ON THE REDUCIBILITY OF AN AFFINELY CONNECTED MANIFOLD

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Introduction

G. de Rham [3] proved an interesting theorem concerning structures of simply-connected, complete and reducible Riemannian manifolds. In this paper I shall first attempt to extend his theorem to affinely connected manifolds. For this purpose I shall define R-reducible manifolds which are regarded as an extension of the notion of reducible Riemannian manifolds. For this manifold, I shall prove that the p-dimensional homotopy group of any of its maximal integral manifolds is isomorphic into the p-dimensional homotopy group of the given manifold under the homomorphism induced by the inclusion map. By virtue of this, it will be shown that a simply-connected R-reducible manifold is equivalent to an affine product. This is nothing but an extension of de Rham's theorem, as mentioned above. Secondly I shall determine structures of R-reducible manifolds whose fundamental groups are cyclic of order two, by the above theorem.

Throughout the whole discussion, I shall adopt the following conventions: I use the word "nbh" for neighborhood. I describe as a path (or a curve) what is usually called a segment of a path (or a curve), including the endpoints, and parameters of paths mean always affine parameters. If X is an affinely connected manifold, I describe as the covering space of X the universal covering space of X with the affine connection induced naturally from X by the covering. Let us suppose that the indices run as follows:

$$a, b, c, d = 1, 2, \ldots, r; \qquad i, j, k, l = r+1, r+2, \ldots, n;$$
$$\alpha, \beta, \gamma = 1, 2, \ldots, n.$$

I wish to note that integral manifolds R and S in this paper can not be intrinsically distinguished and lemmas etc. hold good though we exchange the roles of R and S there.

Furthermore, I wish to note that a part of the idea of this paper owes to A. G. Walker's paper [6] and to express my thanks to Professor S. Sasaki of Tôhoku Univ. for his kind assistance during the preparation of the manuscript.

1. R-reducible manifolds

Let M be an n-dimensional differentiable manifold (of class C^3) with an affine connection without torsion of class C^1 and we assume that M is affinely complete. For the definition of differentiable manifolds, see [5], p. 21, and note that M is connected, separable, and metric. The word "affinely complete" means that any straight line lying on the tangent space at any point $x \in M$ and passing through x can wholly be developed into M.

When the homogeneous holonomy group h at a point o of M fixes an r-dimensional plane T_0 and an $(n-r)$-dimensional plane T_0' complementary to T_0, then M is called a *completely reducible* or briefly, *C-reducible manifold.*

In a C-reducible manifold M, transplant the two planes T_0 and T_0' at 0 to every point $x \in M$ by parallel displacement along a curve $\overset{\frown}{ox}$ of class D^1 and denote the two planes thus obtained by T_x and T_x' respectively, then we get two parallel plane fields T_x, T_x' over M. When we attach at every point x of a coordinate nbh U a *suitable* frame $(e_1, \ldots, e_n)$ whose first r vectors $(e_1, \ldots, e_r)$ span T_x and the remaining $n-r$ vectors $(e_{r+1}, \ldots, e_n)$ span T_x', we may find Pfaffian forms ω^α (class C^1), ω_b^a and ω_j^i such that the connection of M is expressed by

$$dx = \omega^\alpha e_\alpha,$$

$$de_a = \omega_a^b e_b, \qquad de_i = \omega_i^j e_j.$$

As the connection is without torsion, we have

$$(\omega^a)' = [\omega^b \omega_b^a], \;\; (\omega^i)' = [\omega^j \omega_j^i]. \tag{1}$$

The plane field T_x' in U is defined by the system $\omega^a = 0$, and the field T_x by the system $\omega^i = 0$. Since these systems are completely integrable by (1), we may find their first integrals

$$x^{a'} = f^{a'}(x^\alpha), \qquad x^{i'} = f^{i'}(x^\alpha), \tag{2}$$

where we have denoted the coordinates in U by (x^α). As the Jacobian of (2) is not zero, we transform the coordinates (x^α) by (2). Then, for any point $x \in U$ we may find a suitable coordinate nbh V of x covered by the new coordinates $(x^{\alpha'})$. Such new coordinates are called *canonical coordinates* and a nbh covered by canonical coordinates is called a *canonical coordinate nbh.*

In every canonical coordinate nbh V with coordinates (x^α) (we omit "dashes")

$$x^i = \text{const. and } x^a = \text{const.}$$

define the r- and s-dimensional integral manifolds of the two fields T_x and T_x' respectively, where $s = n - r$. We can express the connection of M in terms of natural frames (e^α) in V by

$$dx = dx^\alpha e_\alpha, \qquad de_\alpha = \omega_\alpha^\beta e_\beta.$$

As the planes T_x and T_x' are parallel fields, we see $\omega_a^i = \omega_i^a = 0$, hence we get $\Gamma_{a\alpha}^i = \Gamma_{\alpha a}^i = \Gamma_{i\alpha}^a = \Gamma_{\alpha i}^a = 0$, where we have put $\omega_\alpha^\beta \equiv \Gamma_{\alpha\gamma}^\beta dx^\gamma$. Accordingly, among many components of $\Gamma_{\beta\gamma}^\alpha$, only Γ_{bc}^a and Γ_{ij}^k are non-trivial in general and usually they consist of functions of coordinates $(x^1, \ldots, x^n)$ (cf. [1]). If Γ_{bc}^a are functions of coordinates $(x^1, \ldots, x^r)$ only and Γ_{jk}^i are functions of coordinates $(x^{r+1}, \ldots, x^n)$ only, then M is called an *R-reducible manifold.*

Now, concerning integral manifolds in a C-reducible manifold M we have the following well-known properties (we do not give their proofs here):

a) *Through every point $x \in M$ there pass a pair of two r- and s-dimensional maximal integral manifolds* (cf. [2], p. 94). We shall consider each of them as a differentiable manifold with the system of coordinates and the affine connection induced naturally from a system of canonical coordinates and the affine connection in M. We denote the r- and s-dimensional manifolds by $R(x)$ and $S(x)$ respectively and sometimes we shall use abbreviated notations R and S for them.

b) *The intersection $R(x) \cap S(x)$ is at most countable* (cf. [2], p. 96).

c) *Any path of a maximal integral manifold, say $R(x)$, is a path of M too, and a path of M through x, whose tangent vector at x is contained in the tangent space of $R(x)$ at x, is contained in $R(x)$ and is a path of $R(x)$. Hence $R(x)$ is affinely complete.*

Under these premises we shall discuss structures of R-reducible manifolds.

2. Homotopy groups

DEFINITION 2.1. Let M be a C-reducible manifold. In a maximal integral manifold, say R, of M, a nbh of a point $x \in R$ is called an *intrinsic nbh* in R. In M, a canonical coordinate nbh whose coordinates consist only of all (x^α) satisfying the inequalities $a^\alpha < x^\alpha < b^\alpha$ (a^α, b^α are all const.) is called a *canonical cubic coordinate nbh* or briefly, a *C-nbh*.

DEFINITION 2.2. In an R-reducible manifold M, let U and V be intrinsic coordinate nbhs with coordinates (x^a) and (x^i) in two integral manifolds R and S of M respectively. Let $\Gamma^a_{bc}(x^a)$ and $\Gamma^i_{jk}(x^i)$ be the connection coefficients of R and S in U aud V respectively. Now consider the product $U \times V$ with coordinates (x^a, x^i). We endow the product $U \times V$ with the connection coefficients $\Gamma^\alpha_{\beta\gamma}(x^a, x^i)$ which satisfy the following relations: $\Gamma^a_{bc}(x^a, x^i) \equiv \Gamma^a_{bc}(x^a)$, $\Gamma^i_{jk}(x^a, x^i) \equiv \Gamma^i_{jk}(x^i)$ and the remaining $\Gamma^\alpha_{\beta\gamma}(x^a, x^i)$ are all zero. Then the product $U \times V$ is called the *affine product* of U and V. Moreover, when we cover the product $R \times S$ by a set of affine products $U \times V$, we get a differentiable manifold $R \times S$ with an affine connection. This is also called the *affine product* of R and S.

