IMBEDDING PARTLY ORDERED SETS INTO INFINITELY DISTRIBUTIVE COMPLETE LATTICES

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A partly ordered set P can be imbedded into complete lattices in various ways. The first fundamental theorem (Theorem 2) asserts that if P is imbedded in a complete lattice L J-densely and M-isomorphically (see the definitions (β) and (γ) in § 1) then L is completely isomorphic to the lattice completed by some imbedding operator on P. Imbedding operators on lattices have been discussed by several authors (see reference in [2]). Our definition of imbedding operators on partly ordered sets is a generalization of that on lattices. The second fundamental theorem (Theorem 3) gives a necessary and sufficient condition for the infinite distributivity of the lattice completed by an imbedding operator, which is a generalization of Dilworth and McLaughlin's theorem ([2] Theorem 3). Then we introduce a weak imbedding operator on partly ordered set, and give a necessary and sufficient condition for the infinite distributivity of the lattice completed by the induced imbedding operator.

Using these theorems we obtain some theorems. Among them the followings are typical: Any partly ordered set can be imbedded in an infinitely distributive complete lattice preserving all glb. and all distributive lub. Any infinitely distributive (non-complete) lattice can be imbedded in an infinitely distributive complete lattice preserving all lub. and all glb. (it was noted in [2] that the normal completion does not give the answer). Any upper continuous lattice can be imbedded in an infinitely distributive complete lattice preserving all glb. and all upper continuous limits.

In this paper $\bigcup$ ($\bigcap$) is used for lub. (glb.) in partly ordered sets. While $\vee$($\wedge$) is used for set union (intersection). The other notations are the same as in [1].

1. Preliminaries. A partly ordered set P is said to be imbedded in a complete L if there is a mapping θ, of P into L, which satisfies the condition

(α) *O-isomorphism*: $\theta(a) \geqq \theta(b)$ if and only if $a \geqq b$.

If P is imbedded in L by a mapping θ, then the collection of all elements of L which can be represented as joins of elements of $\theta(P)$ form a complete lattice L^* under the same ordering relation as L. P is imbedded in L^* by the same mapping θ. $\theta: P \rightarrow L^*$ satisfies

(β) *J-density*: any element a^* in L^* can be represented as a join of elements of $\theta(P)$, that is $a^* = \bigcup \theta(a_\lambda), a_\lambda \in P$. Here $\{\lambda\}$ may be any index set.

THEOREM 1. *If a partly ordered set P is imbedded in L J-densely, then* (α)

is equivalent to the condition (γ).

(γ) *M-isomorphism*: $\theta(a) = \bigcap \theta(a_\lambda)$ *in* L *if and only* $a = \bigcap a_\lambda$.

Here and hereafter $a = \bigcap (\bigcup) a_\lambda$ in partly ordered set P means that a is the glb. (lub.) of $\{a_\lambda\}$.

PROOF. Suppose that $a = \bigcap a_\lambda$, then $a \leqq a_\lambda$ for all λ. By O-isomorphism of θ this implies $\theta(a) \leqq \theta(a_\lambda)$ for all λ, that is $\theta(a) \leqq \bigcap \theta(a_\lambda)$. Let $a^* = \bigcap \theta(a_\lambda)$ then a^* is representable as $\bigcup \theta(b_\mu)$ by the J-density of θ. We have $\theta(a_\lambda) \geqq \theta(b_\mu)$ for all λ and μ, and then $a_\lambda \geqq b_\mu$ for all λ and μ. This implies $a = \bigcap a_\lambda \geqq b_\mu$ for all μ. Therefore $\theta(a) \geqq \theta(b_\mu)$ that is $\theta(a) \geqq \bigcup \theta(b_\mu) = a^*$. Thus we have $\theta(a) = \bigcap \theta(a_\lambda)$. That $\theta(a) = \bigcap \theta(a_\lambda)$ implies $a = \bigcap a_\lambda$ comes from the following lemma.

