ON ANGULAR MEASURE IN A METRIC SPACE

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H. Busemann [1] dealt with a metric space called a G-space. If a G-space $\mathfrak{S}$ is of dimension 2, then we can generally define an angular measure Ψ [§ 1]. In this note, we define a function F on figures of $\mathfrak{S}$ which will be called the excess function [§ 2]. When the angular measure Ψ is continuous and the function F is of bounded variation, we define the Gaussian curvature in the general sense which will be called the generalized Gaussian curvature of $\mathfrak{S}$ [§ 2]. If $\mathfrak{G}$ is a G-space with constant curvature in H. Busemann's sense, then the angular measure Φ is introduced [5], [§ 3]. If the excess function F defined by means of the angular measure Φ is of bounded variation, then $\mathfrak{G}$ is a G-space with constant generalized Riemannian curvature [§ 4]. The main purpose of this note is to show that Gauss-Bonnet's theorem holds in $\mathfrak{G}$ and all G-spaces with constant curvature are divided into three classes according as its generalized Riemannian curvature is positive, zero, or negative.

1. In a metric space points will be denoted by small roman letters and the distance between two points x and y by xy. According to H. Busemann [1 ; § 4] the axioms for a space $\mathfrak{G}$ to be a G-space are the following:

A. $\mathfrak{G}$ is metric with distance xy.

B. $\mathfrak{G}$ is finitely compact.

C. $\mathfrak{G}$ is convex metric.

D. Every point x of $\mathfrak{G}$ has a neighborhood $S(x, \alpha(x))$ ($= \{y \mid xy < \alpha(x)\}$) $(\alpha(x) > 0)$ such that for any positive number ε and any two points a and b in $S(x, \alpha(x))$ there exist positive numbers $\delta_i (\leqq \varepsilon)$ $(i = 1, 2)$ for which a point a_1 with $a_1a + ab = a_1b$ and $a_1a = \delta_1$ and another point b_1 with $ab + bb_1 = ab_1$ and $bb_1 = \delta_2$ exist and are unique.

For any two points x and y, the axioms A, B, and C guarantee the existence of a segment $T(x,y)$ from x to y (or $T(y,x)$ from y to x) whose length is equal to the distance xy. The prolongation of a segment is locally possible and unique under the axiom D. The whole prolongation of a segment is said to be an extremal. An extremal $\mathfrak{x}$ has a parametric representation $x(\tau)$, $-\infty < \tau < +\infty$, such that for every τ_0 a positive number $\varepsilon(\tau_0)$ exists such that $x(\tau_1)x(\tau_2) = |\tau_2 - \tau_1|$ for $|\tau_i - \tau_0| \leqq \varepsilon(\tau_0)$ $(i = 1, 2)$. The extremal $\mathfrak{x}$ is said to be a straight line, if its parametric representations have the property: $x(\tau_1)x(\tau_2) = |\tau_2 - \tau_1|$ for any two real numbers τ_1 and τ_2. If every extremal is a straight line, then $\mathfrak{G}$ is said to be a straight line space.

In [1 ; § 4] the number $\eta_\lambda(x)$ $(\lambda \geqq 2)$ and the term "direction" were introduced. $\eta_\lambda(x)$ is defined as the l. u. b. of those β for which every segment with end points in $S(x, \beta)$ is a cocentral subsegment of a segment of length $\lambda\beta$. $\eta_\lambda(x)$ is positive for every point x and every number λ not less than 2. The number $\eta(x)$ is defined as $\min(\eta_6(x), 1)$. Then $\eta(x)$ is regarded as a continuous function of a point x. The segment $T(a, b)$ of length $\eta(a)$ is said to be a direction with the initial point a.

2. Let $\mathfrak{S}$ be a G-space of dimension 2 and p any point of $\mathfrak{S}$. Let $\mathfrak{x}_1$ and $\mathfrak{x}_2$ be two different half extremals issuing from p whose parametric representations are given by $x_1(\tau)$, $0 \leqq \tau < +\infty$, and $x_2(\tau)$, $0 \leqq \tau < +\infty$, respectively. Then $\overline{S(p, \eta(p))}$ is divided by the directions $x_1(\tau)$, $0 \leqq \tau \leqq \eta(p)$, and $x_2(\tau)$, $0 \leqq \tau \leqq \eta(p)$, into two sectors D_1 and D_2. Similarly $\overline{S(p, 2\eta(p))}$ is divided by $x_1(\tau)$, $0 \leqq \tau \leqq 2\eta(p)$, and $x_2(\tau)$, $0 \leqq \tau \leqq 2\eta(p)$, into two sectors D_1' and D_2'. We assume $D_i \subset D_i'(i = 1, 2)$. Then only one of D_1' and D_2' contains all segments $T(x, y)$ with $x \in E[x_1(\tau),\ 0 \leqq \tau \leqq \eta(p)]$[1] and $y \in E[x_2(\tau),\ 0 \leqq \tau \leqq \eta(p)]$, unless $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are opposite. Let D_1' be such a sector. Then D_1' is called a convex sector and D_2 a concave sector. The segments $x_i(\tau)$, $0 \leqq \tau \leqq \eta(p)$, are called the legs of $D_i(i = 1, 2)$. $S(p, \eta(p))$ is said to be the normal neighborhood of p.

At a point p an angular measure Ψ_p is defined as a function on the set of all sectors of $\overline{S(p, \eta(p))}$ which fulfills the following conditions 1°, 2°, and 3°.

