some properties of factors

are closely related with those of their divisors. In this section, we shall study on the property of normalcy and on the property Γ.

The following definition is due to Murray and von Neumann [5].

DEFINITION. A factor M is *normal* if $(N' \cap M)' \cap M = N$ for any W^*-subalgebra N *of* M.

It is known that a factor of type I is normal and a factor of type II is not normal (cf. [2], [5]). Hence, Lemma 4 shows that there are non-normal factors which have normal factors as their divisors. On the other hand, we can prove the following theorem.

THEOREM 1. *A factor which has a non-normal factor as its divisor is not normal.*

PROOF. Let N be a divisor of a factor M and assume that N is not normal and $M = N \otimes P$. The non-normalcy of N implies that there exists a W^*-subalgebra N_1 of N such that

$$(N_1' \cap N)' \cap N \supsetneqq N_1.$$

Let e be the unit of P, then $N_1 \otimes e$ is a W^*-subalgebra of M and moreover we have

$$\begin{aligned}((N_1 \otimes e)' \cap (N \otimes P))' \cap (N \otimes P) &\supset ((N_1' \otimes B) \cap (N \otimes P))' \cap (N \otimes P)\\ &\supset ((N_1' \cap N)' \cap N) \otimes (P' \cap P)\end{aligned}$$

where B is the full operator algebra on the Hilbert space on which P acts. Hence

$$((N_1 \otimes e)' \cap (N \otimes P))' \cap (N \otimes P) \supsetneqq N_1 \otimes e,$$

which proves the theorem.

The type of the direct product of two factors has not yet been determined except when they are both semi-finite or one of them is of type I. It seems probable to the author that a factor of type I has no non-discrete factor as its divisor, the following corollary will give only a partial answer for this.

COROLLARY. *The direct product of two facters is not of type* I *if at least one of them is of type* II.

PROOF. As we have noticed, every factor of type II is not normal, therefore, the direct product of a factor of type II and any other factor is not normal by the theorem. On the other hand every factor of type I is normal and therefore the direct product in question is not of type I.

Next we employ the following definition in [6]:

DEFINITION. A finite factor M has the *property* Γ if it satisfies the following: Given any system $x_1, \ldots, x_m \in M$ and any $\varepsilon > 0$, there exists a unitary element $u \in M$ such that $\tau(u) = 0$ and

$$[[u^{-1}x_k u - x_k]] < \varepsilon \qquad \text{for } k = 1, \ldots, m,$$

where τ is the normalized trace of M and $[[\cdot]]$ is the norm of the prehilbert space generated by M and τ as in a usual way.

THEOREM 2. *Let M be a finite factor. If there exists a divisor of M which has the property Γ, then M itself has this property.*

PROOF. Let N be the divisor of M which has the property Γ and assume that $M = N \otimes P$. By Lemma 4, P is a finite factor. Let $x_1, \ldots, x_m$ be arbitrary elements in M and ε be any positive number. It is known that M is the closure of the algebraical direct product $N \odot P$ by the norm $[[\cdot]]$. Hence there exist

$$\sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i},\ y_{k,i} \in N, \quad z_{k,i} \in P \qquad (k = 1, \ldots, m)$$

with

$$[[x_k - \sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i}]] < \frac{\varepsilon}{3} \qquad (k = 1, \ldots, m).$$

Let τ_1, τ_2 be the normalized traces of N, P and $[[\cdot]]_1$, $[[\cdot]]_2$ be the norms on N, P induced by τ_1, τ_2 respectively. Since N has the property Γ, we can choose a unitary element u in N such that $\tau_1(u) = 0$ and

$$[[u^{-1}y_{k,i}u - y_{k,i}]]_1 < \frac{\varepsilon}{3\mu\rho} \qquad (i = 1, \ldots, n_k,\ k = 1, \ldots, m)$$

where $\mu = \max_k n_k$ and $\rho = \max_{k,i} [[z_{k,i}]]_2$. Let f be the unit of P, then $u \otimes f$ is a unitary element in M. By Theorem 6 in [3], we have

$$\tau(u \otimes f) = \tau_1(u)\tau_2(f) = 0$$

and

$$[[(u \otimes f)^{-1}\Big(\sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i}\Big)(u \otimes f) - \sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i}]]$$

$$= [[\sum_{i=1}^{n_k} (u^{-1}y_{k,i}u - y_{k,i}) \otimes z_{k,i}]] \leqq \sum_{i=1}^{n_k} [[u^{-1}y_{k,i}u - y_{k,i}]]_1\, [[z_{k,i}]]_2 < \varepsilon.$$

Accordingly we have

$$[[(u \otimes f)^{-1} x_k (u \otimes f) - x_k]]$$

$$\leqq [[(u \otimes f)^{-1}(x_k - \sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i})(u \otimes f)]] + [[(u \otimes f)^{-1}\Big(\sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i}\Big)(u \otimes f)$$

$$- \sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i}]] + [[\sum_{i=1}^{n_k} y_{k,i} \otimes z_{k,i} - x_k]] < \varepsilon.$$

This shows that M has the property Γ.