LEMMA 1. *If* P *is imbedded in* L *by a mapping* θ, *then* $\theta(a) = \bigcup \theta(a_\lambda)$ *impies* $a = \bigcup a_\lambda$, *and dually* $\theta(a) = \bigcap \theta(a_\lambda)$ *implies* $a = \bigcap a_\lambda$.

PROOF. $\theta(a) = \bigcup \theta(a_\lambda)$ implies $\theta(a) \geqq \theta(a_\lambda)$ and this implies $a \geqq a_\lambda$ for all λ by O-isomorphism of θ. Let b be any upper bound of $\{a_\lambda\}$, then $\theta(b) \geqq \theta(a_\lambda)$ for all λ, that is $\theta(b) \geqq \bigcup \theta(a_\lambda) = \theta(a)$, which implies $b \geqq a$. Thus we have proved that $a = \bigcup a_\lambda$.

2. Imbedding operators on partly ordered sets. Let P be a given partly ordered set. A mapping ϕ which maps subsets of P into subsets of P is called an *imbedding operator* on P if the following conditions are satisfied:

(2.1) $\phi(A) \supseteqq A$, (2.2) $A \supseteqq B$ implies $\phi(A) \supseteqq \phi(B)$, (2.3) $\phi(\phi(A)) = \phi(A)$, (2.4) $\phi(a) = (a]$, where $\phi(a)$ means $\phi(\{a\})$ and $(a] = \{x, x \in P, x \leqq a\}$, that is the principal ideal generated by a.

A is called ϕ-closed if $\phi(A) = A$. All the ϕ-closed sets form a complete lattice P_ϕ under set inclusion. P_ϕ is called the *completion* of P by the imbedding operator ϕ.

LEMMA 2. *Let* $\Omega = \{A_\lambda\}$ *be the collection of all* ϕ*-closed sets of* P *for an imbedding operator* ϕ *on* P. *Then* Ω *satisfies the following conditions.*

(2.5) *every* A_λ *is an ideal of* P, *i.e., if* $a \in A_\lambda$ *and* $x \leqq a$ *then* $x \in A_\lambda$, (2.6) *every principal ideal is a member of* Ω, (2.7) Ω *is M-complete, i.e., for any subset* $\{B_\mu\}$ *of* Ω, $\wedge B_\mu \in \Omega$, (2.8) $P \in \Omega$.

Conversely if a collection of subsets of $P, \Omega = \{A_\lambda\}$, *satisies these four conditions, then there exists an uniquely determined imbedding operator* ϕ *on* P *such that* Ω *is the collection of all* ϕ*-closed sets.*

It is to be noted that the null set may or may not be included in Ω. The proof of this lemma is simple and so omitted.

THEOREM 2. *Let* ϕ *be an imbedding operator on* P, *then* P *is imbedded in* P_ϕ *J-densely by the mapping* ϕ^* : $\phi^*(a) = (a]$. *Conversely if* P *is imbedded in* L *J-densely by a mapping* $\theta: P \rightarrow L$, *then there is an imbedding operator* ϕ *on* P *such that* $\phi^* = \psi\theta$, *where* $\psi: L \rightarrow P_\phi$ *is a complete isomorphism of* L *onto* P_ϕ.

PROOF. O-isomorphism of ϕ^* is clear. Let $A = \{a_\lambda\}$ be any element of

P_ϕ, then $A = \bigcup \phi^*(a_\lambda)$ as $A = \vee(a_\lambda]$. Thus P is imbedded in P_ϕ J-densely.

Conversely suppose that P is imbedded in a complete lattice L J-densely by a mapping θ. For every element $a^* \in L$ define $\psi(a^*) = \{x;\ x \in P,\ \theta(x) \leqq a^*\}$. Then $\Omega = \{\psi(a^*);\ a^* \in L\}$ satisfies all the conditions of Lemma 2. Thus there is an imbedding operator ϕ on P such that Ω is the collection of all ϕ-closed sets. Now we prove that ψ gives a complete isomorphism of L onto P_ϕ. By J-density of θ $\psi(a^*) \geqq \psi(b^*)$ if and only if $a^* \geqq b^*$. Thus ψ gives a one-one onto mapping of L onto P_ϕ, giving a complete isomorphism.