1. $\Psi_p(D) \geqq 0$ for any sector D.
2. $\Psi_p(D) = \pi$, if and only if the two legs of D are opposite.
3. If two sectors D_1 and D_2 have only one common leg but have no common part, then $\Psi_p(D_1) + \Psi_p(D_2) = \Psi_p(D_1 + D_2)$.

In such a way angular measure Ψ_p is defined at every point p of $\mathfrak{S}$. Then we denote by Ψ the function Ψ_p. The function Ψ is said to be an angular measure on $\mathfrak{S}$. It is easy to see that $\Psi(D) = 0$, if and only if D is a segment.

Let p be a point of $\mathfrak{S}$ and $\{p_\nu\}$ any sequence of points which converges to p. Let D_ν be any sector of each $\overline{S(p_\nu, \eta(p_\nu))}$ such that $\mathrm{Fl}_{\nu\to+\infty} D_\nu = D$[2]. If $\lim_{\nu\to+\infty} \Psi(D_\nu) = \Psi(D)$, then the angular measure Ψ is said continuous at p.

A triangle abc is said to be normal, if the vertices a, b, and c are not collinear and the normal neighborhood of each of these vertices contains the others. Let D be the convex sector of $\overline{S(a, \eta(a))}$ whose legs contain the segments $T(a, b)$ and $T(a, c)$. Then $\Psi(D)$ is called the inside angle of the triangle abc at a and denoted by $b\hat{a}c$ (or $c\hat{a}b$). Similarly $a\hat{b}c$ and $a\hat{c}b$ are defined. From the definition of normal triangles we see that each inside angle is less than π. It is also easily seen that the angle between two segments $T(p, a)$ and $T(p, b)$ is defined. We denote it by $a\hat{p}b$.

To define the excess function F, we put

$$F(\sigma) = b\hat{a}c + c\hat{b}a + a\hat{c}b - \pi$$

for a normal triangle $\sigma(= abc)$. Then F is a function on the set of all

1) $E[x_1(\tau),\ 0 \leqq \tau \leqq \eta(p)]$ means the set of all points of the segment $x_1(\tau), 0 \leqq \tau \leqq \eta(p)$. We use the same notation for half extremals and extremals.

2) Fl means the closed limit introduced by Hausdorff [1], [4].

normal triangles on $\mathfrak{S}$. It is easy to see $F(\sigma) < 2\pi$ for every normal triangle on $\mathfrak{S}$. We assume that F vanishes for empty set. The following property of the function F is clear from the definition.

(2.1) If two normal triangles σ_1 and σ_2 are non-overlapping, namely $\sigma_1^\circ \sigma_2 = \sigma_1 \sigma_2^\circ = \phi$[3] and $\sigma_1 + \sigma_2$ is also a normal triangle σ_3, then

$$F(\sigma_1) + F(\sigma_2) = F(\sigma_3).$$

A set which is expressible as the sum of a finite number of non-overlapping normal triangles is called a figure.

(2.2) If a figure R is expressed as the sum of a finite number of normal triangles in two ways $\sum_{i=1}^{m} \sigma_i$ and $\sum_{i=1}^{n} \sigma_i'$, then the relation

$$\sum_{i=1}^{m} F(\sigma_i) = \sum_{i=1}^{n} F(\sigma_i')$$

holds and this common value is given by

$$F(R) = 2\pi \chi(R) - \pi \chi(R') - \Sigma(\pi - v_i),$$

where R' is the boundary of R, $\chi(R)$ and $\chi(R')$ the Euler characteristics of R and R' respectively and v_i the angle at each vertex a_i measured in R.

(2.2) easily follows from a result obtained by S. Cohn-Vossen [2] for a 2-dimensional Riemannian surface.

From the above the function F is regarded as a function on the set of all figures on $\mathfrak{S}$. F is said to be the excess function on $\mathfrak{S}$. In a 2-dimensional Riemannian space $F(R)$ is the total curvature of a figure R.

On a figure R the upper and lower variations of the function F are denoted by $\overline{W}(F;R)$ and $\underline{W}(F; R)$ respectively. The total variation $\overline{W}(F; R) + |\underline{W}(F; R)|$ is denoted by $W(F; R)$. If $W(F; R) < +\infty$ for any figure R on $\mathfrak{S}$, then the function F is of bounded variation on $\mathfrak{S}$ and we have by Jordan's Decomposition Theorem

(2.3) $\qquad F(R) = \overline{W}(F; R) + \underline{W}(F; R) \quad$ for every figure R.

If $\mathfrak{S}$ is a 2-dimensional Riemannian space, then by Gauss-Bonnet's Theorem F is absolutely continuous.

(2.4) THEOREM. *If the angular measure Ψ is continuous and the excess function F is of bounded variation on $\mathfrak{S}$, then F is continuous at every point.*

PROOF. At first we prove that the absolute variation $W(F; R)$ is continuous.

Suppose that $W(F; R)$ is not continuous at a point p. Then a positive number ε and a sequence of normal triangles $\{\sigma_\nu\}$ which tends to p exist such that

$$W(F; \sigma_\nu) > \varepsilon \quad \text{for each } \nu.$$

We shall show that it is possible to define a sequence of non-overlapping figures $\{R_\nu\}$ such that

3) The interior of a set X is denoted by X°.

(2.5) $\quad p \in R_\nu$ and $|F(R_\nu)| > \varepsilon/2$ for each ν.