REMARK 1. In the above proof, $u \otimes f$ is orthogonal to $e \otimes x$ in the sense of the structure of the prehilbert space generated by M and τ, where e is the unit of M; in fact,

$$\tau((e \otimes x)^*(u \otimes f)) = \tau_1(u)\ \tau_2(x^*f) = 0$$

for any $x \in P$. Accordingly M has the property Γ relative to $e \otimes P$ in the sense of Dixmier [1].

REMARK 2. In [3], we have proved that the direct product of two finite factors is finite too. Hence the set of all finite factors is a semi-group and the factor of type I_1 is the unit factor of this semi-group.

By the Corollary of Theorem 1, we can easily see that the set $\mathfrak{S}_0$ of finite factors of type II is also a semi-group. By the preceding theorem, the unit of $\mathfrak{S}_0$, if it exists, does not have the property Γ since there exists a factor which does not have the propery Γ [6]. Especially the approximately finite factor can not be the unit of $\mathfrak{S}_0$ since it has the property Γ [6, Lemma 6.1.2].

3. The infinite direct product of finite factors. As an application of our theory, we shall consider a finite factor which is the infinite direct product of finite factors.

For this purpose, we shall give a brief consideration to the infinite direct product of factors. The theory of the infinite direct products of factors has been introduced by von Neumann [7] and recently Takeda [8] has generalized it. We shall employ a special kind of infinite direct product of finite factors.

Let M_n $(n = 1, 2, \dots.)$ be finite factors and τ_n $(n = 1, 2, \dots.)$ be their normalized traces. Let $\odot_n M_n$ be the algebraical infinite direct product of M_n and for any element

$$\sum_{j=1}^{n} \alpha_{1,j}x_{1,j} \otimes \alpha_{2,j}x_{2,j} \otimes \dots . \otimes \alpha_{i,j}x_{i,j} \otimes \dots .$$

($\alpha_{i,j}$'s are scalars, $x_{i,j} \in M_i$ and $\alpha_{i,j}x_{i,j}$'s are the identities except finite number of them) in $\odot_n M_n$, put

$$\tau\left(\sum_{j=1}^{n} \alpha_{1,j}x_{1,j} \otimes \dots . \otimes \alpha_{i,j}x_{i,j} \otimes \dots .\right) = \sum_{j=1}^{n}\prod_{i=1}^{\infty} \alpha_{i,j}\tau_i(x_{i,j}).$$

Then τ is well defined and a positive linear functional on the $*$-algebra $\odot_n M_n$. By the usual way, we can construct a Hilbert space $\mathfrak{H}$ by $\odot_n M_n$ and τ. Furthermore $\odot_n M_n$ can be considered as an operator algebra on $\mathfrak{H}$. By $\otimes_n M_n$, we denote the weak closure of $\odot_n M_n$ on $\mathfrak{H}$ and call it the *restricted infinite direct product* of M_n. As Takeda [8] has proved, $\otimes_n M_n$ is a finite factor.

THEOREM 3. *The restricted infinite direct product of finite factors has the property* Γ, *and its fundamental group contains all positive numbers.*

PROOF. Let M be the restricted infinite direct product of finite factors M_n. By Lemma 3, we can represent as

$$M_n = N_n \otimes P_n, \ \ n = 1, 2, \dots.,$$

where N_n are of type I and finite. Lemma 4 shows that every P_n is also finite. Accordingly, there are no difficulties to see that M is the direct product of two restricted infinite direct products of N_n and P_n, that is

$$M = N \otimes P$$

where $N = \otimes_n N_n$ and $P = \otimes_n P_n$. By Corollary 2.1 in [4], N is an approximately finite factor and hence its fundamental group contains all positive numbers [6; Lemma 4.8.4]. This fact and Lemma 7 show that the fundamental group of M contains all positive numbers.

An approximately finite factor has the property Γ and the remainder of the theorem is an immediate consequence of Theorem 4. This completes the proof.

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