That $\phi^* = \psi\theta$ is clear from

$$\psi(\theta(a)) = \{x;\ x \in P,\ \theta(x) \leqq \theta(a)\} = \{x;\ x \in P, x \leqq a\}.$$

By this theorem if we intend to imbed P into complete lattices J-densely we may consider only imbedding operators on P.

3. A condition for the infinite distributivity of $\mathbf{P}_\phi$. In this section we give a necessary and sufficient condition for the infinite distributivity of P_ϕ. Let $S = \{s_\lambda\}$ be any subset of P and let x be any element of P. Define $x \cap S = \{y;\ y \in P,\ y \leqq x,\ y \leqq s_\lambda$ for some $s_\lambda \in S\}$. Thus $x \cap S$ is the intersection of $(x]$ and $(S]$, where $(S]$ is the ideal in P generated by S, i. e., $(S] = \{y;\ y \in P, y \leqq s_\lambda$ for some $s_\lambda \in S\}$.

LEMMA 3. *Let $\phi^*: P \to P_\phi$ be a mapping of P into P_ϕ induced by ϕ, then $\phi(S)$ (as an element of P_ϕ) $= \bigcup \phi^*(s_\lambda)$. Moreover if B is the set union of subsets B_μ's, then $\phi(B) = \bigcup \phi(B_\mu)$.*

LEMMA 4. *Under the same condition as in Lemma 3 $\phi(a \cap b) = \phi^*(a) \cap \phi^*(b)$.*

These lemmas are clear from the definition of ϕ and ϕ^*.

THEOREM 3. *For the infinite distributivity of P_ϕ it is necessary and sufficient that the following condition is satisfied:*

(δ) *$x \cap \phi(S) = \phi(x \cap S)$ for any element x of P and any subset $S = \{s_\lambda\}$ of P.*

A complete lattice is called *infinitely distributive* if $x \cap (\bigcup a_\lambda) = \bigcup (x \cap a_\lambda)$ holds for any element x and any subset $\{a_\lambda\}$ of the lattice.

PROOF. Necessity of the condition. Let us suppose that P_ϕ is infinitely distributive and let $\phi^*: P \to P_\phi$ be the induced mapping. Then

$$\begin{aligned} x \cap \phi(S) &= (x] \wedge \phi(S) \\ &= \phi^*(x) \cap (\bigcup \phi^*(s_\lambda)) && \text{by Lemma 3} \\ &= \bigcup \phi^*(x) \cap \phi^*(s_\lambda)) && \text{by the infinite distributivity of } P_\phi \\ &= \bigcup \phi^*(x \cap s_\lambda) && \text{by Lemma 4} \\ &= \phi(\mathrm{V}(x \cap s_\lambda)) && \text{by Lemma 3} \\ &= \phi(x \cap S). \end{aligned}$$

Sufficiency of the condition. For the proof it is sufficient to prove that $A \wedge (\phi(\vee B_\lambda)) = \phi\vee((A \wedge B_\lambda))$, where A and B_λ's are ϕ-closed sets. Put $B = \vee B_\lambda$, then the above equality becomes $A \wedge \phi(B) = \phi(A \wedge B)$. As $A \wedge \phi(B) \supseteqq \phi(A \wedge B)$ is clear it is sufficient to prove that $A \wedge \phi(B) \subseteqq \phi(A \wedge B)$. Let $x \in A$

$\wedge\ \phi(B)$, then

$$\begin{aligned}(x] &= (x] \wedge \phi(B)\\ &= \phi(x \cap B) \quad \text{by the condition } (\delta)\\ &\subseteqq \phi(A \wedge B) \quad \text{as } x \in A \text{ and } A \text{ and } B \text{ are ideals.}\end{aligned}$$

This implies that $x \in \phi(A \wedge B)$.