The normal triangle σ_1 clealy contains a figure R_1 such that $|F(R_1)| > \varepsilon/2$. If $R_1 \overline{\ni} p$, the figure R_1 satisfies the condition (2.5). If $R_1 \ni p$, then by choosing a suitable triangle σ_1' we have a figure $R_1 \ominus \sigma_1'$ ($= (\overline{R_1 - \sigma_1'})^\circ$) such that $R_1 \ominus \sigma_1' \overline{\ni} p$ and $|F(R_1 \ominus \sigma_1')| > \varepsilon/2$ since the angular measure Ψ is continnous. The figure R_1 ($= (R_1 \ominus \sigma_1')$) satisfies the condition (2.5).

Suppose that the figures $R_1, R_2, \ldots,$ and R_ν have already been chosen and let σ_λ be the first normal triangle of $\{\sigma_\nu\}$ which does not overlap any of the figures $R_1, R_2, \ldots$ and R_ν. In the same way as in the above we can see that σ_λ contains a figure $R_{\nu+1}$ which fulfills (2.5). Thus we have a sequence of figures as described in the above.

Let R be a figure such that $R \supset R_\nu$ for each ν. Then we have

$$W(F;\, R) \geqq \sum_{\nu=1}^{n} W(F;\, R_\nu) > \sum_{\nu=1}^{n} |F(R_\nu)| > n\varepsilon/2.$$

But this contradicts to the fact that the function F is of bounded variation. From this it follows that the upper and lower variations are continuous. Hence, by virtue of Jordan's Decomposition Theorem, (2.4) is proved.

When the function F is of bounded variation, we denote by F^* the additive function of a set induced by F. Then the following (2.6) is obvious. (See [3; Chap. III, § 6]).

(2.6) Under the assumption of (2.4) $F^*(R) = F^*(R^\circ) = F(R)$ for every figure R on $\mathfrak{S}$.

For any subset X of $\mathfrak{S}$ the 2-dimensional Hausdorff measure $\mu(X)$ is defined[4]. We assume that the 2-dimensional Hausdorff measure of every bounded set is finite. Then we have by Lebesgue Decomposition Theorem and Radon-Nykodym's Theorem

$$F^*(X) = T^*(X) + \int_X \frac{1}{k}\, d\mu(x)$$

where X is a Borel set, T^* the function of singularities of F^*, and $1/k$ an integrable function uniquely determined at almost all points on $\mathfrak{S}$. Putting $T^*(R) = T(R)$, we have by (2.6)

$$F(R) = T(R) + \int \frac{1}{k}\, d\mu(x).$$

The function $1/k$ will be said the generalized Gaussian curvature of $\mathfrak{S}$. If the function F is monotone, then F is non-decreasing or non-increasing according as F is non-negative or non-positive. Hence we have

$$\int_\sigma \frac{1}{k}\, d\mu(x) \leqq \widehat{bac} + \widehat{cba} + \widehat{acb} - \pi \qquad \text{or}$$

4) Let X be any subset of S and, for a given ε, Λ the class of all countable coverings $\Sigma\,(X_i = X)$ with $\operatorname{diam} X_i < \varepsilon$. Then the 2-dimensional Hausdorff measure $\mu(X)$ is defined as $(\pi/4)\lim\{\inf_{\Lambda} \Sigma\,(\operatorname{diam} X_i)^2\}$.

$$\int_{\sigma} \frac{1}{k} d\mu(x) \gtreqless \hat{bac} + \hat{cba} + \hat{acb} - \pi$$

for every normal triangle σ ($= abc$) according as K is non-negative or non-positive.

3. In this paragraph we study a G-space with constant curvature in H. Busemann's sense. If in a G-space every point p has a spherical neighborhood $S(p, \rho(p))$ such that the bisector $B(a, a')$[5] of any two distinct points a and a' in $S(p, \rho(p))$ is linear[6] in this neighborhood, then the space is said to be with constant curvature.

In such a space $\mathfrak{G}$, for every point p there exists a positive number $\delta(p)$ $(5\delta(p) = \bar{\delta}(p) < \min(\rho(p), \eta(p))$ which satisfies the following conditions [1; § 15].

(1) The neighborhood $S(p, \delta(p))$ is homeomorphic to the interior of a finite dimensional euclidean sphere; (2) if the dimension of $\mathfrak{G}$ is n ($\geqq 2$), $S(p, \bar{\delta}(p)) \cap B(a, a')$ $(a, a' \in S(p, \bar{\delta}(p))$ is of dimension $(n-1)$ (we put $B_p(a, a') = B(a, a') \cap S(p, \bar{\delta}(p))$ and call this a hyperplane); (3) Every sphere is strictly convex for $0 < \alpha \leqq \bar{\delta}(p)$; (4) every point x of $S(p, \delta(p))$ has a unique foot f on a hyperplane B_p which intersects $S(p, \delta(p))$; (5) a mapping $\Omega(B_p)$ of $S(p, \delta(p))$, which is a motion, is defined as follows:

(a) $x\Omega(B_p) = x$ for every point $x \in B_p$, and

(b) if $x \in S(p, \delta(p)) - B_p$, the point x' ($= x\Omega(B_p)$) is determined by $xf = fx' = xx'/2$.

The mapping $\Omega(B_p)$ is said to be the reflection of $S(p, \delta(p))$ with respect to B_p. All G-spaces with constant curvature are divided into two classes as follows [5]:

I. The class of G-spaces of Type I. If a G-space $\mathfrak{G}$ is of Type I, then the universal covering space $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$ has the following properties:

(1) Every extremal is closed; (2) every extremal through a point p passes through a unique point p' called the conjugate point of p; every extremal subarc from p to p' is a segment of constant length κ; (4) every sphere with radius less than $\kappa/2$ is strictly convex; (5) the bisector of any two distinct points is linear and coincides with a sphere of radius $\kappa/2$.