Let $C(x)$ be a C-nbh of a point x in an R-reducible manifold M. In integral manifolds $R(x)$ and $S(x)$, the connected components containing x of $C(x) \cap R(x)$ and $C(x) \cap S(x)$ are intrinsic nbhs and we denote them together with the coordinates induced naturally from those of $C(x)$ by $C(x)|R$ and $C(x)|S$ respectively. Then, $C(x)|R$ and $C(x)|S$ are intrinsic coordinate nbhs and the following lemma is evident:

LEMMA 2.1. *$C(x)$ is represented by the affine product of $C(x)|R$ and $C(x)|S$.*

Now, under the same notations it follows by applying Whitehead's theorem [7] to M that there exist simple convex intrinsic nbhs $U(x)$ and $V(x)$ of x such that $U(x) \subset C(x)|R$ and $V(x) \subset C(x)|S$, in the geometries of $R(x)$ and $S(x)$

respectively. The word "a simple convex nbh N" means a nbh such that any two points in N are joined by one and only one path which is wholly contained in N. Now, we consider $U(x)$ and $V(x)$ as intrinsic coordinate nbhs in $C(x)|R$ and $C(x)|S$ covering them respectively. Then, we have:

LEMMA 2.2. *Let $W(x)$ be the affine product $U(x) \times V(x)$, then $W(x) \subset C(x)$ and $W(x)$ is a simple convex nbh of M.*

PROOF. It is evident that $W(x) \subset C(x)$ from Lemma 2.1. We shall denote the coordinates of any two points $x_1, x_2 \in W(x)$ by $(x_1^\alpha), (x_2^\alpha)$ and express the unique path in $U(x)$ joining the point $(x_1^a, 0)$ to the point $(x_2^a, 0)$ by $x^a = x^a(t)$, $x^i = 0$ $(0 \leqq t \leqq 1)$ and similarly the unique in $V(x)$ joining the point $(0, x_1^i)$ to the point $(0, x_2^i)$ by $x^a = 0$, $x^i = x^i(t)$ $(0 \leqq t \leqq 1)$. Then it is easily seen that a curve

$$x^a = x^a(t),\ x^i = x^i(t)\ (0 \leqq t \leqq 1)$$

is the unique path in $W(x)$ joining x_1 to x_2.

DEFINITION 2.3. A nbh $W(x)$ such that we defined in Lemma 2.2 is called a *W-nbh* of x.

When a vector v at a point x of M is given, we shall denote by (x, v, c), where c is a constant, the terminal point y of the path obtained by developing the vector cv into M.

LEMMA 2.3. *Let v_0 be a vector tangent to $R(x_0)$ at a point x_0 of an R-reducible manifold M and $v(\tau)$ the vector field parallel to v_0 along a curve $x = x(\tau)$ $(0 \leqq \tau \leqq 1)$ of class C^1 in $S(x_0)$, where $x_0 \equiv x(0)$. Let $u(\tau)$ be a vector at $\sigma = c$ (constant), obtained by parallel displacement of $v(\tau)$ along a path $(x(\tau), v(\tau), \sigma)$ $(0 \leqq \sigma \leqq c)$. Put $y(\tau) \equiv (x(\tau), v(\tau), c)$ and $y_0 \equiv y(0)$. Then, for $0 \leqq \tau \leqq 1$, the following properties are fulfilled:*

a) *$y(\tau) \subset S(y_0)$ and $y(\tau)$ is of class C^1.* b) *$u(\tau)$ is a parallel vector field along the curve $y(\tau)$.* c) *If $x(\tau)$ is a path, so is $y(\tau)$.*

PROOF. A) We shall first prove the lemma in a C-nbh. Let the components of v_0 be $(v_0^a, 0)$ and the coordinates of x_0 be (x_0^a, x_0^i). Express the curve $x = x(\tau)$ by $x^a = x_0^a$, $x^i = x^i(\tau)$, where $x_0^i = x^i(0)$. Now consider *a* differential equations of parallel displacement

$$\frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}\, v^\beta \frac{dx^\gamma}{d\tau} = 0$$

along the curve $x = x(\tau)$. It turns into

$$\frac{dv^a}{d\tau} = 0, \quad \frac{dv^i}{d\tau} + \Gamma^i_{jk}\, v^j \frac{dx^k}{d\tau} = 0.$$

Solve them under the initial conditions $v^a = v_0^a$, $v^i = 0$ when $\tau = 0$, and we get $v^a = v_0^a$, $v^i = 0$. This is the parallel vector field $v(\tau)$. Again solve the differential equations

$$\frac{dx^\alpha}{d\sigma} = v^\alpha, \quad \frac{dv^\alpha}{d\sigma} = -\Gamma^\alpha_{\beta\gamma}\, v^\beta v^\gamma,$$

i.e., $$\frac{dx^a}{d\sigma} = v^a, \ \frac{dv^a}{d\sigma} = -\Gamma^a_{bc}\, v^b v^c;\ \frac{dx^i}{d\sigma} = v^i, \ \frac{dv^i}{d\sigma} = -\Gamma^i_{jk}\, v^j v^k,$$

under the initial conditions $x^a = x_0^a$, $v^a = v_0^a$; $x^i = x^i(\tau)$, $v^i = 0$ when $\sigma = 0$, and we get $x^a = x^a(\sigma, x_0^a, v_0^a)$, $v^a = v^a(\sigma, x_0^a, v_0^a)$, $x^i = x^i(\tau)$, $v^i = 0$. If we put $\sigma = c$ in this solution, we get $y(\tau)$, i. e., $x^a =$ const., $x^i = x^i(\tau)$ and $u(\tau)$, i. e., $v^a = const.$, $v^i = 0$. From these forms, the lemma is easily seen.

B) Next we shall prove our lemma in the large. Let $\widehat{x(\tau)y(\tau)}$ be the path $(x(\tau), v(\tau), \sigma)$ from $\sigma = 0$ to $\sigma = c$. When we cover the path $\widehat{x_0y_0}$ by a finite number of C-nbhs, it follows from A) that there exists $\delta_0 > 0$ such that this lemma holds good for the arc $x = x(\tau)$ $(0 \leqq \tau \leqq \delta_0)$. Now we suppose that the lemma holds good for the arc $x = x(\tau)$ $(0 \leqq \tau < \tau_0)$. Similarly by covering the path $\widehat{x(\tau_0)y(\tau_0)}$ by a finite number of C-nbhs, we see that there exists $\delta_1 > 0$ such that, for $\tau_0 - \delta_1 \leqq \tau \leqq \tau_0$, $y(\tau) \subset S(y(\tau_0 - \delta_1))$, $y(\tau)$ is of class C^1 and b) and c) of the lemma hold good. Hence we may see that the lemma holds good for a curve $x = x(\tau)$ $(0 \leqq \tau \leqq \tau_0)$ too. Summing up these fact, Lemma 2.3 is easily shown.

When X is an affinely connected manifold, we shall denote by $T_X(x)$ the affine space tangent to X at a point $x \in X$. Next, when the terminal point of a curve l_1 coincides with the initial point of another curve l_2, we shall denote by l_1l_2 the curve l_1 followed by l_2.

LEMMA 2.4. *Let C be a C-nbh of an R-reducible manifold M and l: $x^\alpha = x^\alpha(t)$ $(0 \leqq t \leqq 1)$ be a curve of class C^1 in C. Consider two curves l_1: $x^a = x^a(t)$, $x^i = x^i(0)$ and l_2: $x^a = x^a(1)$, $x^i = x^i(t)$ $(0 \leqq t \leqq 1)$, then the closed curve $l_1l_2l^{-1}$ gives rise to the unit element of the holonomy group H at $(x^\alpha(0))$.*

PROOF. Consider the differential equations of developement

$$\frac{dx}{dt} = \frac{dx^\alpha}{dt}e_\alpha, \quad \frac{de_a}{dt} = \Gamma_{ab}^c\frac{dx^b}{dt}e_c, \quad \frac{de_i}{dt} = \Gamma_{ij}^k\frac{dx^j}{dt}e_k \tag{3}$$

and put $x_0 \equiv (x^\alpha(0))$. Solve (3) in $T_M(x_0)$ along l under the initial conditions that x for $t = 0$ takes x_0 and e_α for $t = 0$ coincides with the natural frame $(e_{0\alpha})$ at x_0. We denote the solutions by $x(t)$ and $e_\alpha(t)$ and put $y \equiv x(1)$, $e_{1\alpha} \equiv e_\alpha(1)$.