4. Weak imbedding operators. A mapping ω of subsets of P into subsets of P is called a weak imbedding operator if the following conditions are satisfied:

(4.1) $\omega(A) \supseteqq A$, (4.2) $A \supseteqq B$ implies $\omega(A) \supseteqq \omega(B)$, (4.3) $\omega(\{a\}) = \omega((a]) = (a]$.

Notice that the (4.3) is somewhat stronger than (2.4) and lacks the condition corresponding to (2.3).

Let us call a subset A is ω-closed if $\omega(A) = A$. Then the collection of all ω-closed sets satisfies the conditions of Lemma 2. Thus we have an imbedding operator $\bar{\omega}$ on P, which we call the imbedding operator induced by the weak imbedding operator ω.

For any subset A of P $\bar{\omega}(A)$, the least ω-closed set including A, is constructed transfinitely as follows. Put $\omega^1(A) = \omega(A)$, $\omega^2(A) = \omega(\omega^1(A)), \dots,$ if $\xi = \eta + 1$ put $\omega^\xi(A) = \omega(\omega^\eta(A))$, if ξ is a limit number put $\omega^\xi(A) = \vee_{\eta<\xi}\omega^\eta(A)$. Then for some ordinal number ξ_0 we have $\omega^{\xi_0}(A) = \omega^{\xi_0+1}(A)$. It is clear that $\bar{\omega}(A) = \omega_{\xi_0}(A)$.

LEMMA 5. *The following two conditions for the weak imbedding operator ω and the induced imbedding operator $\bar{\omega}$ on P are eqvivalent*:

(δ 1) $x \cap \omega(A) \subseteqq \bar{\omega}(x \cap A)$ *for any $x \in P$ and any subset A of P.*

(δ 2) $x \cap \bar{\omega}(A) \subseteqq \bar{\omega}(x \cap A)$ *for any $x \in P$ and for any any subset A of P.*

PROOF. Evidently (δ 2) implies (δ 1). Assume (δ 1), then by induction (δ 2) will be proved if $x \cap \omega^\xi(A) \subseteqq \bar{\omega}(x \cap A)$ is proved assuming $x \cap \omega^\eta(A) \subseteqq \bar{\omega}(x \cap A)$ for all $\eta < \xi$. If ξ is a limit number this condition is clearly satisfied. If $\xi = \eta + 1$

$$\begin{aligned}x \cap \omega^\xi(A) &= x \cap \omega(\omega^\eta(A))\\ &\subseteqq \bar{\omega}(x \cap \omega^\eta(A)) \quad \text{by } (\delta 1)\\ &\subseteqq \bar{\omega}(\bar{\omega}(x \cap A)) \quad \text{by the inductive assumption}\\ &= \bar{\omega}(x \cap A).\end{aligned}$$

By Theorem 3 and Lemma 5 we have

THEOREM 4. *Let ω be a weak imbedding operator on P. Then for the infinite distributivity of $P_{\bar{\omega}}$ it is necessary and sufficient that the condition* (δ 1) *is satisfied.*

We shall call an imbedding operator ϕ is *distributive* if P_ϕ is infinitely distributive. Now we cite here some applications of weak imbedding operators. Let ϕ be an imbedding operator on P and let $\mathfrak{S} = \{S_\lambda\}$ be the collection of subsets of P including all one element subsets. Let us define a weak imbedding operator $\phi_{\mathfrak{S}}$ associated with ϕ and $\mathfrak{S}$ as follows: $\phi_{\mathfrak{S}}(A) = \vee\, \phi(S_\mu)$ where set summation is taken under the conditions that $S_\mu \subseteqq A$ and $S_\mu \in \mathfrak{S}$. It is easy to show that $\phi_{\mathfrak{S}}$ is a weak imbedding operator on P.