II. The class of G-spaces of Type II. If a G-space $\mathfrak{G}$ is of Type II, then the universal covering space $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$ has the following properties:

(1) $\tilde{\mathfrak{G}}$ is a straight line space; (2) every sphere is strictly convex; (3) the bisector of any two distinct points is linear and of dimension $(n-1)$.

On account of $(5)_{\mathrm{I}}$ and $(3)_{\mathrm{II}}$, the bisector of two distinct points is said to be a subspace of dimension $(n-1)$.

Let $\mathfrak{S}$ be a G-space with constant curvature and of dimension 2. The angular measure Φ is introduced as follows:

5) The bisector of two distinct points a and a' is defined as the set $\{x \mid ax = a'x\}$.

6) A set E is said linear, if for any two points x and x' of E there exists a segment $T(x, x')$ contained in E.

Let p be any point of $\mathfrak{S}$ and a line[7] $\mathfrak{g}_p$ through p intersect the circle $K(p, \delta(p)/2)$ at points a and a'. The line $B_p(a, a')$ is perpendicular[8] to $T(a, a')$ at p. Let $B_p(a, a')$ intersect $K(p, \delta(p)/2)$ at points b and b'. Then the segments $T(a, a')$ and $T(b\ b')$ divide $\overline{S(p, \delta(p)/2)}$ into four convex sectors $\widehat{apb}$, $\widehat{bpa'}$, $\widehat{a'pb'}$, and $\widehat{b'pa}$. For these sectors we put

$$\Phi_p(\widehat{apb}) = \Phi_p(\widehat{bpa'}) = \Phi_p(\widehat{a'pb'}) = \Phi_p(\widehat{b'pa}) = \pi/4.$$

Let $B_p(a, b)$ intersect $K(p, \delta(p)/2)$ at points c and c' and $B_p(a', b)$ $K(p, \delta(p)/2)$ at points d and d'. Then each of the above four sectors is divided by either $B_p(a, b)$ or $B_p(a', b)$ into two convex sectors. We denote these sectors by $\widehat{apc}$, $\widehat{cpb}$, $\widehat{bpd'}$, $\widehat{d'pa'}$, $\widehat{a'pc'}$, $\widehat{c'pb'}$, $\widehat{b'pd}$ and $\widehat{dpa}$ and put

$$\Phi_p(\widehat{apc}) = \dots = \Phi_p(\widehat{dpa}) = \pi/8.$$

We continue this process. If we denote by A the set of points$\{a, a'; b, b'; c, c'; \dots\}$, then the closure $\overline{A}$ coincides with the circle $K(p, \delta(p)/2)$. For any sector $\widehat{apq}$, by taking a sequence of points $\{q_\nu\}$ $(q_\nu \in A)$ which converges to q, $\Phi_p(\widehat{apq})$ is defined as the limit of the sequence $\Phi_p(\widehat{apq_\nu})$ (See [5] in details).

The function Φ_p thus defined fulfills the conditions $1°, 2°$, and $3°$ in § 2. The definition of the function Φ_p does not depend on any choice of the line $\mathfrak{g}_p$. In snch a way we define the function Φ_p at every point p of $\mathfrak{S}$. Then we denote by Φ the function Φ_p. The angular measure Φ thus obtained is invariant under the reflections with respect to lines.

In the remainder of this note, by means of the angular measure Φ, we study a G-space with constant curvature. For the angle between two segments $T(p, a)$ and $T(p, b)$ we use the same notation $\hat{apb}$ as in § 2.

(3. 1) THEOREM. *The angular measure Φ is continuous.*

PROOF. Let $\{p_\nu\}$ be a sequence of points which converges to a point p, and put $\alpha = \inf \delta(p_\nu)/2$. Then α is positive. Let D_ν be any sector of each $\overline{S(p_\nu, \alpha)}$ such that Fl D_ν coincides with a sector D of $S(p, \alpha)$. Then the legs $T_{1\nu}$ and $T_{2\nu}$ of each D_ν tends to the legs T_1 and T_2 of D respectively. Let $q_{i\nu}$ be the end point of each $T_{i\nu}$ and q_i the end point of $T_i (i = 1, 2)$. Now we prove $\lim_{\nu \to +\infty} q_{1\nu} \hat{p}_\nu q_{2\nu} = q_1 \hat{p} q_2$.

Obviously the sequences of points $\{q_{1\nu}\}$ and $\{q_{2\nu}\}$ converge to the points q_1 and q_2 respectively. Let each $q'_{2\nu}$ be a point on $K(p_\nu, \alpha)$ such that $q_{1\nu} \hat{p}_\nu q'_{2\nu} = q_1 \hat{p} q_2$. If we choose suitably such points $q'_{2\nu}$, then the sequence of points $\{q'_{2\nu}\}$ converges to q_2. Suppose that such points $q'_{2\nu}$ have been chosen. Then there exists a positive integer N such that $S(p, \delta(p)) \supset S(p_\nu, \alpha)$ for every $\nu \geqq N$. Since the angles $q'_{2\nu} \hat{p}_\nu q_{2\nu} (\nu \geqq N)$ are invariant under the reflections with respect to lines which intersect $S(p, \delta(p))$, it follows that, if

7) Let x and x' be two points on $K(p, \delta(p))$. The open segment $T(x, x') - x - x'$ is said a line $\mathfrak{g}_p$.