Again solve (3) in $T_M(x_0)$ along l_1 under the same initial conditions for $t = 0$. We denote the solutions by $x'(t)$ and $e'_\alpha(t)$, then we get $(e'_\alpha(1)) = (e_{1a}, e_{0i})$ and put $y_1 \equiv x'(1)$. Under the above values y_1 and (e_{1a}, e_{0i}) as initial conditions for $t = 0$, solve (3) in $T_M(x_0)$ along l_2. We denote the solutions by $x''(t)$ and $e''_\alpha(t)$, then we get $e''_\alpha(1) = (e_{1a}, e_{1i})$ and put $y_2 \equiv x''(1)$. From this it follows directly that the closed curve $l_1l_2l^{-1}$ gives rise to the unit element of the homogeneous holonomy group h at x_0. On the other hand, we may find that y coincides with y_2. Hence Lemma 2.4 is proved.

We shall here give the following remarks: If l is a path, so are l_1 and l_2. The curve obtained by developing $l_1l_2l^{-1}$ is a triangle $x_0y_1y_2$. Vectors $\overrightarrow{x_0y_1}$ and $\overrightarrow{y_1y_2}$ are equal to the natural projections of a vector $\overrightarrow{x_0y_2}$ into $T_R(x_0)$ and $T_S(x_0)$ respectively.

DEFINITION 2.4. Suppose that through a point x_0 a path l_0 and a curve $\widehat{x_0x_1}$ of class D^1 are given in M. Let v_0 be the vector obtained by developing

l_0 into $T_M(x_0)$ and let v_1 be the vector at x_1, obtained by parallel displacement of v_0 along $\widehat{x_0x_1}$. Again let l_1 be the path obtained by developing v_1 into M. Then l_0 and l_1 are said to be *parallel* along the curve $\widehat{x_0x_1}$.

LEMMA 2.5. *Suppose that a map (not necessarily continuous) f of the square $\{(\sigma, \tau): 0 \leqq \sigma, \tau \leqq 1\}$ into an R-reducible manifold M satisfies the following conditions*:

1) $f(\sigma, 0)$ $(0 \leqq \sigma \leqq 1)$ *is of class* C^1 *and* $f(\sigma, 0) \subset R(o)$, *where* $o \equiv f(0, 0)$. 2) $f(0, \tau)$ $(0 \leqq \tau \leqq 1)$ *is a path and* $f(0, \tau) \subset S(o)$. 3) $f(c, \tau)$ $(0 \leqq \tau \leqq 1)$ *and the path* $f(0, \tau)$ *are parallel along* $f(\sigma, 0)$, *where c is an arbitrary constant.*

Then the following properties are fulfilled:

a) *The closed curve* $l_1 : f(\sigma, 0)f(1, \tau)f(\sigma, 1)^{-1}f(0, \tau)^{-1} (0 \leqq \sigma, \tau \leqq 1)$ *gives rise to the unit element of the holonomy group H at o.* b) *If $f(\sigma, 0)$ is a path, $f(\sigma, 0)$ and $f(\sigma, 1)$ are parallel along $f(0, \tau)$.*

Note that from Lemma 2.3, $f(\sigma, 1)$ $(0 \leqq \sigma \leqq 1)$ is of class C^1.

PROOF. Consider a closed curve

$$l_c : f(\sigma, 0)f(c, \tau)f(\sigma, 1)^{-1}f(0, \tau)^{-1} \ (0 \leqq \sigma \leqq c, 0 \leqq \tau \leqq 1).$$

Cover the path $f(0, \tau)$ by a finite number of C-nbhs. By making use of Lemma 2.3 and 2.4 for every C-nbh in turn. we understand easily that there exists $\delta_0 > 0$ such that a closed curve l_δ for any δ in $0 \leqq \delta \leqq \delta_0$ gives rise to the unit element of H. Now suppose that a closed curve $l_{c'}$ for any c' in $0 \leqq c' < c$ gives rise to the unit element of H. Similarly, cover the path $f(c, \tau)$ by a finite number of C-nbhs, then there exists $\delta_1 > 0$ such that a closed curve

$$l : f(\sigma_1, 0)f(c, \tau)f(\sigma_2, 1)^{-1}f(c - \delta_1, \tau)^{-1}f(\sigma_3, 0)^{-1}$$
$$(0 \leqq \sigma_1 \leqq c, \ c - \delta_1 \leqq \sigma_2 \leqq c, \ 0 \leqq \sigma_3 \leqq c - \delta_1, \ 0 \leqq \tau \leqq 1)$$

gives rise to the unit element of H. Hence it follows that the closed curve $ll_{c-\delta_1}$ i.e., l_c, gives rise to the unit element. Summing up these facts a) is easily proved. If $f(\sigma, 0)$ is a path, we get a parallelogram by developing l_1 into $T_M(o)$. Hence b) is also shown easily.

DEFINITION 2.5. When $x(t)$ $(0 \leqq t \leqq 1)$ is a curve in M on which points $x(t_\lambda)$ $(\lambda = 0, 1, \ldots, m;\ 0 \equiv t_0 < t_1 < \ldots < t_m \equiv 1)$ are specified and curves $x(t)$ $(t_{\nu-1} \leqq t \leqq t_\nu)$ $(\nu = 1, 2, \ldots, m)$ are all paths, then the curve $x(t)$ is called a *broken-path* and the points $x(t_\lambda)$ are called its *vertices*.

In an R-reducible manifold M, let $x'(t)$ be the broken-line obtained by developing a broken-path $x(t)$ of M into $T_M(x_0)$, where $x_0 \equiv x(0)$. Again let $y(t)$ be the broken-path obtained by developing into M the natural projection $y'(t)$ of $x'(t)$ into $T_S(x_0)$ (relative to $T_R(x_0)$), then $y(t) \subset S(x_0)$. In such a case, the broken-path $y(t)$ is called the *natural projection* of $x(t)$ into $S(x_0)$. Then we have:

LEMMA 2.6. *The point $y(t)$ lies on the integral manifold $R(x(t))$.*

PROOF. For $0 \equiv t_0 < t_1 < \ldots < t_k \equiv 1$ we may suppose that every curve $x(t)$ which corresponds to $t_{\nu-1} \leqq t \leqq t_\nu$ $(\nu = 1, \ldots, k)$ is a path contained in a W-nbh W_ν. Put $x_\nu \equiv x(t_\nu)$, $x'_\nu \equiv x'(t_\nu)$ and so on. Denote the path $x(t)$

$(t_{\nu-1} \leqq t \leqq t_\nu)$ by $\widehat{x_{\nu-1}x_\nu}$, the vector $x'(t)$ $(t_{\nu-1} \leqq t \leqq t_\nu)$ by $\overrightarrow{x'_{\nu-1}x'_\nu}$ and so on, where $x_0 = x'_0 = y'_0 = y_0$. In order to prove the lemma we shall make use of Lemmas 2.3, 2.4 and 2.5 repeatedly.