THEOREM 5. *$P_{\phi_{\mathfrak{S}}}$ is infinitely distributive if and only if the following condition is satisfied.*

(δ 3) *$x \cap \phi(S) \subseteqq \overline{\phi}_{\mathfrak{S}}(x \cap S)$ for any element $x \in P$ and any $S \in \mathfrak{S}$.*

PROOF. Necessity of the condition follows from (δ 1) as $\phi(S) = \phi_{\mathfrak{S}}(S)$ for $S \in \mathfrak{S}$. For the proof of the sufficiency of the condition it is sufficient to prove that (δ 3) implies

(δ 3)′ *$x \cap \phi_{\mathfrak{S}}(A) \subseteqq \phi_{\mathfrak{S}}(x \cap A)$ for any $x \in P$ and any subset A of P.*

While

$$\begin{aligned} x \cap \phi_{\mathfrak{S}}(A) &= (x] \wedge [\vee\, \phi(S_\mu);\ S_\mu \subseteqq A,\ S_\mu \in \mathfrak{S}] \\ &= \vee\, [(x] \wedge \phi(S_\mu)] = \vee\, [x \cap \phi(S_\mu)] \\ &\subseteqq \cup\, \overline{\phi}_{\mathfrak{S}}(x \cap S_\mu) \qquad \text{by } (\delta\, 3) \\ &\subseteqq \overline{\phi}_{\mathfrak{S}}(x \cap A) \qquad \text{as } S_\mu \subset A. \end{aligned}$$

As it is clear that $\phi(A) \supset \overline{\phi}_{\mathfrak{S}}(A)$ for all A, and as $x \cap \phi(A) \supseteqq \phi(x \cap A)$ (δ 3) is equivalent to

(δ 4) $x \cap \phi(S) = \overline{\phi}_{\mathfrak{S}}(x \cap S)$ for any $S \in \mathfrak{S}$.

COROLLARY 4.1 *If ϕ is a distributive imbedding operator, then $\overline{\phi}_{\mathfrak{S}}$ is distributive if and only if*

(δ 5) *$\phi(x \cap S) = \overline{\phi}_{\mathfrak{S}}(x \cap S)$ for any $x \in P$ and any $S \in \mathfrak{S}$.*

COROLLARY 4.2 *If ϕ is a distributive imbedding operator and if $\mathfrak{S}$ satisfies the following condition, called meet completeness of $\mathfrak{S}$, then $\overline{\phi}_{\mathfrak{S}}$ is distributive.*

(4.4) *With any $S \in \mathfrak{S}$ and any $x \in P$ there is an $S' \in \mathfrak{S}$ such that the ideal generated by S' coincides with $x \cap S$.*

5. Imbedding a given partly ordered set into infinitely distributive complete lattices. Let P be a given partly ordered set. In P we call $a = \cup\, a_\lambda$ is a distributive join if any b ($\leqq a$) can be represented as $b = \cup\, b_\mu$, where every $b_\mu \leqq a_\lambda$ for some a_λ in $\{a_\lambda\}$.

THEOREM 6. *If P is imbedded in an infinitely distributive complete lattice L J-densely by a mapping θ, then $\theta(a) = \cup\, \theta(a_\lambda)$ implies that $a = \cup\, a_\lambda$ and this is a distributive join.*

PROOF. By Theorem 2 we may replace L by P_ϕ for some imbedding operator ϕ on P and θ by ϕ^*. It was proved in Lemma 1 that $\phi^*(a) = \cup\, \phi^*(a_\lambda)$ implies $a = \cup\, a_\lambda$. Now we prove the distributivity of $a = \cup\, a_\lambda$. Let b be any element such that $b \leqq a = \cup\, a_\lambda$. Put $A = \{a_\lambda\}$, then $(b] \subseteqq (a] = \phi(A)$. Then

$$\phi^*(b) = (b] = (b] \cap \phi(A)$$

$$\begin{aligned} &= b \cap \phi(A) && \\ &= \phi(b \cap A) && \text{by the distributivity of } \phi \\ &= \cap \phi^*(c_\nu), && \text{where } \{c_\nu\} = b \cap A. \end{aligned}$$

This implies that $b = \bigcup c_\nu$ and as $\{c_\nu\} = b \cap A$ we have every $c_\nu \leqq a_\lambda$ for some $a_\lambda \in A$.