8) A line $\mathfrak{g}_p$ is said perpendicular to a set E at a point f, if every point on $\mathfrak{g}_p$ has f as a foot on E.

for a positive number δ, N is sufficiently large, then $q'_{2\nu}\hat{p}_\nu q_{2\nu} < \delta$ for every $\nu \geqq N$. Hence we have

$$|q_{1\nu}\hat{p}_\nu q_{2\nu} - q_1\hat{p}q_2| = |q_{1\nu}\hat{p}_\nu q_{2\nu} - q_{1\nu}\hat{p}_\nu q'_{2\nu}|$$
$$\leqq q'_{2\nu}\hat{p}_\nu q_{2\nu} < \delta \qquad \text{for every } \nu \geqq N.$$

Thus the theorem is proved.

Making use of the angular measure Φ we define the excess function F. Then, by (2. 4) and (3. 1), we have the following:

(3. 2) If the excess function F is of bounded variation, then F is continuous.

Under the assumption of (3. 2), for any bounded subset X there exist a positive integer M and a positive number δ_0 such that every circular disk $\overline{S(p, \gamma)}$ $(p \in X,\ 0 < \gamma \leqq \delta_0)$ is covered by M circular disks with radius $\gamma/5$. Next we shall show this.

Let V be a bounded and connected open set which contains X. If we put $\delta = \inf\limits_{x \in V} \delta(x)$, then δ is positive. Let δ_0 be a positive number not greater than δ. Let a line $\mathfrak{g}_p$ through p $(\in X)$ intersect a sphere $K(p, \gamma)$ $(0 < \gamma \leqq \delta_0)$ at points a and a' and $B_p(a, a')$ $K(p, \gamma)$ at points b and b'. Next divide $T(a, a')$ and $T(b, b')$ into 24 parts of equal length $aa'/24$ $(= bb'/24)$. If δ_0 is sufficiently small, then the lines perpendicular to $T(a, a')$ and $T(b, b')$ at points of the subdivisions form the net composed of 24^2 quadrilaterals P_i $(i = 1, 2, \ldots .\ 24^2)$ such that each P_i is covered by a circular disk with radius $\gamma/5$. This is clear from the continuity of the fnnction F.

For any circular disk $S(q, \delta_0)$ $(q \in X)$ there exists the combination of finite number of reflections with respect to lines by which $S(p, \delta_0)$ is carried onto it. Hence if we put $M = 24^2$, then M and δ_0 are tne numbers which fulfill the condition described above.

Let E be a set contained in a neighborhood $S(x, \delta(x))$. The parameter of regularity $\gamma(E)$ of E is defined as the upper bound of the number $\mu(E)/\mu(\overline{S})$, where $\overline{S}$ denotes any circular disk containing E. Let $\{E_\nu\}$ be a sequence of closed sets on $\mathfrak{S}$ which tends to a point p. If there exists a positive number α such that $\gamma(E_\nu) \geqq \alpha (\nu = 1, 2, \ldots .)$, then the sequence $\{E_\nu\}$ is said to be regular.

Let $\mathfrak{F}$ be a family of closed sets such that the parameter of regularity of each set exceeds a fixed number $\alpha(> 0)$ and for every point x of the set X there exists in $\mathfrak{F}$ a regular sequence of sets $\{W_\nu\}$ $(W_\nu \ni x)$ which tends to x. Theu $\mathfrak{F}$ contains a finite or countable sequence $\{X_\nu\}$ of sets no two of which have common points, such that

$$\mu(X - \sum X) = 0. \tag{3. 3}$$

Next we prove (3. 3). To do this, we suppose that every set of $\mathfrak{F}$ can be covered by a circular disk with radius not greater than $\delta_0/5$.

Choose an arbitrary set X_1 of $\mathfrak{F}$ and suppose that the first λ sets X_1, $X_2, \ldots,$ X_λ no two of which have common points have been chosen. If $X -$

$\sum_{\nu=1}^{\lambda} X_\nu = \phi$, then the theorem is proved. If this is not so, we denote by δ_λ the upper bound of the diameters of all sets which have no common points with $\sum_{\nu=1}^{\lambda} X_\nu$ and choose an arbitrary set $X_{\lambda+1}$ of those sets with diameter exceeding $\delta_\lambda/2$. If $X - \sum_{\nu=1}^{\lambda+1} X_\nu \neq \phi$, then we continue this process.

Suppose that an infinite sequence of sets $\{X_\nu\}$ has been chosen, and put $Y = X - \sum_{\nu=1}^{\infty} X_\nu$. It is sufficient to show that, if $\mu(Y) > 0$, then we arrive at a contradiction. To do this, associate with each set X_ν a circular disk $\bar{S}_\nu$ with radius γ_ν such that $X_\nu \subset \bar{S}_\nu$ and $\mu(X_\nu)/\mu(\bar{S}_\nu) > \alpha/2$, and let $\bar{S}'_\nu$ be the circular disk with the same center as $\bar{S}_\nu$ and the radius $5\gamma_\nu$. We then have

$$(3.4)\qquad \sum_{\nu=1}^{\infty} \mu(\bar{S}'_\nu) < M \sum_{\nu=1}^{\infty} \mu(\bar{S}_\nu) \leqq 2M\alpha^{-1} \sum_{\nu=1}^{\infty} \mu(X_\nu) < +\infty.$$

Hence a positive integer N exists such that $\sum_{\nu=N+1}^{\infty} \mu(\bar{S}'_\nu) < \mu(Y)$. From this it follows that there exists a point $x_0 (\in X)$ not belonging to $\sum_{\nu=N+1}^{\infty} S'_\nu$. By virtue of the assumption there must exist a set $X' (\ni x_0)$ of $\mathfrak{F}$ such that $X' \cap X_\nu = \phi$ for $\nu = 1, 2, \ldots, N$. From (3.4) we see that the radius γ_ν of $\bar{S}_\nu$ tends to zero as $\nu \to +\infty$. Hence X' has common points with at least one of the sets X_ν $(\nu > N)$. Let ν_0 be the smallest integer such that $X' \cap X_{\nu_0} \neq \phi$. The diameter of X' does not exceed $\delta_{\nu_0 - 1}$ $(< 4\gamma_{\nu_0})$. Hence $X' \subset \bar{S}'_{\nu_0}$, which contradicts to the assumption $x_0 \in \sum_{\nu=N+1}^{\infty} \bar{S}'_\nu$. Thus (3.3) is proved.