From $\widehat{x_0x_1} \subset W_1$, it follows that the natural projection $\widehat{y_0y_1}$ of $\widehat{x_0x_1}$ is also contained in W_1. Hence the lemma holds good for $t_0 \leqq t \leqq t_1$. Consider the path $\widehat{y_1x_1}$ in W_1. Take a point x_{12} such that $\widehat{x_1x_{12}}$ is parallel to $\widehat{y_1y_2}$ along $\widehat{y_1x_1}$. Develope the broken-path $\widehat{y_0y_1}\,\widehat{y_1x_1}\,\widehat{x_1x_{12}}$ into $T_M(x_0)$ and we denote the terminal point by x'_{12}. $\overrightarrow{x'_1x'_{12}}$ is equal to $\overrightarrow{y'_1y'_2}$, i. e., the natural projection of $\overrightarrow{x'_1x'_2}$ into $T_S(x_0)$. Hence $\widehat{x_1x_{12}}$ is the natural projection of $\widehat{x_1x_2}$ into $S(x_1)$, and $\widehat{x_1x_{12}} \subset W_2$. Consequently we may show that the lemma holds good for $t_1 \leqq t \leqq t_2$. Next let $\widehat{y_2x_{12}}$ be the path parallel to $\widehat{y_1x_1}$ along $\widehat{y_1y_2}$, and $\widehat{x_{12}x_2}$ be the path in W_2. Then the closed broken-path $\widehat{x_0x_1}\,\widehat{x_1x_2}\,\widehat{x_2x_{12}}\,\widehat{x_{12}y_2}\,\widehat{y_2y_1}\,\widehat{y_1y_0}$ gives rise to the unit element of the holonomy group H at x_0. Take a point x_{23} such that $\widehat{x_2x_{23}}$ is parallel to $\widehat{y_2y_3}$ along the broken-path $\widehat{y_2x_{12}}\,\widehat{x_{12}x_2}$. In the same manner as above, $\widehat{x_2x_{23}}$ is the natural projection of $\widehat{x_2x_3}$ into $S(x_2)$ and $\widehat{x_2x_{23}} \subset W_3$. Hence the lemma holds good for $t_2 \leqq t \leqq t_3$. If we continue this manner, it is evident that Lemma 2.6 is proved.

Let E be the p-cube consisting of points $(t_1, \ldots, t_p)$ in the p-dimensional Euclidean space E^p such that $0 \leqq t_\nu \leqq 1$ $(\nu = 1, \ldots, p)$. In particular, the $(p-1)$-faces defined by $t_p = 0$ and $t_p = 1$ in E are denoted by E_0 and E_1 respectively.

LEMMA 2.7. *Let U be a simple convex nbh of M. Suppose that a map ϕ of E into U satisfies the following conditions:*

1) *ϕ is continuous in $E_0 \cup E_1$.* 2) *When $t_1, \ldots, t_{p-1}$ are regarded as constants, $\phi(t_1, \ldots, t_p)$ $(0 \leqq t_p \leqq 1)$ defines a path. Then, ϕ is a continuous map.*

Since this follows from the theory of differential equations, we do not give its proof here.

LEMMA 2.8. *Suppose that a continuous map ϕ of E into an R-reducible manifold M satisfies the following conditions:* 1) *$\phi(E_0) = x_0$, where x_0 is a fixed point.* 2) *When $t_1, \ldots, t_{p-1}$ are regarded as constants, $\phi(t_1, \ldots, t_p)$ $(0 \leqq t_p \leqq 1)$ is a broken-path which we denote by $\phi_{t_1\ldots t_{p-1}}(t_p)$.* 3) *Vertices of $\phi_{t_1\ldots t_{p-1}}(t_p)$ consist only of points corresponding to $t_p = 0,\ 1/m, \ldots, (m-1)/m,\ 1$.*

Then there exists a continuous map ψ: $E \to S(x_0)$ for which the following properties are fulfilled:

a) $\psi(E_0) = \phi(E_0) = x_0$. b) *Two points, $\psi(t_1, \ldots, t_p)$ and $\phi(t_1, \ldots, t_p)$ for the same value $(t_1, \ldots, t_p)$ lie always on the same integral manifold R.* c) *For $(t_1, \ldots, t_p)$ such that $\phi_{t_1\ldots t_{p-1}}(t_p)$ $(0 \leqq t_p \leqq 1)$ is contained in $S(x_0)$, $\psi(t_1, \ldots, t_p) = \phi(t_1, \ldots, t_p)$.*

PROOF. Let $\phi'_{t_1\ldots t_{p-1}}(t_p)$ be the development of a broken-path $\phi_{t_1\ldots t_{p-1}}(t_p)$ into $T_M(x_0)$. Now consider the map

$$\phi' : E \to T_M(x_0) \qquad ((t_1, \ldots, t_p) \to \phi'_{t_1\ldots t_{p-1}}(t_p)),$$

then we get

$$\phi'(t_1, \ldots, t_p) = (mt_p - \lambda)\phi'(t_1, \ldots, t_{p-1}, (\lambda+1)/m) + (\lambda + 1 - mt_p)\phi'(t_1, \ldots, t_{p-1}, \lambda/m) \tag{4}$$

for $\lambda/m \leqq t_p \leqq (\lambda+1)/m$ ($\lambda = 0, 1, \ldots.. m-1$). From the continuity of ϕ we have

$$\phi_{t_1+\Delta t_1 \ldots t_{p-1}+\Delta t_{p-1}}(t_p) \to \phi_{t_1 \ldots t_{p-1}}(t_p) \ (\Delta t_\nu \to 0;\ \nu = 1, \ldots, p-1).$$

Hence, $\quad \phi'_{t_1+\Delta t_1, \ldots, t_{p-1}+\Delta t_{p-1}}(\lambda/m) \to \phi'_{t_1 \ldots t_{p-1}}(\lambda/m)$ ($\lambda = 0. 1, \ldots, m$),

i. e., $\quad \phi'(t_1 + \Delta t_1, \ldots, t_{p-1} + \Delta t_{p-1}, \lambda/m) \to \phi'(t_1, \ldots, t_{p-1}, \lambda/m)$.

Consequently $\phi'(t_1, \ldots, t_{p-1}, \lambda/m)$ is continuous. From this and (4), $\phi'(t_1, \ldots, t_p)$ is also continuous.

Next, let $\psi'_{t_1, \ldots t_{p-1}}(t_p)$ be the natural projection of a broken-line $\phi'_{t_1 \ldots t_{p-1}}(t_p)$ into $T_S(x_0)$. Then it follows directly that the map

$$\psi' : E \to T_S(x_0) \quad ((t_1, \ldots, t_p) \to \psi'_{t_1 \ldots t_{p-1}}(t_p))$$

is continuous. Again let $\psi_{t_1 \ldots t_{p-1}}(t_p)$ be the development of a broken-line $\psi'_{t_1 \ldots t_{p-1}}(t_p)$ into $S(x_0)$. Consider the map

$$\psi : E \to S(x_0) \qquad ((t_1, \ldots, t_p) \to \psi_{t_1 \ldots t_{p-1}}(t_p)).$$

By the similar manner, it is possible to deduce that the map ψ is continuous. It follows directly that ψ satisfies a) and c), and b) holds good by virtue of Lemma 2. 6.

THEOREM 1. *Let f be a continuous map of the boundary ∂E of E into a maximal intergal manifold, say S, of an R-reducible manifold M. If f is homotopic in M to a constant map, then it is homotopic in S to a constant map.*

PROOF. We shall suppose $f(E_0) = x_0$ and $f(E_1) = x_1$, where $x_0, x_1 \in S$. This assumption does not lose its generality of our theorem. Since M has a metric independent of the connection, we denote the distance between x and y by $d(x, y)$. From the given conditions, we may extend the map f to a continuous map $E \to M$ and denote such a map again by f. Put $D \equiv f(E)$, then D is a compact subset of M. Next, in a nbh $W(x)$ at a point x there exists always the greatest positive number (or infinity) δ such that $W(x) \supset \{y : d(x, y) < \delta\}$. δ is called the *radius* of $W(x)$.