By this theorem we see that when P is imbedded in an infinitely distributive complete lattice J-densely, the joins in P (i. e. lub.) which can be preserved are only distributive joins. While we can imbed P in an infinitely distributive complete lattice J-densely preserving all distributive joins, which we are now proving in the next.

A subset S of P is called δ-closed if the following condition is satisfied:

(5.1) If $a = \bigcup s_\lambda$ is a distributive join with $s_\lambda \in S$ for all λ and if $b \leqq a$, then $b \in S$.

The collection of all δ-closed sets satisfies the conditions of Lemma 2, as is easily seen, and so we have an imbedding operator δ on P. Now we prove that P_δ is infinitely distributive and the induced mapping δ^* preserves all the distributive joins.

Let A be any subset of P. Define $\omega(A) = \{x; x \leqq \bigcup a_\lambda,\ a_\lambda \in A,\ \bigcup a_\lambda$ is a distributive join$\}$, then ω is a weak imbedding operator. Clearly $\delta = \overline{\omega}$. For the infinite distributivity of P_δ it is sufficient to prove that $x \cap \omega(A) \subseteqq \delta(x \cap A)$ for any $x \in P$ and any subset A of P, by Theorem 5. If $y \in x \cap \omega(A)$, then $y \leqq x, y \leqq \bigcup a_\lambda$ for some distributive join $\bigcup a_\lambda$ with $a_\lambda \in A$. By the distributivity of $\bigcup a_\lambda$, y is represented as $y = \bigcup b_\mu$ where $\{b_\mu\} = y \cap A$. As $y \leqq x$ we have $y \cap A \subseteqq x \cap A$, and $\bigcup b_\mu$ is a distributive join as will be shown in the next.

LEMMA 6. *Let* $a = \bigcup a_\lambda$ *be a distributive join in* P *and let* $b \leqq a$, *then* $b = \bigcup b_\mu$, *where* $\{b_\mu\} = b \cap \{a_\lambda\}$, *is a distributive join.*

PROOF. Let $c \leqq b$, then $c \leqq a$. As $a = \bigcup a_\lambda$ is a distributive join c can be represented as $c = \bigcup c_\nu$, where $\{c_\nu\} = c \cap \{a_\lambda\}$. As $c \leqq b$ we have $\{c_\nu\} = c \cap \{a_\lambda\} \subseteqq b \cap \{a_\lambda\} = \{b_\mu\}$, and then we have every $c_\nu \leqq b_\mu$ for some μ in $\{\mu\}$.

By this lemma we have showed that $y \in x \cap \omega(A)$ implies that $y \in \delta(x \cap A)$.

THEOREM 7. *Any partly ordered set can be imbedded in an infinitely distributive complete lattice J-densely, preserving all the distributive joins (and only those joins).*

LEMMA 7. *When* P *is a lattice* L *the distributivity of* $a = \bigcup a_\lambda$ *is equivalent to the condition:*

(5.2) *for any* $x \in L$ $\bigcup(x \cap a_\lambda)$ *is defined and equal to* $x \cap (\bigcup a_\lambda)$.