By use of (3.3) it is easily proved that Vitali's Covering Theorem holds on $\mathfrak{S}$, i.e., if a set X is covered by a family $\mathfrak{F}$ of closed sets in the sense of Vitali, there exists in $\mathfrak{F}$ a finite or countable sequence $\{X_\nu\}$ of sets no two of which have common points such that (3.3) holds. For any subset Z there exists a $(G)_\delta$ set G such that $Z \subset G$ and $\mu(Z) = \mu(G)$. By virtue of this property and Vitali's Covering Theorem it is easily seen that the additive function of a set F^* is derivable at almost all points [3]. Taking account of the reflection with respect to lines, F is derivable at every point and its derivative is equal to a constant number $1/k$. Hence the excess function F is derivable at every point, i.e., for any regular sequence of normal triangles $\{\sigma_\nu\}$ which tends to a point p $\lim_{\nu\to+\infty} F(\sigma_\nu)/\mu(\sigma_\nu)$ exists and is equal to $1/k$.

Next we prove the following

(3.5) THEOREM. *If the excess function F is of bounded variation, then F is absolutely continuous and its derivative is equal to a constant number $1/k$. For any normal triangle $\sigma(= abc)$ the function F is given by*

$$\mu(\sigma)/k = \widehat{bac} + \widehat{cba} + \widehat{acb} - \pi.$$

PROOF. It is sufficient to prove that the function T^* of sigularities of F^* vanishes on $\mathfrak{S}$. It is easy to see that the derivative of the function T^* is equal to zero at every point. We prove the theorem only in the case where the function T^* is non-negative, since its upper and lower variations are finite.

Let p be any point on $\mathfrak{S}$, and put $m\delta = \delta(p)$, where m is a positive integer. Let a line $\mathfrak{g}_p$ through p intersect $K(p, \delta)$ at points a and a' and $\mathfrak{h}_p$ and $\mathfrak{h}'_p$ be the supporting lines of $K(p, \delta)$ at a and a' respectively. Then $\mathfrak{g}_p$ divides $S(p, \delta(p))$ into two domains. We denote by D one of these domains.

Let $\{p_\nu\}$ and $\{p'_\nu\}$ be the sequence of points in D such that $p_\nu \in E[\mathfrak{h}_p]$, $p'_\nu \in E[\mathfrak{h}'_p]$, and $ap_\nu = a'p'_\nu = \delta/2^\nu$ for each ν and $\mathfrak{g}_{p,\nu}$ the line which contains each $T(p_\nu, p'_\nu)$ as a subsegment. Then

$$\mathfrak{g}_{p,\nu} \cap \mathfrak{g}_p = \phi \qquad \text{for every } \nu. \tag{3.6}$$

Next we subdivide the segment $T(a, a')$ by points $a_\nu^{(1)}, a_\nu^{(2)}, \ldots, a_\nu^{(\bar{\nu})}$ as follows:

Take on $\mathfrak{g}_p$ the point $a_\nu^{(1)}$ such that $aa_\nu^{(1)} = ap_\nu$ and let $p_\nu^{(1)}$ be the point at which the line perpendicular to $\mathfrak{g}_p$ at $a_\nu^{(1)}$ intersects the line $\mathfrak{g}_{p,\nu}$. Further take on $\mathfrak{g}_p$ the point $a_\nu^{(2)}$ such that $a_\nu^{(1)}a_\nu^{(2)} = a_\nu^{(1)}p_\nu^{(1)}$. Then we can determine the point $p_\nu^{(2)}$ as above. If $a_\nu^{(1)}a' \leqq a_\nu^{(1)}p_\nu^{(1)}$, we end this process. If $a_\nu^{(1)}a' > a_\nu^{(1)}p_\nu^{(1)}$, then we continue this process. On account of (3.6), after finite steps we arrive at a point $a_\nu^{(\bar{\nu})}$ such that $a_\nu^{(\bar{\nu})}a' \leqq a_\nu^{(\bar{\nu})}p_\nu^{(\bar{\nu})}$ and $a_\nu^{(\bar{\nu})} \in E[T(a, a')]$. Then we take on $\mathfrak{g}_p$ and $\mathfrak{g}_{p,\nu}$ the points $a_\nu^{(\bar{\nu}+1)}$ and $p_\nu^{(\bar{\nu}+1)}$ in the same way as above respectively.

Thus we have $\bar{\nu}+1$ quadrilaterals $a_\nu^{(i)}a_\nu^{(i+1)}p_\nu^{(i+1)}p_\nu^{(i)}(i = 0, 1, 2, \ldots,)$ for each ν, where $a_\nu^{(0)} = a$. We denote by $P_\nu^{(i)}$ each quadrilateral $a_\nu^{(i)}a_\nu^{(i+1)}p_\nu^{(i+1)}p_\nu^{(i)}$. By virtue of the continuity of the function F, each inside angle of $P_\nu^{(i)}$ tends to $\pi/2$ as $\nu \to +\infty$. Hence it follows that for every point x of $T(a, a')$ there exists a regular sequence of quadrilaterals $\{P_\nu^{(i_\nu)}\}$ $(P_\nu^{(i_\nu)} \ni x)$ tending to x.