Choose at every point x of D a W-nbh of x such that the greatest lower bound of these radii is a positive number. This is possible because D is compact. We denote the W-nbh by $W(x)$ and the greatest lower bound by δ_0. Once more, choose at every point x of D a W-nbh of x, contained in a nbh $\{y : d(x, y) < \delta_0/2\}$, such that the greatest lower bound of these radii takes a positive number. This is also possible and we denote the W-nbh by $w(x)$ and the greatest lower bound by δ_1. Next, at a point t of E, when there exists the greatest p-cube with the center t, whose $(p-1)$-faces are respectively parallel to those of E and its interior is wholly contained in $f^{-1}(w(f(t)) \cap D) \cup (E^p - E)$, we denote the length of the side by $\rho(t)$. If the p-cube does not exist, put $\rho(t) \equiv 2$. Then it follows easily that the greatest lower bound ρ_0 of $\rho(t)$ for all $t \in E$ is a positive number.

Moreover, take a positive integer m such that $1/m < \rho_0$ and divide E into m^p p-cubes, whose sides are of the same length $1/m$ and their faces are respectively parallel to those of E. We call every one of the p-cubes a *small p-cube* and its $(p-1)$-faces *small $(p-1)$-faces*. We denote by $A_{q_1\dots q_p}$ a small *p-cube* i. e., the set of points $(t_1, \dots, t_p)$ satisfying $q_\nu/m \leqq t_\nu \leqq (q_\nu+1)/m$ $(\nu = 1, \dots, p;\ q_\nu = 0, 1, \dots, m-1)$, and by $o_{q_1\dots q_p}$ its center. Put $x_{q_1\dots q_p} \equiv f(o_{q_1\dots q_p})$, then we have

$$f(A_{q_1\dots q_p}) \subset w(x_{q_1\dots q_p}).$$

In two small $(p-1)$-faces $t_p = q_p/m$ and $t_p = (q_p+1)/m$ of $A_{q_1\dots q_p}$, take points $(t_1, \dots, t_{p-1}, q_p/m)$ and $(t_1, \dots, t_{p-1}, (q_p+1)/m)$ respectively and consider in $w(x_{q_1\dots q_p})$ only one path $l(t_1, \dots, t_p)$ with the parameter t_p $(q_p/m \leqq t_p \leqq (q_p+1)/m)$, joining a point $f(t_1 \dots, t_{p-1}, q_p/m)$ to a point $f(t_1, \dots, t_{p-1}, (q_p+1)/m)$. Then from Lemma 2.7 we get a continuous map

$$\phi_{q_1\dots q_p}: A_{q_1\dots q_p} \to w(x_{q_1}\dots{}_{q_p}) \quad ((t_1, \dots, t_p) \to l(t_1, \dots, t_p)). \tag{5}$$

Choose another small p-cube A' whose t_p-coordinates satisfy $q_p/m \leqq t_p \leqq (q_p+1)/m$ and suppose $N \equiv A' \cap A_{q_1\dots q_p} \neq 0$. We denote any point of N by $(t'_1, \dots, t'_{p-1}, t_p)$ and put $w' \equiv w(f(o'))$, where o' is the center A'. Let l and l' be two paths joining a point $f(t'_1, \dots, t'_{p-1}, q_p/m)$ to another point $f(t'_1, \dots, t'_{p-1}, (q_p+1)/m)$ in $w(x_{q_1\dots q_p})$ and w' respectively. Let y_0 be a point of $f(N)$ and y be an arbitrary point of $w(x_{q_1\dots q_p})$ then

$$d(y, y_0) \leqq d(y, x_{q_1\dots q_p}) + d(x_{q_1\dots q_p}, y_0) < \delta_0/2 + \delta_0/2 = \delta_0.$$

Hence, $\qquad w(x_{q_1\dots q_p}) \subset W(y_0).$

Similarly, $\qquad w' \subset W(y_0).$

However, since $W(y_0)$ is a simple convex nbh, we have $l = l'$. Consequently if $\phi': A' \to w'$ is the continuous map analogus to (5) and t is any point of N, we get

$$\phi_{q_1\dots q_p}(t) = \phi'(t).$$

From this and (5), we get a continuous map ϕ_{q_p} of the part $\{(t_1, \dots, t_p): q_p/m \leqq t_p \leqq (q_p+1)/m\}$ of E into M, regarded as the union of maps $\phi_{q_1\dots q_p}$ with $q_p = \text{const}$. Then, we have $\phi_{q_p}(t_1, \dots, t_{p-1}, q_p/m) = f(t_1, \dots, t_{p-1}, q_p/m)$ and $\phi_{q_p}(t_1, \dots, t_{p-1}, (q_p+1)/m) = f(t_1, \dots, t_{p-1}, (q_p+1)/m)$ for $0 \leqq t_1, \dots, t_{p-1} \leqq 1$.

Again if we make the map $\phi: E \to M$ as the union of maps $\phi_{q_p}(q_p = 0, 1, \dots, m-1)$, ϕ is evidently continuous and satisfies

$$f(t_1, \dots, t_{p-1}, \lambda/m) = \phi(t_1, \dots, t_{p-1}, \lambda/m) \quad (\lambda = 0, 1, \dots, m). \tag{6}$$

In the next place, we take a small $(p-1)$-face contained in ∂E, such that $q_p/m \leqq t_p \leqq (q_p+1)/m$, for example $B_{q_2\dots q_p} \equiv \{(0, t_2, \dots, t_p): q_2/m \leqq t_2 \leqq (q_2+1)/m, \dots, q_p/m \leqq t_p \leqq (q_p+1)/m\}$. $B_{q_2\dots q_p}$ is a small $(p-1)$-face of $A_{0q_2\dots q_p}$. Now we have $f(B_{q_2\dots q_p}) \subset S$ from the assumption of f. On the other hand both $f(B_{q_2\dots q_p})$ and $\phi(B_{q_2\dots q_p}) \subset w(x_{0q_2\dots q_p})$. Consequently $f(B_{q_2\dots q_p})$ and $\phi(B_{q_2\dots q_p})$ are contained in a simple convex intrinsic nbh V, i. e. a connected component of $w(x_{0q_2\dots q_p}) \cap S$ in S, by virtue of Definition 2.3 and (6). Take any point $(0, t_2, \dots, t_p) \in B_{q_2\dots q_p}$ and make in $w(x_{0q_2\dots q_p})$ a path $l(0, t_2, \dots, t_p, \tau)$

$(0 \leqq \tau \leqq 1)$ such that $l(0, t_2, \ldots, t_p, 0) = f(0, t_2, \ldots, t_p)$ and $l(0, t_2, \ldots, t_p, 1) = \phi(0, t_2, \ldots, t_p)$. From Lemma 2.7 we get a continuous map

$$l_{q_2 \cdots q_p}: B_{q_2 \cdots q_p} \times I \to V \qquad ((0, t_2, \ldots, t_p, \tau) \to l(0, t_2, \ldots, t_p, \tau)), \tag{7}$$

where $I = \{\tau: 0 \leqq \tau \leqq 1\}$. Again choose another small $(p-1)$-face B, contained in ∂E, such that $q_p/m \leqq t_p \leqq (q_{p+1})/m$ and $B \cap B_{q_2 \cdots q_p} \neq 0$. Let A be the small p-cube containing B and o be the center of A. Put $w = w(f(o))$ and let $(0, t_2, \ldots, t_p)$ be a point of $B_{q_2 \cdots q_p} \cap B$. Let l and l' be paths joining $f(0, t_2, \ldots, t_p)$ to $\phi(0, t_2, \ldots, t_p)$ in $w(x_{0 q_2 \cdots q_p})$ and w respectively. Then we get $l = l'$, because $w(x_{0 q_2 \cdots q_p})$ and w are contained in a W-nbh. Here we note that, if $A = A_{0 q_2 \cdots q_p}$, we have $l = l'$ directly. Then, as the union of maps (7) of all small $(p-1)$-faces is contained in the part $(\partial E)_{q_p}$ of ∂E such that $q_p/m \leqq t_p \leqq (q_p + 1)/m$, we have a continuous map

$$l_{q_p}: (\partial E)_{q_p} \times I \to S,$$

where l_{q_p} in $t_p = q_p/m$ and $t_p = (q_p + 1)/m$ is independent of τ from (6), $l_{q_p} = f$ in $\tau = 0$ and $l_{q_p} = \phi$ in $\tau = 1$. Consequently we have a continuous map

$$g: \partial E \times I \to S \tag{8}$$

by making the union of maps l_{q_p} $(q_p = 0, 1, \ldots, m-1)$. g satisfies $g(E_0 \times I) = x_0$, $g(E_1 \times I) = x_1$ and $g(t \times 0) = f(t)$, $g(t \times 1) = \phi(t)$ for $t \in \partial E$. From (8), $f|\partial E$ is homotopic to $\phi|\partial E$ in S, leaving x_0 and x_1 fixed.