PROOF. If $a = \bigcup a_\lambda$ is a distributive join in L, then $a \cap x \leqq a$ implies $a \cap x = \bigcup b_\mu$, where $\{b_\mu\} = (a \cap x) \cap \{a_\lambda\} = x \cap \{a_\lambda\}$. Then every $b_\mu \leqq x \cap a_\lambda$ for some λ and as every $x \cap a_\lambda \leqq x \cap a$ we have $x \cap a = \bigcup b_\mu = \bigcup (x \cap a_\lambda)$. Conversely if the conditon (5.2) is satisfied for $a = \bigcup a_\lambda$, then $b \leqq a$ implies

$b = a \cap b = \bigcup (b \cap a_\lambda)$ with $b \cap a_\lambda \leqq a_\lambda$. This shows the distributivity of $\bigcup a_\lambda$.

In a lattice L $a = \bigcup a_\lambda$ is called a *distributive join* if (5.2) is satisfied. Then by Theorem 7 we have

COROLLARY 7.1. *Any lattice can be imbedded J-densely in an infinitely distributive complete lattice preserving all distributive joins and only those joins.*

If in a lattice, not necessarily complete, all joins are distributive then the lattice is called *infinitely distributive.*

COROLLARY 7.2. *Any infinitely distributive lattice can be imbedded in an infinitely distributive complete lattice J-densely preserving all joins.*

COROLLARY 7.3. *Any upper continuous lattice can be imbedded in an infinitely distributive complete lattice J-densely, preserving all upper continuous limits.*

Here a lattice L (not necessarily complete) is called upper continuous if $a_\lambda \uparrow a$ implies $x \cap a_\lambda \uparrow x \cap a$ for every upper continuous limit $a_\lambda \uparrow a$ and any $x \in L$. The proof of this corollary follows from the fact that if $a_\lambda \uparrow a$ implies $x \cap a_\lambda \uparrow a \cap x$ for any $x \in L$ then $a = \bigcup a_\lambda$ is a distributive join.

6. Set of distributive imbedding operators. In this section we prove that the set of all distributive imbedding operators forms a complete sublattice of the (complete) lattice of all imbedding operators.

Let Π be the set of all imbedding operators on P. For any two imbedding operators ϕ and ψ define $\phi \geqq \psi$ if and only if

(6.1) $\phi(A) \supset \psi(A)$ for any subset A of P.

As is easily seen (6.1) is equivalent to the condition

(6.2) every ϕ-closed set is ψ-closed.

Π forms a complete lattice under the above defined ordering relation. The strongest imbedding operator in Π is the normal imbedding operator ν and the weakest imbedding operator is the ideal imbedding operator ι. Here ν-closed sets are normal ideals and ι-closed sets are ideals in P.

LEMMA 8. *Let $\{\phi_\lambda\}$ be a set of imbedding operators on P. then $\psi = \bigcap \phi_\lambda$ is defined as $\psi(A) = \wedge \phi_\lambda(A)$.*

PROOF. First ψ defined by the above definition is an imbedding operator: $\psi(A) \supseteqq A$, $A \supseteqq B$ implies $\psi(A) \supseteqq \psi(B)$, $\psi(a) = (a]$, $\psi(\psi(A)) = \wedge \phi_\lambda(\psi(A)) \subseteqq \wedge \phi_\lambda(A) = \psi(A)$ and as the reverse inclusion relation is clearly satisfied we have $\psi(\psi(A)) = \psi(A)$. It is clear that $\psi \leqq \phi_\lambda$ for all λ. Let ψ' be any imbedding operator on P such that $\psi' \leqq \phi_\lambda$ for all λ, then $\psi'(A) \subseteqq \wedge \phi_\lambda(A) = \psi(A)$. Thus we have $\psi = \bigcap \phi_\lambda$.

LEMMA 9. *Let $\{\phi_\lambda\}$ be a collection of imbedding operators on P, then $\omega(A) = \vee \phi_\lambda(A)$ is a weak imbedding operator on P and $\overline{\omega} = \bigcup \phi_\lambda$.*

PROOF. That ω is a weak imbedding operator is easily followed by the definition of $\omega(A)$. That $\bar{\omega} = \bigcup \phi_\lambda$ follows from the following.