Now we prove that for an arbitrary positive number ε there exists a positive integer N such that

$$T^*(P_\nu^{(i_\nu)}) < \varepsilon\mu(P_\nu^{(i_\nu)}) \text{ for each } \nu \geqq N \text{ and each } i \ (0 \leqq i \leqq \bar{\nu}). \tag{3.7}$$

If this is not so, then we should have a sequence of positive integers $\{\lambda\}$ $(\subset\{\nu\})$ such that

$$T^*(P_\lambda^{(i_\lambda)}) \geqq \varepsilon\mu(P_\lambda^{(i_\lambda)}) \tag{3.8}$$

for each λ and a positive integer i_λ $(0 \leqq i_\lambda \leqq \bar{\lambda})$. Let $\mathfrak{h}_{p,\lambda}$ be the line perpendicular to $\mathfrak{g}_p$ at the midpoint of the segment $T(a, a_\lambda^{(i\lambda)})$. Then $a_\lambda^{(i\lambda)}\,\Omega(\mathfrak{h}_{p,\lambda}) = a$. Hence each quadrilateral $P_\lambda^{(i_\lambda)}$ is carried by $\Omega(\mathfrak{h}_{p,\lambda})$ onto a quadrilateral P'_λ with the vertex a. The sequence of quadrilaterals $\{P'_\lambda\}$ is regular and tends to a. Hence there exists a positive integer N' such that

$$T^*(P'_\lambda) < \varepsilon\mu(P'_\lambda) \quad \text{for every } \lambda \geqq N'.$$

Obviously $T^*(P_\lambda^{(i\lambda)}) = T^*(P'_\lambda)$ and $\mu(P_\lambda^{(i\lambda)}) = \mu(P'_\lambda)$ for every λ, but this contradicts to (3.8). Thus (3.7) is proved.

Between the lines $\mathfrak{g}_p$ and $\mathfrak{g}_{p,N}$, there exist $\bar{N}+1$ quadrilaterals $P_N^{(0)}, P_N^{(1)}, \ldots,$ and $P_N^{(\bar{N})}$ which fulfill the condition (3.7). We put

$$\overline{P}_N^{(i)} = P_N^{(i)}\Omega(\mathfrak{g}_{p,N}) \qquad (i = 0, 1, 2, \ldots, N).$$

Then it is easy to see that the two vertices of each $\overline{P}_N^{(i)}$ lie on the line $\mathfrak{g}_p\Omega(\mathfrak{g}_{p,N})$. Hence, in such a way, we get the figure R composed of a finite number of non-overlapping quadrilaterals P_ν such that $S(p, \delta(p))\supset R\supset S(p, \delta)$ for a sufficiently large positive integer m and each P_ν fulfills the condition (3.8). The figure R is expressed as the sum ΣP_ν. From this it follows that

$$T^*(S(p, \delta)) < T^*(R) = \Sigma T^*(P_\nu)$$
$$< \varepsilon\Sigma\mu(P_\nu)$$
$$= \varepsilon\mu(R), \text{ and}$$
$$\mu(R) > \mu(S(p, \delta)).$$

Therefore we conclude that $T^*(S(p, \delta)) = 0$, since ε is arbitrary. From this we see that T^* vanishes on $\mathfrak{S}$. Thus the theorem is proved.

4. Let $\mathfrak{G}$ be a G-space with constant curvature and of dimension $n(\geqq 2)$ and $\widetilde{\mathfrak{G}}$ the universal covering space of $\mathfrak{G}$. In a subspace of dimension $(n-1)$ of $\widetilde{\mathfrak{G}}$ the bisector of any two distinct points is linear and of dimension $(n-2)$. We call this a subspace of dimension $(n-2)$. Repeating this, in a subspace of dimension 2, the bisector of any two distinct points is an extremal [5].

The generalized Gaussian curvature of every subspace of dimension 2 is equal to a constant number $1/k$. If G is Riemannian, then the number $1/k$ is its Riemannian curvature. The number $1/k$ will be said the generalized Riemannian curvature of $\mathfrak{G}$.

(4.1) THEOREM. *If the space $\mathfrak{G}$ is of type I, then its generalized Riemannian curvature is positive.*

PROOF. In $\widetilde{\mathfrak{G}}$, every subspace $\widetilde{\mathfrak{S}}$ of dimension 2 is compact and covered by a finite number of triangles $\sigma_i (i = 1, 2, \ldots, m)$. From (2.4) it follows that

$$\sum_{i=1}^{m} F(\sigma_i) = 4\pi$$

since $\chi(\widetilde{\mathfrak{S}}) = 2$. Therefore we have by (3.5)

$$\sum_{i=1}^{m} \mu(\sigma_i)/k = 4\pi.$$

Since $\sum_{i=1}^{m} \mu(\sigma_i) = \mu(\widetilde{\mathfrak{S}}) > 0$, the number $1/k$ is positive. Thus the theorem is proved.

On $\widetilde{\mathfrak{S}}$ the following properties can easily be proved by classical arguments.