Hence it is sufficient to show that $\phi|\partial E$ is homotopic to a constant map in S. In fact the continuous map $\phi: E \to M$ satisfies wholly the conditions of Lemma 2.8. Moreover $\phi(E_1) = x_1 \in S$ and $\phi(\partial E) \subset S$. Hence we have the continuous map

$$\psi: E \to S. \tag{9}$$

For any point $t \in \partial E - E_1 + \partial E_1$, $\psi(t) = \phi(t)$, hence $\psi(\partial E_1) = x_1$. On the other hand, $\psi(E_1) \subset R(x_1)$, hence $\psi(E_1) \subset S \cap R(x_1)$. Consequently $\psi(E_1) = x_1$, from b) of § 1. Since we have $\psi(t) = \phi(t)$ for $t \in \partial E$, it follows from (9) that $\phi|\partial E$ is homotopic in S to a constant map.

COROLLARY. *The p-dimensional homotopy group of any maximal integral manifold of an R-reducible manifold M is isomorphic into the p-dimensional homotopy group of M under the homomorphism induced by the inclusion map.*

PROOF. We shall attempt the proof with respect to an integral manifold S. Consider the inclusion map $i: S \to M$ and we get the homomorphism $i_*: \pi_p(S) \to \pi_p(M)$ induced by i. Let N be the kernel of i_*. Since any element of N is mapped to the unit element of $\pi_p(M)$ under i_*, N is of the unit element of $\pi_p(S)$ from Theorem 1. Consequently our Corollary is proved.

3. Simply-connected R-reducible manifolds

S. Sasaki [4] proved that any two points of M cannot necessarily be joined by a path, but we have:

LEMMA 3.1. *Any two points x and y of M can be joined by a broken-path.*

PROOF. Consider a curve l joining x and y and cover l by a finite number of simple convex nbhs. Then we can make a broken-path joining x and y.

LEMMA 3.2. *Let x and y be any two points of an R-reducible manifold M, then $R(x)\cap S(y) \neq 0$.*

This is evident from Lemmas 2.6 and 3.1.

DEFINITION 3.1. Let $v(x)$ be a vector field over an integral manifold S of a C-reducible manifold M, where $x\in S$. If $v(x_1)$ and $v(x_2)$ at any two points x_1 and x_2 are parallel regardless of curves $\widehat{x_1x_2}$ of class D^1 in S, $v(x)$ is called a *parallel vector field* over S.

LEMMA 3.3. *Let v_0 be a vector at x_0, tangent to $R(x_0)$ of an R-reducible manifold M. When $S(x_0)$ is simply-connected, there exists a vector field $v(x)$ over $S(x_0)$ parallel to v_0, where $x\in S(x_0)$.*

PROOF. Consider a closed curve l of class D^1, with the endpoint x_0 in $S(x_0)$ and let v_1 be the vector at the terminal point x_0, obtained by parallel displacement of v_0 along l. From the proof of Lemma 2.3, it follows that v_1 is tangent to $R(x_0)$. Suppose $v_1 \neq v_0$. Then there exists $c>0$ such that $y_0 \neq y_1$, where $y_0 \equiv (x_0, v_0, c)$ and $y_1 \equiv (x_0, v_1, c)$. From Lemma 2.3, $y_0, y_1 \in R(x_0)\cap S(y_0)$. Contract l to x_0 and we get a curve $\widehat{y_1y_0}$ as the locus of y_1. Here $\widehat{y_1y_0}\subset R(x_0)\cap S(y_0)$. This is contradictory to b) of §1. Hence $v_1 = v_0$. From this, Lemma 3.3 is easily shown.

LEMMA 3.4 *Under the same assumption and notations as Lemma 3.3 put $y\equiv(x, v(x), c)$ and $y_0 \equiv (x_0, v_0, c)$, where c is a constant. If $S(y_0)$ is simply-connected too, $S(x_0)$ is equivalent to $S(y_0)$ under the map*

$$f:\ S(x_0)\to S(y_0) \qquad (x\to y).$$

The word "equivalent" in such a case means the equivalence as affinely connected manifolds.

PROOF. Let $u(y)$ be the vector at y, obtained by parallel displacement of $v(x)$ along a path $(x, v(x), t)$ $(0\leqq t\leqq c)$. For two distinct points x_1 and x_2 in $S(x_0)$, y_1 and y_2 are also distinct, where $y_1 \equiv (x_1, v(x_1), c)$ etc. In fact, if $y_1 = y_2$, we have the closed curve l in $S(y_0)$ as the image under f of a curve $\widehat{x_1x_2}$ of class D^1 in $S(x_0)$. From Lemma 2.3 $u(y_1)$ and $u(y_2)$ are parallel along l. However since $S(y_0)$ is simply-connected, $u(y_1) = u(y_2)$ by virtue of Lemma 3.3. Hence we get $x_1 = x_2$, because x_1 and x_2 are represented as $(y_1, u(y_1), -c)$. This is contradictory to the fact that x_1 and x_2 are distinct points. Consequently, when we put $S' \equiv f(S(x_0))$, then $S(x_0)$ and S' correspond one-to-one under f to each other, where $S'\subset S(y_0)$. Moreover $S(x_0)$ and S' are equivalent under f. In fact if we cover a path $\widehat{xy} = (x, v(x), t)$ $(0\leqq t\leqq c)$ by a finite number of C-nbhs, we get in $S(x_0)$ and S' two intrinsic nbhs of x and y respectively, equivalent under f. From this fact the equivalence of $S(x_0)$ and S' is easily shown.

Hence it is sufficient to show $S(y_0) = S'$. Take a point $y_1\in S(y_0)$ and make a curve $\widehat{y_0y_1}$ of class D^1 in $S(y_0)$. We get a vector $u(y_1)$ at y_1, by parallel displacement of $u(y_0)$ along $\widehat{y_0y_1}$. Put $x_1 \equiv (y_1, u(y_1), -c)$. From Lemma 2.3,

$x_1 \in S(x_0)$ and $(x_1, v(x_1), c) = y_1$. Hence $S(y_0) = S'$.

DEFINITION 3.2. In an R-reducible manifold M, let l_1 be a broken-path $\widehat{x_0x_1}\,\widehat{x_1x_2}\dots\widehat{x_{h-1}x_h}$ in $R(x_0)$ with the vertices $x_0, x_1, \dots, x_h$ and let l be a curve $\widehat{x_0y_0}$ of class D^1 in $S(x_0)$. First displace $\widehat{x_0x_1}$ parallelly along l, and we get a path $\widehat{y_0y_1}$ at y_0 and a curve $\widehat{x_1y_1}$ as the locus of x_1. Again displace $\widehat{x_1x_2}$ parallelly along $\widehat{x_1y_1}$, and we get a path $\widehat{y_1y_2}$ at y_1 and a curve $\widehat{x_2y_2}$ as the locus of x_2. Continuing this process successively, we get a broken-path $\widehat{y_0y_1}\,\widehat{y_1y_2}\dots\widehat{y_{h-1}y_h}$ and a curve $\widehat{x_hy_h}$. The broken-path $\widehat{y_0y_1}\,\widehat{y_1y_2}\dots\widehat{y_{h-1}y_h}$ is called to be *parallel* to l_1 along l.