LEMMA 10. *Let ϕ and ω be respectively an imbedding operator and a weak imbedding operator on P such that $\phi(A) \supseteqq \omega(A)$ for every subset A of P, then $\phi \geqq \bar{\omega}$.*

This follows from the definition of $\bar{\omega}$.

THEOREM 8. *All the distributive imbedding operators on P form a complete sublattice Π_d of Π.*

For the proof it is sufficient to prove that $\psi = \bigcap \phi_\lambda$ and $\bar{\omega} = \bigcup \phi_\lambda$ in Lemmas 8 and 9 are distributive assuming all the ϕ_λ's are distributive.

For ψ: $a \cap \psi(A) = (a] \wedge (\wedge \phi_\lambda(A)) = \wedge((a] \wedge \phi_\lambda(A))$
$= \wedge(\phi_\lambda(a \cap A))$ by the distributivity of ϕ_λ
$= \psi(a \cap A)$.

For ω: By Theorem 4 it is suufficient to prove that $a \cap \omega(A) \subseteqq \bar{\omega}(a \cap A)$.

$a \cap \omega(A) = (a] \wedge (\vee \phi_\lambda(A)) = \vee((a] \wedge \phi_\lambda(A))$
$= \vee \phi_\lambda(a \cup A)$ by the distributivity of ϕ_λ
$= \omega(a \cap A) \subseteqq \bar{\omega}(a \cap A)$.

7. Similar imbedding operators. Two imbedding operators ϕ and ψ on the same partly ordered set P are called similar if the induced mappings $\phi^*: P \to P_\phi$ and $\psi^*: P \to P_\psi$ preserves the same joins, i. e.

(7.1) $\phi^*(a) = \bigcup \phi^*(a_\lambda)$ if and only if $\psi^*(a) = \bigcup \psi^*(a_\lambda)$.

This condition is equivalent to the condition

(7.2) $(a] = \phi(A)$ if and only if $(a] = \psi(A)$, where $A = \{a_\lambda\}$.

Clearly similarity is an equivalence relation and each equivalence class is M-complete and convex in Π, that is if $\{\phi_\lambda\}$ is a set mutually similar imbedding operators then $\bigcap \phi_\lambda$ is similar to those operators and if $\phi \geqq \psi$, $\phi \sim \psi$ and if $\phi \geqq \theta \geqq \psi$ then θ is similar to $\phi(\psi)$.

THEOREM 9. *Any similar equivalence class contains at most one distributive imbedding operator.*

The proof of this theorem is rooted in the following

LEMMA 11. *If $\phi \geqq \psi$, $\phi \sim \psi$ and if ϕ is distributive, then $\phi = \psi$.*

PROOF. Let A be any subset of P and let $x \in \phi(A)$, then

$(x] = (x] \wedge \phi(A) = x \cap \phi(A)$
$= \phi(x \cap A)$ by the distributivity of ϕ
$= \psi(x \cap A)$ as $\phi \sim \psi$
$\subseteqq \psi(A)$.

This implies $x \in \psi(A)$. Thus we have $\phi(A) \subseteqq \psi(A)$, that is $\phi = \psi$ as the reverse inclusion relation is an assumption.

PROOF OF THEOREM 8. Let ϕ and ψ be two similar distributive imbedding operators. Then $\phi \cap \psi$ is similar to ϕ and ψ by M-completeness of similar class. By Lemma 11 $\phi = \phi \cap \psi = \psi$.

THEOREM 10. *The imbedding operator* δ *introduced in Theorem* 7 *is the strongest distributive imbedding operator.*

This fact comes from Theorem 6 and Theorem 9.

REFERENCES

[1] G. BIRKHOFF, Lattice theory, rev. ed., New York, 1949.
[2] R. P. DILWORTH AND J. E. MCLAUGHLIN, Distributivity in lattices, Duke Math. J., 19(1952), 683-693.