(4.2) (i) Let abc be a rectangular triangle with $\widehat{acb} = \pi/2$ and $m(a, b)$ the midpoint of the segment $T(a, b)$. Then the distance between $m(a, b)$ and

$E[T(b,c)]$ is greater than the half of ac. (ii) Every equidistant curve to an extremal $\mathfrak{x}$ turns its convexity toward $\mathfrak{x}$. (iii) For an extremal subarc $x(\tau)$, $\alpha \leqq \tau \leqq \beta$, and an extremal $\mathfrak{x}$, the function $f(\tau) = x(\tau)E[\mathfrak{x}]$ ($= \inf_{x\in E[\mathfrak{x}]} x(\tau)x$) is a concave function.

Next we prove the following

(4.3) THEOREM. *If the space* $\mathfrak{G}$ *is of Type II, then its generalized Riemannian curvature is non-positive.*

PROOF. It is sufficient to prove that, if $1/k$ is positive, we arrive at a contradiction. Let $\tilde{\mathfrak{S}}$ be a subspace of dimension 2 of $\tilde{\mathfrak{G}}$, and let p be any point on $\tilde{\mathfrak{S}}$ and f the foot of p on an extremal $\mathfrak{x}$ $(\ni p)$. Further let $x(\tau)$, $-\infty < \tau < +\infty$, be the parametric representation of $\mathfrak{x}$ such fhat $x(0) = f$ and τ_0 a fixed positive number. Then, for any positive number $\tau (> \tau_0)$, we have by putting $x(\tau) = b$ and $x(-\tau) = b'$

$$(4.4) \qquad \mu(pbb')/k = \hat{pbb'} + \hat{pb'b} - (\pi - \hat{bpb'}) > \mu(paa')/k,$$

where $x(\tau_0) = a$ and $x(-\tau_0) = a'$. We can easily see that, on $\tilde{\mathfrak{S}}$, for any positive number ε there exist two positive numbers α and β such that, for any three points x, y and z which satisfy the conditions $xy = xz = \alpha$ and $yz \geqq 2\alpha(1-\beta)$, the inequality $\hat{yxz} \geqq \pi - \varepsilon/2$ holds.

Assume that $\mu(paa') > k\varepsilon > 0$, and put $\lambda(\tau) = pb$ ($= px(\tau)$) and $2\beta = \delta$. The function $f(\tau) = \lambda(\tau) - \tau + \delta\tau$ is continuous on the interval $\tau_0 \leqq \tau < +\infty$, and $\lim_{\tau\to+\infty} f(\tau) = +\infty$. Hence $f(\tau)$ attains its minimum at some value $\bar{\tau}$ on $\tau_0 \leqq \tau < +\infty$ and fulfills the condition

$$\lambda(\bar{\tau}+\sigma) - (\bar{\tau}+\bar{\sigma}) + \delta(\tau+\sigma) - \{\lambda(\bar{\tau}) - \bar{\tau} + \delta\bar{\tau}\} \geqq 0 \text{ for } \sigma \geqq 0.$$

Therefore

$$(4.5) \qquad \lambda(\bar{\tau}+\sigma) - \lambda(\bar{\tau}) \geqq \sigma(1-\delta).$$

Put $c = x(\bar{\tau})$ and $c' = x(-\bar{\tau})$, and let d and e be the points on $T(p,c)$ and on the prolongation of the segment $T(f,c)$ respectively such that $cd = ce = \alpha$. If we put $\sigma = \alpha$ in (4.5), we then have

$$de \geqq 2\,\alpha(1-\beta)$$

since $de - \alpha > \lambda(\bar{\tau}+\alpha) - \lambda(\bar{\tau})$. Hence we see $pcd \geqq \pi - \varepsilon/2$. Let d' be the point on the prolongation of the segment $T(f,c')$ such that $c'd' = \alpha$. Then we see $\hat{pc'd'} = \pi - \varepsilon/2$ since on $\tilde{\mathfrak{S}}$ the bisector property holds in the large. On the other hand $\pi - \hat{cpc'} \geqq 0$ is obvious. Hence we see from (4.4)

$$\begin{aligned} \mu(\hat{paa'})/k < \mu(\hat{pcc'})/k &\leqq \hat{pcc'} + \hat{pc'c} \\ &= (\pi - \hat{pcd}) + (\pi - \hat{pc'd'}) \\ &\leqq \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

which contradicts to the assumption $0 < k\varepsilon < \mu(\hat{paa'})$. Thus the theorem is proved.

If the number $1/k$ is equal to zero, then $\tilde{\mathfrak{S}}$ has the same property as a euclidean plane, i. e., the theorem in plane geometry holds on $\tilde{\mathfrak{S}}$. We can introduce Lebesgue measure which coincides with Hausdorff measure μ. If $1/k < 0$, then the following properties of $\tilde{\mathfrak{S}}$ is also easily proved by classical arguments.

(4. 6) (i) Let abc be a rectangular triangle with $\widehat{acb} = \pi/2$ and $m(a, b)$ the midpoint of the segment $T(a, b)$. Then the distance between $m(a, b)$ and $E[T(b, c)]$ is less than the half of ac. (ii) Every equidistant curve to an extremal $\mathfrak{x}$ turns its concavity toward $\mathfrak{x}$. (iii) For an extremal subarc $x(\tau), \alpha \leqq \tau \leqq \beta$, and an extremal $\mathfrak{x}$ the function $f(\tau) = x(\tau)E[\mathfrak{x}]$ is a convex function.

In virtue of (3. 5), (4. 1), (4. 2), (4. 3) and (4. 6), if $\mathfrak{G}$ is a G-space with constant curvature and the excess function is of bounded variation, the space $\mathfrak{G}$ is of Type I or Type II according as its constant generalized Riemannian curvature is positive or non-positive. Specially, if $1/k = 0$, the space $\tilde{\mathfrak{G}}$ is regarded as an n-dimensional euclidean space.

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