It follows that the broken-path $\widehat{y_0y_1}\,\widehat{y_1y_2}\dots\widehat{y_{h-1}y_h}$ coincides with the development of the broken-line at y_0 parallel to the development of the given broken-path l_1 and $\widehat{x_\nu y_\nu} \subset S(x_\nu)$ $(\nu = 1, 2, \dots, h)$ from Lemmas 2.3 and 2.5. Moreover when l is a broken-path, the curve $\widehat{x_hy_h}$ coincides with the broken-path obtained by parallel displacement of l along l_1.

LEMMA 3.5. *When all S of an R-reducible manifold M are simply connected and a broken-path $\widehat{x_0y_0}$ of M is given in $R(x_0)$, we have*: a) *There exists over $S(x_0)$ a broken-path field parallel to $\widehat{x_0y_0}$.* b) *If y is the terminal point of its broken-path at any point x of $S(x_0)$, $S(x_0)$ and $S(y_0)$ are equivalent under the map*

$$f:\ S(x_0) \to S(y_0) \qquad (x \to y).$$

This is obvious from Lemmas 3.3 and 3.4.

DEFINITION 3.3. We call such a map f as is defined in Lemma 3.5 an *equivalent map with respect to a broken-path $\widehat{x_0y_0}$.*

LEMMA 3.6. *Suppose that all R and S of an R-reducible manifold M are simply-connected and g_0 and h_0 are any two broken-paths in $R(x_0)$ joining x_0 to y_0. Then the equivaleut map with respect to g_0 coincides with the one with respect to h_0.*

PROOF. Let $g(x)$ and $h(x)$ be two broken-path fields over $S(x_0)$ parallel to g_0 and h_0 respectively, where $x \in S(x_0)$. It is sufficient to show that the terminal point y_1 of an element $g(x_1)$ coincides with the terminal point y_2 of an element $h(x_1)$. If we consider a broken-path $\widehat{x_0x_1}$, y_1 is also regarded as the terminal point of the broken-path at y_0, parallel to $\widehat{x_0x_1}$ along g_0 and so is y_2, along h_0. Since any R is simply-connected, y_1 coincides with y_2 from Lemma 3.5.

THEOREM 2. *When all R and S of an R-reducible manifold M are simply-connected, the affine product $\widetilde{M} \equiv R(o) \times S(o)$, $o \in M$, is equivalent to the covering space of M.*

PROOF. We put $R_0 \equiv R(o)$ and $S_0 \equiv S(o)$. A point $\tilde{x}$ of $\widetilde{M}$ is always represented by (y, z), where $y \in R_0$ and $z \in S_0$. Let $\widehat{oy}$ be a broken path in R_0 joining o to y and $\widehat{oz}$ a broken-path in S_0 joining o to z. Let x be the terminal point of the broken-path at z, obtained by parallel displacement of $\widehat{oy}$ along $\widehat{oz}$ in M, and $x \in S(y) \cap R(z)$. From Lemma 3.6 we see that the point x does not depend on the broken-paths $\widehat{oy}$ and $\widehat{oz}$, but does depend upon the points y and z. Now consider a map

$$f: \widetilde{M} \to M \qquad (\widetilde{x} \to x).$$

A) Let x be a point of M. We can take a point y of $R_0 \cap S(x)$, for $R_0 \cap S(x) \neq 0$ by virtue of Lemma 3.2. Let z be the terminal point of the broken-path at x, obtained by parallel displacement of $\widehat{oy}^{-1}$ along $\widehat{yx}$, where $\widehat{oy}$ and $\widehat{yx}$ are arbitrary broken-paths in R_0 and $S(x)$ respectively. Then $z \in S_0$. Now if we denote by $\widetilde{x}$ a point (y, z) in $\widetilde{M}$, $f(\widetilde{x}) = x$. Consequently $f(\widetilde{M}) = M$.

B) Let y_λ, $\lambda \in J$, be all points of $R_0 \cap S(x)$ for $x \in M$, where J is the index-set. Let z_λ be a point determined from y_λ in the same manner as A), then $z_\lambda \in R(x) \cap S_0$. Make two broken-paths $\widehat{oy_\lambda}$ and $\widehat{oz_\lambda}$ in R_0 and S_0 respectively. On the other hand, consider a W-nbh $W(x)$. By virtue of Definition 2.3, $W(x)$ is necessarily represented by the affine product $U(x) \times V(x)$, where $U(x) \subset R(x)$ and $V(x) \subset S(x)$. Let $U(y_\lambda)$ be the image of $U(x)$, obtained by the equivalent map with respect to $\widehat{oz_\lambda}^{-1}$. Let $V(z_\lambda)$ be the analogous image of $V(x)$ with respect to $\widehat{oy_\lambda}^{-1}$. Denote by $\widetilde{x}_\lambda$ a point (y_λ, z_λ) in $\widetilde{M}$, then the product $\widetilde{W}_\lambda \equiv U(y_\lambda) \times V(z_\lambda)$ is regarded as a nbh of $\widetilde{x}_\lambda$ and is equivalent to W under f.

C) We have $f^{-1}(x) = \bigcup_{\lambda \in J} \widetilde{x}_\lambda$ from B). Now we shall verify

$$\widetilde{W}_\lambda \cap \widetilde{W}_\mu = 0, \qquad \lambda, \mu \in J \quad (\lambda \neq \mu).$$

In fact, suppose that $\widetilde{W}_\lambda \cap \widetilde{W}_\mu \neq 0$, then there is a point $\widetilde{u} \in \widetilde{W}_\lambda \cap \widetilde{W}_\mu$. Put $u \equiv f(\widetilde{u})$ and $u \in W(x)$. Let $\widehat{\widetilde{u}\,\widetilde{x}_\lambda}$ be one and only one path in $\widetilde{W}_\lambda$, and $\widehat{\widetilde{u}\,\widetilde{x}_\mu}$ in $\widetilde{W}_\mu$. Since $f(\widehat{\widetilde{u}\,\widetilde{x}_\lambda})$ and $f(\widehat{\widetilde{u}\,\widetilde{x}_\mu})$ are contained in $W(x)$, these are the same path $\widehat{ux}$. Hence the directions at $\widetilde{u}$ of two paths $\widehat{\widetilde{u}\,\widetilde{x}_\lambda}$ and $\widehat{\widetilde{u}\,\widetilde{x}_\mu}$ can not coincide, because $\widetilde{x}_\lambda \neq \widetilde{x}_\mu$. Consequently there exist two distinct points $\widetilde{u}_1$ and $\widetilde{u}_2$ in $\widetilde{W}_\lambda$ such that $\widetilde{u}_1 \in \widehat{\widetilde{u}\,\widetilde{x}_\lambda}$, $\widetilde{u}_2 \in \widehat{\widetilde{u}\,\widetilde{x}_\mu}$ and $f(\widetilde{u}_1) = f(\widetilde{u}_2) \in W(x)$. This contradicts to the equivalence of $\widetilde{W}_\lambda$ and $W(x)$ under f. Hence $\widetilde{W}_\lambda \cap \widetilde{W}_\mu = 0$.

Summing up the above results, we see that the map $f: \widetilde{M} \to M$ is a covering.

COROLLARY. *When an R-reducible manifold M is simply-connected, M is equivalent to the affine product $R(o) \times S(o)$, where $o \in M$.*

It follows directly from Corollary of Theorem 1 and Theorem 2. This Corollary is an extension of de Rham's theorem referred in the introduction.

4. R-reducible manifolds whose fundamental groups are cyclic of order two

DEFINITION 4.1. Let $p(x)$ be the number of points contained in $R(x) \cap S(x)$

for a point x of a C-reducible manifold M. $p(x)$ is called the *multiplicity* at x of M. Especially if $p(x)$ is constant over M, the number p is called the *multiplicity* of M. It may be finite or infinite.

LEMMA 4.1. *When all R and S of an R-reducible manifold M are simply-connected, $p(x)$ is constant over M, where $x \in M$.*

PROOF. Consider the affine product $\widetilde{M} \equiv R(o) \times S(o)$, where $o \in M$. From Theorem 2, $\widetilde{M}$ is equivalent to the covering space of M. Denote its covering by f, and the number p of points contained in $f^{-1}(x)$ is independent of x. From the proof of Theorem 2, p is also the number of points contained in $R(o) \times S(x)$. From this it is easily proved that $p(x)$ is constant over M.

LEMMA 4.2. *Let $\widetilde{M}$ be the covering space of an R-reducible manifold M. When f is its covering and $\tilde{o}$ is any point of $\widetilde{M}$, the following properties are fulfilled:*

a) *$\widetilde{M}$ is an R-reducible manifold and equivalent to the affine product $\widetilde{R}_0 \times \widetilde{S}_0$, where $\widetilde{R}_0$ and $\widetilde{S}_0$ are the r- and s-dimensional maximal integral manifolds through $\tilde{o}$ respectively.* b) *Any maximal integral manifold, say $\widetilde{R}_0$, is the covering space of $R(o)$ and f is its covering, where $o \equiv f(\tilde{o})$.*

PROOF. It follows that $\widetilde{M}$ is separable (since $\pi_1(M)$ is at most countable) and metric. Thus a) is easily shown. Hence, it is sufficient to show $f(\widetilde{R}_0) = R_0$, because $\widetilde{R}_0$ is simply-connected. For a point $\tilde{y} \in \widetilde{R}_0$, consider a curve $\widehat{\tilde{o}\tilde{y}}$ in $\widetilde{R}_0$, then $f(\widehat{\tilde{o}\tilde{y}}) \subset R(o)$. Hence $f(\widetilde{R}_0) \subset R(o)$. Next, for a point $y \in R(o)$, consider a curve $\widehat{oy}$ in $R(o)$. We get a curve $\widehat{\tilde{o}\tilde{y}}$ in $\widetilde{R}_0$, such that $f(\widehat{\tilde{o}\tilde{y}}) = \widehat{oy}$. Hence $f(\widetilde{R}_0) \supset R_0$. Consequently $f(\widetilde{R}_0) = R(o)$.

LEMMA 4.3. *When an R-reducible manifold M has multiplicity one, M is equivalent to the affine product $R(o) \times S(o)$, where $o \in M$.*

This is easily proved.

In the following we shall adopt the following convention: For any manifold X, $\pi_1(X) = 1$ means that X is simply-connected, and $\pi_1(X) = 2$ means that the fundamental group of X is cyclic of order two.

THEOREM 3. *When $\pi_1(M) = 2$ for an R-reducible manifold M, M has either one of the following structures:*

a) *$\pi_1(R) = 1$, $\pi_1(S) = 1$ for any R, S of M and M has multiplicity two.*

b) *$\pi_1(R) = 1$, $\pi_1(S) = 2$ for any R, S of M and M is equivalent to the affine product $R(o) \times S(o)$, where $o \in M$.*

c) *$\pi_1(R) = 1$ for any R of M, and all S of M are divided into two non-vacuous classes, one satisfying $\pi_1(S) = 1$ and the other satisfying $\pi_1(S) = 2$. The multiplicity at a point of S is two or one according to $\pi_1(S) = 1$ or 2.*

Similarly the structures obtained by exchanging R and S do also exist.

Conversely there exist R-reducible manifolds M with any one of the structures mentioned above and $\pi_1(M) = 2$.

PROOF. From Corollary of Theorem 1, $\pi_1(R)$ and $\pi_1(S)$ for any R and S

are of order one or two, because $\pi_1(M)=2$. Consequently we have only the following cases:

A) $\pi_1(R)=1$ and $\pi_1(S)=1$ for any R and S.

B) $\pi_1(R)=1$ for any R and $\pi_1(S)=2$ for any S.

C) $\pi_1(R)=1$ for any R and there exist at least two S_1 and S_2 such that $\pi_1(S_1)=1$ and $\pi_1(S_2)=2$.

D) There exists at least a pair R_0 and S_0 such that $\pi_1(R_0)=2$, and $\pi_1(S_0)=2$. Here we do not enumerate cases obtained by exchanging R and S.

The case A). By virtue of Lemma 4.1, there exists the multipicity p of M. Suppose $p\neq 2$. From Theorem 2, the affine product $\widetilde{M}\equiv R(o)\times S(o)$ is the covering space of M and let f be its covering, where $o\in M$. For a point $x\in M$, $f^{-1}(x)$ does not consist of two points. Hence $\pi_1(M)\neq 2$, so we have arrived at a contradiction. Consequently p must be two.

We shall show the existence of the case A) by an example:

Let $\widetilde{R}$ and $\widetilde{S}$ be r-and s-dimensional spheres respectively. For a point $\in\widetilde{R}$ let $f(y)$ be its antipodal point, and similarly, for a point $z\in\widetilde{S}$ let $f(z)$ be its antipodal point. Define the isometric map of the metric product $\widetilde{R}\times\widetilde{S}$ onto itself by

$$(y,z)\rightarrow(f(y),f(z)).$$

We denote this map by f again and put $\widetilde{M}\equiv\widetilde{R}\times\widetilde{S}$. In $\widetilde{M}$ if we identify any point $x\in\widetilde{M}$ with $f(x)$, we get a reducible Riemannian manifold M. It follows easily that M satisfies a) of Theorem 3.

The case B). Suppose that the multiplicity at a point o is not one, and $R(o)\cap S(o)$ contains at least a point x distinct from o. We shall use notations of Lemma 4.2 and consider $\tilde{o}$ as a point of $f^{-1}(o)$. By virtue of $\pi_1(R)=1$, $f:\widetilde{R}_0\rightarrow R(o)$ is an equivalent map. Hence there exists one and only one point $\tilde{x}\in\widetilde{R}_0$ such that $f(\tilde{x})=x$. For a curve $\widehat{xo}$ in $S(x)$ consider a curve of $f^{-1}(\widehat{xo})$ with the initial point $\tilde{x}$, then it follows that there exists a point $\tilde{o}_2\in\widetilde{S}(\tilde{x})$ such that $f(\tilde{o}_2)=o$, where $\widetilde{S}(\tilde{x})$ is the s-dimensional maximal integral manifold through $\tilde{x}$ of $\widetilde{M}$.

On the other hand, since $\widetilde{S}_0$ is the covering space of $S(o)$ and $\pi_1(S(o))=2$, there exists a point $\tilde{o}_1\in\widetilde{S}_0$ such that $f(\tilde{o}_1)=o$, distinct from $\tilde{o}$. Hence $f^{-1}(o)$ contains at least three point $\tilde{o}$, $\tilde{o}_1$ and $\tilde{o}_2$. This is contradictory to the fact that $\widetilde{M}$ is the covering space of M. Consequently the multiplicity of M exists and it is one. From Lemma 4.3, M is equivalent to the affine product $R(o)\times S(o)$ and satisfies b) of Theorem 3.

The case C). It is shown by Lemma 4.2 that the multiplicity at a point of S is two or one according to $\pi_1(S)=1$ or 2. By an example, we shall show the existence of the case C). Let $\widetilde{R}$ be the r-dimensional Euclidean space and o be a point of $\widetilde{R}$. Let $\widetilde{S}$ be an s-dimensional sphere. Let $f(y)$ be the symmetric point of $y \in \widetilde{R}$ with respect to o and $f(z)$ the antipodal point of $z \in \widetilde{S}$. Consider the metric product $\widetilde{M} \equiv \widetilde{R} \times \widetilde{S}$ and denote again by f the isometric map of $\widetilde{M}$ onto itself, such that

$$(y, z) \rightarrow (f(y), f(z)).$$

In $\widetilde{M}$ if we identify any point $x \in \widetilde{M}$ with $f(x)$, we get a reducible Riemannian manifold M. It follows easily that M satisfies c) of Theorem 3.

The case D). By Lemma 4.2, we can show that this case does not occur.

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